Math 215B: Homework 1

Due Thursday, January 16, 2018

(1) Show that the Poincareé Duality theorem implies that if F is a field and M^n is a closed F-oriented manifold with fundamental class $[M^n] \in H_n(M^n; F)$, then the pairing

$$H^k(M^n; F) \times H^{n-k}(M^n; F) \longrightarrow F \qquad (1)$$
$$\phi \times \psi \to \langle \phi \cup \psi, [M^n] \rangle$$

is nonsingular for every $k = 0, \cdots, n$.

(2) Show that

$$H_c^*(\mathbb{R}^n; G) \cong \tilde{H}^*(S^n; G)$$

and more generally if X is a topological space so that in its one-point compactification $X \cup \infty$, the point ∞ has a contractible open neighborhood, then

$$H_c^*(X; G) \cong H^*(X \cup \infty; G).$$

Here H_c^* is the cohomology with compact supports.

(3) Let $\pi : \tilde{X} \to X$ be a covering space. Let Φ be a smooth structure on X. Prove that there is a smooth structure $\tilde{\Phi}$ on $\tilde{X}$ so that $\pi : (\tilde{X}, \tilde{\Phi}) \to (X, \Phi)$ is an immersion.

(4) Prove that the following are submanifolds of the space of $n \times n$ matrices, $Mat_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Compute their dimensions.

1. $GL_n(\mathbb{R})$
2. $SL_n(\mathbb{R})$
3. $SO(n)$.

(5). (a) Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \to \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.

(b) Use the above to show that $\mathbb{RP}^n$ is compact.

(c) Prove that an n -dimensional vector bundle ζ has n - linearly independent sections if and only if ζ is trivial.