Professor R.L. Cohen June 3 2008

Math 120
Final Exam

Instructions. This is a take - home examination. Write out complete solutions to the following problems, while explaining all your steps. Make sure you justify all your arguments and statements. You may use your textbook, class notes, and may use or quote any result discussed in class or in the book. However you may not consult any other books or materials. You must work alone on this examination. The exam is due on Monday, June 7 at Bob Hough's office, 380-G by 9:30 am. No late examinations will be accepted. There are 200 points possible. *Good luck!*

Name ________________________________

1. (20)	____________	6. (25)	____________
2. (10)	____________	7. (20)	____________
3. (20)	____________	8. (20)	____________
4. (20)	____________	9. (20)	____________
5. (25)	____________	10. (20)	____________

Total ______________

1. (20 points) Let G be a finite group acting on a finite set A. In your second writing assignment you proved that if N is the number of orbits of this group action, then

$$N = \frac{1}{|G|} \sum_{g \in G} F(g)$$

where $F(g)$ is the number of elements in A that are fixed by g.

a. Suppose that g_1 and g_2 are conjugate elements in G. Prove that $F(g_1) = F(g_2)$.

b. A bracelet is made from five beads mounted on a circular wire. How many different bracelets can we manufacture if we have red, blue, and yellow beads at our disposal?

2. (10 points) Decompose the dihedral group D_{10} into conjugacy classes and write down the class equation for this group.

3. (20 points) Suppose that H and K are normal subgroups of a group G and that $H \cap K = \{1\}$.

a. Prove that if $h \in H$ and $k \in K$, then $hk = kh$.

b. Suppose that $G = HK$. Prove that G is isomorphic to $H \times K$.

4. (20 points) Give examples of the following types of groups or rings. Justify your assertions.

1. A non-abelian group of order 8.
2. Three non-isomorphic abelian groups of order 8.
3. A group of order 27 having the property that every element besides the identity has order 3.
4. A ring with precisely 4 units.

5. (25 points) a. Let p and q be primes with the property that $q = p + 2$. Such pairs of primes are called "twin primes". Let G be a group of order p^3q. Prove that G has a unique subgroup of order p^3, and a unique subgroup of order q.

b. Find generators for a Sylow p-subgroup of the symmetric group S_{2p}, where p is an odd prime. Show that this is an abelian group of order p^2.

6. (25 points) a. In each of parts (1) - (3), give the number of nonisomorphic abelian groups of the specified order - do not list the groups: **(1)** order 100 **(2)**order 576 **(3)** order 1155.

b. In each of parts (1) - (3), give the lists of invariant factors for all abelian groups of the specified order: **(a)** order 270, **(b)** order 9801, **(c)** order 320.

7. (20 points) a. The *center* of a ring R is $\{z \in R \,|\, zr = rz \,\text{for all}\, r \in R\}$. Prove that the center of a ring is a subring that contains the identity. Prove that the center of a division ring is a field.

b. Let G be a finite group that consists of the set $\{g_1, \cdots, g_n\}$. Prove that the element $N = g_1 + g_2 + \cdots + g_n$ is in the center of the group ring RG.

8. (20 points) a. Prove that the 2×2 matrix ring $M_2(\mathbb{R})$ contains a subring that is isomorphic (as rings) to $\mathbb{C}$.

b. Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

9. (20 points) Let $\phi : R \to S$ be a ring homomorphism.

a. Prove that if J is an ideal of S then $\phi^{-1}(J)$ is an ideal of R.

b. Prove that if ϕ is surjective and I is an ideal of R then $\phi(I)$ is an ideal of S. Give an example where this fails if ϕ is not surjective.

10. (20 points) Let $x^4 - 16$ be an element of the polynomial ring $E = \mathbb{Z}[x]$ and use the bar notation ot denote passage to the quotient ring $\mathbb{Z}[x]/(x^4 - 16)$.

a. Find a polynomial of degree ≤ 3 that is congruent to $7x^{13} - 11x^9 + 5x^5 - 2x^3 + 3$ modulo $(x^4 - 16)$.

b. Prove that $\overline{x-2}$ and $\overline{x+2}$ are zero divisors in $\bar{E}$.