Math 120 Homework 5 Solutions

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6.) Let H be a characteristic subgroup of G. Then H is mapped to itself by all automorphisms of G. In particular H is mapped to itself by all inner automorphisms, hence is normal. To see that a normal subgroup need not be characteristic, consider the subgroup $\langle i \rangle < Q_8$, where Q_8 denotes the group of quaternions (see Dummit and Foote p. 36 for a definition). We have $|i| = 4$ so $\langle i \rangle$ is an index two subgroup of Q_8, hence is normal. However, by the symmetry of the group definition, we see that there is an automorphism σ of Q_8 with $\sigma(i) = j$, $\sigma(j) = k$, $\sigma(k) = i$. The automorphism σ carries $\langle i \rangle$ to $\langle j \rangle$, so $\langle i \rangle$ is not characteristic.

8.) a. Let $g \in G$ and denote by σ_g the automorphism of G obtained via conjugation by g. Since K is normal in G, by proposition 13 (page 133) σ_g restricts to an automorphism of K. Then σ_g fixes H because all automorphisms of K fix H. It follows that conjugation by any $g \in G$ fixes H so H is normal in G.

b. Let $\sigma \in Aut(G)$. Then K characteristic in G implies σ fixes K, hence restricts to an automorphism of K. Since H is characteristic in K, it follows that the restriction of σ to K fixes H. Hence all automorphisms of G fix H and H is characteristic in G. To prove that V_4 is characteristic in S_4, take $G = S_4$, $K = A_4$ and $H = V_4$ in the above. Now any automorphism of S_4 must map the 3-cycles to themselves, since these are the only elements of order 3 in S_4. But the 3-cycles generate A_4, so any automorphism of S_4 maps A_4 to A_4, since it just permutes the generators. To see that V_4 is characteristic in A_4, note that in A_4, $(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$ are the only elements of order 2, since the 2-cycles have odd order. This implies that an automorphism of A_4 maps the generators of V_4 among themselves, hence maps V_4 to itself.

c. Let $G = S_4$, $K = V_4$ and $H = \{1, (1\ 2)(3\ 4)\}$. Then by the above, V_4 is characteristic in S_4 and H is normal in V_4 because it is index 2. But H is not normal in G. Indeed, we know that all elements of the same cycle type are conjugate so we can conjugate H to the cyclic subgroup generated by $(1\ 3)(2\ 4)$ (for instance, by conjugating by $(2\ 3)$).

9.) The subgroup $\langle r \rangle$ is normal in D_{2n} because it is index 2. Let $\langle r^i \rangle$ be a subgroup of $\langle r \rangle$. By a theorem on cyclic groups (Theorem 7, page 58 in Dummit and Foote), $\langle r^i \rangle$ is the unique subgroup of its order contained in $\langle r \rangle$, hence is characteristic in $\langle r \rangle$ (i.e. because the image of $\langle r^i \rangle$ under any automorphism must be a subgroup of $\langle r \rangle$ and have the same cardinality). It follows that $\langle r^i \rangle$ is normal in D_{2n} by part a of the previous problem.

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5.) Let p^e be the largest power of p dividing $2n$. Since p does not divide 2, p^e also divides n. Write $n = p^e m$ with $(p, m) = 1$. Then since r has order $p^e m = n$, r^m has order p^e, so $\langle r^m \rangle$ is a group of order p^e, hence is a cyclic Sylow-p subgroup of D_{2n}. Since all Sylow-p subgroups are conjugate, all are isomorphic, hence all cyclic. Since all elements of form sr^i have order 2 and so cannot generate a cyclic subgroup of order p^e, it follows that all Sylow-p subgroups of D_{2n} are generated by a power of r, hence are contained in $\langle r \rangle$. But $\langle r \rangle$ is cyclic, hence has only one subgroup of any given order, so that $\langle r^m \rangle$ is the only subgroup of D_{2n} of order p^e, and hence is normal by Sylow's theorem.

14.) Note $312 = 8 \times 3 \times 13$. We'll show that a group of order 312 has a Sylow-13 subgroup that is normal. Note that the number of Sylow-13 subgroups is 1 mod 13 and divides $312/13 = 24$. The only numbers that are 1 mod 13 less than 24 are 1 and 14. Of these, only 1 divides 24, so the number of Sylow-13's is 1 and it is normal by Sylow's theorem.

24.) Note $231 = 3 \times 7 \times 11$. Let G be a group of order 231. The number of Sylow-11 subgroups in G is 1 mod 11 and divides 21. The numbers less than or equal to 21 which are 1 mod 11 are 1 and 12. Of these, only 1 divides 21, so there is 1 Sylow-11 subgroup of G and it is normal. Call it H. To see that H is contained in the center of G, let G act on H by conjugation. This gives a homomorphism $\phi : G \to Aut(H)$. The quotient of G by $\ker \phi$ is congruent to the image of ϕ in $Aut(H)$. Hence the order of the image divides the order of $G = 231$. But $Aut(H) \cong (Z_{11})^\times$ and this has order $\phi(11) = 10$ by Dummit and Foote's Proposition 16, p. 135. Since the image of ϕ is a subgroup of $Aut(H)$, it has order dividing 10. It follows that the order of $im(\phi)$ divides $(231, 10) = 1$, so the image is just the identity and ϕ is trivial. It follows that all elements of G commute with all elements of H, so $H \leq Z(G)$.

To prove that G has a normal 7-Sylow, note that the number of Sylow-7 subgroups of G is 1 mod 7 and divides 33. The four divisors of 33 are 1, 3, 11 and 33. Of these, only 1 is 1 mod 7, so there is a single 7-Sylow and it is normal.

30.) Note $168 = 8 \times 3 \times 7$. Thus the number of 7-Sylows contained in a simple group G of order 168 is 1 mod 7 and divides 24, but is greater than 1 because otherwise the single 7-Sylow would be normal. The numbers that are 1 mod 7, greater than 1 and less than 24 are 8, 15, and 22, and of these, only 8 divides 24. Thus there are 8 7-Sylows. Each 7-Sylow has order 7, hence is cyclic, having six non-identity elements of order 7 and the identity of order 1. If two 7-Sylow's have non-identity x in their intersection, then x has order 7, hence generates both 7-Sylows, so that they are equal. It follows that the 8 distinct 7-Sylows intersection pairwise only at the identity, so that they contribute $8 \times 6 = 48$ elements of order 7. But any element of order 7 generates a cyclic group of order 7, that is, a 7-Sylow, so all elements of order 7 have been accounted for as belonging to one of the 8 7-Sylows. Thus G has exactly 48 elements of order 7.

46.) The order of S_{p^2} is $(p^2)!$. There are $p-1$ factors of p in $(p^2)!$ accounted for by the numbers $p, 2p, \ldots, (p-1)p$ and two more factors of p accounted for by p^2. It follows that $(p^2)!$ has exactly p^{p+1} factors of p, and the size of a Sylow-p subgroup of S_{p^2} is p^{p+1}. To find such a subgroup, first consider the subgroup H generated by the following p, p-cycles:

$$\sigma_1 = (1\ 2\ \ldots\ p),\ \sigma_2 = (p+1\ \ldots 2p),\ \ldots,\ \sigma_p = (p^2 - p + 1\ \ldots\ p^2).$$

These p-cycles are disjoint, so commute and thus H is abelian. Note that σ_i is not contained in $\langle \sigma_1, \ldots, \sigma_{i-1} \rangle$ because each of $\sigma_1, \ldots, \sigma_{i-1}$ fixes ip while σ_i does not, so that an easy induction argument gives $|\langle \sigma_1, \ldots, \sigma_i \rangle| = p^i$ and $|H| = p^p$.

Now let τ be the product of p cycles

$$\tau = (1\ p+1\ \ldots\ p^2 - p + 1)(2\ p+2\ \ldots\ p^2 - p + 2) \ldots (p\ 2p\ \ldots\ p^2).$$

We have

$$\tau \sigma_i \tau^{-1} = \tau(ip - p + 1\quad ip - p + 2\ \ \ldots\ \ ip)\tau^{-1} = (ip+1\ \ ip+2\ \ \ldots\ \ ip+p) = \sigma_{i+1},$$

and similarly

$$\tau^{-1} \sigma_i \tau = \sigma_{i-1}$$

where the subscripts are taken modulo p. In particular, τ is contained in the normalizer of H, since conjugation by τ just permutes the generators of H. But then $\langle \tau \rangle H$ is a subgroup of G, of order $\frac{|\tau||H|}{|H \cap \langle \tau \rangle|}$. The order of τ is p. The order of $H \cap \langle \tau \rangle$ divides p, but is less than p since $\tau \notin H$, (i.e. because τ does not commute with the generators of H and H is abelian). It follows that $\langle \tau \rangle H$ is a group of order p^{p+1}, so is a p-Sylow. It is generated by $\tau, \sigma_1, \ldots, \sigma_p$, and it is non-abelian.