Math 215B HW 1 Solutions

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1 Poincare Duality

We first note the following relation between cup and cap products:

$$\langle \phi \frown \psi; \sigma \rangle = \langle \sigma \smile \phi; \psi \rangle \quad \to \quad \langle \sigma \smile \cdot; [M^n] \rangle = \langle \cdot \frown [M^n]; \sigma \rangle \tag{1}$$

Now consider the sequence of maps

$$H^{n-k}(M;F) \xrightarrow{h} \text{Hom}(H_{n-k}(M;F), F) \xrightarrow{D^*} \text{Hom}(H^k(M;F), F)\,, \tag{2}$$

where definitions of h and D^* are explained in Hatcher's book. By the relation between cup and cap products,

$$D^* \circ h(\sigma) = D^*(\langle \sigma; \cdot \rangle) = \langle \sigma; [M^n] \frown \cdot \rangle = \pm \langle \sigma \smile \cdot; [M^n] \rangle\ . \tag{3}$$

Hence showing that the pairing is nonsingular is equivalent to showing that $D^* \circ h$ is an isomorphism. But since F is a field, h is an isomorphism by UCT, and D^* is an isomorphism because it is the hom dual of the isomorphism map that appears in Poincare duality.

2 Compactly Supported Cohomology

We derive the result by the following chain of isomorphisms

$$\begin{aligned} H_c^*(X;G) &\simeq \varinjlim H^*(X, X - K_r; G) \simeq \varinjlim H^*(X \cup \infty, X \cup \infty - K_r; G) \\ &\simeq H^*(X \cup \infty, \infty; G) = \tilde{H}^*(X \cup \infty; G) \end{aligned} \tag{4}$$

where the first and fourth isomorphisms are by definition, the second isomorphism is by excision of ∞ [1] and the third isomorphism is by contractibility of $X \cup \infty - K_r$ for sufficiently large r.

[1] Since $X \cup \infty - K_r$ is contractible, the interior of $X \cup \infty - K_r$ contains the closure of ∞ for any sequence of K_r. Excision hence applies.

3 Covering Space Smooth Structure

We first note that the covering space of a manifold is still a manifold (i.e. the covering space inherits second-countability+Hausdorff)[2]. To get a smooth structure, we simply take a set of charts (ϕ_i, U_i) for X with $\{U_i\}$ evenly covered, and lift them to charts $\{\tilde{U}_{i,a}\}$ in $\tilde{X}$ with a labeling the sheets of the covering space. Smoothness of transition maps follows immediately and π is an immersion because it is a local diffeomorphism with the smooth structure defined above.

4 Submanifold Structure

(a) View $Mat_{n,n}(\mathbb{R})$ as $\mathbb{R}^{n^2}$. Define $\det : \mathbb{R}^{n^2} \to \mathbb{R}$. det is a polynomial (in particular continuous). Therefore $\det^{-1}(U)$ is open whenever U is open. In particular, $\det^{-1}(\mathbb{R} - \{0\}) = GL_n(\mathbb{R})$ is open. But open subsets of $\mathbb{R}^k$ are automatically submanifolds of the same dimension. Therefore, $GL_n(\mathbb{R})$ is a smooth submanifold of dimension n^2.

(b) $SL_n(\mathbb{R}) = \det^{-1}(1)$. For an arbitrary $A \in SL_n(\mathbb{R})$, we can compute the directional derivative

$$\nabla_A A = \lim_{\epsilon \to 0} \frac{\det A(1+\epsilon)^n - \det A}{\epsilon} = \det A \lim_{\epsilon \to 0} \frac{1 + n\epsilon + O(\epsilon^2) - 1}{\epsilon} = n \det A = n\,. \tag{5}$$

Hence, for all A, the map $\det_* : T\mathbb{R}^{n^2} \to T\mathbb{R}$ (differential/pushforward of det) is surjective. Hence 1 is a regular value, and by the regular value theorem, $SL_n(\mathbb{R})$ is a submanifold of codimension 1 in $\mathbb{R}^{n^2}$.

(c) We work instead with $O(n) = \{M \in Mat_{n,n}(\mathbb{R}) | MM^T = I\}$. Since $SO(n)$ is just one of the two path components of $O(n)$, if we can show $O(n)$ is a submanifold, then $SO(n)$ is a submanifold of the same dimension. Now consider the map $f : \mathbb{R}^{n^2} \approx \mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2} \approx \mathrm{Sym}_{n,n}(\mathbb{R})$ defined by $f(A) = AA^T$. At some arbitrary element A with $f(A) = I$, we have:

$$\nabla_X f(A) = \lim_{\epsilon \to 0} \frac{(A+\epsilon X)(A+\epsilon X)^T - AA^T}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon(tAX^T + tXA^T)}{\epsilon} = AX^T + XA^T \tag{6}$$

By taking $X = \frac{SA}{2}$ for some symmetric S, $\nabla_X f(A)$ spans $\mathrm{Sym}_{n,n}(\mathbb{R})$ for all $A \in f^{-1}(I)$. Therefore, I is a regular value of f and $O(n)$ has codimension equal to $\dim(\mathrm{Sym}_{n,n}(\mathbb{R}) = \frac{n(n+1)}{2}$. Hence, $\dim SO(n) = \dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

[2] if you want to know the proof, consult for example Lee's monograph on smooth manifolds

5 Real Projective Space

(a) Let $x \in S^n$ label the element $[x] \in \mathbb{RP}^n$. Collect the components $f_{ij}([x])$ into a matrix f. Then

$$f_{ij}([x]) = x_i x_j = f_{ji}([x]) \quad \sum_j f_{ij}([x]) f_{jk}([x]) = x_i x_k \sum_j x_j^2 = x_i x_k = f_{ik}([x]) \quad \operatorname{tr} f = \sum_i x_i^2 = 1\,. \tag{7}$$

This means $f(\mathbb{RP}^n)$ is a subset of $C = \{m \in \mathrm{Sym}_{n+1,n+1}(\mathbb{R}) | m^2 = m, \operatorname{tr} m = 1\}$. Now we show f is bijective. For injective, suppose $f([y]) = f([x])$, then diagonal components give constraints $y_i^2 = x_i^2 \forall i$, implying $y_i = \pm x_i$. But suppose that $y_i = x_i, y_j = -x_j$, then $x_i x_j \neq y_i y_j$ leading to a contradiction. Hence, $\pm$ needs to be consistently chosen for all i. However, since representatives x, y should only be taken from a half-sphere, we conclude that there is in fact a unique sign choice and f is injective. To show surjectivity, note that for any m with $m^2 = m$ and $\operatorname{tr} m = 1$, $m = O^T \Lambda O$ where $O \in SO(n+1)$ and $\Lambda = \mathrm{diag}(1, 0, \dots, 0)$. By carrying out the matrix multiplication, we see that $m = f([x])$ where $x_i = O_{1i}$. $O \in SO(n+1)$ guarantees that $\sum_i O_{1i}^2 = 1$, and $x \in S^n$. This concludes surjectivity. In fact, this computation gives an explicit map $g : C \to S^n$ such that $\pi \circ g$ is the inverse of f. By construction, $f \circ \pi, g$ are polynomial functions of $x \in \mathbb{R}^n$ and hence smooth. By the general theory of smooth manifolds, when we restrict the domain of $f \circ \pi$ to a submanifold S^n we get a smooth map. Similarly, when we restrict the range of g to a submanifold S^n, we get a smooth map (it is important that S^n is embedded. Restriction of the range to an immersed submanifold generally kills smoothness). Since $\pi : S^n \to \mathbb{RP}^n$ is a covering map (hence a local diffeomorphism) and f is bijective, we conclude that f is in fact a diffeomorphism.

(b) For every $A \in C$, $\sum_{ij} A_{ij}^2 = \operatorname{tr} A^2 = \operatorname{tr} A = 1$. Hence C has bounded Euclidean distance from the origin of $\mathbb{R}^{(n+1)^2}$ and must be bounded. C is obviously closed since the three defining properties are preserved under taking limits. By Heine-Borel, C is compact. Since compactness is preserved under diffeomorphisms, $\mathbb{RP}^n$ is compact.

(c) Suppose $\zeta \xrightarrow{\pi} B$ is trivial, then there exists a bundle isomorphism $\Phi : \mathbb{R}^n \times B \to \zeta$. Define a set of sections $\sigma_i(x) = (e_i, x)$ where $\{e_i\}$ is a complete basis in $\mathbb{R}^n$. Then $\{\sigma_i(x)\}$ inherits linear independence from $\{e_i\}$. Conversely, if there is a set of linearly independent sections $\sigma_i(x) : B \to \zeta$, then define $\Phi : \mathbb{R}^n \times B \to \zeta$ via $\Phi(v, x) = \sum_i v_i \sigma_i(x)$. Fiberwise, this is a linear isomorphism since $\{\sigma_i(x)\}$ is linearly independent. Hence Φ is a bundle isomorphism.