(1) For a smooth manifold \( M \), define the \emph{diagonal} to be the space
\[
\Delta_M = \{(x, y) \in M \mid x = y\}.
\]
The smooth structure on \( M \) induces a smooth structure on \( \Delta_M \) via the homeomorphism
\[
M \xrightarrow{\sim} \Delta_M, \ x \mapsto (x, x)
\]
(a) Show that with this smooth structure, \( \Delta_M \hookrightarrow M \times M \) is a smooth embedding.

(b) Prove that there is a diffeomorphism \( \Delta_T M \cong T(\Delta_M) \), i.e. the tangent bundle of the diagonal is diffeomorphic to the diagonal of the tangent bundle.

(c) Let \( A \subset M \) be a submanifold, and let \( f : M \rightarrow N \) be a smooth map. Prove that \( f \) is transverse to \( A \) if and only if the product map
\[
f \times \iota_A : N \times A \rightarrow M \times M
\]
is transverse to \( \Delta_M \), where \( \iota_A \) is the inclusion of \( A \).

\textbf{Solution.} (a) Since the smooth structure on the diagonal is obtained by identification with \( M \), it is enough to show that the map \( \delta_M : M \rightarrow M \times M, x \mapsto (x, x) \) is a smooth embedding.

The smooth maps from a manifold \( Z \) into a product of two manifolds \( X \times Y \) are exactly the maps of the form \((\varphi_X, \varphi_Y) : z \mapsto (\varphi_X(z), \varphi_Y(z))\), where \( \varphi_X : Z \rightarrow X \), and \( \varphi_Y : Z \rightarrow Y \) are smooth. This can be checked locally on charts, and thus reduces to the case of euclidean spaces, where it is true by definition. In particular the map \( \delta_M = (\text{id}_M, \text{id}_M) \) is smooth.

Note that \( \pi_1 \circ \delta_M = \text{id}_M \), where \( \pi_1 : M \times M \rightarrow M \) is the projection onto the first factor. Since \( \pi_1 \) is continuous, this implies that \( \delta_M \) is a homeomorphism onto its image. Since \( \pi_1 \) is also smooth, it follows by the chain rule that \( \delta_M \) is an immersion. Thus it is a smooth embedding.

(b) We have \( \Delta_T M \cong T M \), by choice of the smooth structure. By functoriality of the tangent space construction the diffeomorphism \( M \xrightarrow{\sim} \Delta_M \) induces a diffeomorphism \( T M \xrightarrow{\sim} T(\Delta_M) \). Thus \( \Delta_T M \cong T M \cong T(\Delta_M) \).

(c) Suppose that \( f \) is transverse to \( A \), i.e. for every \( x \in M \cap f^{-1}(A) \), we have \( T_yN = T_yA + Tf_x(T_xM) \), where \( y = f(x) \). Let now \((x, y) \in N \times A \cap (f \times \iota_A)^{-1}(\Delta_M) \) and \((w_1, w_2) \in T_{f(x)}M \times T_yM \cong T_{(f(x), y)}(M \times M) \) be arbitrary. Then in particular \( y = f(x) \). Since \( f \) and \( A \) are transversal, we can write \( w_1 - w_2 = u + Tf_x(v) \) for \( u \in T_yA \) and \( v \in T_xN \). Then \( w_1 - Tf_x(v) = w_2 + u \) call this vector \( w \), so that \( w_1 = w + Tf_x(v) \) and \( w_2 = w - u \). Then \((w_1, w_2) = (w, w) + (Tf_x(v), -u) = (w, w) + (f \times \iota_A)(x, y)(v, -u) \in T_{(y, y)}\Delta_M + T(f \times \iota_A)(x, y)\).
This shows that $\Delta_M$ is transverse to $f \times \iota_A$.

Now we prove the converse by assuming transversality of $\Delta_M$ and $f \times \iota_A$. Let $x \in N$ be arbitrary, $y = f(x)$, and consider any $w \in T_y M$. Consider $(w, 0) \in T_{(y,y)} (M \times M)$. By assumption, we can write $(w, 0) = (w_1, w_1) + (T_x f(v), u)$ for some $w_1 \in T_y M, v \in T_x N$ and $u \in T_y A$. Since $w_1 + u = 0$, we have $w_1 \in T_y A$. Thus $w = w_1 + T_x f(v) \in T_y A + T f_x (T_x N).

This shows that $f$ and $A$ are transverse.

**Hirsch, p.27 Exercise 4:** Any product of spheres can be embedded into Cartesian space of one dimension higher.

**Solution.** We claim that for all integers $r \geq 1$ and $n_1, \ldots, n_r \geq 0$, there exists a smooth embedding $S^{n_1} \times \cdots \times S^{n_r} \to \mathbb{R}^{n_1 + \cdots + n_r + 1}$.

The idea is to use the embeddings $\nu_n : S^n \times \mathbb{R} \to \mathbb{R}^{n+1}, (x, t) \mapsto e^t x$ successively on all the spheres. This already covers the $r = 1$ case. We prove the general case by induction on $r$. So suppose $r > 1$ and we are given $n_1, \ldots, n_r \geq 0$. For brevity, let $n := n_1 + \ldots + n_r$. By our inductive assumption, applied to the last $r-1$ factors, there exists an embedding $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r} \to S^{n_1} \times \mathbb{R}^{n-n_1+1}$.

But then $\nu_{n_1} \times \text{id}_{\mathbb{R}^{n-n_1}}$ gives an embedding $S^{n_1} \times \mathbb{R}^{n-n_1+1} \to \mathbb{R}^{n_1+1} \times \mathbb{R}^{n-n_1} \simeq \mathbb{R}^{n+1}$.

Composing these two embeddings gives us what we wanted.

**Hirsch, p.32 Exercise 4:** Let $A \subset N$ be a neat $C^r$ submanifold, $r > 0$. Let $f : (M, \partial M) \to (N, \partial N)$ be a $C^r$ map. Suppose every point of $A$ is a regular value of $f$. Then $f^{-1}(A)$ is a neat $C^r$ submanifold. (Note: The condition that $f : \partial M \to \partial N$ has regular values along $\partial A$ as required in the book is redundant, it would be relevant if we only required that $f$ is transversal to $A$.)

**Solution.** Say $\dim M = m, \dim N = n$ and $\dim A = k$. The statement can be checked locally around every $x \in f^{-1}(A)$. Suppose there exists such an $x$. By passing to suitable charts at $x$ and $y := f(x)$, we can immediately reduce to the case that $x$ is the origin in $\mathbb{R}^m$, that $M$ is either $\mathbb{R}^m$ or the half space $\mathbb{R}^{m-1} \times \mathbb{R}_{\geq 0}$ and similarly that $N = \mathbb{R}^n$ or $N = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$, that $y$ is the origin of $\mathbb{R}^n$ and that $A = \mathbb{R}^k \times \{0\}^{n-k} \cap N$ (here we use that $A$ is neat). Now by definition of a smooth function on a subset of $\mathbb{R}^n$, even if $M$ is only a half space, there exists an open ball $B$ around the origin in $\mathbb{R}^m$ to which $f$ can be extended to a smooth function $B \to \mathbb{R}^n$. Fix one such extension. The assumption that $y$ is a regular value of $f$ implies that the extended function will have $x$ as a regular point. Since $f$ takes boundary to boundary, clearly $x \partial M$ implies $y \in \partial N$. On the other hand, by regularity, $f$ maps a small neighborhood around $x$ onto a neighborhood of $y$ (use for example the implicit function theorem). It follows that $x$ lies on the boundary if and only if $y$ does.

In the case that both of them are interior points we are done by the implicit function theorem.
Now suppose that they lie on the boundary, which by our setup implies that $M$ and $N$ are half spaces. Since $x$ is a regular point of the extension of $f$ to $B$, by possibly shrinking $B$, we can assume it is a submersion. Thus $f|B^{-1}(\partial N)$ is an $n-1$-dimensional submanifold of $B$, and contains $B\cap \partial M$, which is also an $n-1$ dimensional submanifold. Thus by shrinking $B$ further if necessary, we can assume that $f^{-1}(\partial N) = B \cap \partial M$. Note that this implies that the interior of $M \cap B$ is exactly one connected component of $B \setminus \partial M$.

By the implicit function theorem, there is a smaller ball $x \in B' \subset B$, and a diffeomorphism $\Phi : B \cong V \subset \mathbb{R}^m$, with $\Phi(x) = 0$, and such that $f \circ \Phi^{-1} : V \to \mathbb{R}^n$ is the projection onto the last $n$ coordinates. By construction, $\Phi(\partial M) = (f \circ \Phi^{-1})^{-1}(\partial N) = (\{0\}^{m-1} \times \mathbb{R}) \cap V$. Since $\Phi$ is a diffeomorphism, $V \setminus \Phi(\partial M)$ has two connected components, one of which is the image of the interior of $M$ and we see that it consists exactly of the points of $V$ with positive last coordinate. Thus $\Phi(M) = \mathbb{R}^{m-1} \times \mathbb{R}_{\geq 0} \cap V$, and the preimage of $A$ under $f \circ \Phi^{-1}$ is exactly $\mathbb{R}^{m-n} \times \mathbb{R}^k \times \{0\}^{m-n+k} \cap V$. This shows that $f^{-1}(A)$ is a neat submanifold locally around $x$. 

3