Math 215B

Take-home Final Exam March 5, 2020

Instructions. You are welcome to use the results from the books or class. If you have any questions about the exam, you may e-mail me or ask me in person. Discussion of the problems with anyone else is not permitted. Please send your solutions in a pdf file to me via email (rlc@stanford.edu) by 5:00 pm, Thursday, March 12.

Good luck!

Name ______________________________

(1). (10 pts) Prove the following theorem. (**Note.** This theorem was stated in class but it was not proved. Your job is to prove it now.)

Theorem 1. *Let $Q^q \subset M^n$ be a smooth, closed, oriented submanifold of the closed, oriented manifold M^n. Let $f : P^p \to M^n$ be a smooth map, where P^p is also a smooth, closed, oriented manifold. Suppose furthermore that $f \pitchfork Q^q$. Let*

$$D_M[Q] \in H^{n-q}(M^n;\mathbb{Z})$$

be the Poincaré dual of $[Q] \in H_q(M^n;\mathbb{Z})$. Then

$$f^*(D_M[Q]) \cap [P^p] = [f^{-1}(Q)] \in H_{p+q-n}(P^p;\mathbb{Z}).$$

(2). (a) (10 pts) A theorem of Hopf states that if X is a path connected space of the homotopy type of a CW complex, and it is endowed with a basepoint, then there is an isomorphism,

$$\begin{aligned}[X, S^1]_\bullet &\xrightarrow{\cong} H^1(X;\mathbb{Z}) \\ f &\to f^*(\sigma)\end{aligned}$$

where $\sigma \in H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ is a generator, and $[X, S^1]_\bullet$ denotes the based homotopy classes of basepoint preserving maps from X to S^1.

Let M^n be a closed, oriented, connected n-dimensional manifold with basepoint $x_0 \in M^n$. Suppose $\alpha \in H^1(M;\mathbb{Z})$. Let $f_\alpha : M \to S^1$ represent α via Hopf's theorem. Let $N = f_\alpha^{-1}(t)$ where $t \in S^1$ is a regular value of f_α. Show that the homology class $[N] \in H_{n-1}(M)$ is Poincaré dual to $\alpha \in H^1(M)$.

(b) (10 pts) Prove, using Hopf's theorem, the following theorem of Thom: If M^n is a closed, orientable manifold, then any homology class in $H_{n-1}(M^n)$ is represented by the fundamental class of a smooth codimension one, closed, oriented submanifold.

(3). Let M^n be a smooth, closed, oriented manifold of dimension n. Consider the diagonal embedding,

$$\begin{aligned}\Delta_M : M &\hookrightarrow M \times M \\ x &\longrightarrow (x, x).\end{aligned}$$

Let ν_{Δ_M} be the normal bundle of this embedding.

(a). (10 pts) Show that there is an isomorphism of vector bundles over M,

$$\nu_{\Delta_M} \cong TM,$$

where TM is the tangent bundle of M.

(b). (10 pts) Let $\tau : M \times M \to T(\nu_{\Delta_M})$ be the Thom collapse map. Here $T(\nu_{\Delta_M})$ is the Thom space of the normal bundle. Consider the composition map in homology,

$$\phi : H_p(M^n;\mathbb{Z}) \times H_q(M^n;\mathbb{Z}) \xrightarrow{\times} H_{p+q}(M \times M;\mathbb{Z}) \xrightarrow{\tau_*} H_{p+q}(T(\nu_{\Delta_M});\mathbb{Z}) \xrightarrow{\cap u} H_{p+q-n}(M^n;\mathbb{Z}).$$

The first map in this sequence is the cross product, and the last map in this sequence is the Thom isomorphism in homology, given by capping with the Thom class. Show that this composition map ϕ is equal, up to sign, to the intersection product:

$$\phi(\alpha, \beta) = \pm \alpha \cdot \beta.$$

(4). Recall that $\mathbb{RP}^n = \{(x_1, \cdots x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}\}/\sim$ where $(x_1, \cdots x_{n+1}) \sim -(x_1, \cdots x_{n+1})$. We denote an equivalence class using square brackets $[x_1, \cdots x_{n+1}] \in \mathbb{RP}^n$.

Define a smooth function

$$f : \mathbb{RP}^n \to \mathbb{R}$$

by

$$f([x_1, \cdots x_{n+1}]) = \sum_{k=1}^{n+1} k x_k^2$$

(a) (10 pts) Show that the critical points of f are $u_1, \cdots u_{n+1}$, where $u_i = [0, \cdots 0, 1, 0, \cdots 0]$, where the 1 occurs in the i^{th} coordinate.

(**Hint.** First construct charts U_i, $i = 1, \cdots, n+1$, where $U_i = \{[x_1, \cdots x_{n+1}] : x_i \neq 0\}$, by proving that there are diffeomorphisms $\psi_i : U_i \cong B_1^n$, where B_i^n the unit open ball around the origin in $\mathbb{R}^n$. ψ_i given by

$$\psi_i[x_1, \cdots x_{n+1}] = (x_1, \cdots x_{i-1}, x_{i+1}, \cdots, x_n).$$

Then compute the differential of the composition

$$B_i^n \xrightarrow{\psi_i^{-1}} U_i \subset \mathbb{RP}^n \xrightarrow{f} \mathbb{R}.$$

Use this to show that the only critical point of f in U_i is u_i.

(b). (10 pts) Compute the index of each critical point.

(c). (10 pts) Show that $f : \mathbb{RP}^n \to \mathbb{R}$ is a Morse function.

(d). (10 pts) Using parts (a) - (c) to show that the Euler characteristic of $\mathbb{RP}^n$ is 0 if n is odd and 1 if n is even.

(e) (10 pts) Prove that if n is even, $\mathbb{RP}^n$ does not admit a nowhere zero vector field.