Math 215B: Homework 2

Due Thursday, January 23, 2020

(1). Show that the Grassmannian $Gr_k(\mathbb{R}^n)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ (or k-planes of $\mathbb{R}^n$) can be given an atlas as follows. Let $E \subset \mathbb{R}^n$ be a k-plane and $E^\perp$ its orthogonal complement. Identify $\mathbb{R}^n$ with $E \times E^\perp$. Every k-plane near enough to E is the graph of a unique linear map $E \to E^\perp$. In this way a neighborhood of $E \in Gr_k(\mathbb{R}^n)$ is mapped homeomorphically onto an open set in the vector space of linear maps $E \to E^\perp$. Show that this makes $Gr_k(\mathbb{R}^n)$ a differentiable manifold of dimension $k(n-k)$.

(2). Define the complex projective space $\mathbb{CP}^n$ to be the space of equivalence classes $[z_0, ..., z_n]$ of $(n+1)$-tuples of complex numbers, not all 0. The equivalence relation is: $[z_0, ..., z_n] \sim [wz_0, ..., wz_n]$ if w is a nonzero complex number. The topology is the natural quotient space topology. Show that $\mathbb{CP}^n$ has an atlas $(\{\phi_i, U_i), i = 0, ..., n\}$ defined as follows. Let U_i be the set of equivalence classes whose i^{th} entry is nonzero. Map U_i into $\mathbb{C}^n$ by $[z_0, ..., z_n] \to (z_0/z_i, \cdots, z_{i-1}/z_i, z_{i+1}/z_i, \cdots z_n/z_i)$. Under the natural identification of complex n-space $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, then show that these maps form an atlas on $\mathbb{CP}^n$, making it into a $2n$-dimensional differentiable manifold.

(3). Describe an atlas for Quaternionic projective n-space $\mathbb{HP}^n$, constructed as in Exercise 2, using quaternions instead of complex numbers. Show that it is a differentiable $4n$-dimensional manifold.

(4). Prove that the two definitions of tangent bundle given on p. 43 of the text are equivalent, when the manifold M^n is a submanifold of $\mathbb{R}^L$. By "equivalent" I mean that the vector bundles they define are isomorphic.

(5). Let $Vect^n(X)$ be the set of isomorphism classes of n-dimensional real vector bundles over X, and let $Prin^{GL_n(\mathbb{R})}(X)$ be the set of isomorphism classes of principal $GL_n(\mathbb{R})$-bundles over X. Show that $Vect^n(X)$ and $Prin^{GL_n(\mathbb{R})}(X)$ are in bijective correspondence.

(6). (a) Show that the manifold $Gr_2(\mathbb{R}^3)$ of 2-dimensional subspaces of $\mathbb{R}^3$ is diffeomorphic to real projective space $\mathbb{RP}^2$.

(b) Show that $\mathbb{RP}^3$ is diffeomorphic to $SO(3)$.

(c) Show that the manifold of oriented 2-dimensional subspaces of $\mathbb{R}^4$ (supply the definition) is diffeomorphic to $S^2 \times S^2$.