6 Floer theory for the projective space of a polarized vector space and $\mathcal{L}\mathbb{CP}^n$

We now explain what the method described in the previous section gives in some infinite dimensional examples. Motivated by the fact that the compactification of the flow category of the area functional on $\mathcal{L}\mathbb{CP}^n$, constructed in §4, is the flow category of a function on an infinite dimensional projective space, we begin by considering projective spaces.

Example 6.1 — Real projective space
Let V be the real vector space of sequences $x = \{x_n\}_{n\in\mathbb{Z}}$ with only a finite number of non-zero terms, topologized as the direct limit of its finite dimensional subspaces. We use the usual Hilbert norm $\|-\|$ on V; of course V is not complete in this norm. Let $S(V)$ be the sphere in V, and consider the function $f : S(V) \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=-\infty}^{\infty} n x_n^2.$$

This descends to a function

$$f : \mathbb{P}(V) \to \mathbb{R}$$

with critical points $c_i = [\delta_i]$, $i \in \mathbb{Z}$, where δ_i is the i-th element in the standard basis for V. The gradient flow of f, with respect to the Hilbert norm, is the flow on $\mathbb{P}(V)$ defined by the linear flow ψ on V where $\psi_t(\delta_n) = e^{nt}\delta_n$. We could replace V by a space of sequences of suitably rapid decay, but this does not make any real difference.

The unstable manifold $W^u(c_i)$ and the stable manifold $W^s(c_i)$ are given by

$$\begin{aligned} W^u(c_i) &= \{[x] \in \mathbb{P}(V) : x_i \neq 0, x_j = 0 \quad \text{if } j < i\} \\ W^s(c_i) &= \{[x] \in \mathbb{P}(V) : x_i \neq 0, x_j = 0 \quad \text{if } j > i\}. \end{aligned}$$

Neither $W^u(c_i)$ nor $W^s(c_i)$ is finite dimensional, but the intersection $W^u(c_i) \cap W^s(c_j)$ is transverse, finite dimensional, and

$$\dim W^s(c_i) \cap W^u(c_j) = j - i \qquad \text{if } j \geq i.$$

From this it is not difficult to identify the flow category $\mathcal{C} = \mathcal{C}_f$ explicitly.

Now consider the question of whether this category is framed in the sense of §3. Fix a pair of integers $a < b$ and let $\mathcal{C}_a^b$ be the full subcategory generated by the critical points c_i with $a \leq i \leq b$. This category is the flow category of the function

$$f(x) = \sum_{n=a}^{b} n x_n^2$$

on $\mathbb{RP}^{b-a} = \mathbb{P}(V_a^b)$, where V_a^b is the finite dimensional subspace of V with basis δ_i, $a \leq i \leq b$, and we are using the natural homogeneous coordinates

$x = [x_a, \ldots, x_b]$ with $\sum x_n^2 = 1$ on $\mathbb{P}(V_a^b)$. Therefore, $\mathcal{C}_a^b$ is a framed category and each of the spaces of morphisms $F(j,i)$ with $a \leq i < j \leq b$ inherits a framing φ_a^b from the flow category $\mathcal{C}_a^b$. The framing φ_a^b comes from embedding $F(j,i)$ in the unstable sphere of the critical point c_j in $\mathbb{RP}^{b-a}$, which, since c_j has index $j-a$ in $\mathbb{RP}^{b-a}$, is a sphere of dimension $j-a-1$. In particular, this shows that the framings φ_a^b and $\varphi_a^{b'}$ are identical. However, the framings φ_a^b and $\varphi_{a'}^b$ are not the same, as we will show.

Let us work in the stable category $\mathcal{S}$, described in the Appendix, and define

$$|(\mathcal{C}_a^b, \varphi)| = S^{a-b}|Z|$$

where $Z : \mathcal{J}_a^b \to \mathcal{T}_*$ is the functor defined by the category $\mathcal{C}_a^b$ equipped with the framing φ. This has the effect of removing the suspensions which occur in the statement of Proposition (5.1) and it simplifies notation; it is a straightforward matter to keep track of the suspensions, if necessary.

The manifold $F(i+1,i)$ has dimension zero and, using φ_a^b, is framed in S^{i-a}, the unstable sphere of the critical point c_{i+1} in $\mathbb{RP}^{b-a}$. Thus it gives a map of spheres $S^{i-a} \to S^{i-a}$. This map is the relative attaching map between the $(i-a+1)$-cell and the $(i-a)$-cell in $|(\mathcal{C}_a^b, \varphi_a^b)| = \mathbb{RP}^{b-a}$. Therefore it has degree $1-(-1)^{i-a}$. Using the framing φ_{a-1}^b of $F(i+1,i)$ in the unstable S^{i-a+1} of the critical point c_{i+1} in $\mathbb{RP}^{b-a+1}$ we get a map $S^{i-a+1} \to S^{i-a+1}$ which is the relative attaching map between the $(i-a+2)$-cell and the $i+1-a$-cell of $|(\mathcal{C}_{a-1}^b, \varphi_{a-1}^b)| = \mathbb{RP}^{b-a+1}$; therefore it has degree $1-(-1)^{i-a+1}$. So the framings φ_a^b and φ_{a-1}^b produce different maps.

For $a' < a$ the framings $\varphi_{a'}^b$ and φ_a^b differ because the normal bundle to $\mathbb{RP}^{b-a}$ in $\mathbb{RP}^{b-a'}$, that is $(a-a')\eta$ where η is the real Hopf line bundle, is non-trivial. Furthermore, it is straightforward to check that

$$|(\mathcal{C}_a^b, \varphi_{a'}^b)| = (\mathbb{RP}^{b-a})^{(a-a')\eta}.$$

Since the framings $\varphi_{a'}^{b'}$ and $\varphi_{a'}^b$ agree it follows that

$$|(\mathcal{C}_a^b, \varphi_{a'}^{b'})| = (\mathbb{RP}^{b-a})^{(a-a')\eta}$$

whenever $a' \leq a < b \leq b'$.

From the construction of the realization there are maps

$$|(\mathcal{C}_a^b, \varphi_a^b)| \to |(\mathcal{C}_a^{b'}, \varphi_a^{b'})|, \qquad |(\mathcal{C}_{a'}^b, \varphi_{a'}^b)| \to |(\mathcal{C}_a^b, \varphi_{a'}^b)|$$

for $a' \leq a < b \leq b'$ which we now identify. The first is the inclusion

$$\mathbb{RP}^{b-a} \to \mathbb{RP}^{b'-a}$$

and the second is the map

$$\mathbb{RP}^{b-a'} \longrightarrow (\mathbb{RP}^{b-a})^{(a-a')\eta}$$

obtained from the Pontryagin-Thom construction applied to the embedding $\mathbb{RP}^{b-a} \to \mathbb{RP}^{b-a'}$; we describe this construction briefly.

Suppose we have an embedding $P \to M$ of compact manifolds. Let ν be the normal bundle of the embedding and let N_P be an open tubular neighbourhood of P in M. Then the inclusion $N_P \to M$ is an open embedding and so it gives a map $N_P^+ \leftarrow M$ where N_P^+ is the one-point compactification of N_P. Now N_P^+ is just the Thom complex of ν and so the embedding $N \to M$ gives a map $M \to P^\nu$. Applied to the embedding $\mathbb{RP}^{b-a} \to \mathbb{RP}^{b-a'}$, with normal bundle $(a-a')\eta$, this gives the required map.

We now explain how to assemble the spaces $|(\mathscr{C}_a^b, \varphi_{a'}^{b'})|$ into a single object, a pro-spectrum, which correctly reflects the relation between the different spaces. To do this, we use the maps we have just described. The bundle $(a-a')\eta$ extends over $\mathbb{RP}^{b-a}$ and so, using the theory of Thom spaces of virtual bundles described in the Appendix, we can convert the map $\mathbb{RP}^{b-a'} \to (\mathbb{RP}^{b-a})^{(a-a')\eta}$ into a stable map

$$(\mathbb{RP}^{b-a'})^{-(a-a')\eta} \longrightarrow \mathbb{RP}^{b-a}.$$

Therefore we can construct the sequence

$$\mathbb{RP}^{b-a} \longleftarrow (\mathbb{RP}^{b-a+1})^{-\eta} \longleftarrow (\mathbb{RP}^{b-a+2})^{-2\eta} \longleftarrow \dots$$

in the stable category of compact spaces. As explained in the Appendix, this sequence defines a pro-spectrum.

Using the inclusions $\mathbb{RP}^{b-a} \to \mathbb{RP}^{b'-a}$, we can take the limit over b and then, using Thom spaces of virtual vector bundles over CW complexes of finite type (see the Appendix), we get the pro-spectrum defined by the sequence of Thom spectra

$$\mathbb{RP}^\infty \longleftarrow (\mathbb{RP}^\infty)^{-\eta} \longleftarrow (\mathbb{RP}^\infty)^{-2\eta} \longleftarrow \cdots.$$

This pro-spectrum is the final output of the construction; it is the *Floer homotopy type* associated to the function $f : \mathbb{P}(V) \to \mathbb{R}$. In fact, this pro-spectrum, which is usually denoted by $\mathbb{RP}^\infty_{-\infty}$, is exactly the pro-spectrum which occurs in the theorem of Lin mentioned in the introduction.

Floer's method of associating a chain complex to the function $f : \mathbb{P}(V) \to \mathbb{R}$ gives the chain complex C_* with $C_p = \mathbb{Z}$ for all $p \in \mathbb{Z}$, and the boundary operator $\partial_p : C_p \to C_{p-1}$ is multiplication by $1 + (-1)^p$. It is easy to check that the homology of this chain complex is the same as the homology of the pro-spectrum $\mathbb{RP}^\infty_{-\infty}$ for any coefficient group. So the Floer groups do compute the homology of $\mathbb{RP}^\infty_{-\infty}$.

Example 6.2 — Complex projective space
Now consider the complex analogue of the previous example. Let W be the complex vector space of sequences $z = \{z_n\}_{n\in\mathbb{Z}}$ with only a finite number of non-zero terms, equipped with the direct limit topology, and Hilbert norm $\|-\|$. This time the function $S(W) \longrightarrow \mathbb{R}$ defined by

$$z \mapsto \sum_{n=-\infty}^{\infty} n|z_n|^2$$

descends to a function $f : \mathbb{P}(W) \longrightarrow \mathbb{R}$, and the flow of this function is exactly the flow $\Phi^{(0)}$ on the projective space $\mathbb{P}(W)$ which arose in §4. The construction used in the case of the real projective space $\mathbb{P}(V)$ shows that the Floer homotopy type associated to this function is the pro-spectrum $\mathbb{CP}^{\infty}_{-\infty}$ defined by the sequence of Thom spectra

$$\mathbb{CP}^{\infty} \leftarrow (\mathbb{CP}^{\infty})^{-\zeta} \leftarrow (\mathbb{CP}^{\infty})^{-2\zeta} \leftarrow \cdots$$

where ζ is the complex Hopf line bundle. Once more we find by direct computation that the Floer chain complex of f does indeed compute the cohomology of this pro-spectrum.

Example 6.3 — The area function on $\mathscr{L}\mathbb{CP}^n$
To associate a Floer homotopy type to the area functional $\mathscr{A}$ on $\mathscr{L}\mathbb{CP}^n$, we first compactify the flow category $\mathscr{C}_{\mathscr{A}}$ to give the flow category of the flow $\Phi^{(n)}$ on the projective space $\mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}[z, z^{-1}])$, as in §4. Now the method of (6.1) and (6.2) gives the pro-spectrum defined by the sequence of Thom spectra

$$\mathbb{CP}^{\infty} \leftarrow (\mathbb{CP}^{\infty})^{-(n+1)\zeta} \leftarrow (\mathbb{CP}^{\infty})^{-2(n+1)\zeta} \leftarrow \cdots$$

where ζ is the complex Hopf line bundle. This pro-spectrum is $\mathbb{CP}^{\infty}_{-\infty}$.

The reason why $(n+1)\zeta$ appears in this construction is that the category $\mathscr{C}_a^b$ which occurs in this example is the flow category of a Morse-Bott-Smale function on $\mathbb{CP}^{(n+1)(b-a)}$, and the normal bundle to the embedding of $\mathbb{CP}^{(n+1)(b-a)}$ in $\mathbb{CP}^{(n+1)(b-(a-1))}$ is $(n+1)\zeta$.

As explained in the Appendix, the cohomology of $\mathbb{CP}^{\infty}_{-\infty}$ with integral coefficients is $\mathbb{Z}[u, u^{-1}]$, the ring of Laurent polynomials in u, where u has degree 2. Thus, as a group, this is one copy of $\mathbb{Z}$ in every even dimension. Computing the first Chern class of $\mathbb{CP}^n$ shows that the Floer homology $\mathrm{HF}_*(\mathscr{L}\mathbb{CP}^n)$ is $\mathbb{Z}/(2n+2)$ graded, and Floer [12] shows that these groups are $\mathbb{Z}$ in even degrees and 0 in odd degrees.

The relation between these groups is as follows. Let $e((n+1)\zeta)$ be the Euler class of the bundle $(n+1)\zeta$, which is the bundle which naturally occurs in the above sequence of Thom spectra; of course $e((n+1)\zeta) = u^{n+1}$. If we now set $e((n+1)\zeta)$ to be 1, we get Floer's groups with their $\mathbb{Z}/(2n+2)$ grading:

$$\mathrm{HF}_*(\mathscr{L}\mathbb{CP}^n) = \frac{\mathbb{Z}[u, u^{-1}]}{(u^{n+1} - 1)} = \frac{H^*(\mathbb{CP}^{\infty}_{-\infty})}{(e((n+1)\zeta) - 1)}.$$

The Floer cohomology of $\mathcal{L}\mathbb{CP}^n$ has a ring structure using the «pair of pants» product and what is more the above isomorphisms are isomorphisms of rings.

We close this section with some final comments.

(i) It seems very likely that the same method will work for the area functional on $\mathcal{L}Gr_k(\mathbb{C}^n)$ where $Gr_k(\mathbb{C}^n)$ is the Grassmannian of complex k planes in $\mathbb{C}^n$. In this case the ring strucuture of Floer cohomology ring has been computed, by Witten; it is the «deformed cohomology ring» of the Grassmannian. Once more, we expect the Floer homotopy type to come from an inverse system of Thom spectra, and to find the Floer cohomology ring is given by a formula similar to the one which arises in the case of $\mathbb{CP}^n$.

(ii) In the projective space examples, the pro-spectra $\mathbb{RP}^{\infty}_{-\infty}$ and $\mathbb{CP}^{\infty}_{-\infty}$ are indeed the natural candidates for the semi-infinite homotopy type of the polarized manifolds $\mathbb{P}(V)$ and $\mathbb{P}(W)$. The polarization of these projective spaces is defined by the natural polarization of V and W. For example the construction of $\mathbb{RP}^{\infty}_{-\infty}$ can be phrased so that it only depends on the polarization $V = V^- \oplus V^0 \oplus V^+$ where V^- has basis $\{\delta_i\}_{i<0}$, V^+ has basis $\{\delta_i\}_{i>0}$ and V^0, the «finite dimensional ambiguity», is the one dimensional space spanned by δ_0.

Let W be a finite dimensional subspace of V such that $W = W^- \oplus W^0 \oplus W^+$ where $W^\pm = W \cap V^\pm$ and $W^0 = W \cap V^0$. We refer to $W^0 \oplus W^+$ as the positive part of W and W^- as the negative part. To W we associate the Thom space $\mathbb{P}(W)^{-\xi}$ of the virtual bundle $-\xi$ where ξ is the vector bundle over $\mathbb{P}(W)$ defined by W^-. If we choose bases then we get an isomorphism $\xi \cong (\dim W^-)\eta$ where η is the Hopf line bundle. In the stable category $\mathbb{P}(W)^{-\xi}$ has one i-cell for each i with $-\dim W_- \leq i \leq \dim W - \dim W^-$.

Now suppose we have an inclusion $W_1 \to W_2$ which preserves the decompositions. If it is an isomorphism on the negative part, i.e. increases the positive part, then we get an obvious map

$$\mathbb{P}(W_1)^{-\xi_1} \to \mathbb{P}(W_2)^{-\xi_2}.$$

On the other hand, if it is an isomorphism on the positive part, i.e. increases the negative part, then we get a map

$$\mathbb{P}(W_1)^{-\xi_1} \leftarrow \mathbb{P}(W_2)^{-\xi_2}.$$

This map is constructed as follows. The bundle ξ_1 is a sub-bundle of ξ_2 restricted to $\mathbb{P}(W_1)$ and therefore the virtual bundle $-\xi_2$ restricted to $\mathbb{P}(W_1)$ is a sub-virtual-bundle of $-\xi_1$. This gives a map (in the stable category) of Thom spaces $\mathbb{P}(W_1)^{-\xi_1} \leftarrow \mathbb{P}(W_3)^{-\xi_2}$.

Thus we get a system of Thom spectra indexed by subspaces W of V. An inclusion which is an isomorphism on the negative part (i.e increases the positive part) gives a map in the same direction whereas an inclusion which is an isomorphism on the positive part (i.e. increases the negative part) gives a map in the other direction. Using the basis for V, we can use the subspaces V_a^b to reduce this system to one indexed by pairs of integers, and this gives the pro-spectrum $\mathbb{RP}^{\infty}_{-\infty}$. This construction fits in rather well with the coordinate free theory of spectra described in [19], where the spaces defining the spectrum are indexed by finite dimensional subspaces of an infinite dimensional real vector space, rather than the integers. Here we have spaces indexed by the finite dimensional subspaces of a polarized vector space.

Therefore, we are able to associate a «semi-infinite homotopy type to $\mathbb{P}(V)$ which depends only on the polarization of V, and the Floer function $f : \mathbb{P}(V) \to \mathbb{R}$ does indeed compute the «semi-infinite» cohomology of $\mathbb{P}(V)$. However the construction depends very heavily on special features of $\mathbb{P}(V)$.

Appendix — Spectra and pro-spectra

In stable homotopy theory it is convenient, and it greatly simplifies many arguments, to be able to work in a suitable stable category of spaces. The *stable category* $\mathcal{S}$ of finite CW complexes is defined, essentially, by inverting the suspension functor on the category of finite CW complexes. The objects of $\mathcal{S}$ are defined to be S^nX where X is a finite CW complex, and $n \in \mathbb{Z}$. If n is positive then S^nX is just the n-th suspension of X, but by allowing n to be negative we have introduced formal desuspensions of X. More precisely, the objects of $\mathcal{S}$ are pairs (X, n), where X is a finite CW complex and $n \in \mathbb{Z}$, modulo the equivalence relation generated by identifying (X, n) with $(S^nX, 0)$ if n is positive, and S^nX is the equivalence class of (X, n). A map from $S^nX \to S^mY$ is defined by giving a map $S^{n+k}X \to S^{m+k}Y$ where k is chosen large enough so that both $S^{n+k}X$ and $S^{m+k}Y$ are genuine CW complexes. Two maps are identified if they are equal after a suitably large number of suspensions. Thus,

$$\operatorname*{Map}_{\mathcal{S}}(S^nX, S^mY) = \lim_{k\to\infty} \operatorname{Map}(S^{n+k}X, S^{m+k}Y)$$

where the maps in the direct system are given by suspension.

The main difficulty with this stable category is that it is not closed under direct and inverse limits. The category of *spectra* $\mathcal{SP}$ is defined, essentially, to be the category one obtains by formally adjoining direct limits of sequences in the stable category. Thus a spectrum $\mathbf{K}$ is given by a sequence of CW complexes $\{K_p\}$ and maps $SK_p \to K_{p+1}$. One thinks of the spectrum $\mathbf{K}$ as the direct limit of the sequence

$$K_0 \to S^{-1}K_1 \to \ldots \to S^{-p}K_p \to S^{-p+1}K_{p+1} \to \cdots$$

in $\mathcal{S}$. Note that it is only necessary to define the spaces K_{p_n} and the maps $S^{p_{n+1}-p_n}K_{p_n} \to K_{p_{n+1}}$ for a strictly increasing sequence of integers p_n; for if

$p_n < m < p_{n+1}$ we can define $K_m = S^{m-p_n} K_{p_n}$ and use the identity map $SK_m \to K_{m+1}$ if $m + 1 \neq p_{n+1}$.

It takes some work to define maps between spectra and set up a good category of spectra. The essential difficulty is the usual one when dealing with maps from a direct limit **K** to a direct limit **L**; it is certainly the case that a map of direct systems defines such a map but there are many different direct systems with limits **K** and **L**. However the technicalities involved in the definition of a suitable category of spectra are well-understood; see [1] and [19]. In particular [19] gives the definition from the «coordinate-free» point of view which leads to a category of spectra with all the good properties one could expect.

Spectra define generalised cohomology theories: if X is a finite CW complex

$$h^p(X; \mathbf{K}) = \lim_{k \to \infty} [S^{k-p}X; K_k].$$

The maps in the direct system are defined by

$$[S^{k-p}X, K_k] \to [S^{k-p+1}X, SK_k] \to [S^{k-p+1}X, K_{k+1}],$$

where the first map is given by suspension, and the second by the structure maps of the spectrum **K**. In the literature this group $h^p(X; \mathbf{K})$ is often denoted by $\mathbf{K}^p(X)$. Furthermore every generalised cohomology theory arises in this way. This is one of the main justifications for introducing spectra.

For some purposes the category of spectra is still not big enough. In the study of the Segal conjecture it becomes clear that one also needs pro-spectra, inverse systems of spectra. The most convincing argument for the necessity of pro-spectra is given in [2], pages 5–6. By definition, a *pro-spectrum* is a doubly indexed family of finite CW complexes $\{X_{p,q}\}$ equipped with maps

$$SX_{p,q} \to X_{p,q+1}, \qquad X_{p,q} \to X_{p,q-1}.$$

Thus if we fix p the sequence of spaces $X_{p,q}$ with structure maps $SX_{p,q} \to X_{p,q+1}$ form a spectrum $\mathbf{X}_p$ and the maps $X_{p,q} \to X_{p-1,q}$ give a map of spectra $\mathbf{X}_p \to \mathbf{X}_{p-1}$; so we have an inverse system of spectra

$$\cdots \leftarrow \mathbf{X}_{p-1} \leftarrow \mathbf{X}_p \leftarrow \cdots.$$

It is only necessary to define these spaces and maps for a strictly increasing sequences of integers p_n and q_n. Furthermore, to define a pro-spectrum it is sufficient to give the structure maps in the stable category $\mathcal{S}$.

The first example of a pro-spectrum which arises in the main text occurs in §5, where for any pair of integers a and b with $a < b$, we have spaces $|Z|_a^b$, and for $a' \leq a < b \leq b'$, we have maps

$$S^{b'-b}|Z|_a^b \to |Z|_a^{b'}, \qquad |Z|_{a'}^b \to S^{a-a'}|Z|_a^b.$$

The corresponding pro-spectrum is defined by $X_{a,b} = S^a|Z|_a^b$ and the natural structure maps. Note that since a can be negative, these spaces and maps really do lie in the stable category; as noted above, they nonetheless define a pro-spectrum.

A good example which illustrates these ideas and is of considerable importance in our approach to Floer homotopy type arises from the theory of Thom spaces. Recall that the Thom space X^E of a vector bundle E over a compact space X is defined to be the one-point compactification E^+ of the total space of E. Now consider the problem of defining the Thom space of a virtual vector bundle ξ over X. A virtual vector bundle is an element $\xi \in KO(X)$ and its dimension, which is an integer, is defined by the homomorphism $KO(X) \to KO(\mathrm{pt}) = \mathbb{Z}$ given by the inclusion of a point in X. We are assuming, of course, that X is connected. By standard properties of K-theory we can find a genuine vector bundle E over X such that

$$\xi = E - k \in KO(X),$$

where k is a trivial bundle of dimension k. Now X^ξ is defined to be the object $S^{-k}X^E$ in the stable category $\mathcal{S}$. It is not difficult to check that in $\mathcal{S}$ the homotopy type of X^ξ does not depend on the choice of E and k.

Now suppose that X is CW complex of finite type (this means that the n-skeleton $X^{(n)}$ of X is a finite CW complex for each n) and ξ is a virtual vector bundle on X. In this case we cannot necessarily choose a vector bundle E such that $\xi = E - k$, but we can choose vector bundles $E^{(n)}$ over $X^{(n)}$ such that

$$\xi^{(n)} = E^{(n)} - k_n \in KO(X^{(n)}),$$

where $\xi^{(n)}$ is the restriction of ξ to $X^{(n)}$. Furthermore these bundles can be chosen so that there is a bundle map $E^{(n)} + k_m - k_n \to E^{(m)}$, covering the inclusion of $X^{(n)} \to X^{(m)}$, which is an isomorphism on fibres. Thus we get maps

$$S^{k_{n+1}-k_n}(X^{(n)})^{E^{(n)}} \to (X^{(n+1)})^{E^{(n+1)}},$$

and the spaces $\{(X^{(n)})^{E^{(n)}}\}$ with these maps define a spectrum. This is the Thom spectrum of ξ, denoted by X^ξ. Once more the homotopy type of X^ξ does not depend on the choices made in its definition. For example, the MU-spectrum used in §5 is the Thom spectrum of the universal bundle over BU.

On a compact space X, if E is a sub-vector-bundle of F, we get a map of Thom complexes $X^E \to X^F$. Similarily if X has finite type and ξ is a sub-virtual-vector-bundle of η, which means that there is a genuine vector bundle E such that $\xi + E = \eta$, then we get a map of Thom spectra

$$X^\xi \to X^\eta.$$

In particular, suppose that X has finite type, and E is a vector bundle over X. Then $-E$ is a virtual vector bundle over X, and if $k \geq 0$ then $-kE$ is a sub-virtual-vector-bundle of $(-k+1)E$. So we get maps $X^{-kE} \to X^{(-k+1)E}$ and we can form the inverse system of Thom spectra

$$X \leftarrow X^{-E} \leftarrow X^{-2E} \leftarrow \cdots$$

with k-th term X^{-kE} where $k \geq 0$. This inverse system of Thom spectra is a pro-spectrum which we will denote by $X^{-\infty E}$. The examples which occur in the main text are the cases where $X = \mathbb{RP}^{\infty}$ and E is the real Hopf line bundle; $X = \mathbb{CP}^{\infty}$ and E is the complex Hopf line bundle. These pro-spectra are denoted by $\mathbb{RP}^{\infty}_{-\infty}$ and $\mathbb{CP}^{\infty}_{-\infty}$ respectively.

The cohomology of a pro-spectrum $\mathbf{X}$ is defined as follows. The pro-spectrum is an inverse system of spectra

$$\cdots \leftarrow \mathbf{X}_{p-1} \leftarrow \mathbf{X}_p \leftarrow \cdots$$

and then

$$H^*(\mathbf{X}) = \lim_{p \to \infty} H^*(\mathbf{X}_p).$$

In good cases, that is where there is no $\lim^1$-term, we get

$$H^*(\mathbf{X}) = \lim_{p \to \infty} \lim_{\infty \leftarrow q} H^*(X_{p,q})$$

where the spaces $\{X_{p,q}\}$, with the appropriate structure maps, define the pro-spectrum $\mathbf{X}$.

In the case of the pro-spectrum $X^{-\infty E}$ defined by a vector bundle over a CW complex of finite type, it is straightforward to compute cohomology. If E is orientable it has an Euler class $e(E) \in H^{\dim(E)}(X;\mathbb{Z})$, and

$$H^*(X^{-\infty E};\mathbb{Z}) = H^*(X;\mathbb{Z})[e(E)^{-1}].$$

If E is not orientable then it has an Euler class in mod 2 cohomology, and the mod 2 cohomology of $X^{-\infty E}$ is given by the same formula. For $\mathbb{RP}^{\infty}_{-\infty}$ and $\mathbb{CP}^{\infty}_{-\infty}$ this shows that

$$H^*(\mathbb{RP}^{\infty}_{-\infty};\mathbb{Z}/2) = \mathbb{Z}/2[x, x^{-1}]$$
$$H^*(\mathbb{CP}^{\infty}_{-\infty};\mathbb{Z}) = \mathbb{Z}[u, u^{-1}].$$

Here x has degree 1 and corresponds to the Euler class of the real Hopf line bundle in $H^1(\mathbb{RP}^{\infty};\mathbb{Z}/2)$; u has degree 2 and corresponds to the Euler class of the complex Hopf line bundle in $H^2(\mathbb{CP}^{\infty};\mathbb{Z})$.

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