On the homotopy invariance of string topology

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Abstract

Let $M^n$ be a closed, oriented, $n$-manifold, and $LM$ its free loop space. In [3] a commutative algebra structure in homology, $H_*(LM)$, and a Lie algebra structure in equivariant homology $H_{S^1}^*(LM)$, were defined. In this paper we prove that these structures are homotopy invariants in the following sense. Let $f : M_1 \rightarrow M_2$ be a homotopy equivalence of closed, oriented $n$-manifolds. Then the induced equivalence, $Lf : LM_1 \rightarrow LM_2$ induces a ring isomorphism in homology, and an isomorphism of Lie algebras in equivariant homology. The analogous statement also holds true for any generalized homology theory $h_*$ that supports an orientation of the $M_i$'s.

Introduction

The term “string topology” refers to multiplicative structures on the (generalized) homology of spaces of paths and loops in a manifold. Let $M^n$ be a closed, oriented, smooth $n$-manifold. The basic “loop homology algebra” is defined by a product

$$\mu : H_*(LM) \otimes H_*(LM) \rightarrow H_*(LM)$$

of degree $-n$, and and the “string Lie algebra” structure is defined by a bracket

$$[\cdot, \cdot] : H_{S^1}^*(LM) \otimes H_{S^1}^*(LM) \rightarrow H_{S^1}^*(LM)$$

of degree $2 - n$. These were defined in [3]. Here $H_{S^1}^*(LM)$ refers to the equivariant homology, $H_{S^1}^*(LM) = H_*(ES^1 \times S, LM)$. More basic structures on the chain level were also studied in [3]. Furthermore, these structures were shown to exist for any multiplicative homology theory $h_*$ that supports an orientation of $M$ (see [8]). Alternative descriptions of the basic structure were given in [8] and [4], but in the end they all relied on various perspectives of intersection theory of chains and homology classes.

The existence of various descriptions of these operations leads to the following:

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**Question.** To what extent are the string topology operations sensitive to the smooth structure of the manifold, or even the homeomorphism structure?

The main goal of this paper is to settle this question. We will prove more: we will show that the loop homology algebra and string Lie algebra structures are homotopy invariants.

To state the main result, let $h_*$ be a multiplicative homology theory that supports an orientation of $M$. Being a multiplicative theory means that the corresponding cohomology theory, $h^*$, admits a cup product, or more precisely, the representing spectrum of the theory is required to be a ring spectrum. An $h_*$-orientation of a closed $n$-manifold $M$ can be viewed as a choice of fundamental class $[M] \in h_n(M)$ that induces a Poincaré duality isomorphism.

**Theorem 1.** Let $M_1$ and $M_2$ be closed, $h_*$-oriented $n$-manifolds. $f : M_1 \to M_2$ be an orientation preserving homotopy equivalence. Then the induced homotopy equivalence of loop spaces, $Lf : LM_1 \to LM_2$ induces a ring isomorphism of loop homology algebras,

$$(Lf)_* : h_*(LM_1) \xrightarrow{\cong} h_*(LM_2)$$

and an isomorphism of graded Lie algebras,

$$(Lf)_* : h^{S^1}_*(LM) \xrightarrow{\cong} h^{S^1}_*(LM).$$

Evidence for the above theorem came from the results of [8] and [7] which said that for simply connected manifolds $M$, there is an isomorphism of graded algebras,

$$H_*(LM) \cong H^*(C^*(M), C^*(M)),$$

where the right hand side is the Hochschild cohomology of $C^*(M)$, the differential graded algebra of cochains on $M$, with multiplication given by cup product. The Hochschild cohomology algebra is clearly a homotopy invariant.

However, the above isomorphism is defined in terms of the Pontrjagin-Thom construction arising from the diagonal embedding $M^S \subset M^T$ associated with each surjection of finite sets $T \to S$. Consequently, since the Pontrjagin-Thom construction uses the smooth structure, this isomorphism a priori seems to be sensitive to the smooth structure. Without an additional argument, one can only conclude from this isomorphism that the loop homology algebra of two homotopy equivalent simply connected closed manifolds are abstractly isomorphic. In summary, to prove homotopy invariance in the sense of Theorem 1, one needs a different argument.

The argument we present here, which does not need the simple connectivity hypothesis, uses the description of the loop product $\mu$ in terms of a Pontrjagin-Thom collapse map of an embedding

$$LM \times_M LM \hookrightarrow LM \times LM$$
given in [8]. Here $LM \times_M LM$ is the subspace of $LM \times LM$ consisting of those pairs of loops $(\alpha, \beta)$, with $\alpha(0) = \beta(0)$. In this description we are thinking of the loop space as the space of piecewise smooth maps $[0, 1] \to M$ whose values at 0 and 1 agree. This is a smooth, infinite dimensional manifold. The differential topology of such manifolds is discussed in [9] and [4].

This description quickly reduces the proof of the theorem to the question of whether the homotopy type of the complement of this embedding, $(LM \times LM) - (LM \times_M LM)$ is a stable homotopy invariant when considered as “a space over” $LM \times LM$. By using certain pullback properties, the latter question is then further reduced to the question of whether the complement of the diagonal embedding, $\Delta : M \to M \times M$, or somewhat weaker, the complement of the embedding

$$\Delta_k : M \to M \times M \times D^k$$
$$x \to (x, x, 0)$$

is a homotopy invariant when considered as a space over $M \times M$. For this we develop the notion of relative smooth and Poincare embeddings. This is related to the classical theory of Poincare embeddings initiated by Levitt [15] and Wall [19], and further developed by the second author in [10] and [11]. However, for our purposes, the results we need can be proved directly by elementary arguments. The results in Section 2 on relative embeddings are rather fundamental, but don’t appear in the literature. These results may be of independent interest, and furthermore, by proving them here, we make the paper self contained.

Early on in our investigation of this topic, our methods led us to advertise the following question, which is of interest independent of its applications to string topology.

Let $F(M, q)$ be the configuration space of $q$-distinct, ordered points in a closed manifold $M$.

**Question.** Assume that $M_1$ and $M_2$ be homotopy equivalent, simply connected closed $n$-manifolds. Are $F(M_1, q)$ and $F(M_2, q)$ homotopy equivalent?

One knows that these configuration spaces have isomorphic cohomologies ([2]), stable homotopy types ([1], [5]) and have homotopy equivalent loop spaces ([6], [16]). But the homotopy invariance of the configuration spaces themselves is not yet fully understood. For example, when $q = 2$ and the manifolds are 2-connected, then one does have homotopy invariance ([16], [1]). On the other hand, the simple connectivity assumption in the above question is a necessity: a recent result of Longoni and Salvatore [17] shows that for the homotopy equivalent lens spaces $L(7, 1)$ and $L(7, 2)$, the configuration spaces $F(L(7, 1), 2)$ and $F(L(7, 2), 2)$ have distinct homotopy types.

This paper is organized as follows. In Section 1 we will reduce the proof of the main theorem to a question about the homotopy invariance of the complement of the diagonal embedding, $\Delta_k : M \to M \times M \times D^k$. In Section 2 we develop the theory of relative smooth and Poincare embeddings, and then apply it to prove the homotopy invariance of these configuration spaces, and complete the proof of
Conventions. A finitely dominated pair of spaces \((X, \partial X)\) is a Poincare pair of dimension \(d\) if there exists a pair \((\mathcal{L}, [X])\) consisting of a rank one abelian local coefficient system \(\mathcal{L}\) on \(X\) and a "fundamental class" \([X] \in H_n(X, \partial X; \mathcal{L})\) such that the cap product homomorphisms

\[
\cap [X] : H^*(X; M) \to H_{d-*}(X, \partial X; \mathcal{L} \otimes M)
\]

and

\[
\cap [\partial X] : H^*(\partial X; M) \to H_{d-1-*}(\partial X; \mathcal{L} \otimes M)
\]

are isomorphisms for all local coefficient bundles \(M\) on \(X\) (respectively on \(\partial X\)). Here \([\partial X] \in H_{d-1}(\partial X; \mathcal{L})\) denotes the image of \([X]\) under the evident boundary homomorphism. If such a pair \((\mathcal{L}, [X])\) exists, then it is unique up to unique isomorphism.

1 A question about configuration spaces

We begin the process of proving Theorem 1 by reducing the question of the homotopy invariance of the loop homology product, to a question about certain configuration spaces.

Let \(M\) be a closed \(n\)-manifold as above, and let \(h_*\) be a multiplicative generalized homology theory that supports an orientation of \(M\). Of course, ordinary homology with coefficients in a commutative ring is such a multiplicative theory.

Recall that a \(q\)-dimensional vector bundle \(\zeta \to X\) is \(h^*\)-orientable if there is a (Thom) class \(u \in h^q(X^\zeta)\) such that cup product with it yields a Thom isomorphism,

\[
\cup u : h^*(X) \xrightarrow{\cong} h^{*+q}(X^\zeta),
\]

where \(X^\zeta\) denotes the Thom space of \(\zeta\). As usual, we say that \(M\) is \(h^*\)-orientable if the tangent bundle \(TM \to M\) is.

Recall from [8] that the loop homology product \(\mu : h_*(LM) \times h_*(LM) \to h_*(LM)\) can be defined in the following way. Consider the pullback square

\[
\begin{array}{ccc}
LM \times_M LM & \xrightarrow{\iota} & LM \times LM \\
\downarrow \varepsilon & & \downarrow \varepsilon_0 \\
M & \xrightarrow{\Delta} & M \times M
\end{array}
\]

where \(\varepsilon_0 : LM \to M\) is the fibration given by evaluation at the basepoint: \(\varepsilon_0(\gamma) = \gamma(0)\). Let \(\eta(\Delta)\) be a tubular neighborhood of the diagonal embedding of \(M\), and let \(\eta(\iota)\) be the inverse image of this neighborhood in \(LM \times LM\). The normal bundle of \(\Delta\) is the tangent bundle, \(TM\). Recall that the evaluation map \(\varepsilon_0 : LM \to M\) is a locally trivial fiber bundle [14]. (If \(X\) is a CW-complex but not a manifold, the best one can say is that \(\varepsilon_0 : LX \to X\) is a Serre fibration.) Therefore the tubular
neighborhood \( \eta(i) \) of \( \iota : LM \times_M LM \hookrightarrow LM \times LM \) is homeomorphic to total space of the pullback of the tangent bundle, \( e^*(TM) \). We therefore have a Pontrjagin-Thom collapse map,

\[
\tau : LM \times LM \longrightarrow LM \times LM / ((LM \times LM) - \eta(i)) \cong (LM \times_M LM)^{TM}
\]

where \((LM \times_M LM)^{TM}\) is the Thom space of the pullback \( e^*(TM) \to LM \times_M LM \). We are then able to define the \textit{umkehr homomorphism} in homology as the composition,

\[
\iota_! : h_*(LM \times LM) \xrightarrow{\tau_*} h_*(LM \times_M LM) \xrightarrow{\cong} h_{*-n}(LM \times_M LM)
\]

where the second map is the Thom isomorphism, which exists by our orientability assumptions.

Now as pointed out in [3], there is a natural map \( j : LM \times_M LM \to LM \) given by sending a pair of loops \((\alpha, \beta)\) with the same starting point, to the concatenation of the loops, \( \alpha * \beta \).

The loop homology product is then defined to be the composition

\[
\mu_* : h_*(LM \times LM) \xrightarrow{\iota_!} h_{*-n}(LM \times_M LM) \xrightarrow{\bar{j}_*} h_*(LM).
\]

Now, since \( j_* : LM \times_M LM \to LM \) is clearly a natural transformation with respect to smooth maps of the underlying manifold, the homotopy invariance of the loop product \( \mu_* \) will be implied if one can prove the naturality of the umkehr homomorphism, \( \iota_! \), with respect to homotopy equivalences of the underlying manifold. But by the definition of this umkehr map (2), this will be implied if one can show that for any homotopy equivalence \( f : M_1 \to M_2 \) of closed, \( h_* \)-oriented manifolds, there is a homotopy equivalence

\[
\bar{f} : LM_1 \times LM_1 / ((LM_1 \times LM_1) - \eta_1(\iota)) \xrightarrow{\cong} LM_2 \times LM_2 / ((LM_2 \times LM_2) - \eta_2(\iota))
\]

making the diagram of projection maps

\[
\begin{array}{ccc}
LM_1 \times LM_1 & \xrightarrow{\tau} & LM_1 \times LM_1 / ((LM_1 \times LM_1) - \eta_1(\iota)) \\
\downarrow Lf \times Lf & & \downarrow \bar{f} \\
LM_2 \times LM_2 & \xrightarrow{\tau} & LM_2 \times LM_2 / ((LM_2 \times LM_2) - \eta_2(\iota))
\end{array}
\]

homotopy commute. Here \( \eta_i(\iota) \) are the tubular neighborhoods of \( \iota_i : LM_i \times_M LM_i \hookrightarrow LM_i \times LM_i, \) \( i = 1, 2 \), described above.

Notice that we can relax this condition somewhat by only asking \( \bar{f} \) be a \textit{stable} homotopy equivalence, (i.e an equivalence of an iterated suspension of both sides). This is because we are only interested in the induced map in homology, which is invariant under suspensions.

We will reduce this criterion one step further, by proving the following theorem, which is the main result of this section.
Consider the embedding $\Delta_k$ described in the introduction.

$$\Delta_k : M \xrightarrow{\Delta \times 0} M \times M \times D^k.$$ 

Let $F_{D^k}(M, 2)$ be the complement of a tubular neighborhood of this embedding, $F_{D^k}(M, 2) = M \times M \times D^k - \eta(\Delta_k(M))$. Notice that $F_{D^k}(M, 2)$ is homotopy equivalent to the complement, $M \times M \times D^k - \Delta_k(M)$.

**Theorem 2.** Assume $M_1$ and $M_2$ are closed $h_*$-oriented manifolds and that $f : M_1 \to M_2$ is an orientation preserving homotopy equivalence. Assume furthermore that there is a homotopy equivalence $f_k : F_{D^k}(M_1, 2) \to F_{D^k}(M_2, 2)$ such that each of the following squares homotopy commute:

\[
\begin{array}{ccc}
M_1 \times M_1 \times S^{k-1} & \xrightarrow{f \times f \times 1} & M_2 \times M_2 \times S^{k-1} \\
\cap & \downarrow & \cap \\
F_{D^k}(M_1, 2) & \xrightarrow{f_k \cong} & F_{D^k}(M_2, 2) \\
\cap & \downarrow & \cap \\
M_1 \times M_1 \times D^k & \xrightarrow{f \times f \times 1 \cong} & M_2 \times M_2 \times D^k.
\end{array}
\]

Then

$$Lf_* : h_*(LM_1) \to h_*(LM_2)$$

is a ring isomorphism of the loop homology algebras, and

$$Lf_* : h_*^{S^1}(LM_1) \to h_*^{S^1}(LM_1)$$

is Lie algebra isomorphism with respect to the string bracket. That is, Theorem 1 follows.

**Proof.** Observe that for $i = 1, 2$ there are homotopy equivalences,

$$LM_i \times LM_i \times D^k / (LM_i \times LM_i \times D^k) - (\eta_i(\iota) \times D^k) \cong \Sigma^k ((LM_i \times LM_i) / (LM_i \times LM_i) - (\eta_i(\iota))) .$$

So by the argument above (5), to prove this theorem it suffices to show that there is an induced homotopy equivalence of the complements of these embeddings,

$$\tilde{f} : (LM_1 \times LM_1 \times D^k) - (\eta_1(\iota) \times D^k) \xrightarrow{\cong} (LM_2 \times LM_2 \times D^k) - (\eta_2(\iota) \times D^k)$$

that makes the following diagram homotopy commute:

\[
\begin{array}{ccc}
(LM_1 \times LM_1 \times D^k) - (\eta_1(\iota) \times D^k) & \xrightarrow{\cong} & LM_1 \times LM_1 \times D^k \\
\uparrow f & \downarrow & \uparrow Lf \times Lf \times 1 \\
(LM_2 \times LM_2 \times D^k) - (\eta_2(\iota) \times D^k) & \xrightarrow{\cong} & LM_2 \times LM_2 \times D^k
\end{array}
\] (6)
We now produce the homotopy equivalence \( \tilde{f} \).

First observe that for \( i = 1, 2 \), the following diagrams are pullback squares, where the vertical maps are fibrations:

\[
\begin{array}{ccc}
(LM_i \times LM_i \times D^k) - (\eta_i \times D^k) & \longrightarrow & LM_i \times LM_i \times D^k \\
\downarrow \text{ cof } \downarrow \text{ cof } \downarrow \text{ cof } \downarrow \text{ cof } \downarrow \\
F_{D^k}(M_1, 2) & \longrightarrow & M_i \times M_i \times D^k
\end{array}
\]  

(7)

Now consider the following two maps

\[
g_1, g_2 : (LM_1 \times LM_1 \times D^k) - (\eta_1 \times D^k) \rightarrow M_2 \times M_2 \times D^k
\]

defined as follows: the map \( g_1 \) is defined to be the composition

\[
LM_1 \times LM_1 \times D^k \xrightarrow{\eta_1 \times D^k} LM_2 \times LM_2 \times D^k \xrightarrow{\text{cof}} M_2 \times M_2 \times D^k.
\]

Similarly, the map \( g_2 \) is defined to be the composition

\[
LM_1 \times LM_1 \times D^k \xrightarrow{\eta_1 \times D^k} LM_2 \times LM_2 \times D^k \xrightarrow{\text{cof}} M_2 \times M_2 \times D^k.
\]

A homotopy \( H : F_{D^k}(M_1, 2) \times I \rightarrow M_2 \times M_2 \) making the bottom square in the statement of Theorem 2 commute determines a homotopy between \( g_1 \) and \( g_2 \). Since square (7) is a pullback square of fibrations, the homotopy lifting property then defines a map

\[
\tilde{f} : (LM_1 \times LM_1) - (\eta_1 \times D^k) \rightarrow (LM_2 \times LM_2) - (\eta_2 \times D^k)
\]

that is a restriction of \( Lf \times Lf \times 1 : LM_1 \times LM_1 \times D^k \rightarrow LM_2 \times LM_2 \times D^k \) and that lifts the equivalence \( f_k : F_{D^k}(M_1, 2) \rightarrow F_{D^k}(M_2, 2) \). In other words, \( \tilde{f} \) gives us a map between the pullback squares (7) for \( i = 1, 2 \), where the maps between the other corners of the square are \( Lf \times Lf \times 1 : LM_1 \times LM_1 \times D^k \rightarrow LM_2 \times LM_2 \times D^k \); \( f_k : F_{D^k}(M_1, 2) \rightarrow F_{D^k}(M_2, 2) \), and \( f \times f \times 1 : M_1 \times M_1 \times D^k \rightarrow M_2 \times M_2 \times D^k \).

Since these three maps are all homotopy equivalences, so is \( \tilde{f} \). As argued above, the existence of such an equivalence \( \tilde{f} \), implies that

\[
L \tilde{f}_* : h_*(LM_1) \rightarrow h_*(LM_2)
\]

is a ring isomorphism of the loop homology algebras.

To prove that \( Lf \) induces an isomorphism of the string Lie algebras, recall the definition of the Lie bracket from [3]. Given \( \alpha \in h^S_p(LM) \) and \( \beta \in h^S_q(LM) \), then the bracket \( [\alpha, \beta] \) is the image of \( \alpha \times \beta \) under the composition,

\[
\begin{array}{ccc}
h^S_p(LM) \times h^S_q(LM) & \xrightarrow{\text{tr}_{S^1} \times \text{tr}_{S^1}} & h_{p+1}(LM) \times h_{q+1}(LM) \\
\downarrow \text{loop product} & & \downarrow \text{loop product} \\
h_{p+q+2-n}(LM) & \xrightarrow{\text{loop product}} & h_{p+q+2-n}(LM).
\end{array}
\]

(8)
Here $tr_{S^1} : h_{0}^{S^1}(LM) \rightarrow h_{\ast+1}(LM)$ is the $S^1$ transfer map (called “M” in [3]), and $j : h_{\ast}(LM) \rightarrow h_{\ast}^{S^1}(LM)$ is the usual map that descends nonequivariant homology to equivariant homology (called “E” in [3]).

We now know that $Lf$ preserves the loop product, and since it is an $S^1$-equivariant map, it preserves the transfer map $tr_{S^1}$ and the map $j$. Therefore it preserves the string bracket.

\[ \square \]

2 Relative embeddings and the proof of Theorem 1

Theorem 2 reduces the proof of the homotopy invariance of the loop product and the string bracket (Theorem 1) to the homotopy invariance of the spaces $F_{D^k}(M,2)$, which are complements of embeddings. We will study this by introducing the notion of a relative embedding.

The following theorem implies that the assumptions of Theorem 2 hold.

**Theorem 3.** Let $f : M_1 \rightarrow M_2$ be a homotopy equivalence of closed $n$-manifolds. Then for $k$ sufficiently large there is a homotopy equivalence $f_2 : F_{D^k}(M_1,2) \rightarrow F_{D^k}(M_2,2)$ so that both squares in the following diagram homotopy commute:

\[
\begin{array}{ccc}
M_1 \times M_1 \times S^{k-1} & \xrightarrow{f \times f \times 1} & M_2 \times M_2 \times S^{k-1} \\
\cap & \sim & \cap \\
F_{D^k}(M_1,2) & \xrightarrow{f_2} & F_{D^k}(M_2,2) \\
\cap & \sim & \cap \\
M_1 \times M_1 \times D^k & \xrightarrow{f \times f \times 1} & M_2 \times M_2 \times D^k.
\end{array}
\]

Thus the proof of Theorem 1 is finished once we prove Theorem 3. This is the goal of the present section.

2.1 Relative smooth embeddings

Let $N$ be a compact smooth manifold of dimension $n$ whose boundary $\partial N$ comes equipped with a smooth manifold decomposition

\[
\partial N = \partial_0 N \cup \partial_1 N
\]

in which $\partial_0 N$ and $\partial_1 N$ are glued together along their common boundary

\[
\partial_{01} N := \partial_0 N \cap \partial_1 N.
\]

Assume that $K$ is a space obtained from $\partial_0 N$ by attaching a finite number of cells. Hence we have a relative cellular complex

\[(K, \partial_0 N).\]
It then makes sense to speak of the *relative dimension*

\[ \dim(K, \partial_0 N) \leq k \]

as being the maximum dimension of the attached cells.

Let

\[ f : K \to N \]

be a map of spaces which extends the identity map of \( \partial_0 N \).

**Definition 1.** We call these data, \((K, \partial_0 N, f : K \to N)\), a *relative smooth embedding problem*

**Lemma 4.** If \( 2k < n \), then there is a codimension zero compact submanifold

\[ W \subset N \]

such that \( \partial W \cap \partial N = \partial_0 N \) and this intersection is transversal. Furthermore, there is a homotopy of \( f \), fixed on \( \partial_0 N \), to a map of the form

\[ K \xrightarrow{\sim} W \xrightarrow{c} N \]

in which the first map is a homotopy equivalence.

**Definition 2.** The output of Lemma 4 is called a *solution* to the relative smooth embedding problem. The lemma says that the problem can always be solved if \( 2k < n \).

We remark that Lemma 4 is essentially a simplified version of a result of Hodgson [Ho] who strengthens it by \( r \) dimensions when the map \( f \) is \( r \)-connected.

**Proof of the Lemma 4.** First assume that \( K = \partial_0 N \cup D^k \) is the effect of attaching a single \( k \)-cell to \( \partial_0 N \). Then the restriction of \( f \) to the disk gives a map

\[ (D^k, S^{k-1}) \to (N, \partial_0 N) \]

and, by transversality, we can assume that its restriction \( S^{k-1} \to \partial_0 N \) is a smooth embedding. Applying transversality again, the map on \( D^k \) can be generically deformed relative to \( S^{k-1} \) to a smooth embedding. Call the resulting embedding \( g \). Let \( W \) be defined by taking a regular neighborhood of \( \partial_0 N \cup g(D^k) \subset N \). Then \( g \) and \( W \) give the desired solution in this particular case.

The general case is by induction on the set of cells attached to \( \partial_0 N \). The point is that if a solution \( W \subset N \) has already been achieved on a subcomplex \( L \) of \( K \) given by deleting one of the top cells, then removing the interior of \( W \) from \( N \) gives a new manifold \( N' \), such that \( \partial N' \) has a boundary decomposition. The attaching map \( S^{k-1} \to L \) can be deformed (again using transversality) to a map into \( \partial_0 N' \). Then we have reduced to a situation of solving the problem for a map of the form \( D^k \cup \partial_0 N' \to N' \), which we know can be solved by the previous paragraph. \( \square \)
We now thicken the complex $K$ by crossing $\partial_0 N$ with a disk. Namely, for an integer $j \geq 0$, define the space

$$K_j \simeq K \cup_{\partial_0 N} (\partial_0 N) \times D^j,$$

where we use the inclusion $\partial_0 N \times 0 \subset (\partial_0 N) \times D^j$ to form the amalgamated union. Then $(K, \partial_0 N) \subset (K_j, (\partial_0 N) \times D^j)$ is a deformation retract, and the map $f: K \to N$ extends in the evident way to a map

$$f_j: K_j \to N \times D^j$$

that is fixed on $(\partial_0 N) \times D^j$.

**Theorem 5.** Let $f: K \to N$ be as above, but without the dimension restrictions. Then for sufficiently large $j \geq 0$, the embedding problem for the map $f_j: K_j \to N \times D^j$ admits a solution.

**Proof.** The relative dimension of $(K_j, (\partial_0 N) \times D^j)$ is $k$, but for sufficiently large $j$ we have $2k \leq n + j$. The result follows from the previous lemma. \qed

### 2.2 Relative Poincaré embeddings

Now suppose more generally that $(N, \partial N)$ is a (finite) Poincaré pair of dimension $n$ equipped with a boundary decomposition such that $\partial_0 N$ is a smooth manifold. Assume also that the inclusion $\partial_{01} N \to N$ is a 2-connected map. As above, let

$$f: K \to N$$

be a map which is fixed on $\partial_0 N$. We will assume that the relative dimension of $(K, \partial_0 N)$ is at most $n - 3$. Call these data a *relative Poincaré embedding problem*.

**Definition 3.** A *solution* of a relative Poincaré embedding problem as above consists of

- a Poincaré pair $(W, \partial W)$, and a Poincaré decomposition
  
  $$\partial W = \partial_0 N \cup \partial_1 W$$
  
  such that $\partial_1 W \to W$ is 2-connected;
- a Poincaré pair $(C, \partial C)$ with Poincaré decomposition
  
  $$\partial C = \partial_1 W \cup \partial_1 N;$$
- a weak equivalence $h: K \to W$ which is fixed on $\partial_0 N$;
- a weak equivalence
  
  $$e: W \cup_{\partial_1 W} C \to N$$
  
  which is fixed on $\partial N$, such that $e \circ h$ is homotopic to $f$ by a homotopy fixing $\partial_0 N$.  

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The above is depicted in the following schematic homotopy decomposition of $N$:

![Schematic diagram](image)

The space $C$ is called the complement, which is a Poincaré space with boundary $\partial_1 W \cup \partial_1 N$. The above spaces assemble to give a strictly commutative square which is homotopy cocartesian:

$$(\partial_1 W, \partial_{01} N) \longrightarrow (C, \partial_1 N) \quad (9)$$

$$(W, \partial_0 N) \longrightarrow (N, \partial N)$$

(compare [11]). From here through the rest of the paper we refer to such a commutative square as a “homotopy pushout”.

As above, we can construct maps $f_j : K_j \rightarrow N \times D^j$, which define a family of relative Poincare embedding problems. Our goal in this section is to prove the analogue of Theorem 5 that shows that for sufficiently large $j$ one can find solutions to these problems.

We begin with the following result, comparing the smooth to the Poincare relative embedding problems.

**Lemma 6.** Assume that is $M$ is a compact smooth manifold equipped with a boundary decomposition. Let

$$\phi: (N; \partial_0 N, \partial_1 N) \rightarrow (M, \partial_0 M, \partial_1 M)$$

be a homotopy equivalence whose restriction $\partial_0 N \rightarrow \partial_0 M$ is a diffeomorphism.

Then the relative Poincaré embedding problem for $f$ admits a solution if the relative smooth embedding problem for $\phi \circ f$ admits a solution.

**Proof.** A solution of the smooth problem together with a choice of homotopy inverse for $h$ extending the inverse diffeomorphism on $\partial_0 M$ gives solution to the relative Poincaré embedding problem. \qed
Now suppose that \( D(\nu) \to \partial N \) is the unit disk bundle of the normal bundle of an embedding of \( \partial N \) into codimension \( \ell \) Euclidean space, \( \mathbb{R}^{n+\ell} \). Let \( D(\nu|_{\partial_0 N}) \to \partial_0 N \) be its restriction to \( \partial_0 N \). The zero section then gives an inclusion \( \partial_0 N \subset D(\nu|_{\partial_0 N}) \). Set

\[
K_\nu := K \cup_{\partial_0 N} D(\nu|_{\partial_0 N}).
\]

Clearly, \( K_\nu \) is canonically homotopy equivalent to \( K \).

Assuming \( \ell \) is sufficiently large, there exists Spivak normal fibration \([18]\)

\[
S(\xi) \to N
\]

whose fibers have the homotopy type of an \( \ell - 1 \) dimensional sphere. Then, by the uniqueness of the Spivak fibration \([18]\), we have a fiber homotopy equivalence over \( \partial N \)

\[
S(\nu) \simeq_{\to} S(\xi|_{\partial N}).
\]

Let \( D(\xi) \) denote the fiberwise cone fibration of \( S(\xi) \to N \). Then we have a canonical map

\[
f_\nu: K_\nu \to D(\xi)
\]

which is fixed on the \( D(\nu|_{\partial_0 N}) \). Note that

\[
\partial D(\xi) = D(\nu|_{\partial_0 N}) \cup S(\xi)
\]

is a decomposition of Poincaré spaces such that \( D(\nu|_{\partial_0 N}) \) has the structure of a smooth manifold.

Let us set \( \partial_0 D(\xi) := D(\nu|_{\partial_0 N}) \) and \( \partial_1 D(\xi) := S(\xi) \).

Then the classical construction of the Spivak fibration (using regular neighborhood theory in Euclidean space) shows that there is a homotopy equivalence

\[
(D(\xi); \partial_0 D(\xi), \partial_1 D(\xi)) \sim (M; \partial_0 M, \partial_1 M)
\]

in which \( M \) is a compact codimension zero submanifold of some Euclidean space. Furthermore, the restriction \( \partial_0 D(\xi) \to \partial_0 M \) is a diffeomorphism.

Consequently, by Lemma 4 and Theorem 6 we obtain

**Proposition 7.** If the rank of \( \nu \) is sufficiently large, then the relative Poincaré embedding problem for \( f_\nu \) has a solution.

Let \( \eta \) denote a choice of inverse for \( \xi \) in the Grothendieck group of reduced spherical fibrations over \( N \). For simplicity, we may assume that the fiber of \( \xi \) is a sphere of dimension \( \dim N - 1 \). Then there is a fiber homotopy equivalence

\[
S(\tau\partial N \oplus \epsilon) \simeq S(\eta|_{\partial N}),
\]
where \( \epsilon \) is the trivial line bundle. Since \( \xi \oplus \eta \) is trivializable, for some integer \( j \) there is a homotopy equivalence

\[
(D(\xi \oplus \eta); D(\nu \oplus \tau_{|\partial_1 N}) \cup S(\xi \oplus \eta)) \to (N \times D^j; (\partial_0 N) \times D^j, (\partial_1 N) \times D^j \cup N \times S^{j-1})
\]

which restricts to a diffeomorphism \( D(\nu \oplus \tau) \to (\partial_0 N) \times D^j \).

Now a choice of solution of the relative Poincaré embedding problem for \( f_{\nu} \), as given by Proposition 7, guarantees that the relative problem for \( f_{\nu \oplus \tau} \) has a solution. But clearly, the latter is identified with the map \( f_j : K_j \to N \times D^j \). Consequently, we have proven the following.

**Theorem 8.** If \( j \gg 0 \) is sufficiently large, then the relative Poincaré embedding problem for \( f_j : K_j \to N \times D^j \) has a solution.

### 2.3 Application to diagonal maps and a proof of Theorem 1

We now give a proof of Theorem 3. By the results of section 1, this will complete the proof of Theorem 1.

Let \( f : M_1 \to M_2 \) be a homotopy equivalence of closed smooth manifolds. Using an identification of the tangent bundle \( \tau_{M_1} \) with the normal bundle of the diagonal, \( \Delta : M_1 \to M_1 \times M_1 \), we have an embedding

\[
D(\tau_{M_1}) \subset M_1 \times M_1,
\]

which is identified with a compact tubular neighborhood of the diagonal. The closure of its complement will be denoted \( F(M_1, 2) \). Notice that the inclusion \( F(M_1, 2) \subset M_1^{\times 2} - \Delta \) is a homotopy equivalence of spaces over \( M_1^{\times 2} \). Notice also that we have a decomposition

\[
M_1^{\times 2} = D(\tau_{M_1}) \cup S(\tau_{M_1}) F(M_1, 2).
\]

Making the same construction with \( M_2 \), we also have a decomposition

\[
M_2^{\times 2} = D(\tau_{M_2}) \cup S(\tau_{M_2}) F(M_2, 2).
\]

Notice that since \( f : M_1 \to M_2 \) is a homotopy equivalence, the composite

\[
D(\tau_{M_1}) \xrightarrow{\text{projection}} M_1 \xrightarrow{f} M_2 \xrightarrow{\text{zero section}} D(\tau_{M_2})
\]

is also a homotopy equivalence. Let \( T \) be the mapping cylinder of this composite map. Then we have a pair

\[
(T, D(\tau_{M_1}) \amalg D(\tau_{M_2})).
\]

The map \( f^{\times 2} : M_1^{\times 2} \to M_2^{\times 2} \) also has a mapping cylinder \( T^{(2)} \) which contains the manifold

\[
\partial T^{(2)} := M_1^{\times 2} \amalg M_2^{\times 2}.
\]
Then $(T^{(2)}, \partial T^{(2)})$ is a Poincaré pair. Furthermore,

$$\partial T^{(2)} = (D(\tau_{M_1}) \amalg D(\tau_{M_2})) \cup (F(M_1, 2) \amalg F(M_2, 2))$$

is a manifold decomposition. Let us set $\partial_0 T^{(2)} = D(\tau_{M_1}) \amalg D(\tau_{M_2})$ and $\partial_1 T^{(2)} = (F(M_1, 2) \amalg F(M_2, 2))$.

The diagonal maps of $M_1$ and $M_2$ extend to give a map

$$g: T \rightarrow T^{(2)}$$

which extends the identity map of $\partial_0 T^{(2)}$. In other words, $g$ is a relative Poincaré embedding problem.

By Proposition 7, there exists an integer $j \gg 0$ such that the associated relative Poincaré embedding problem

$$g_j: T_j \rightarrow T^{(2)} \times D^j$$

has a solution. Here,

$$T_j := T \cup (\partial_0 T^{(2)}) \times D^j,$$

and

$$\partial_0 (T^{(2)} \times D^j) := (D(\tau_{M_1}) \times D^j) \amalg (D(\tau_{M_2}) \times D^j) \quad \partial_1 (T^{(2)} \times D^j) := F_{D^j}(M_1, 2) \amalg F_{D^j}(M_2, 2).$$

This makes $T^{(2)} \times D^j$ a Poincaré space with boundary decomposition

$$\partial(T^{(2)} \times D^j) = \partial_0 (T^{(2)} \times D^j) \cup \partial_1 (T^{(2)} \times D^j).$$

By definition 8, a solution to this Poincaré embedding problem yields Poincaré pairs $(W, \partial W)$ and $(C, \partial C)$, with the following properties.

- $\partial W = \partial_0 (T^{(2)} \times D^j) \cup \partial_1 W$, where $\partial_0 (T^{(2)} \times D^j) = (D(\tau_{M_1}) \times D^j) \amalg (D(\tau_{M_2}) \times D^j)$ and $\partial_1 W \rightarrow W$ is 2-connected,
- $\partial C = \partial_1 W \cup \partial_1 (T^{(2)} \times D^j)$, where $\partial_1 (T^{(2)} \times D^j) = F_{D^j}(M_1, 2) \amalg F_{D^j}(M_2, 2)$. Notice that $\partial_{0, 1} (T^{(2)} \times D^j) = \partial(D(\tau_{M_1} \times D^j)) \amalg \partial(D(\tau_{M_2} \times D^j))$.
- There is a weak equivalence, $h: T_j \xrightarrow{\cong} W$, fixed on $(D(\tau_{M_1}) \times D^j) \amalg (D(\tau_{M_2}) \times D^j)$.
- There is a weak equivalence $e: W \cup_{\partial_1 W} C \rightarrow T^{(2)} \times D^j$

which is fixed on $\partial(T^{(2)} \times D^j)$, such that $e \circ h$ is homotopic to $g_j: T_j \rightarrow T^{(2)} \times D^j$ by a homotopy fixing $(D(\tau_{M_1}) \times D^j) \amalg (D(\tau_{M_2}) \times D^j)$. 

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The above homotopy decomposition of $T^{(2)} \times D^j$ is indicated in the following schematic:

Furthermore, the complement $C$ of the solution sits in a commutative diagram

Here each (horizontal) arrow marked with $\sim$ is a homotopy equivalence. From this diagram it follows that to complete the proof of Theorem 3, it suffices to show that the horizontal arrows in the second row are homotopy equivalences. By symmetry, it will suffice to prove that the left map in the second row, $F_{D^j}(M_1, 2) \to C$, is a homotopy equivalence.

To do this, consider the following commutative diagram.
Lemma 9. Each of the commutative squares in diagram 10 is a homotopy pushout.

Before we prove this lemma, we show how we will use it to complete the proof of Theorem 3, and therefore Theorem 1.

Proof of Theorem 3. By the lemma, since each of the squares of this diagram is a homotopy pushout, then so is the outer diagram,

\[ F_{D^j}(M_1, 2) \rightarrow C \]
\[ M_1^{x^2} \times D^j \rightarrow T^{(2)} \times D^j. \]

Now recall that \( T^{(2)} \) is the mapping cylinder of the homotopy equivalence, \( f^{x^2} : M_1^{x^2} \rightarrow M_2^{x^2} \). Therefore the inclusion, \( M_1^{x^2} \rightarrow T^{(2)} \) is an equivalence, and hence so is the bottom horizontal map in this pushout diagram, \( M_1^{x^2} \times D^j \rightarrow T^{(2)} \times D^j \). Furthermore, the inclusion \( F_{D^j}(M_1, 2) \rightarrow M_1^{x^2} \times D^j \) is 2-connected, assuming the dimension of \( M \) is 2 or larger. Therefore by the pushout property of this square and the Blakers-Massey theorem, we conclude that the top horizontal map in this diagram, \( F_{D^j}(M_1, 2) \hookrightarrow C \) is a homotopy equivalence.

As described before, this is what was needed to complete the proof of Theorem 3.

We have therefore reduced the proof of Theorem 1 to the proof of Lemma 9.

Proof of Lemma 9. We first consider the right hand commutative square. By the properties of the solution to the relative embedding problem given above in (2.3), we know that \( e : C \rightarrow T^{(2)} \times D^j \) extends to an equivalence, \( e : C \cup_{\partial_1 W} W \rightarrow T^{(2)} \times D^j \). Now notice that the intersection of \( \partial_1 W \) with \( F_{D^j}(M_1, 2) \) is the boundary, \( \partial(D(\tau_{M_1}) \times D^j) \). But

\[ (F_{D^j}(M_1, 2)) \cup_{(D(\tau_{M_1}) \times D^j)} W = (M_1^{x^2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} W. \]

This proves that the right hand square is a homotopy pushout.

We now consider the left hand diagram. Again, by using the properties of the solution of the relative embedding problem given above in (2.3), we know that the homotopy equivalence \( h : T_j \rightarrow W \) extends to a homotopy equivalence,

\[ h : (M_1^{x^2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} T_j \rightarrow (M_1^{x^2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} W. \]

But by construction, \( T_j \) is homotopy equivalent to the mapping cylinder of the composite homotopy equivalence, \( D(\tau_{M_1}) \xrightarrow{\text{project}} M_1 \xrightarrow{f} M_2 \xrightarrow{\text{zero section}} D(\tau_{M_2}) \). This implies that the inclusion \( D(\tau_{M_1}) \times D^j \hookrightarrow T_j \) is a homotopy equivalence, and so \( (M_1^{x^2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} T_j \) is homotopy equivalent to \( M_1^{x^2} \times D^j \). Thus the inclusion given by the bottom horizontal map in the square in question, \( M_1^{x^2} \times D^j \rightarrow (M_1^{x^2} \times D^j) \cup_{(D(\tau_{M_1}) \times D^j)} W \) is also a homotopy equivalence. Since the top horizontal map is the identity, this square is also a homotopy pushout. This completes the proof of Lemma 9, which was the last step in the proof of Theorem 1. \( \square \)
References


