Math 120 Homework 2 Solutions

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10.) By theorem 7, $\langle\overline{30}\rangle = \langle\overline{(30,54)}\rangle = \langle\overline{6}\rangle$ and since $\overline{30}$ and $\overline{6}$ generate the same cyclic group, they must have equal orders. But $|\overline{6}| = 9$ since all smaller multiples of 6 lie between 0 and 54 while $9 \cdot \overline{6} = \overline{54} = \overline{0}$.

13.) The proof is the same in both a and b: The groups $\mathbb{Z} \times Z_2$ and $\mathbb{Q} \times Z_2$ each contain the element $(0,1)$ of order 2, while neither $\mathbb{Z}$ nor $\mathbb{Q}$ contains an element of order 2 (such an element would be equal to its negative, hence would be 0 and have order 1).

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6.) The collection of elements $id, (1\ 2), (3\ 4), (1\ 2)(3\ 4)$ forms a group (as can be checked by making the group multipication table) and contains (1 2) and (1 2)(3 4). Any group containing these elements must contain both the identity and (3 4), hence this is the group generated by (1 2) and (1 2)(3 4). The exhibited group has order 4, but every non-identity element has order 2, hence the group is not cyclic.

14.) a. Let A consist of the set of elements in the group (this is finite). Then these elements generate a subgroup of G, since they are contained in G, but any subgroup containing all of them contains G. Hence $\langle A\rangle = G$ and G is finitely generated.

b. $\mathbb{Z} = \langle 1\rangle$ since for $m \in \mathbb{Z}$, $m = m \cdot 1$ if $m > 0$ and $m = |m| \cdot (-1)$ if $m < 0$ (while $0 = 1 - 1$).

c. Let $H = \langle q_1, ..., q_r\rangle \leq \mathbb{Q}$ and write $q_i = m_i/n_i$ with $(m_i, n_i) = 1$. Let $k = n_1 \ldots n_r$. We claim $H \leq \langle \frac{1}{k}\rangle$. Indeed, any element x of H is generated by $q_1, \ldots, q_r$, and as $\mathbb{Q}$ is abelian, is of form $c_1 q_1 + \ldots + c_r q_r$ with $c_i \in \mathbb{Z}$ for each i. Hence $x = \sum_i \frac{c_i m_i}{n_i} = \sum_i \frac{c_i m_i \hat{n}_i}{k} = \frac{1}{k}\sum_i c_i m_i \hat{n}_i$ where $\hat{n}_i = \frac{k}{n_i}$. But $\sum_i c_i m_i \hat{n}_i \in \mathbb{Z}$ so $x \in \langle \frac{1}{k}\rangle$. Then H is a subgroup of a cycle group, hence is cyclic.

d. We'll show that $\mathbb{Q}$ is not cyclic. Suppose $\mathbb{Q} = \langle q\rangle$ with $q = \frac{m}{n}$. Then for all $x \in \mathbb{Q}$ we would have $x = cq = \frac{cm}{n}$ so that the denominator of x (in lowest terms) divides n. Take $x = \frac{1}{n+1}$ for a contradiction.

15.) Take H the subgroup of reduced fractions having odd denominators. H is a subgroup of $\mathbb{Q}$ (see HW 1, p.21, #6a). H is not cyclic for the same reason that $\mathbb{Q}$ is not cyclic, i.e. because the denominators of fractions in H are not bounded.

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7.) Every element of D_{16} may be (uniquely) represented as $x = s^i r^n$ with $i = 0$ or 1 and $0 \leq n < 8$. First consider x with $i = 1$. Then $rx = rsr^n = sr^{-1}r^n = sr^{n-1}$ while $xr = sr^n r = sr^{n+1}$ so $rx \neq xr$ and $x \notin Z$. Next consider $x = r^n$. Then x commutes with r. We have $sx = sr^n$ while $xs = r^n s = sr^{-n}$ so if $x \in Z$ then $r^n = r^{-n}$ which implies $r^{2n} = 1$ or $n = 0$ or $n = 4$. Since r^n commutes with both r and s if $n = 0$ or 4, it is in the center (i.e. because r, s generate) so $Z = \{1, r^4\}$.

12.) Every subgroup of order 2 in A is cyclic, so is generated by a single element of order two, hence is one of $\langle a \rangle$, $\langle ab^2 \rangle$ or $\langle b^2 \rangle$. The only subgroup of order 1 is $\langle 1 \rangle$. Since $|A| = 8$ every subgroup of A has order 1, 2, 4, or 8. Thus given the three subgroups of order 4, we know all the subgroups. They are: $\langle 1 \rangle$, of order 1, $\langle a \rangle, \langle ab^2 \rangle, \langle b^2 \rangle$, of order 2, $\langle a, b^2 \rangle \cong V_4$, $\langle b \rangle \cong Z_4$ and $\langle ab \rangle \cong Z_4$ of order 4 and $A = \langle a, b \rangle$ of order 8. Now $\langle ab \rangle$ and $\langle b \rangle$ share the lone element b^2 of order 2, hence $\langle b^2 \rangle$ is the only subgroup of order 2 contained in these groups. Meanwhile, $\langle a, b^2 \rangle$ contains a, ab^2 and b^2 so it contains all three subgroups of order 2. It follows that in the lattice diagram, we have A at the top. In the next level below A are $\langle a, b^2 \rangle$, $\langle ab \rangle$ and $\langle b \rangle$. In the third level, $\langle a \rangle$, $\langle ab^2 \rangle$ and $\langle b^2 \rangle$ are all below $\langle a, b^2 \rangle$ while $\langle b^2 \rangle$ is also below $\langle ab \rangle$ and $\langle b \rangle$. Finally $\langle 1 \rangle$ is in the fourth level, under all three subgroups of order 2.

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1.) Take $x, y \in \phi^{-1}(E)$ with $x = \phi^{-1}(x')$ and $y = \phi^{-1}(y')$. Then $\phi(xy^{-1}) = \phi(x)\phi(y)^{-1} = x'(y')^{-1} \in E$ since $x', y' \in E$ and E is a subgroup. Hence $xy^{-1} \in \phi^{-1}(E)$ so $\phi^{-1}(E)$ is a subgroup by the subgroup criterion. Now suppose that E is normal in H. Take $g \in G$ and $x \in \phi^{-1}(E)$ with $\phi(x) = x' \in E$. Then $\phi(gxg^{-1}) = \phi(g)x'\phi(g)^{-1} \in E$ since $x' \in E$ and E normal. Hence $gxg^{-1} \in \phi^{-1}(E)$ so $g\phi^{-1}(E)g^{-1} \subset \phi^{-1}(E)$. To show the equality, observe that for $y \in \phi^{-1}(E)$, $g^{-1}yg = x \in \phi^{-1}(E)$ (by above) so $gxg^{-1} = y$ and $g\phi^{-1}(E)g^{-1} \supset \phi^{-1}(E)$. Thus $\phi^{-1}(E)$ is normal. Since $\ker \phi = \phi^{-1}(0)$ and 0 is normal in H, we get $\ker \phi$ normal in G.

3.) B is normal since A is abelian. Take $x, y \in A$. Then $\overline{x} + \overline{y} = \overline{x + y} = \overline{y + x} = \overline{y} + \overline{x}$, where the middle equality is computed in A. This proves A/B is abelian.

For an example of non-abelian G with abelian quotient, take $G = D_8$ and $N = \langle r^2, s \rangle \cong V_4$. Then N is normal because it is index 2 (the argument is, there is only one choice for both its left and right cosets). The group G/N has order 2, hence is cyclic, abelian. In fact, N is abelian so we have shown that we may have non-abelian G with normal subgroup N with both N and G/N abelian.