Math 120 Homework 4 Solutions

May 1, 2008

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7.) Let G denote the group of rigid motions of a tetrahedron. Label the locations of the four vertices l_1, l_2, l_3 and l_4. Label the four vertices v_1, v_2, v_3, v_4. For a given orientation O of the tetrahedron, let $\pi_O : \{1,2,3,4\} \to \{1,2,3,4\}$ be defined by $\pi(i)$ is the label of the location at which vertex i resides. Then we can define a map $\Pi : G \to S_4$ by $\Pi(g) = \pi_O$ where O is the orientation induced on the tetrahedron when rigid motion g is applied to it, starting from the initial position in which vertex i is in location i for each i. Suppose that we apply successively rigid motion g_1 and then g_2 (which results in $g_2 g_1$). The location of vertex i after the first motion is $\Pi(g_1)(i)$. After the second motion, v_i is carried to the point that $v_{\Pi(g_1)(i)}$ would be under g_2, that is $\Pi(g_2) \circ \Pi(g_1)(i)$, and so we see that Π is a homomorphism. A rigid motion is uniquely determined by where it carries the four vertices, and hence we have that G is isomorphic to its image under Π in S_4.

We can place the tetrahedron in $\mathbb{R}^3$ by setting $l_1 = (1,0,0)$, $l_2 = (-1/2, \sqrt{3}/2, 0)$, $l_3 = (-1/2, -\sqrt{3}/2, 0)$, $l_4 = (0,0,\sqrt{2})$. Then note that under the identity rigid motion,

$$[(v_2 - v_1) \times (v_3 - v_1)] \cdot (v_4 - v_1) > 0$$

where $\times$ denotes cross product and $\cdot$ denotes dot product. Moreover, this product is an invariant of the rigid motion. To see this, note that the product is six times the signed volume of the tetrahedron. We can imagine moving the tetrahedron through space. Our product varies continuously with motion of the tetrahedron, but it can take on only two values, six times the volume and its negative. Since it varies continuously but its range is discrete, it must be constant. Now if we interchange only the locations of v_1 and v_2, the volume product becomes negative. Hence (1 2) is not in the image of G under Π. It follows that $Im(\Pi)$ is a proper subgroup of S_4. Furthermore, all 3-cycles appear, i.e. by fixing one vertex and rotating the other three. Two non-commuting 3-cycles generate A_4 (see the lattice diagram of A_4 in the text) and hence the image contains A_4. Since A_4 is index 2, it is not contained in a proper subgroup. Since the image of Π is proper, it must be exactly A_4.

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4.) Let $a_1 = (1,1), a_2 = (1,2), a_3 = (1,3), a_4 = (2,1), a_5 = (2,2), a_6 = (2,3), a_7 = (3,1), a_8 = (3,2)$ and $a_9 = (3,3)$. We have $a_1 \sim a_5 \sim a_9$ since $(1\ 2)a_1 = a_5$ and $(1\ 3)a_1 = a_9$. It is impossible for a_1 to be carried to any other tuple under the action since both elements of a_1 will be mapped to the same value by a permutation, while all other tuples will have two distinct value. The remaining a_i fall into the same orbit. To see this, suppose $i_1 \neq i_2$ and $j_1 \neq j_2$. Let π satisfy $\pi(i_1) = j_1$, $\pi(i_2) = j_2$. Then $\pi(i_1, i_2) = (j_1, j_2)$ so $(i_1, i_2) \sim (j_1, j_2)$.

We have the following association of elements of S_3 with elements of S_9:

S_3	S_9
id	id
(1 2)	(1 5)(2 4)(3 6)(7 8)
(1 3)	(1 9)(2 8)(3 7)(4 6)
(2 3)	(2 3)(4 7)(5 9)(6 8)
(1 2 3)	(1 5 9)(2 6 7)(3 4 8)
(1 3 2)	(1 9 5)(2 7 6)(3 8 4)

From the orbit of doubled pairs, the stabilizer of $(1, 1)$ is those permutations fixing 1, that is $\{id, (2\ 3)\}$. From the orbit of distinct pairs, the stabilizer of $(1, 2)$ is those permutations fixing 1 and 2, that is $\{id\}$.

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2.) The representation in S_6 is given as follows

S_3	S_6
id	id
(1 2)	(1 2)(3 5)(4 6)
(1 3)	(1 4)(2 5)(3 6)
(2 3)	(1 3)(2 6)(4 5)
(1 2 3)	(1 5 6)(2 4 3)
(1 3 2)	(1 6 5)(2 3 4)