1. (a) (12 pts) For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

   i. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
   This is a subspace of $\mathbf{F}^3$. To handle this and part iv) at the same time, let us consider the set

$$\{(x_1, x_2, x_3) \in \mathbf{F}^3 \ : \ ax_1 + bx_2 + cx_3 = 0\}$$

   and let us check that it is a subspace for any scalars a, b, c. $(0, 0, 0)$ is in this set because $a0 + 0b + 0c = 0$. If (x_1, x_2, x_3) and (y_1, y_2, y_3) are two vectors in this set, then

$$\begin{aligned} a(x_1 + y_1) + b(x_2 + y_2) + c(x_3 + y_3) &= (ax_1 + bx_2 + cx_3) + (ay_1 + by_2 + cy_3) \\ &= 0 + 0 = 0 \end{aligned}$$

   shows that it is closed under addition. Finally, let α be a scalar. This way,

$$a(\alpha x_1) + b(\alpha x_2) + c(\alpha x_3) = \alpha(ax_1 + bx_2 + cx_3) = \alpha \cdot 0 = 0$$

   and it is also closed under scalar multiplication.

   ii. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \ : \ x_1 + 2x_2 + 3x_3 = 4\}$.
   This is not a subspace, given that $(0, 0, 0)$ is not an element of this set.

   iii. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \ : \ x_1x_2x_3 = 0\}$
   This is not a subspace, given that it is not closed under addition. $(1, 1, 0)$ and $(0, 1, 1)$ both belong to this set, but their sum $(1, 2, 1)$ does not.

   iv. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 \ : \ x_1 = 5x_3\}$
   This is a subspace according to the analysis on part i).

   (b) (4 pts) Give an example of a nonempty subset $U \subseteq \mathbf{R}^2$ such that U is closed under addition and under taking inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

   Solution: Let

$$U = \{(a, b) \in \mathbf{R}^2 \mid a \in \mathbf{Z} \ \text{ and } \ b = 0\}$$

   that is, U is the set of elements of the form $(0, 0), (1, 0), (-1, 0), (2, 0), (-2, 0), \ldots$ This set is closed under addition and additive inverses (since these statements hold for $\mathbf{Z}$) but is not a subspace of $\mathbf{R}^2$. The reason being, that it is not closed under scalar multiplication:

$$\frac{1}{2} \in \mathbf{F}, \ \ (1, 0) \in U \ \text{ but } \ \frac{1}{2} \cdot (1, 0) = (1/2, 0) \notin U.$$

   □

2. (15 pts) Let $\mathcal{P}_n(\mathbf{F})$ be the space of all polynomials over $\mathbf{F}$ of degree less than or equal to n. Prove or disprove: there is a basis (p_0, p_1, p_2, p_3) of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

   Solution: This is true. Recall that $dim(\mathcal{P}_3(\mathbf{F})) = 4$ with natural basis $\{1, z, z^2, z^3\}$, so if we can find a list of 4 linearly independent polynomials of degree at most 3, this will give us a basis for $\mathcal{P}_3(\mathbf{F})$. Notice that if v_1, v_2, v_3, v_4 are linearly independent vectors in a vector space V, then so are $v_1, v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4$ (this is true not only for 4 but for any integer). To see this, suppose that there are scalars $a_1, \ldots, a_4 \in \mathbf{F}$ so that

$$0_V = a_1v_1 + a_2(v_1 + v_2) + a_3(v_1 + v_2 + v_3) + a_4(v_1 + v_2 + v_3 + v_4)$$

   This way

$$\begin{aligned} 0_V &= a_1v_1 + a_2(v_1 + v_2) + a_3(v_1 + v_2 + v_3) + a_4(v_1 + v_2 + v_3 + v_4) \\ &= (a_1 + a_2 + a_3 + a_4)v_1 + (a_2 + a_3 + a_4)v_2 + (a_3 + a_4)v_3 + a_4v_4 \end{aligned}$$

   and since $v_1, \ldots v_4$ are linearly independent, this yields the system of equations

$$\begin{aligned} 0 &= a_1 + a_2 + a_3 + a_4 \\ 0 &= a_2 + a_3 + a_4 \\ 0 &= a_3 + a_4 \\ 0 &= a_4 \end{aligned}$$

   which can be easily solved and implies that $a_1 = \cdots = a_4 = 0$.
   With this in mind, take $v_1 = z^3, v_2 = z^2, v_3 = z, v_4 = 1$. Given that $\{v_1, \ldots, v_4\}$ is a basis for $\mathcal{P}_3(\mathbf{F})$ then so is

$$\{z^3 \ , \ z^3 + z^2 \ , \ z^3 + z^2 + z \ , \ z^3 + z^2 + z + 1\}$$

   and the claim is proven since all of these polynomials have degree 3. □

3. (15 pts) Let V and W be vector spaces over $\mathbf{F}$, with V being finite dimensional. Suppose that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap Null(T) = \{0\}$ and $Range(T) = \{Tu : u \in U\}$.

   Solution: Since $Null(T)$ is a subspace of the finite dimensional space V, then there it has a linear complement U. That is, U is a subspace of V so that $Null(T) \oplus U = V$. This way, $Null(T) \cap U = \{0\}$ and every $v \in V$ can be written as $v = n + u$ for $n \in Null(T)$ and $u \in U$. Thus

$$\begin{aligned} Range(T) &= \{T(v) : v \in V\} \\ &= \{T(u+n) : n \in Null(T) \text{ and } u \in U\} \\ &= \{T(u) + T(n) : n \in Null(T) \text{ and } u \in U\} \\ &= \{T(u) : n \in Null(T) \text{ and } u \in U\} \\ &= \{T(u) : u \in U\} \end{aligned}$$

□

4. (16 pts)

(a) Let U be a the subspace of $\mathbf{F}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = x_2 = x_3, \text{ and } x_5 = 6x_4\}$$

Find a basis for U and compute its dimension.

Solution: Notice that $(x_1, \ldots, x_5) \in U$ if and only if

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= (x_1, x_1, x_1, x_4, 6x_4) \\ &= x_1(1, 1, 1, 0, 0) + x_4(0, 0, 0, 1, 6))\end{aligned}$$

and therefore

$$U = Span\{(1, 1, 1, 0, 0), (0, 0, 0, 1, 6)\}$$

For the set of vectors $(1, 1, 1, 0, 0), (0, 0, 0, 1, 6)$ to be a basis for U, the only thing we need to make sure is that they are linearly independent. Since neither of them is a scalar multiple of the other, we get that they are a basis for U and therefore $dim(U) = 2$. □

(b) Let U be as in part (a). Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose Null space is U.

Solution: Let us proceed by contradiction assuming that there exists a linear map $T : \mathbf{F}^5 \longrightarrow \mathbf{F}^2$, so that $Null(T) = U$. If this were the case, by the rank-Nullity theorem we would have that

$$\begin{aligned}dim(\mathbf{F}^5) &= dim(Null(T)) + dim(Range(T)) \\ 5 &= 2 + dim(Range(T))\end{aligned}$$

and therefore

$$3 = dim(Range(T)) \leq dim(\mathbf{F}^2) = 2$$

which is a contradiction. □

5. (18 pts) Let V and W be vector spaces over $\mathbf{F}$, with W being finite dimensional. Suppose that $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that the composition $S \circ T$ is the identity map on V.

Solution: For the first implication, let us assume that $T \in \mathcal{L}(V, W)$ is an injective linear transformation. Our goal is to construct a linear map $S \in \mathcal{L}(W, V)$ so that $S \circ T$ is the identity on V. Since by definition

$$T : V \longrightarrow Range(T)$$

is surjective, then by hypothesis it is also injective, and therefore invertible (Notice the change in the target space). Let $L : Range(T) \longrightarrow V$ be such inverse. Since W is finite dimensional, then by a homework problem it is possible to extend L to a linear transformation

$$S : W \longrightarrow V$$

so that $S(w) = L(w)$ for all $w \in Range(T)$. Notice that on $Range(T)$, S operates as the inverse of T and therefore

$$S \circ T(v) = L \circ T(v) = v$$

for all $v \in V$.

Let us now assume that there exists $S \in \mathcal{L}(W, V)$ so that $S \circ T$ is the identity on V and let us show that T must be injective. Recall that injectivity is the same as $Null(T) = \{0\}$. If $v \in Null(T)$ then $T(v) = 0_W$ and therefore

$$v = S \circ T(v) = S(Tv) = S(0_W) = 0_V$$

which shows that T is injective. $\square$