1. (a) (12 pts) For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

i. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

This is a subspace of $\mathbf{F}^3$. To handle this and part iv) at the same time, let us consider the set

$\{(x_1, x_2, x_3) \in \mathbf{F}^3 : ax_1 + bx_2 + cx_3 = 0\}$

and let us check that it is a subspace for any scalars $a, b, c$. $(0, 0, 0)$ is in this set because $a0 + 0b + 0c = 0$. If $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ are two vectors in this set, then

\[
a(x_1 + y_1) + b(x_2 + y_2) + c(x_3 + y_3) = (ax_1 + bx_2 + cx_3) + (ay_1 + by_2 + cy_3) = 0 + 0 = 0
\]

shows that it is closed under addition. Finally, let $\alpha$ be a scalar. This way,

\[
a(\alpha x_1) + b(\alpha x_2) + c(\alpha x_3) = \alpha(ax_1 + bx_2 + cx_3) = \alpha \cdot 0 = 0
\]

and it is also closed under scalar multiplication.

ii. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

This is not a subspace, given that $(0, 0, 0)$ is not an element of this set.

iii. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$

This is not a subspace, given that it is not closed under addition. $(1, 1, 0)$ and $(0, 1, 1)$ both belong to this set, but their sum $(1, 2, 1)$ does not.

iv. $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

This is a subspace according to the analysis on part i).

(b) (4 pts) Give an example of a nonempty subset $U \subseteq \mathbf{R}^2$ such that $U$ is closed under addition and under taking inverses (meaning $-u \in U$ whenever $u \in U$), but $U$ is not a subspace of $\mathbf{R}^2$.

Solution: Let

\[
U = \{(a, b) \in \mathbf{R}^2 \mid a \in \mathbf{Z} \text{ and } b = 0\}
\]

that is, $U$ is the set of elements of the form $(0, 0), (1, 0), (-1, 0), (2, 0), (-2, 0), \ldots$. This set is closed under addition and additive inverses (since these statements hold for $\mathbf{Z}$) but is not a subspace of $\mathbf{R}^2$. The reason being, that it is not closed under scalar multiplication:

\[
\frac{1}{2} \in \mathbf{F}, \ (1, 0) \in U \text{ but } \frac{1}{2} \cdot (1, 0) = (1/2, 0) \notin U.
\]

\[\Box\]
2. (15 pts) Let $P_n(F)$ be the space of all polynomials over $F$ of degree less than or equal to $n$. Prove or disprove: there is a basis $(p_0, p_1, p_2, p_3)$ of $P_3(F)$ such that none of the polynomials $p_0, p_1, p_2, p_3$ has degree 2.

Solution: This is true. Recall that $\dim(P_3(F)) = 4$ with natural basis \{1, $z$, $z^2$, $z^3$\}, so if we can find a list of 4 linearly independent polynomials of degree at most 3, this will give us a basis for $P_3(F)$. Notice that if $v_1, v_2, v_3, v_4$ are linearly independent vectors in a vector space $V$, then so are $v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4$ (this is true not only for 4 but for any integer). To see this, suppose that there are scalars $a_1, \ldots, a_4 \in F$ so that

$$0_V = a_1 v_1 + a_2(v_1 + v_2) + a_3(v_1 + v_2 + v_3) + a_4(v_1 + v_2 + v_3 + v_4)$$

This way

$$0_V = a_1 v_1 + a_2(v_1 + v_2) + a_3(v_1 + v_2 + v_3) + a_4(v_1 + v_2 + v_3 + v_4) = (a_1 + a_2 + a_3 + a_4)v_1 + (a_2 + a_3 + a_4)v_2 + (a_3 + a_4)v_3 + a_4 v_4$$

and since $v_1, \ldots v_4$ are linearly independent, this yields the system of equations

$$0 = a_1 + a_2 + a_3 + a_4$$
$$0 = a_2 + a_3 + a_4$$
$$0 = a_3 + a_4$$
$$0 = a_4$$

which can be easily solved and implies that $a_1 = \cdots = a_4 = 0$.

With this in mind, take $v_1 = z^3, v_2 = z^2, v_3 = z, v_4 = 1$. Given that $\{v_1, \ldots, v_4\}$ is a basis for $P_3(F)$ then so is

$$\{z^3, z^3 + z^2, z^3 + z^2 + z, z^3 + z^2 + z + 1\}$$

and the claim is proven since all of these polynomials have degree 3. □
3. (15 pts) Let $V$ and $W$ be vector spaces over $F$, with $V$ being finite dimensional. Suppose that $T \in L(V, W)$. Prove that there exists a subspace $U$ of $V$ such that $U \cap \text{Null}(T) = \{0\}$ and $\text{Range}(T) = \{Tu : u \in U\}$.

Solution: Since $\text{Null}(T)$ is a subspace of the finite dimensional space $V$, then there it has a linear complement $U$. That is, $U$ is a subspace of $V$ so that $\text{Null}(T) \oplus U = V$. This way, $\text{Null}(T) \cap U = \{0\}$ and every $v \in V$ can be written as $v = n + u$ for $n \in \text{Null}(T)$ and $u \in U$. Thus

$$\text{Range}(T) = \{T(v) : v \in V\}$$

$$= \{T(u + n) : n \in \text{Null}(T) \text{ and } u \in U\}$$

$$= \{T(u) + T(n) : n \in \text{Null}(T) \text{ and } u \in U\}$$

$$= \{T(n) : n \in \text{Null}(T) \text{ and } u \in U\}$$

$$= \{T(u) : u \in U\}$$

□
4. (16 pts)

(a) Let $U$ be a the subspace of $\mathbb{F}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = x_2 = x_3, \text{ and } x_5 = 6x_4\}$$

Find a basis for $U$ and compute its dimension.

**Solution:** Notice that $(x_1, \ldots, x_5) \in U$ if and only if

$$(x_1, x_2, x_3, x_4, x_5) = (x_1, x_1, x_1, x_4, x_4)$$

$$= x_1(1, 1, 1, 0, 0) + x_4(0, 0, 0, 1, 6)$$

and therefore

$$U = \text{Span}\{(1, 1, 1, 0, 0), (0, 0, 0, 1, 6)\}$$

For the set of vectors $(1, 1, 1, 0, 0), (0, 0, 0, 1, 6)$ to be a basis for $U$, the only thing we need to make sure is that they are linearly independent. Since neither of them is a scalar multiple of the other, we get that they are a basis for $U$ and therefore $\text{dim}(U) = 2$. □

(b) Let $U$ be as in part (a). Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose Null space is $U$.

**Solution:** Let us proceed by contradiction assuming that there exists a linear map $T : \mathbb{F}^5 \rightarrow \mathbb{F}^2$, so that $\text{Null}(T) = U$. If this were the case, by the rank-Nullity theorem we would have that

$$\text{dim}(\mathbb{F}^5) = \text{dim}(\text{Null}(T)) + \text{dim}(\text{Range}(T))$$

$$5 = 2 + \text{dim}(\text{Range}(T))$$

and therefore

$$3 = \text{dim}(\text{Range}(T)) \leq \text{dim}(\mathbb{F}^2) = 2$$

which is a contradiction. □
5. (18 pts) Let $V$ and $W$ be vector spaces over $F$, with $W$ being finite dimensional. Suppose that $T \in \mathcal{L}(V, W)$. Prove that $T$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that the composition $S \circ T$ is the identity map on $V$.

Solution: For the first implication, let us assume that $T \in \mathcal{L}(V, W)$ is an injective linear transformation. Our goal is to construct a linear map $S \in \mathcal{L}(W, V)$ so that $S \circ T$ is the identity on $V$. Since by definition

$$T : V \longrightarrow \text{Range}(T)$$

is surjective, then by hypothesis it is also injective, and therefore invertible (Notice the change in the target space). Let $L : \text{Range}(T) \longrightarrow V$ be such inverse. Since $W$ is finite dimensional, then by a homework problem it is possible to extend $L$ to a linear transformation

$$S : W \longrightarrow V$$

so that $S(w) = L(w)$ for all $w \in \text{Range}(T)$. Notice that on $\text{Range}(T)$, $S$ operates as the inverse of $T$ and therefore

$$S \circ T(v) = L \circ T(v) = v$$

for all $v \in V$.

Let us now assume that there exists $S \in \mathcal{L}(W, V)$ so that $S \circ T$ is the identity on $V$ and let us show that $T$ must be injective. Recall that injectivity is the same as $\text{Null}(T) = \{0\}$. If $v \in \text{Null}(T)$ then $T(v) = 0_W$ and therefore

$$v = S \circ T(v) = S(Tv) = S(0_W) = 0_V$$

which shows that $T$ is injective. □