Math 120 Midterm Solutions

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1.) a. $N_G(S) = \{g \in G : gSg^{-1} = S\}$.

b. A normal subgroup is a subgroup H such that $N_G(H) = G$.

c. $G_a = \{g \in G : ga = a\}$.

2.) Proper subgroups of D_6 have order dividing 6 by Lagrange's theorem. Hence all proper subgroups have order 1, 2 or 3. The first case is the identity. The latter two cases have prime order, hence are cyclic. Thus the only proper subgroups of D_6 are cyclic. We have $(sr^i)^2 = s^2r^{-i}r^i = 1$ so these elements generate subgroups of order 2 consisting of the identity and the element. Meanwhile $\langle r\rangle = \langle r^2\rangle = \{1, r, r^2\}$ so the set of subgroups is

$$\{1, \langle r\rangle, \langle s\rangle, \langle sr\rangle, \langle sr^2\rangle, D_6\}.$$

3.) a. H contains 1 so H is not empty. Suppose $x, y \in H$ with $|x| = m$, $|y| = n$, $m, n < \infty$. We know $|y| = |y^{-1}|$ e.g. because $1 = y^k(y^{-1})^k$ so that $y^k = 1$ if and only if $(y^{-1})^k = 1$. Now then $(xy^{-1})^{mn} = x^{mn}y^{-mn}$ since G is abelian. But $x^{mn}y^{-mn} = (x^m)^n(y^{-n})^m = 1^n1^m = 1$. It follows that $|xy^{-1}|$ divides mn, and in particular, is finite. Thus $xy^{-1} \in H$ so H is a subgroup by the subgroup criterion.

b. Again, $1 \in K$ so K is non-empty. Our proof above shows that if $x, y \in K$ then $|xy^{-1}|$ divides $|x||y|$. Now $x, y \in K$ implies $|x||y|$ is odd. All factors of an odd number are odd, so $|xy^{-1}|$ is odd, and hence $xy^{-1} \in K$. It follows that K is a subgroup by the subgroup criterion.

c. E is not a subgroup because it does not contain 1.

4.) We have $\sigma(1) = 4$, $\sigma(4) = 2$, $\sigma(2) = 1$ so this forms one cycle. Also $\sigma(3) = 5$, $\sigma(5) = 6$, $\sigma(6) = 3$ so this forms the second cycle: $\sigma = (1\ 4\ 2)(3\ 5\ 6)$. The order of σ is the lcm of its cycle lengths, hence is 3.

5.) Since $A \cap B$ is a subgroup of A and B, $|A \cap B|$ divides the gcd of $|A|$ and $|B|$, by Lagrange's theorem. Hence $|A \cap B|$ is 1 or 3. Now we have the identity $|AB| = \frac{|A||B|}{|A\cap B|}$ (proposition 13, p. 93 of Dummit and Foote) so $|AB| = \frac{15\times 21}{|A\cap B|}$. An upper bound for $|AB|$ is $|G| = 105$ so we have $|A \cap B| \geq 3$. Hence $|A \cap B| = 3$ and $|AB| = 105$ which implies $G = AB$ and every element of G may be expressed as a product ab with $a \in A$ and $b \in B$.

6.) a. Let $g \in G$. The size of the conjugacy class $K(g)$ is $|G : C_G(g)|$ where $C_G(g)$ is the centralizer of g. Now by Lagrange's theorem, $|G : C_G(g)| = \frac{|G|}{|C_G(g)|}$. Note that every element of the center is contained in the centralizer of G, so $|C_G(g)| \geq |Z(G)|$ and hence $\frac{|G|}{|C_G(g)|} \leq \frac{|G|}{|Z(G)|} = |G : Z(G)| = n$.

b. We may write σ as $(1\ \sigma(1)\ \sigma^2(1)\ \sigma^3(1)\ \sigma^4(1))$ and hence its square is given by

$$(1\ \sigma^2(1)\ \sigma^4(1)\ \sigma^6(1)\ \sigma^8(1)) = (1\ 3\ 5\ 2\ 4),$$

where we use $\sigma^5 = 1$, while its inverse is given by

$$(1\ \sigma^{-1}(1)\ \sigma^{-2}(1)\ \sigma^{-3}(1)\ \sigma^{-4}(1)) = (1\ 5\ 4\ 3\ 2),$$

here using $\sigma^{-1} = \sigma^4$ since $\sigma^5 = 1$. Then by the identity

$$\tau\sigma\tau^{-1} = (\tau(1)\ \tau(\sigma(1))\ \tau(\sigma^2(1))\ \tau(\sigma^3(1))\ \tau(\sigma^4(1)))$$

we may choose τ mapping 1,2,3,4,5 to 1,3,5,2,4 to get $\tau\sigma\tau^{-1} = \sigma^2$, and we may choose τ mapping 1,2,3,4,5 to 1,5,4,3,2 to achieve $\tau\sigma\tau^{-1} = \sigma^{-1}$.