Math 215B HW 2 Solutions

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1 Grassmannian Smooth Structure

Let $E \in Gr_k(\mathbb{R}^n)$ be a k-plane. Introduce the Euclidean metric and consider the canonical decomposition $E \times E^\perp \simeq \mathbb{R}^n$ given in the problem. Any k-plane close to E is the graph of a linear function $f_E : E \to E^\perp$. Clearly, the space of such functions can be identified with $\mathrm{Mat}_{k,n-k}(\mathbb{R}) \approx \mathbb{R}^{k(n-k)}$. Therefore, for each E, we have constructed a neighborhood U_E and maps f_E for which $f : U_E \to \mathbb{R}^{k(n-k)}$ is a homeomorphism. All transition maps are just linear changes of coordinates [1]. Smoothness follows from linearity and we are done.

2 Complex Projective Space

This problem comes down to evaluating the transition maps. Suppose we have two charts U_i, U_j (i-th entry nonzero and j-th entry nonzero respectively). We want to compute the chart map $\phi_i \circ \phi_j^{-1}$ on the intersection $U_i \cap U_j$ (i-th and j-th entry nonzero). This is an explicit computation:

$$\begin{aligned}
&\phi_i \circ \phi_j^{-1}(w_0, w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n) \\
&= \phi_i[w_0, w_1, \ldots, w_{j-1}, 1, w_{j+1}, \ldots, w_n] \\
&= (w_0/w_i, w_1/w_i, \ldots w_{i-1}/w_i, w_{i+1}/w_i, \ldots, w_{j-1}/w_i, 1/w_i, w_{j+1}/w_i, \ldots, w_n/w_i)
\end{aligned} \tag{1}$$

This function is clearly well-defined and smooth when $w_i \neq 0$ (which is always true for $[w_0, \ldots, w_n] \in U_i \cap U_j$). We can do the exact same computation for $\phi_j \circ \phi_i^{-1}$ and conclude that the inverse map is smooth. The set of charts (U_i, ϕ_i) thus define a smooth atlas for $\mathbb{CP}^n$. Taking the maximal smooth atlas containing this atlas gives the smooth structure.

3 Quaternionic Projective Space

Recall that the quaternionic projective space consists of equivalence classes under the following equivalence relation: $[q_0, \ldots, q_n] \sim [cq_0, \ldots, cq_n]$ for all $c \in \mathbb{H}$ and $c \neq 0$. We can build charts as for

[1] To see this, we can take a passive perspective, and regard two planes $E, E' \in U_E \cap U_{E'}$ as the same plane expressed in different coordinates

the complex case:

$$\phi_i : U_i \to \mathbb{H}^n \quad \phi_i([q_0, \ldots, q_n]) = (\frac{q_0}{q_i}, \ldots, \frac{q_{i-1}}{q_i}, \frac{q_{i+1}}{q_i}, \ldots, \frac{q_n}{q_i}) \tag{2}$$

The transition maps are algebraic functions of the quaternionic variables and hence smooth functions of the coordinates in $\mathbb{R}^{4n}$ identified with $\mathbb{H}^n$. This is the desired smooth structure.

4 Two Definitions of a Tangent Bundle

Denote by TM_1 the bundle defined via equivalence classes of triples and denote by TM_2 the bundle defined as a submanifold of $\mathbb{R}^L \times \mathbb{R}^L$. For any triple $[x, \alpha, v]$, we can find a curve $\gamma_v : [-\epsilon, \epsilon] \to \mathbb{R}^n$ such that $\phi_\alpha^{-1}(\gamma_v(0)) = x$, $\frac{d\gamma_v}{dt}(0) = v$. We can construct the bundle map

$$f : TM_1 \to TM_2 \quad f([x, \alpha, v]) = (x, \frac{d\phi_\alpha^{-1}\gamma_v}{dt}(0)) \tag{3}$$

Where the derivative on the RHS makes sense if we consider $\phi_\alpha^{-1}\gamma_v$ as a curve embedded in $\mathbb{R}^L$. If we picked a different representative $[x, \beta, u]$, then we must have

$$\frac{d\gamma_u}{dt}(0) = u = D(\phi_\beta\phi_\alpha^{-1})_{\phi_\alpha(x)}(v) = D(\phi_\beta\phi_\alpha^{-1})_{\phi_\alpha(x)}(\frac{d\gamma_v}{dt}(0)) \tag{4}$$

But that immediately implies

$$\frac{d\phi_\beta^{-1}\gamma_u}{dt}(0) = D(\phi_\alpha^{-1})_{\phi_\alpha(x)}\frac{d\gamma_v}{dt}(0) = \frac{d\phi_\alpha^{-1}\gamma_v}{dt}(0) \tag{5}$$

f is hence well-defined. Next we show that f is an isomorphism on the fibers. Linearity is obvious by linearity of derivatives. To construct an inverse, we define γ to be a curve $[-\epsilon, \epsilon] \to M \subset \mathbb{R}^L$ with $\gamma(0) = x$ and $\frac{d\gamma}{dt} = v$. Now let

$$f^{-1}(x, v) = [x, \alpha, \frac{d\phi_\alpha\gamma}{dt}(0)]\,. \tag{6}$$

It is clear that $f \circ f^{-1} = f^{-1} \circ f = I$. Hence f induces a vector bundle isomorphism.

5 $\mathbf{Vect}^n(X)$ and $\mathbf{Prin}^{GL_n(\mathbb{R})}(X)$

Suppose that $p : E \to B$ is a n-dimensional vector bundle over X with trivializations $\{U_\alpha, \phi_\alpha\}$. The clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ of the vector bundle can then be used as clutching functions of the principal bundle E_{GL} with total space $E_{GL} = \bigcup_\alpha U_\alpha \times GL_n(\mathbb{R})/\sim$. The equivalence relation on the principal bundle identifies $(x, g) \in U_\alpha \times GL_n(\mathbb{R})$ and $(x, g\phi_{\alpha,\beta}(x)) \in U_\beta \times GL_n(\mathbb{R})$. Conversely, if we start with a principal bundle $p : E_{GL} \to B$ with trivializations $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. We can define the associated vector bundle $E = \bigcup_\alpha U_\alpha \times \mathbb{R}^n/\sim$ where $\sim$ identifies $(x, v) \in U_\alpha \times \mathbb{R}^n$ with $(x, v\psi_{\alpha,\beta}(x)) \in U_\beta \times \mathbb{R}^n$. This canonical correspondence proves a bijective correspondence between isomorphism classes of real vector bundles and principal $GL_n(\mathbb{R})$ bundles.

6 Several Diffeomorphisms

(a) $Gr_2(\mathbb{R}^3)$ is the collection of 2-planes in $\mathbb{R}^3$. Equip $\mathbb{R}^3$ with the Euclidean metric, then 2-planes (through the origin) in $\mathbb{R}^3$ are in bijective correspondence with normal lines passing through the origin. Since $\mathbb{RP}^2$ is precisely the space of lines through the origin, we conclude that $\mathbb{RP}^2$ and $Gr_2(\mathbb{R}^3)$ are diffeomorphic.

(b) We first show that $\mathbb{RP}^3$ is diffeomorphic to $\tilde{D}^3$, the solid ball in $\mathbb{R}^3$ with antipodal points on the boundary identified. Since $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$, we can represent each point in $\mathbb{RP}^3$ uniquely as a point in the union of the upper hemisphere ($z_0 > 0$) and the equatorial S^2 with antipodal points identified. Now foliate D^3 as a family of S^2's, each at a distance $0 \leq r < \pi$ away from the origin. Consider the map $f(z_0, z_1, z_2, z_3) = (r = \sqrt{\pi^2 - z_0^2}, z_1, z_2, z_3)$. Since $z \in S^3$, we have $z_1^2 + z_2^2 + z_3^2 = \pi^2 - z_0^2$. Hence, f is a diffeomorphism between the open D^3 and the interior of $\mathbb{RP}^3$. The boundary of $\mathbb{RP}^3$ consists precisely of points $(0, z_1, z_2, z_3)$ with antipodal points identified. Therefore, f also maps the boundary diffeomorphically to the boundary $\partial D^3 \approx S^2$ with antipodal points identified.

On the other hand, $SO(3)$ is the space of all possible rotations of the sphere S^2. Each rotation can be described as a rotation of angle θ around an axis v (represented as a unit vector) through the origin. We now consider the map $g : SO(3) \to \tilde{D}^3$ sending $(v, \theta) \to (\theta, \theta v)$ with inverse $g^{-1}(\theta, u) = (\frac{u}{\theta}, \theta)$. One can check that g, g^{-1} are both smooth, making g a diffeomorphism.

(c) Many of you noticed that $Gr_2^+(\mathbb{R}^4)$ admits a natural action of $SO(4)$ (via left multiplication) with isotropy group $SO(2) \times SO(2)$ that are independent rotations of the first two and last two coordinates. Since the $SO(4)$ action is transitive, $Gr_2^+(\mathbb{R}^4)$ is diffeomorphic to $SO(4)/SO(2) \times SO(2)$. After this step, most people did some handwaving and claimed that this quotient is diffeomorphic to $S^2 \times S^2$. Some of you identified $SO(4)$ with $SU(2) \times SU(2)/\{(e,e), (-e,-e)\}$ and then claimed that

$$\frac{SO(4)}{SO(2) \times SO(2)} = \frac{SU(2) \times SU(2)}{U(1) \times U(1)} \tag{7}$$

as if the quotient by $\{(e,e), (-e,-e)\}$ didn't exist, and then obtained two copies of S^2 by two Hopf fibrations. I don't think this is correct. Below I outline two solutions that I think are more justifiable.

(c1)[2] Every oriented plane $Gr_2^+(\mathbb{R}^4)$ is specified uniquely by two orthonormal unit vectors $u, v \in \mathbb{R}^4$ which are identified with unit quaternions in $\mathbb{H}$. Given u, v, we can show that $I = v\bar{u}$ satisfies a few nice properties. (1) Since $v\bar{v} = \bar{v}v = u\bar{u} = \bar{u}u = 0$, and since u, v are orthogonal, we have that $(v+u)(\bar{v}+\bar{u}) = 2 + u\bar{v} + v\bar{u} = 2$. Hence $u\bar{v} = -v\bar{u}$. Using this relation we see that $Iu = v\bar{u}u = v$ and $Iv = v\bar{u}v = -u\bar{v}v = -u$. Hence, I is a $\pi/2$ rotation that preserves the oriented plane defined by (u, v). (2) $I^2 = v\bar{u}v\bar{u} = -u\bar{v}v\bar{u} = -u\bar{u} = -1$. In fact, $I = v\bar{u}$ is the unique pure quaternion (no real

[2] This solution is due to a math stackexchange answer:
https://math.stackexchange.com/questions/1828484/grassmanian-2-4-homeomorphic-to-s2-times-s2.
Here I flesh out the stackexchange answer a bit more for clarity.

component) with $I^2 = -1$ that fixes the oriented plane (u, v). Since the space of pure quaternions satisfying $I^2 = -1$ is diffeomorphic to S^2, we thus obtain a map from $Gr_2^+(\mathbb{R}^4) \to S^2$. The fiber of this map consists of all oriented planes that invariant under I, which is precisely $\mathbb{CP}^1 \approx S^2$. Hence, the map induces on $Gr_2^+(\mathbb{R}^4)$ a fiber bundle structure of S^2 over S^2. To prove this bundle is trivial, we provide an explicit mapping.

First consider the following map $f : S^3 \times S^2 \to Gr_2^+(\mathbb{R}^4)$ given by $f(x, I) = (x, Ix)$. Here x is a unit quaternion and I is a pure quaternion with $I^2 = -1$. By our previous argument, this map is surjective but not injective. $f(x, I) = f(y, J)$ implies $(x, Ix) = (y, Jy)$ which means that $y \in (x, Ix)$ and $J = I$. Hence, we need to quotient out by an S^1 in S^3 in order to make this map injective. This quotient is precisely the Hopf fibration and in the end we get a map:

$$f([x, I]) = (x, Ix) \quad f : S^2 \times S^2 \to Gr_2^+(\mathbb{R}^4) \tag{8}$$

This map is bijective and smooth by construction. Hence it is the desired diffeomorphism.

(c2) This alternative solution is mostly due to Alec Lau. It is pretty clever and doesn't involve quaternions at all (I think it's helpful to include a correct solution that only uses elementary methods). Every plane in $\mathbb{R}^4$ can be represented by a 2×4 matrix with linearly independent rows. We define the oriented Grassmannian space $Gr_2^+(\mathbb{R}^4)$ to be the set of all equivalence classes $[A]$ under the relation $A \sim B$ if $A = CB$ and C is a 2×2 matrix with $\det C > 0$. Note that in the standard Grassmannian, we only require $\det C \neq 0$.

We now introduce the notation

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} = [v_1, v_2, v_3, v_4] \tag{9}$$

where v_i represents the i-th column of the matrix representative of $Gr_2^+(\mathbb{R}^4)$. Denote by $|v_i, v_j|$ the determinant of the 2×2 matrix formed by v_i, v_j. One can check that if $[w_1, w_2, w_3, w_4] = C[v_1, v_2, v_3, v_4]$, then $|w_i, w_j| = \det C|v_i, v_j|$. With this property in mind, we can write down the following smooth map from $Gr_2^+(\mathbb{R}^4) \to S^5 \subset \mathbb{R}^6$

$$f([v_1, v_2, v_3, v_4]) = \frac{(|v_1, v_2|, |v_3, v_4|, |v_1, v_3|, |v_2, v_4|, |v_1, v_4|, |v_2, v_3|)}{\sqrt{|v_1, v_2|^2 + |v_3, v_4|^2 + |v_1, v_3|^2 + |v_2, v_4|^2 + |v_1, v_4|^2 + |v_2, v_3|^2}}\,. \tag{10}$$

If $w = Av$ with $\det A > 0$, then $f(w) = \frac{\det A}{\det A} f(v) = f(v)$. Hence the map is well-defined. If $f(w) = f(v)$, then $w_{ij} = \lambda v_{ij}$ for some $\lambda > 0$. After some algebra, one can find a matrix A with $\det A = \lambda$ such that $w = Av$. We therefore conclude that f is injective. But this map is not quite surjective onto S^5. We need a further constraint. Note that for any v, we always have

$$|v_1, v_2| \cdot |v_3, v_4| - |v_1, v_3| \cdot |v_2, v_4| + |v_1, v_4| \cdot |v_2, v_3| = 0\,. \tag{11}$$

Therefore, the image of f is contained in the subset $\Gamma = \{(x, y, z, w, u, v)|x^2+y^2+z^2+w^2+u^2+v^2 = 1, xy - zw + uv = 0\}$. Let's now show that f surjects onto Γ. By definition, at least one of the six coordinates are nonvanishing. WLOG, let $x \neq 0$. Then it is easy to see that:

$$f\left(\left[\begin{pmatrix} x & 0 & -v & -w \\ 0 & 1 & z/x & u/x \end{pmatrix}\right]\right) = (x, \frac{-uv + wz}{x}, z, w, u, v) = (x, y, z, w, u, v) \tag{12}$$

Therefore f is in fact a bijection between $Gr_2^+(\mathbb{R}^4)$ and Γ. This map is smooth because it descends from a smooth map from $M_{2,4} \to \Gamma$ via a smooth projection. The inverse is smooth for the same reason. Therefore, f is a diffeomorphism. Now we just have to show that $\Gamma = S^2 \times S^2$. That can be achieved by a simple change of coordinates:

$$2x = a + b \quad 2y = a - b \quad 2z = d + c \quad 2w = d - c \quad 2u = e + f \quad 2v = e - f \tag{13}$$

Plugging this change of coordinates into the defining equation for Γ, we find two constraints

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = 2 \quad a^2 + c^2 + e^2 = b^2 + d^2 + f^2 \tag{14}$$

Rearranging these equations, we recover $a^2 + c^2 + e^1 = b^2 + d^2 + f^2 = 1$. This is precisely the defining equation of $S^2 \times S^2 \subset \mathbb{R}^6$.