Ralph L. Cohen

Bundles, Homotopy, and Manifolds

An introduction to graduate level algebraic and differential topology

Contents

Introduction

Differential Topology is the study of the topology of differentiable manifolds and differentiable mappings between them. Algebraic Topology studies topological properties of more general spaces using a variety of algebraic and homotopy theoretic techniques. These subjects are of central importance throughout much of mathematics, especially those areas with a geometric perspective such as Differential Geometry, Geometric Analysis, Symplectic Geometry, and Algebraic Geometry.

The goal of this book is to introduce the reader to various topics of Algebraic and Differential Topology that (s)he might find useful in their further study and research in these many areas of mathematics. The main philosophy of the presentation here is that there is no clear dividing line between these important areas of topology. A modern study of Differential Topology relies on the techniques of Algebraic Topology, and many important questions in Algebraic Topology come from the study of differentiable manifolds. We will present topics and methods of Differential and Algebraic Topology, going from very basic discussions to more specific topics of recent research. Our objective is that the reader will obtain a literacy in these topics, so that the interested reader can then pursue these topics in more depth.

This book can be viewed as a hybrid of a text book and more advanced lecture notes. In various basic areas of study, such as the basics of differentiable manifolds, fiber bundles, characteristic classes, homotopy theory, cobordism theory, and Morse theory, a rather full account with details is given. In discussions about more advanced topics, including topics of recent research, results and techniques are often sketched, but references and citations are carefully given.

In this book we will assume the reader is familiar with the basics of algebraic topology, such as the fundamental group, homology, and cohomology. The text by Hatcher [67] is an excellent reference for these topics. Perhaps the most basic theorem concerning the algebraic topology of manifolds is the Poincaré Duality theorem. Because not every student having completed a first course in algebraic topology will have seen the Poincaré Duality theorem, we begin these notes with a brief discussion of this topic in chapter 1, where we basically summarize the approach to this important theorem contained in Section 3.3 of [67].

The topics covered in these notes include the following:

- The basics of differentiable manifolds (tangent spaces, vector fields, tensor fields, differential forms)
- Fiber bundles in general, Lie groups, principal bundles, vector bundles and their classification via universal bundles, automorphisms of principal bundles (gauge transformations) and their classifying spaces
- Characteristic classes of vector bundles and their calculation
- Embeddings, immersions, tubular neighborhoods, and normal bundles. This includes a discussion of Smale-Hirsch theory and the resulting algebraic topological approach to the study of immersions of manifolds.
- Basic homotopy theory including homotopy groups, Serre fibrations, obstruction theory, Eilenberg-MacLane spaces, and spectral sequences
- Transversality and Intersection theory using Poincaré duality
- Stable homotopy theory
- Cobordism theory including the Pontrjagin-Thom construction and calculations of various cobordism rings, a study of framed cobordism including surgery theory techniques and a discussion of the Kervaire invariant including the recent dramatic solution of the "Kervaire invariant one problem", and finally as a discussion of the topology of cobordism categories, their relation to diffeomorphisms of manifolds, and a discussion of the recent solution of the "Mumford conjecture" about the cohomology of moduli spaces of Riemann surfaces, and its generalizations.
- Morse theory, including flow categories and their classifying spaces.

These notes emanated from a variety of graduate courses the author has given over the years at Stanford University. The author is grateful to the students in these courses for their inspiration and for their feedback. The author is particularly grateful to his former PhD students Kevin Iga and Paul Norbury, for their help with several aspects of this book.

1

Topological Manifolds and Poincaré Duality

The subject of much of this book is the topology of manifolds. Manifolds of dimension n are topological spaces that have a well defined local topology (they are locally homeomorphic to $\mathbb{R}^n$), but globally, two n-dimensional manifolds may have very different topologies.

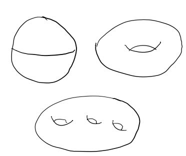


FIGURE 1.1
These surfaces are all 2-dimensional manifolds, as they are all locally homeomorphic to $\mathbb{R}^2$. However their global topologies are quite different.

Nonetheless we will find that the homological structure of manifolds is quite striking. In particular they satisfy an important, unifying property, called "Poincaré Duality". The discussion and proof of this property is the subject of this chapter. As the reader will see, this property will be used throughout the book, and is used in a basic way in many areas of topology and geometry. Our proof follows the exposition of Hatcher [67], We refer the reader to that source for a fuller discussion of this important topic.

Throughout this book, unless otherwise stated what we will mean by a "space" is a topological space of the homotopy type of a CW-complex.

Definition 1.1. *An n-dimensional (topological) manifold is a second countable Hausdorff space M^n that is locally homeomorphic to $\mathbb{R}^n$. That is, each point $x \in M^n$ has an open neighborhood U_x which is homeomorphic to $\mathbb{R}^n$, or equivalently, to the open ball $B^n = \{v \in \mathbb{R}^n : |v| < 1\}$. A specific homeomorphism $\phi : U_x \to \mathbb{R}^n$ is called a chart around x. An open cover of M^n consisting of charts is called an atlas.*

1.0.1 Orientations

We observe that the local Euclidean property of manifolds has a manifestation homologically. Namely, suppose M^n is a connected, n-dimensional manifold, and let $x \in M^n$. Let U be an open neighborhood of x that is homeomorphic To $\mathbb{R}^n$. Then we have the following calculation of the relative homology:

$$\begin{aligned}
H_q(M^n, M - \{x\}) &\cong H_q(U, U - \{x\}) \quad \text{by excision} \\
&\cong H_q(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \quad \text{by the local-Euclidean property} \\
&\cong \tilde{H}_{q-1}(\mathbb{R}^n - \{0\}) \ \text{ by the exact sequence of the homology} \\
&\text{of a pair and the fact that } \mathbb{R}^n \text{ is contractible} \\
&\cong \begin{cases} \mathbb{Z} & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In particular, observe that the dimension n, is determined homologically.

Definition 1.2. *Let M^n be an n-dimensional manifold. A local orientation* of M^n at x is a choice of generator of $H_n(M^n, M^n - \{x\}) \cong \mathbb{Z}$.

Notice that there are two choices of local orientations at any point $x \in M^n$, and a choice of orientation is equivalent to choosing an isomorphism $\Phi_x : H_n(M^n, M^n - \{x\}) \xrightarrow{\cong} \mathbb{Z}$.

Definition 1.3. *A manifold M^n is orientable,* if there is a continuous choice of local orientations at each point $x \in M^n$. A specific choice of such a continuous choice of local orientations is called a (global) orientation of M^n.

Of course this definition is not yet complete, because we have not yet defined what is meant by a "continuous choice of local orientations". To make this precise, we use the theory of covering spaces.

For $x \in M^n$, let $Or_x(M^n)$ be the set of local orientations of M^n at x. That is, it is the set of generators of $H_n(M^n, M^n - x)$. As observed above, this is a set with two elements, as there are two possible choices of generators for the infinite cyclic group. Said another, but equivalent way, $Or_x(M^n)$ is the set of isomorphisms, $\sigma : H_n(M^n, M^n - x) \xrightarrow{\cong} \mathbb{Z}$.

Let $Or(M^n)$ be the space of all local orientations on M^n. That is, as a set,

$$Or(M^n) = \bigcup_{x \in M^n} Or_x(M^n). \tag{1.1}$$

Proposition 1.1. *There is a natural topology on $Or(M^n)$ with respect to which the map $p : Or(M^n) \to M^n$ defined by $p(v) = x$ if and only if $v \in Or_x(M^n)$, is a two-fold covering space.*

Before we prove this proposition, we note that we can, as a result, define what we mean by a "continuous choice of local orientations". That is, such a continuous choice would simply be a continuous cross section $\sigma : M^n \to Or(M^n)$ of this covering space. This means that σ is a continuous map with the property that $p(\sigma(x)) = x$ for all $x \in M^n$. Notice that such a continuous section $x \to \sigma(x) \in Or_x(M^n)$ is precisely a continuous choice of local orientation as x varies over all points of $x \in M^n$. The continuity is reflected by the topology of $Or(M^n)$ stated in Proposition 1.1.

We now prove Proposition 1.1.

Proof. Let $\mathcal{U} = \{(U_\alpha, \phi_{U_\alpha}) : \alpha \in \Lambda\}$ be an open cover of M^n by charts. That is, $M = \bigcup_{\alpha \in \Lambda} U_\alpha$, and each $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. Notice that for each pair $\alpha, \beta \in \Lambda$, there is a continuous map

$$\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(\phi_{U_\alpha}(U_\alpha \cap U_\beta);\ \phi_{U_\beta}(U_\alpha \cap U_\beta))$$

where the target is the space of homeomorphisms between these two open subspaces of $\mathbb{R}^n$. This space of homeomorphisms is endowed with the compact-open topology. Each such homeomorphism determines an isomorphism

$$H_n(\phi_\alpha(U_\alpha \cap U_\beta); \phi_\alpha(U_\alpha \cap U_\beta) - \{\phi_\alpha(x)\}) \xrightarrow{\cong} H_n(\phi_\beta(U_\alpha \cap U_\beta); \phi_\beta(U_\alpha \cap U_\beta) - \{\phi_\beta(x)\}).$$

By excision, this in turn determines a self-isomorphism

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Notice that since $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$, The group of such self isomorphisms consists of the identity and minus the identity. That is, this isomorphism group is $\mathbb{Z}/2$.

Thus $\psi_{\alpha,\beta}$ determines a continuous locally constant (i.e constant on each path component) map

$$\Psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}/2 = \{\pm 1\}.$$

We then give an alternate definition,

$$Or(M^n) = \coprod_{\alpha \in \Lambda} U_\alpha \times Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))/\sim \tag{1.2}$$

where $Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is the two-point set of generators of this homology group, and the equivalence relation $\sim$ is defined by the following: If $x \in U_\alpha \cap U_\beta$ and $\gamma \in Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$, then

$$(x, \gamma) \sim (x, \Psi_{\alpha,\beta}(x)(\gamma))$$

where $(x, \gamma) \in U_\alpha \times Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$ and $(x, \Psi_{\alpha,\beta}(x)(\gamma)) \in U_\beta \times Gen(H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$.

$Or(M^n)$, as defined by (1.2) and it is given the quotient topology.

□

Exercise. Finish the proof of Proposition 1.1. Specifically show that as sets, the two definitions of $Or(M^n)$ given in (1.1) and (1.2) are the same, and that the map

$$p : Or(M^n) \to M^n$$
$$(x, \gamma) \to x$$

is a two-fold covering map.

Notice that if M^n is orientable, which is to say, the orientation double cover admits a section, $\sigma : M^n \to Or(M^n)$, then it has another orientation, called the *opposite orientaion,* and written $-\sigma$, whose value on a point $x \in M^n$ is the unique point in $Or_x(M^n)$ that is *not* equal to $\sigma(x)$.

Corollary 1.2. *A manifold M^n admits an orientation if and only if the orientation double covering $p : Or(M^n) \to M^n$ is trivial. That is, it admits an isomorphism of covering spaces, to the trivial double covering space, $\pi : M \times \mathbb{Z}/2 \to M$ defined by projecting onto the first coordinate.*

Proof. Suppose M^n is orientable. Then the orientation double cover $p : Or(M^n) \to M^n$ admits a continuous section $\sigma : M^n \to Or(M^n)$. We can then define a trivialization Θ of the covering space

$$\begin{array}{ccc} M^n \times \mathbb{Z}/2 & \xrightarrow{\Theta} & Or(M^n) \\ \pi \downarrow & & \downarrow p \\ M^n & \xrightarrow[=]{} & M^n \end{array}$$

by $\Theta(x,1) = \sigma(x)$, and $\Theta(x,-1) = -\sigma(x)$.

Conversely, assume that $Or(M^n)$ is trivial. That is, $Or(M^n)$ is isomorphic to $M \times \mathbb{Z}/2$ as covering spaces. Since $\pi : M^n \times \mathbb{Z}/2 \to M^n$ clearly admits two distinct sections, then so does $p : Or(M^n) \to M^n$. □

It will be quite helpful to have the following homological implications of orientability.

Theorem 1.3. *Let M^n be an n-manifold and $A \subset M^n$ a compact subspace. Then*

1. *If $\alpha : M^n \to Or(M^n)$ is a section of the orientation double cover (i.e an orientation of M^n), then there exists a unique homology class $\alpha_A \in H_n(M, M - A)$ whose image in $H_n(M, M - x)$ is $\alpha(x)$ for every $x \in A$.*
2. *$H_i(M, M - A) = 0$ for $i > n$.*

Observation. A compact manifold is often called "closed". Notice that if M^n is a closed oriented manifold, we can let $A = M^n$ and then the above theorem implies that exists a unique *"orientation class"* or *"fundamental class"* $[M^n] = \alpha_M \in H_n(M) \cong \mathbb{Z}$ with the property that the restriction of $[M^n]$ to $H_n(M^n, M^n - x)$ is the value of the orientation $\alpha(x)$.

Proof. We sketch the proof here. We refer the reader to Hatcher [67] Lemma 3.27.

The idea of the proof follows a theme that is often followed in studying homological properties of manifolds. Namely, one proves the theorem first for $\mathbb{R}^n$, which will imply a local version of the theorem for every manifold, and then use "patching arguments" such as the Mayer-Vietoris sequence, to prove the theorem for general manifolds.

We break down the proof of this theorem into four steps.

Step 1. We first observe that if the theorem is true for A and B (both compact), as well as $A \cap B$, then the theorem is true for $A \cup B$.

Consider the following Mayer-Vietoris sequence:

$$0 \to H_n(M, M - (A \cup B)) \xrightarrow{\Phi} H_n(M, M - A) \oplus H_n(M, M - B) \\ \xrightarrow{\Psi} H_n(M, M - (A \cap B)) \to \cdots$$

Here we are using the facts that $(M - A) \cup (M - B) = M - (A \cap B)$ and $(M - A) \cap (M - B) = M - (A \cup B)$.

Notice that the zero on the left side is the assumption that $H_{n+1}(M, M - (A \cap B)) = 0$.

Notice that $\Psi(\alpha_A \oplus \alpha_B) = 0$, since by assumption, α_A and α_B restrict to the same class in $H_n(M, M - (A \cap B))$. Using the fact that Φ is a monomorphism, one can conclude that there is a unique class $\alpha_{A\cup B} \in H_n(M^n, M^n - (A \cup B))$ that restricts to α_A in $H_n(M^n, M^n - A)$ and to α_B in $H_n(M^n, M^n - B)$. This completes Step 1.

Step 2. Assume the theorem is true for $M^n = \mathbb{R}^n$. We then prove the theorem for general n-manifolds M^n.

Notice that a compact set $A \subset M^n$ can be written as a finite union $A = A_1 \cup \cdots \cup A_k$, where each A_i is a subspace of a chart $A_i \subset U_i$. We apply the result of Step 1 to $(A_1 \cup \cdots \cup A_{k-1})$ and A_k. Notice that the intersection of these two spaces is $(A_1 \cap A_k) \cup \cdots \cup (A_{k-1} \cap A_k)$. This is a union of $k-1$ compact subspaces, each of which is contained in a chart. By induction, we could conclude the validity of the result in this step, if we knew it to be true for $k = 1$, i.e compact subsets A that are contained in a chart, $A \subset U$. But in this case,

$$H_n(M^n, M^n - A) \cong H_n(U, U - A)$$

by excision, which is isomorphic to $H_n(\mathbb{R}^n, \mathbb{R}^n - C)$, where C is a compact subspace of $\mathbb{R}^n$. But by the assumptions of this step, we know the theorem to be true in this case.

We are therefore reduced to proving the theorem for $M^n = \mathbb{R}^n$.

Step 3. Assume $M^n = \mathbb{R}^n$, and prove the theorem for the case $A = A_1 \cup \cdots \cup A_k$ where each A_i is convex. The same argument as was used to prove Step 2 reduces this to the case when A is itself convex. In this case

$$H_*(\mathbb{R}^n, \mathbb{R}^n - A) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - x)$$

since A is contractible with a canonical contraction to any $x \in A$. In particular $\mathbb{R}^n - A \simeq \mathbb{R}^n - x$.

We leave the general case of an arbitrary compact subspace $A \subset \mathbb{R}^n$ to the reader. This argument is carried out in detail in Hatcher's book [67].

□

We observe that if R is any commutative ring with unit, we could have done the entire discussion above using homology with R-coefficients. That is, we may define a covering space

$$p : Or(M^n; R) \to M^n$$

with the property that

$$p^{-1}(x) = Or_x(M^n; R) = Gen(H_n(M^n, M^n - x; R)).$$

By $Gen(H_n(M^n, M^n - x; R)$ we mean the following. By choosing a chart U around x, one has an isomorphism $H_n(M^n, M^n - x; R) \cong H_n(U, U - x; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - x; R) \cong R$. A generator of R is an element $u \in R$ such that $R \cdot u = R$. $Gen(H_n(M^n, M^n - x; R)$ is the preimage of the group of generators of R under this isomorphism. We observe that this group of "generators" is well defined. That is, it is independent of the choice of chart, even though the chart is what defines the isormorphism of

$$Gen(H_n(M^n, M^n - x; R)$$

with $Gen(R)$.

Definition 1.4. *If R is a commutative ring with unit, then an R-orientation of an n-dimensional manifold M^n is a section of the "R-orientation covering space" $p : Or(M^n; R) \to M^n$.*

Observations.

1. By sending $1 \in \mathbb{Z}$ to $1 \in R$, there is always a canonical ring homomorphism $\mathbb{Z} \to R$. This induces a map of covering spaces $Or(M^n) \to Or(M^n; R)$. Thus if M^n is $\mathbb{Z}$ - orientable, it is orientable with respect to any commutative ring with unit R. In fact a choice of $\mathbb{Z}$ - orientation of M^n induces an R-orientation.
2. Let $R = \mathbb{Z}/2$. Then since $Gen(\mathbb{Z}/2) = \{1\}$ is the trivial, one-element group, then the covering space $p : Or(M^n; \mathbb{Z}/2) \to M^n$ is a homeomorphism. Thus it has a unique section. So every manifold is $\mathbb{Z}/2$-orientable, and has a unique $\mathbb{Z}/2$-orientation.
3. Finally observe that Theorem 1.3 can be generalized to a statement about R-orientations for any commutative ring R. In particular when $R = \mathbb{Z}/2$ one has the following consequence.

Corollary 1.4. *Let M^n be a connected, closed n-dimensional manifold. Then*

$$H_n(M^n; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

1.0.2 Poincaré Duality

Poincaré duality states that for a closed, orientable n-dimensional manifold M^n, the k^{th}-cohomology group and the $(n-k)^{th}$ homology group are isomorphic. The isomorphism is given by the "cap product" with the fundamental, or orientation class $[M^n] \in H_n(M)$. Before we state the Poincaré Duality theorem more carefully, and in more generality, we recall the cap product operation. We refer the reader to any introductory text in algebraic topology for details.

Let X be any topological space, and let R be a commutative ring with unit. The cap product operation is an operation of the form

$$\cap : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R) \quad \text{for } k \geq \ell.$$

Let $[v_0, \cdots v_k]$ represent the k- simplex spanned by vectors $v_0, \cdots v_k \in \mathbb{R}^N$, where N is large. Let $\sigma \in C_k(X;R)$, and $\phi \in C^\ell(X;R)$. Then one defines

$$\sigma \cap \phi = \phi(\sigma_{|_{[v_0,\cdots,v_\ell]}}) \cdot \sigma_{|_{[v_\ell,\cdots,v_k]}}. \tag{1.3}$$

One will then find that the boundary of this cap product chain is given by

$$\partial(\sigma \cap \phi) = (-1)^\ell(\partial\sigma \cap \phi - \sigma \cap \delta\phi) \tag{1.4}$$

where $\partial : C_r(X;R) \to C_{r-1}(X;R)$ is the boundary operator and $\delta : C^p(X;R) \to C^{p+1}(X;R)$ is the coboundary operator. Notice that this formula quickly implies that the cap product of a cycle with a cocycle is a cycle, and hence induces an operation

$$\cap : H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R). \tag{1.5}$$

And indeed it gives operations on relative (co)homology:

$$\begin{aligned} \cap : &H_k(X,A;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X,A;R) \\ &H_k(XA;R) \times H^\ell(X,A;R) \longrightarrow H_{k-\ell}(X;R) \end{aligned} \tag{1.6}$$

The reader can check that the cap product satisfies the following rather odd naturality property:

$$f_*(\alpha) \cap \phi = f_*(\alpha \cap f^*(\phi)). \tag{1.7}$$

This property becomes more reasonable (and easier to remember) when one realizes that it simply says that if $f : X \to Y$ is a continuous map, then the following diagram commutes:

$$\begin{array}{ccc} H_k(X) \times H^\ell(X) & \xrightarrow{\cap} & H_{k-\ell}(X) \\ \downarrow f_* \quad \uparrow f^* & & \downarrow f_* \\ H_k(Y) \times H^\ell(Y) & \xrightarrow[\cap]{} & H_{k-\ell}(Y) \end{array}$$

Exercise. Show that the cap product is adjoint to the cup product in cohomology. That is, prove that for $\phi \in H^\ell(X;R)$, $\sigma \in H_k(X;R)$, and $\psi \in H^{k-\ell}(X;R)$, then

$$\langle \psi \cup \phi; \sigma \rangle = \pm \langle \psi, \sigma \cap \phi \rangle. \tag{1.8}$$

Here $<,>$ represents the evaluation pairing of cohomology on homology.

The following is the basic statement of Poincaré Duality:

Theorem 1.5. *(Poincaré Duality) If M^n is a closed, R-oriented n-dimensional manifold with fundamental class $[M^n] \in H_n(M^n; R)$, then the map*

$$D = [M^n] \cap \_ : H^k(M^n : R) \to H_{n-k}(M^n; R)$$

is an isomorphism for all k.

Exercise. Show that the Poincaré Duality theorem implies that if F is a field and M^n is a closed F-oriented manifold with fundamental class $[M^n] \in H_n(M^n; F)$, then the pairing

$$\begin{aligned} H^k(M^n; F) \times H^{n-k}(M^n; F) &\longrightarrow F \\ \phi \times \psi &\to \langle \phi \cup \psi, [M^n] \rangle \end{aligned} \tag{1.9}$$

is nonsingular for every $k = 0, \cdots, n$.

In order to prove the Poincaré Duality theorem for compact manifolds, it actually is useful to generalize the theorem to the setting of noncompact manifolds. In this setting, however, one must use the notion of "cohomology with compact supports".

Roughly, a cochain with compact supports is one which is zero on chains living outside some compact set. More carefully,

$$C^i_c(X; G) = \bigcup_{K \text{ compact}} C^i(X, X - K; G).$$

(Strictly speaking, by the union sign we mean the colimit.) The ordinary coboundary map defines a cochain complex

$$\cdots \to C^i_c(X; G) \xrightarrow{\delta} C^{i+1}_c(X; G) \xrightarrow{\delta} \cdots \tag{1.10}$$

The resulting cohomology is written as $H^*_c(X; G)$.

Exercise. Show that

$$H^*_c(\mathbb{R}^n; G) \cong \tilde{H}^*(S^n; G)$$

and more generally that

$$H^*_c(X; G) \cong \tilde{H}^*(X \cup \infty; G)$$

where $X \cup \infty$ is the one-point compactification of X. Here we must assume that the point at infinity in the one-point compactification has a contractible open neighborhood.

Notice that by Theorem 1.3, that if M^n is an R-orientable n-manifold with orientation α, then for every compact subspace $K \subset M^n$, there is a well-defined orientation class $\alpha_K \in H_n(M^n; M^n - K; R)$ that restricts to the R-orientation $\alpha(x) \in H_n(M^n, M^n - \{x\}; R)$. Consider the cap product

$$H^k(M^n, M^n - K; R) \times H_n(M^n, M^n - K; R) \to H_{n-k}(M^n; R).$$

Capping with α_K defines an operation

$$\cap\alpha_K : H^k(M^n, M^n - K; R) \to H_{n-k}(M^n; R).$$

Taking the colimit over K defines a duality operation from the cohomology with compact supports:

$$D_{M^n} : H^k_c(M^n; R) \to H_{n-k}(M^n; R).$$

The following is the generalized form of Poincaré duality that we will prove:

Theorem 1.6. *Let M^n be an R-oriented manifold. Then the duality map*

$$D_{M^n} : H^k_c(M^n; R) \to H_{n-k}(M^n; R).$$

is an isomorphism for all k.

The proof of Theorem 1.6 (and thereby Theorem 1.5) involves a "patching" argument, for which we will need a lemma involving the Mayer Vietoris sequence.

Notice that if K and L are compact subspaces of M, we have the set theoretic properties,

$$(M - K) \cup (M - L) = M - (K \cap L) \quad \text{and}$$
$$(M - K) \cap (M - L) = M - (K \cup L).$$

So in cohomology there is a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H^k(M;M - (K \cap L)) \to H^k(M, M - K) \oplus H^k(M, M - L) \qquad (1.11) \\ \to H^k(M, M - (K \cup L)) \to H^{k+1}(M, M - (K \cap L)) \to \cdots \end{aligned}$$

Now suppose $M^n = U \cup W$, where both U and W are open subsets. By taking a limit over compact subsets, Mayer-Vietoris sequence (1.11) yields the following Mayer-Vietoris sequence of cohomologies with compact supports:

$$\cdots \to H^k_c(U \cap W) \to H^k_c(U) \oplus H^k_c(W) \to H^k_c(M^n) \to H^{k+1}_c(U \cap W) \to \cdots$$

We leave to the reader to check the following lemma.

Lemma 1.7. *Let M^n be an R- oriented n-manifold with $M = U \cup W$, where both U and W are open subsets. Then there is a commutative diagram of Mayer-Vietoris sequences:*

$$\begin{array}{ccccccccc} H^k_c(U \cap W) & \longrightarrow & H^k_c(U) \oplus H^k_c(W) & \longrightarrow & H^k_c(M^n) & \longrightarrow & H^{k+1}_c(U \cap W) & \longrightarrow \\ \downarrow D_{U\cap W} & & \downarrow D_U \oplus D_W & & \downarrow D_{M^n} & & \downarrow D_{U\cap V} & \\ H_{n-k}(U \cap W) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(W) & \longrightarrow & H_{n-k}(M^n) & \longrightarrow & H_{n-k-1}(U \cap W) & \longrightarrow \end{array}$$

Here all (co)homologies are taken with R-coefficients.

We now sketch the proof of Theorem 1.6. We again refer the reader to [67] for a more detailed exposition.

Proof. This proof has several steps.

Step 1. If $M^n = U \cup W$, and D_U, D_W and $D_{U\cap W}$ are isomorphisms, then so is D_M,

This follows from the above Lemma 1.7 and the five lemma.

Step 2. The theorem holds for $M^n = \mathbb{R}^n$.

Proof. Think of $\mathbb{R}^n$ as the interior of the closed unit ball around the origin, B_1. For any positive real number r, let B_r be the closed ball around origin in $\mathbb{R}^n$ of radius r. For r a number strictly between 0 and 1, notice that

$$H_n(B_1, B_1 - B_r) = H_n(B_r, \partial B_r) \cong H_n(B_1, \partial B_1) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}.$$

Since any compact set $K \subset \mathbb{R}^n = interior(B_1)$ is a subset of B_R for some $R > 0$, we see that $H^*_c(\mathbb{R}^n) \cong H^*(B_1, \partial B_1)$, and the reader can readily check that taking the cap product with the generator of $H_n(B_1, \partial B_1)$ gives the evaluation map

$$H^n(B_1, \partial B_1) \cong Hom(H_n(B_1, \partial B_1), \mathbb{Z}) \xrightarrow{\cong} Hom(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$$

where the last isomorphism is given by evaluating on a generator of $H_n(B_1, \partial B_1)$, which is to say, its fundamental class.

Step 3. The theorem holds for M^n an arbitrary open subset of $\mathbb{R}^n$.

Proof. Write M^n as a countable union of convex open sets in $\mathbb{R}^n$.

$$M^n = \bigcup_j U_j.$$

Let $V_i = \bigcup_{j<i} U_j$. Notice that both V_i and $V_i \cap U_i$ are unions of $i-1$ convex open sets. So we may make an inductive assumption that the theorem holds for manifolds that are the union of less than or equal to $i-1$ convex open sets in $\mathbb{R}^n$. So D_{V_i} and $D_{V_i\cap U_i}$ are isomorphisms. Then Step 1 implies that $D_{V_i\cup U_i}$ is an isomorphism. But $V_i \cup U_i = V_{i+1}$. This completes the inductive step.

Step 4. The theorem holds if M^n is a countable union of open sets U_i each homeomorphic to $\mathbb{R}^n$.

Proof. This follows by the same argument as in Step 3, with "open set in $\mathbb{R}^n$" replacing "convex open set in $\mathbb{R}^n$". We leave the details to the reader.

We are now done for manifolds that can be expressed as a countable union of charts. We now prove the general case.

Step 5. The general case.

Proof. Consider the collection of open sets $U \subset M^n$ for which D_U is an isomorphism. This collection is partially ordered by inclusion. Notice that the union of every totally ordered subcollection is again in this collection, by the argument in Step 3.

Zorn's Lemma implies that there is a maximal open set U for which this theorem holds. We claim that $U = M^n$. If $U \neq M^n$, let $x \in M^n - U$, and let V be a chart around x. Since V is homeomorphic to $\mathbb{R}^n$, the theorem holds for V by Step 2. It also holds for $U \cap V$ by Step 3. Therefore by Step 1, the theorem holds for $U \cup V$. This contradicts the maximality of U, so we must conclude that $U = M^n$. □

2

Fiber Bundles

In this chapter we define our basic object of study: locally trivial fibrations, or "fiber bundles". We discuss many examples, including spaces, vector bundles, and principal bundles. We also describe various constructions on bundles, including pull-backs, sums, and products.

As mentioned earlier in this book, throughout all that follows, all spaces will be assumed to be of the homotopy type of CW-complexes.

2.1 Definitions and examples

Let B be connected space with a basepoint $b_0 \in B$, and $p : E \to B$ be a continuous map.

Definition 2.1. *The map* $p : E \to B$ *is a* **locally trivial fibration, or fiber bundle,** with fiber **F** if it satisfies the following properties:

1. $p^{-1}(b_0) = F$
2. $p : E \to B$ is surjective
3. For every point $x \in B$ there is an open neighborhood $U_x \subset B$ and a "fiber preserving homeomorphism" $\Psi_{U_x} : p^{-1}(U_x) \to U_x \times F$, that is a homeomorphism making the following diagram commute:

$$\begin{array}{ccc} p^{-1}(U_x) & \xrightarrow[\cong]{\Psi_{U_x}} & U_x \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle proj} \\ U_x & = & U_x \end{array}$$

Some examples:

- The projection map $X \times F \longrightarrow X$ is the *trivial* fibration over X with fiber F.

- Let $S^1 \subset \mathbb{C}$ be the unit circle with basepoint $1 \in S^1$. Consider the map $f_n : S^1 \to S^1$ given by $f_n(z) = z^n$. Then $f_n : S^1 \to S^1$ is a locally trivial fibration with fiber a set of n distinct points (the n^{th} roots of unity in S^1).
- Let $exp : \mathbb{R} \to S^1$ be the covering space given by
$$exp(t) = e^{2\pi it} \in S^1.$$
Then exp is a locally trivial fibration with fiber the integers $\mathbb{Z}$.
- Recall that the n - dimensional real projective space $\mathbb{RP}^n$ is defined by
$$\mathbb{RP}^n = S^n / \sim$$
where $x \sim -x$, for $x \in S^n \subset \mathbb{R}^{n+1}$.
Let $p : S^n \to \mathbb{RP}^n$ be the projection map. This is a locally trivial fibration with fiber the two point set.
- As the reader has undoubtedly observed, the previous three examples are covering spaces. Indeed all covering spaces are examples of locally trivial fibrations, or fiber bundles.
- The complex analogue of the last example. Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$. Recall that the complex projective space $\mathbb{CP}^n$ is defined by
$$\mathbb{CP}^n = S^{2n+1} / \sim$$
where $x \sim ux$, where $x \in S^{2n+1} \subset \mathbb{C}^n$, and $u \in S^1 \subset \mathbb{C}$. Then the projection $p : S^{2n+1} \to \mathbb{CP}^n$ is a locally trivial fibration with fiber S^1.
- Consider the Moebeus band $M = [0,1] \times [0,1]/ \sim$ where $(t,0) \sim (1-t,1)$. Let C be the "center circle" $C = \{(1/2, s) \in M\}$ and consider the projection
$$\begin{aligned} p : M &\to C \\ (t,s) &\to (1/2, s). \end{aligned}$$
This map is a locally trivial fibration with fiber $[0,1]$.

Given a fiber bundle $p : E \to B$ with fiber F, the space B is called the *base space* and the space E is called the total space. We will denote this data by a triple (F, E, B).

Definition 2.2. *A map (or "morphism") of fiber bundles $\Phi : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ is a pair of basepoint preserving continuous maps $\bar{\phi} : E_1 \to E_2$ and $\phi : B_1 \to B_2$ making the following diagram commute:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{\phi}} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\phi]{} & B_2 \end{array}$$

Notice that such a map of fibrations determines a continuous map of the fibers, $\phi_0 : F_1 \to F_2$.

A map of fiber bundles $\Phi : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ is an isomorphism if there is an inverse map of fibrations $\Phi^{-1} : (F_2, E_2, B_2) \to (F_1, E_1, B_1)$ so that $\Phi \circ \Phi^{-1}$ $\Phi^{-1} \circ \Phi = 1$ are both the identity maps.

Finally we say that a fibration (F, E, B) is *trivial* if it isomorphic to the trivial fibration $B \times F \to B$.

Exercise. Verify that all of the above examples of fiber bundles are all nontrivial except for the first one.

The notion of a locally trivial fibration or fiber bundle is quite general and includes examples of many types. For example we already noticed that **covering spaces** are examples of locally trivial fibrations. Covering spaces are locally trivial fibrations with discrete fibers. Two other very important classes of examples of fiber bundles are *vector bundles* and *principal bundles.* We now describe these notions in some detail.

2.1.1 Vector Bundles

Definition 2.3. *An n- dimensional vector bundle with fiber an n - dimensional* $\mathbf{k}$ *- vector space V over a field* $\mathbf{k}$*, is a locally trivial fibration $p : E \to B$ satisfying the following properties:*

1. *For each $x \in B$, the fiber $p^{-1}(x) \subset E$ has the structure of an n-dimensional* $\mathbf{k}$ *- vector space,*
2. *the local trivializations*

$$\psi : p^{-1}(U) \to U \times V$$

induce $\mathbf{k}$ *- linear isomorphisms on each fiber. That is, restricted to each $x \in U$, ψ defines a* $\mathbf{k}$ *- linear isomorphism*

$$\psi : p^{-1}(x) \xrightarrow{\cong} \{x\} \times V.$$

It is common to denote the data (V, E, B) defining an n - dimensional vector bundle by a Greek letter, e.g ζ.

A "map" or "morphism" of vector bundles $\Phi : \zeta \to \xi$ is a map of fiber bundles as defined above, with the added requirement that when restricted to each fiber, $\bar{\phi}$ is a $\mathbf{k}$ - linear transformation.

Examples

- Given an n - dimensional **k** vector space V, then $B \times V \to B$ is the corresponding trivial bundle over the base space B. Notice that all n - dimensional trivial bundles over B are isomorphic, and we denote its isomorphism class by ϵ_n.

- Consider the " Moebeus line bundle" μ defined to the the one dimensional real vector bundle ("line bundle") over the circle given as follows. Let $E = [0,1] \times \mathbb{R}/\sim$ where $(0,t) \sim (1,-t)$. Let C be the "middle" circle $C = \{(s,0) \in E\}$. Then μ is the line bundle defined by the projection

$$\begin{aligned} p &: E \to C \\ (s,t) &\to (s,0). \end{aligned}$$

- Define the real line bundle γ_1 over the projective space $\mathbb{RP}^n$ as follows. Let $x \in S^n$. Let $[x] \in \mathbb{RP}^n = S^n/\sim$ be the class represented by x. Then $[x]$ determines, and is determined by the line through the origin in $\mathbb{R}^{n+1}$ going through x. It is well defined since both representatives of $[x]$ (x and $-x$) determine the same line. Thus $\mathbb{RP}^n$ can be thought of as the space of lines through the origin in $\mathbb{R}^{n+1}$. Let $E = \{([x],v) : [x] \in \mathbb{RP}^n, v \in [x]\}$. Then γ_1 is the line bundle defined by the projection

$$\begin{aligned} p &: E \to \mathbb{RP}^n \\ ([x],v) &\to [x]. \end{aligned}$$

  Exercise. Verify that the $\mathbb{RP}^1$ is a homeomorphic to a circle, and the line bundle γ_1 over $\mathbb{RP}^1$ is isomorphic to the Moebeus line bundle μ.

- By abuse of notation we let γ_1 also denote the *complex* line bundle over $\mathbb{CP}^n$ defined analogously to the *real* line bundle γ_1 over $\mathbb{RP}^n$ above.

- Let $Gr_k(\mathbb{R}^n)$ (respectively $Gr_k(\mathbb{C}^n)$) be the space whose points are k - dimensional subvector spaces of $\mathbb{R}^n$ (respectively $\mathbb{C}^n$). These spaces are called *"Grassmannian"* manifolds, and are topologized as follows. Let $V_k(\mathbb{R}^n)$ denote the space of *injective* linear transformations from $\mathbb{R}^k$ to $\mathbb{R}^n$. Let $V_k(\mathbb{C}^n)$ denote the analogous space of injective linear transformations $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$. These spaces are called *"Stiefel manifolds"*, and can be thought of as spaces of $n \times k$ matrices of rank k. These spaces are given topologies as subspaces of the appropriate vector space of matrices. To define $Gr_k(\mathbb{R}^n)$ and $Gr_k(\mathbb{C}^n)$, we put an equivalence relation on $V_k(\mathbb{R}^n)$ and $V_k(\mathbb{C}^n)$ by saying that two transformations A and B are equivalent if they have the same image in $\mathbb{R}^n$ (or $\mathbb{C}^n$). If viewed as matrices, then $A \sim B$ if and only if there is an element $C \in GL(k,\mathbb{R})$ (or $GL(k,\mathbb{C})$) so that $A = BC$. Then the equivalence classes of these matrices are completely determined by their image in $\mathbb{R}^n$ (or $\mathbb{C}^n$), i.e the equivalence class is determined completely by a k - dimensional subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Thus we define

$$Gr_k(\mathbb{R}^n) = V_k(\mathbb{R}^n)/\sim \quad \text{and} \quad Gr_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim$$

with the corresponding quotient topologies.

Consider the vector bundle γ_k over $Gr_k(\mathbb{R}^n)$ whose total space E is the subspace of $Gr_k(\mathbb{R}^n) \times \mathbb{C}^n$ defined by

$$E = \{(W, \omega) : W \in Gr_k(\mathbb{R}^n) \, \text{and} \, \omega \in W \subset \mathbb{R}^n\}.$$

Then γ_k is the vector bundle given by the natural projection

$$\begin{aligned} E &\to Gr_k(\mathbb{R}^n) \\ (W, \omega) &\to W \end{aligned}$$

For reasons that will become apparent later, the bundles γ_k are called the "universal" or "canonical" k - dimensional bundles over the Grassmannians.

- Notice that the universal bundle γ_k over the Grassmanians $Gr_k(\mathbb{R}^n)$ and $Gr_k(\mathbb{C}^n)$ come equipped with embeddings (i.e injective vector bundle maps) in the trivial bundles $Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$ and $Gr_k(\mathbb{C}^n) \times \mathbb{C}^n$ respectively. We can define the orthogonal complement bundles $\gamma_k^\perp$ to be the $n - k$ dimensional bundles whose total spaces are given by

$$E_k^\perp = \{(W, \nu) \in Gr_k(\mathbb{R}^n) \times \mathbb{R}^n : \nu \perp W\}$$

and similarly over $Gr_k(\mathbb{C}^n)$. Observe that the natural projection to the Grassmannian defines $n - k$ dimensional vector bundles (over $\mathbb{R}$ and $\mathbb{C}$ respectively).

Exercises

1. Verify that γ_k is a k -dimensional real vector bundle over $Gr_k(\mathbb{R}^n)$.
2. Define the analogous bundle (which by abuse of notation we also call γ_k) over $Gr_k(\mathbb{C}^n)$. Verify that it is a k-dimensional complex vector bundle over $Gr_k(\mathbb{C}^n)$.
3. Verify that $\mathbb{RP}^{n-1} = Gr_1(\mathbb{R}^n)$ and that $\mathbb{CP}^{n-1} = Gr_1(\mathbb{C}^n)$.

An important notion associated to vector bundles (and in fact all fibrations) is the notion of a (cross) section. We've already encountered this notion when the fiber bundle is a covering space in our discussion of orientations in Chapter 1.

Definition 2.4. *Given a fiber bundle*

$$p : E \to B$$

a section s is a continuous map $s : B \to E$ such that $p \circ s = identity : B \to B$.

Notice that every vector bundle has a section, namely the *zero section*

$$\begin{aligned} z : B &\to E \\ x &\to 0_x \end{aligned}$$

where 0_x is the origin in the vector space $p^{-1}(x)$. However most geometrically interesting sections have few zero's. Indeed as we will see later, an appropriate count of the number of zero's of a section of an n - dimensional bundle over an n - dimensional manifold is an important topological invariant of that bundle (called the "Euler number"). In particular an interesting geometric question is to determine when a vector bundle has a nowhere zero section, and if it does, how many linearly independent sections it has. (Sections $\{s_1, \cdots, s_m\}$ are said to be linearly independant if the vectors $\{s_1(x), \cdots, s_m(x)\}$ are linearly independent for every $x \in B$.) These questions are classical in the case where the vector bundle is the tangent bundle, as we will see later in our discussion of differentiable manifolds. A section of the tangent bundle is called a **vector field.**The question of how many linearly independent vector fields exist on the sphere S^n was answered by J.F. Adams [4] in the early 1960's using sophisticated techniques of homotopy theory.

Exercises (from [121])

1. Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \to \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.

2. Use exercise 1 to show that $\mathbb{RP}^n$ is compact.

3. Prove that an n -dimensional vector bundle ζ has n - linearly independent sections if and only if ζ is trivial.

2.1.2 Principal Bundles

Principal bundles are basically parameterized families of topological groups, and often Lie groups. (A Lie group is a topological group with a compatible differentiable structure. Such structures will be discussed in Chapter 3.) In order to define the notion of a principal bundle carefully we first review some basic properties of group actions.

Recall that a right action of topological group G on a space X is a map

$$\begin{aligned} \mu : X \times G &\to X \\ (x, g) &\to xg \end{aligned}$$

satisfying the basic properties

1. $x \cdot 1 = x$ for all $x \in X$
2. $x(g_1 g_2) = (xg_1)g_2$ for all $x \in X$ and $g_1,\, g_2 \in G$.

Notice that given such an action, every element g acts as a homeomorphism, since action by g^{-1} is its inverse. Thus the group action μ defines a map

$$\mu : G \to Homeo(X)$$

where $Homeo(X)$ denotes the group of homeomorphisms of X. The two conditions listed above are equivalent to the requirement that $\mu : G \to Homeo(X)$ be a group homomorphism.

Let X be a space with a right G - action. Given $x \in X$, let $xG = \{xg : g \in G\} \subset X$. This is called the *orbit* of x under the G - action. The isotropy subgroup of x, $Iso(x)$, is defined by $Iso(x) = \{g \in G : xg = x\}$ Notice that the map

$$G \to xG$$

defined by sending g to xg defines a homeomorphism from the coset space to the orbit

$$G/Iso(x) \xrightarrow{\cong} xG \subset X.$$

A group action on a space X is said to be *transitive* if the space X is the orbit of a single point, $X = xG$. Notice that if $X = x_0 G$ for some $x_0 \in X$, then $X = xG$ for *any* $x \in X$. Notice furthermore that the transitivity condition is equivalent to saying that for any two points $x_1, x_2 \in X$, there is an element $g \in G$ such that $x_1 = x_2 g$. Finally notice that if X has a transitive G - action, then the above discussion about isotropy subgroups implies that there exists a subgroup $H < G$ and a homeomorphism

$$G/H \xrightarrow{\cong} X.$$

Of course if X is smooth, G is a Lie group, and the action is smooth, then the above map would be a diffeomorphism.

A group action is said to be *(fixed point) free* if the isotropy groups of every point x are trivial,

$$Iso(x) = \{1\}$$

for all $x \in X$. Said another way, the action is free if and only if the only time there is an equation of the form $xg = x$ is if $g = 1 \in G$. That is, if for $g \in G$, the fixed point set $Fix(g) \subset X$ is the set

$$Fix(g) = \{x \in X : xg = x\},$$

then the action is free if and only if $Fix(g) = \emptyset$ for all $g \neq 1 \in G$.

We are now able to define principal bundles.

Definition 2.5. *Let G be a topological group. A principal G bundle* is a fiber bundle $p : E \to B$ with fiber $F = G$ satisfying the following properties.

1. The total space E has a free, fiberwise right G action. That is, it has a free group action making the following diagram commute:

$$\begin{array}{ccc} E \times G & \xrightarrow{\mu} & E \\ {\scriptstyle p\times\epsilon}\downarrow & & \downarrow{\scriptstyle p} \\ B \times \{1\} & = & B \end{array}$$

   where ϵ is the constant map.

2. The induced action on fibers

$$\mu : p^{-1}(x) \times G \to p^{-1}(x)$$

   is free and transitive.

3. There exist local trivializations

$$\psi : p^{-1}(U) \xrightarrow{\cong} U \times G$$

   that are equivariant. That is, the following diagrams commute:

$$\begin{array}{ccc} p^{-1}(U) \times G & \xrightarrow[\cong]{\psi\times 1} & U \times G \times G \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle 1\times \text{ mult.}} \\ p^{-1}(U) & \xrightarrow[\psi]{\cong} & U \times G. \end{array}$$

Notice that in a principal G - bundle, the group G acts freely on the total space E. It is natural to ask if a free group action suffices to induce a principal G - bundle. That is, suppose E is a space with a free, right G action, and define B to be the orbit space

$$B = E/G = E/\sim$$

where $y_1 \sim y_2$ if and only if there exists a $g \in G$ with $y_1 = y_2 g$ (i.e if and only if their orbits are equal: $y_1 G = y_2 G$). Define $p : E \to B$ to be the natural projection, $E \to E/G$. Then the fibers are the orbits, $p^{-1}([y]) = yG$. So for $p : E \to B$ to be a principal bundle we must check the local triviality condition.

An important example of this situation is the following (taken from the notes on principal bundles by S. Mitchell [122]): Consider the additive group of real numbers, $\mathbb{R}$, and its subgroup of rational numbers, $\mathbb{Q}$. As a subgroup of $\mathbb{R}$, $\mathbb{Q}$ acts freely on the right by translation:

$$\begin{aligned} \mathbb{R} \times \mathbb{Q} &\to \mathbb{R}. \\ (t, q) &\to t + q \end{aligned}$$

However the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is clearly *not* a principal $\mathbb{Q}$- bundle. For if it had locally trivial neighborhoods, then since $\mathbb{R}/\mathbb{Q}$ has the trivial topology, it would have to be globally trivial. But clearly $\mathbb{R}$ is not homeomorphic to $\mathbb{R}/\mathbb{Q} \times \mathbb{Q}$.

To avoid this type of example we simply define a subgroup H of G to be *admissible* if the quotient $G \to G/H$ is a principal bundle, i.e it has locally trivial neighborhoods. Clearly any subgroup of a discrete group is admissible. It is also known that any closed subgroup of a Lie group is admissible [133].

Proposition 2.1. *Suppose $G \to P \to B$ is a principal G-bundle. Suppose $H < G$ is an admissible subgroup. Then*

$$H \to P \to P/H$$

is a principal H-bundle.

Proof. . For any subgroup H we have that

$$P/H = P \times_G G/H$$

where the right side is the quotient of the diagonal action of G on $P \times G/H$. The fact that

$$P = P \times_G G \to P \times_G G/H = P/H$$

has local trivializations follows from that facts that $P \to P/G$ and $G \to G/H$ have local trivializations. □

Examples.

- The projection map $p : S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1 - bundle.
- Let $V_k(\mathbb{R}^n)$ be the Stiefel manifold of rank k $n \times k$ matrices described above. Then the projection map

  $$p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$$

  is a principal $GL(k, \mathbb{R})$ - bundle. Similarly the projection map

  $$p : V_k(\mathbb{C}^n) \to Gr_k(\mathbb{C}^n)$$

  is a principal $GL(k, \mathbb{C})$ - bundle.
- Let $V_k(\mathbb{R}^n)^O$ denote those $n \times k$ real matrices whose k - columns are orthonormal n - dimensional vectors. This is the Stiefel manifold of orthonormal k - frames in $\mathbb{R}^n$. Then the induced projection map

  $$p : V_k(\mathbb{R}^n)^O \to Gr_k(\mathbb{R}^n)$$

  defined by sending a k-frame in $\mathbb{R}^n$ to the subspace that they span, is a principal $O(k)$ - bundle. Similarly, if $V_k(\mathbb{C}^n)^U$ is the space of orthonormal k - frames in $\mathbb{C}^n$ (with respect to the standard Hermitian inner product), then the projection map

$$p : V_k(\mathbb{C}^n)^U \to Gr_k(\mathbb{C}^n)$$

is a principal $U(n)$ - bundle.

- There is a homeomorphism

$$\rho : U(n)/U(n-1) \xrightarrow{\cong} S^{2n-1}$$

and the projection map $U(n) \to S^{2n-1}$ is a principal $U(n-1)$ - bundle.

To see this, notice that $U(n)$ acts transitively on the unit sphere in $\mathbb{C}^n$ (i.e S^{2n-1}). Moreover the isotropy subgroup of the point $e_1 = (1, 0, \cdots, 0) \in S^{2n-1}$ are those elements $A \in U(n)$ which have first column equal to $e_1 = (1, 0, \cdots, 0)$. Such matrices also have first row $= (1, 0, \cdots, 0)$. That is, A is of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where A' is an element of $U(n-1)$. Thus the isotropy subgroup $Iso(e_1) \cong U(n-1)$ and the result follows.

Notice that a similar argument gives a diffeomorphism $SO(n)/SO(n-1) \cong S^{n-1}$.

- There is a homeomorphism

$$\rho : U(n)/U(n-k) \xrightarrow{\cong} V_k(\mathbb{C}^n)^U.$$

The argument here is similar to the above, noticing that $U(n)$ acts transitively on $V_k(\mathbb{C}^n)^U$, and the isotopy subgroup of the $n \times k$ matrix

$$e = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

consist of matrices in $U(n)$ of them form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \vdots & & \ddots & & \vdots \\ 0 & 0 & 1 & \cdots & 0 & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ & & (0) & & & & (B) & & & \end{pmatrix}$$

where B is an $(n-k) \times (n-k)$ dimensional unitary matrix.

- A similar argument shows that there are homeomorphisms

$$\rho : U(n)/\left(U(k) \times U(n-k)\right) \xrightarrow{\cong} Gr_k(\mathbb{C}^n)$$

and

$$\rho : O(n)/\left(O(k) \times O(n-k)\right) \xrightarrow{\cong} Gr_k(\mathbb{R}^n)$$

Principal bundles define other fiber bundles in the presence of group actions. Namely, suppose $p : E \to B$ be a principal G - bundle and F is a space with a cellular right group action. Then the product space $E \times F$ has the "diagonal" group action $(e, f)g = (eg, fg)$. Consider the orbit space, $E \times_G F = (E \times F)/G$. Then the induced projection map

$$p : E \times_G F \to B$$

is a locally trivial fibration with fiber F.

For example we have the following important class of fiber bundles.

Proposition 2.2. *Let G be a compact topological group and $K < H < G$ closed subgroups. Then the projection map of coset spaces*

$$p : G/K \to G/H$$

is a fiber bundle with fiber H/K.

Proof. Observe that $G/K \cong G \times_H H/K$ where H acts on H/K in the natural way. Moreover the projection map $p : G/K \to G/H$ is the projection can be viewed as the projection

$$G/K = G \times_H H/K \to G/H$$

and so is the H/K - fiber bundle induced by the H - principal bundle $G \to G/H$ via the action of H on the coset space H/K. □

Example

We know by the above examples, that $U(2)/U(1) \cong S^3$, and that $U(2)/U(1) \times U(1) \cong Gr_1(\mathbb{C}^2) = \mathbb{CP}^1 \cong S^2$. Therefore there is a principal $U(1)$ - fibration

$$p : U(2)/U(1) \to U(2)/U(1) \times U(1),$$

or equivalently, a principal $U(1) = S^1$ fibration

$$p : S^3 \to S^2.$$

This fibration is the well known "Hopf fibration", and is of central importance in both geometry and algebraic topology. In particular, as we will see later, the map from S^3 to S^2 gives an nontrivial element in the homotopy group $\pi_3(S^2)$, which from the naive point of view is quite surprising. It says, that, in a sense that can be made precise, there is a "three dimensional hole" in S^2 that cannot be filled. Many people (eg. Whitehead, see [160]) refer to this discovery as the beginning of modern homotopy theory.

The fact that the Hopf fibration is a locally trivial fibration also leads to an interesting geometric observation. First, it is not difficult to see directly (and we will prove this later) that by removing closed balls around the l "north and south" poles of S^2, one gets an open cover of the sphere consisting of two open sets homeomorphic to the open disk D^2. We call them D^2_+ and D^2_-. The Hopf fibration is trivial over these open sets. That is, there are local trivializations,

$$\psi_+ : D^2_+ \times S^1 \to p^{-1}(D^2_+)$$

and

$$\psi_- : D^2_- \times S^1 \to p^{-1}(D^2_-).$$

Putting these two local trivializations together yields the following classical result:

Theorem 2.3. *The sphere S^3 is homeomorphic to the union of two solid tori $D^2 \times S^1$ whose intersection is their common torus boundary, $S^1 \times S^1$.*

As another example of fiber bundles induced by principal bundles, suppose that

$$\rho : G \to GL(n, \mathbb{R})$$

is a representation of a topological group G, and $p : E \to B$ is a principal G bundle. Then let $\mathbb{R}^n(\rho)$ denote the space $\mathbb{R}^n$ with the action of G given by the representation ρ. Then the projection

$$E \times_G \mathbb{R}^n(\rho) \to B$$

is a vector bundle.

Exercise.

Let $p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ be the principal bundle described above. Let $\mathbb{R}^n$ have the standard $GL(n, \mathbb{R})$ representation. Proved that the induced vector bundle

$$p : V_k(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$$

is isomorphic to the universal bundle γ_k described in the last section.

In the last section we discussed sections of vector bundles and in particular vector fields. For principal bundles, the existence of a section (or lack thereof) completely determines the triviality of the bundle.

Theorem 2.4. *A principal G - bundle $p : E \to B$ is trivial if and only if it has a section.*

Proof. If $p : E \to B$ is isomorphic to the trivial bundle $B \times G \to B$, then clearly it has a section. So we therefore only need to prove the converse.

Suppose $s : B \to E$ is a section of the principal bundle $p : E \to B$. Define the map

$$\psi : B \times G \to E$$

by $\psi(b, g) = s(b)g$ where multiplication on the right by g is given by the right G - action of G on E. It is straightforward to check that ψ is an isomorphism of principal G - bundles, and hence a trivialization of E. □

2.1.3 Clutching Functions and Structure Groups

Let $p : E \to B$ be a fiber bundle with fiber F. Cover the basespace B by a collection of open sets $\{U_\alpha\}$ equipped with local trivializations $\psi_\alpha : U_\alpha \times F \xrightarrow{\cong} p^{-1}(U_\alpha)$. Let us compare the local trivializations on the intersection: $U_\alpha \cap U_\beta$:

$$U_\alpha \cap U_\beta \times F \xrightarrow[\cong]{\psi_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow[\cong]{\psi_\alpha^{-1}} U_\alpha \cap U_\beta \times F.$$

For every $x \in U_\alpha \cap U_\beta$, $\psi_\alpha^{-1} \circ \psi_\beta$ determines a homeomorphism of the fiber F. That is, this composition determines a map $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(F)$. These maps are called the *clutching functions* of the fiber bundle. When the bundle is a real n - dimensional vector bundle then the clutching functions are of the form

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

Similarly, complex vector bundles have clutching functions that take values in $GL(n, \mathbb{C})$.

If $p : E \to B$ is a G - principal - bundle, then the clutching functions take values in G:

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G.$$

In general for a bundle $p : E \to B$ with fiber F, the group in which the clutching values take values is called the *structure group* of the bundle. If no group is specified, then the structure group is the homeomorphism group $Homeo(F)$.

The clutching functions and the associated structure group completely determine the isomorphism type of the bundle. Namely, given an open covering of a space B, and a compatible family of clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$, and a space F upon which the group acts, we can form the space

$$E = \bigcup_\alpha U_\alpha \times F / \sim$$

where if $x \in U_\alpha \cap U_\beta$, then $(x, f) \in U_\alpha \times F$ is identified with $(x, f\phi_{\alpha,\beta}(x)) \in U_\beta \times F$. E is the total space of a locally trivial fibration over B with fiber F and structure group G. If the original data of clutching functions came from locally trivializations of a bundle, then notice that the construction of E above yields a description of the total space of the bundle. Thus we have a description of the total space of a fiber bundle completely in terms of the family of clutching functions.

Suppose ζ is an n - dimensional vector bundle with projection map $p : E \to B$ and local trivializations $\psi_\alpha : U_\alpha \times \mathbb{R}^n \to p^{-1}(U_\alpha)$. Then, as observed above, the clutching functions take values in the general linear group

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

So the total space E has the form $E = \bigcup_\alpha U_\alpha \times \mathbb{R}^n / \sim$ as above. We can then form the corresponding principal $GL(n, \mathbb{R})$ bundle with total space

$$E_{GL} = \bigcup_\alpha U_\alpha \times GL(n, \mathbb{R}) / \sim$$

with the same clutching functions. That is, for $x \in U_\alpha \cap U_\beta$, $(x, g) \in U_\alpha \times GL(n, \mathbb{R})$ is identified with $(x, g \cdot \phi_{\alpha,\beta}(x)) \in U_\beta \times GL(n, \mathbb{R})$. The principal bundle

$$p : E_{GL} \to B$$

is called the *associated principal bundle* to the vector bundle ζ, or sometimes is referred to as the *associated frame* bundle.

Observe also that this process is reversable. Namely if $p : P \to X$ is a principal $GL(n, \mathbb{R})$ - bundle with clutching functions $\theta_{\alpha,\beta} : V_\alpha \cap V_\beta \to GL(n, \mathbb{R})$, then there is an associated vector bundle $p : P_{\mathbb{R}^n} \to X$ where

$$P_{\mathbb{R}^n} = \bigcup_\alpha V_\alpha \times \mathbb{R}^n$$

where if $x \in V_\alpha \cap V_\beta$, then $(x, v) \in V_\alpha \times \mathbb{R}^n$ is identified with $(x, v \cdot \theta_{\alpha,\beta}(x)) \in V_\beta \times \mathbb{R}^n$.

This correspondence between vector bundles and principal bundles proves the following result:

Theorem 2.5. *Let $Vect_n^{\mathbb{R}}(X)$ and $Vect_n^{\mathbb{C}}(X)$ denote the set of isomorphism classes of real and complex n - dimensional vector bundles ovr X respectively. For a Lie group G let $Prin_G(X)$ denote the set of isomorphism classes of principal G - bundles. Then there are bijective correspondences*

$$Vect_n^{\mathbb{R}}(X) \xrightarrow{\cong} Prin_{GL(n,\mathbb{R})}(X)$$
$$Vect_n^{\mathbb{C}}(X) \xrightarrow{\cong} Prin_{GL(n,\mathbb{C})}(X).$$

This correspondence and Theorem 2.4 allows for the following method of determining whether a vector bundle is trivial:

Corollary 2.6. *A vector bundle $\zeta : p : E \to B$ is trivial if and only if its associated principal $GL(n)$ - bundle $p : E_{GL} \to B$ admits a section.*

Clutching functions and structure groups are also useful in studying structures on principal bundles and their associated vector bundles.

Definition 2.6. *Let $p : P \to B$ be a principal G - bundle, and let $H < G$ be a subgroup. P is said to have a reduction of its structure group to H if and only if P is isomorphic to a bundle whose clutching functions take values in H:*

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to H < G.$$

Let $P \to X$ be a principal G - bundle. Then P has a reduction of its structure group to $H < G$ if and only if there is a principal H - bundle $\tilde{P} \to X$ and an isomorphism of G bundles,

$$\begin{array}{ccc} \tilde{P} \times_H G & \xrightarrow{\cong} & P \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

Definition 2.7. *Let $H < GL(n, \mathbb{R})$. Then an H - structure on an n - dimensional vector bundle ζ is a reduction of the structure group of its associated $GL(n, \mathbb{R})$ - principal bundle to H.*

Examples.

- A $\{1\} < GL(n, \mathbb{R})$ - structure on a vector bundle (or its associated principal bundle) is a trivialization or *framing* of the bundle. A framed manifold is a manifold with a framing of its tangent bundle.
- Given a $2n$ - dimensional real vector bundle ζ, an *almost complex structure* on ζ is a $GL(n, \mathbb{C}) < GL(2n, \mathbb{R})$ structure on its associated principal bundle. An almost complex structure on a manifold is an almost complex structure on its tangent bundle.

We now study two examples of vector bundle structures in some detail: Euclidean structures, and orientations.

Example 1: $\mathbf{O}(n)$ - structures and Euclidean structures on vector bundles.

Recall that a Euclidean vector space is a real vector space V together with a positive definite quadratic function

$$\mu : V \to \mathbb{R}.$$

Specifically, the statement that μ is quadratic means that it can written in the form

$$\mu(v) = \sum_i \alpha_i(v)\beta_i(v)$$

where each α_i and $\beta_i : V \to \mathbb{R}$ is linear. The statement that μ is positive definite means that

$$\mu(v) > 0 \quad \text{for } v \neq 0.$$

Positive definite quadratic functions arise from, and give rise to inner products (i.e symmetric bilinear pairings $(v, w) \to v \cdot w$) defined by

$$v \cdot w = \frac{1}{2}(\mu(v + w) - \mu(v) - \mu(w)).$$

Notice that if we write $|v| = \sqrt{v \cdot v}$ then $|v|^2 = \mu(v)$. So in particular there is a metric on V.

This notion generalizes to vector bundles in the following way.

Definition 2.8. *A Euclidean vector bundle is a real vector bundle $\zeta : p : E \to B$ together with a map*

$$\mu : E \to \mathbb{R}$$

which when restricted to each fiber is a positive definite quadratic function. That is, μ induces a Euclidean structure on each fiber.

Exercise.

Show that an $O(n)$ - structure on a vector bundle ζ gives rise to a Euclidean structure on ζ. Conversely, a Euclidean structure on ζ gives rise to an $O(n)$ - structure.

Hint. Make the constructions directly in terms of the clutching functions.

In the next chapter we will discuss *smooth*, or *differentiable* manifolds. As we will see, every such manifold comes equipped with a tangent bundle $\tau M \to M$. The following concept is fundamental in Differential Geometry.

Definition 2.9. *A smooth Euclidean structure on the tangent bundle $\mu : \tau M \to \mathbb{R}$ is called a* Riemannian structure on M.

Exercises.

1. *Existence theorem for Euclidean metrics.* Using a partition of unity, show that any vector bundle over a paracompact space can be given a Euclidean structure.

2. *Isometry theorem.* Let μ and μ' be two different Euclidean structures on the same vector bundle $\zeta : p : E \to B$. Prove that there exists a self map of vector bundles $f : E \to E$ which is a homeomorphism on the total space, and which carries each fiber isomorphically onto itself, so that the composition $\mu \circ f : E \to \mathbb{R}$ is equal to μ'. (*Hint.* Use the fact that every positive definite matrix A can be expressed uniquely as the square of a positive definite matrix $\sqrt{A}$. The power series expansion

$$\sqrt{(tI + X)} = \sqrt{t}(I + \frac{1}{2t}X - \frac{1}{8t^2}X^2 + - \cdots),$$

is valid providing that the characteristic roots of $tI + X = A$ lie between 0 and $2t$. This shows that the function $A \to \sqrt{A}$ is continuous, and in fact smooth.)

Example 2: S$L(n, \mathbb{R})$ - structures and orientations.

Recall that an orientation of a real n - dimensional vector space V is an equivalence class of basis for V, where two bases $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$ are equivalent (i.e determine the same orientation) if and only if the change of basis matrix $A = (a_{i,j})$, where $w_i = \sum_j a_{i,j} v_j$ has positive determinant, $det(A) > 0$. Let $Or(V)$ be the set of orientations of V. Notice that $Or(V)$ is a two point set.

For a vector bundle $\zeta : p : E \to B$, an orientation is a continuous choice of orientations of each fiber. Said more precisely, we may define the "orientation double cover" $Or(\zeta)$ to be the two - fold covering space

$$Or(\zeta) = E_{GL} \times_{GL(n,\mathbb{R})} Or(\mathbb{R}^n)$$

where E_{GL} is the associated principal bundle, and where $GL(n, \mathbb{R})$ acts on $Or(\mathbb{R}^n)$ by matrix multiplication on a basis representing the orientation.

Definition 2.10. ζ *is orientable* if the orientation double cover $Or(\zeta)$ admits a section. A choice of section is an orientation of ζ.

This definition is reasonable, in that a continuous section of $Or(\zeta)$ is a continuous choice of orientations of the fibers of ζ.

Recall that $SL(n, \mathbb{R}) < GL(n, \mathbb{R})$ and $SO(n) < O(n)$ are the subgroups consisting of matrices with positive determinants. The following is now straightforward.

Theorem 2.7. *An n - dimensional vector bundle ζ has an orientation if and only if it has a $SL(n, \mathbb{R})$ - structure. Similarly a Euclidean vector bundle is orientable if and only if it has a $SO(n)$ - structure. Choices of these structures are equivalent to choices of orientations.*

2.2 Pull Backs and Bundle Algebra

In this section we describe the notion of the pull back of a bundle along a continuous map. We then use it to describe constructions on bundles such as direct sums, tensor products, symmetric and exterior products, and homomorphisms.

2.2.1 Pull Backs

Let $p : E \to B$ be a fiber bundle with fiber F. Let $A \subset B$ be a subspace. The restriction of E to A, written $E_{|_A}$ is simply given by

$$E_{|_A} = p^{-1}(A).$$

The restriction of the projection $p : E_{|_A} \to A$ is clearly still a locally trivial fibration with fiber F.

This notion generalizes from inclusions of subsets $A \subset B$ to general maps $f : X \to B$ in the form of the *pull back* bundle over X, $f^*(E)$. This bundle is defined by

$$f^*(E) = \{(x, u) \in X \times E : f(x) = p(u)\}.$$

Proposition 2.8. *The map*

$$\begin{aligned} p_f : f^*(E) &\to X \\ (x, u) &\to x \end{aligned}$$

is a locally trivial fibration with fiber F. Furthermore if $\iota : A \hookrightarrow B$ is an inclusion of a subspace, then the pull-back $\iota^*(E)$ is equal to the restriction $E_{|_A}$.

Proof. Let $\{U_\alpha\}$ be a collection of open sets in B and $\psi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ local trivializations of the bundle $p : E \to B$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X, and the maps

$$\psi_\alpha(f) : f^{-1}(U_\alpha) \times F \to p_f^{-1}(f^{-1}(U_\alpha))$$

defined by $(x, y) \to (x, \psi_\alpha(f(x), y))$ are clearly local trivializations.

This proves the first statement in the proposition. The second statement is obvious. □

We now use the pull back construction to define certain algebraic constructions on bundles.

Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be fiber bundles with fibers F_1 and F_2 respectively. Then the cartesian product

$$p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$$

is clearly a fiber bundle with fiber $F_1 \times F_2$. In the case when $B_1 = B_2 = B$, we can consider the pull back (or restriction) of this cartesian product bundle via the diagonal map

$$\begin{aligned} \Delta : B &\hookrightarrow B \times B \\ x &\to (x, x). \end{aligned}$$

Then the pull-back $\Delta^*(E_1 \times E_2) \to B$ is a fiber bundle with fiber $F_1 \times F_2$. It is called the *internal product*, or *Whitney sum* of the fiber bundles E_1 and E_2. It is written

$$E_1 \oplus E_2 = \Delta^*(E_1 \times E_2).$$

Notice that if E_1 and E_2 are G_1 and G_2 principal bundles respectively, then $E_1 \oplus E_2$ is a principal $G_1 \times G_2$ - bundle. Similarly, if E_1 and E_2 are n and m dimensional vector bundles respectively, then $E_1 \oplus E_2$ is an $n + m$ - dimensional vector bundle. Observe that the clutching functions of $E_1 \oplus E_2$ naturally lie in $GL(n, \mathbb{R}) \times GL(m, \mathbb{R})$ which is thought of as a subgroup of $GL(n + m, \mathbb{R})$ consisting of $(n + m) \times (n + m)$ - dimensional matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in GL(n, \mathbb{R})$ and $B \in GL(m, \mathbb{R})$.

We now describe other algebraic constructions on vector bundles. The first is a generalization of the fact that given a subspace of a vector space, the ambient vector space splits as a direct sum of the subspace and the quotient space.

Let $\eta : E^\eta \to B$ be a k - dimensional vector bundle and let $\zeta : E^\zeta \to B$ be an n - dimensional bundle. Let $\iota : \eta \hookrightarrow \zeta$ be a linear embedding of vector bundles. So on each fiber ι is a linear embedding of a k - dimensional vector space into an n - dimensional vector space. Define ζ/η to be the vector bundle whose fiber at x is E^ζ_x / E^η_x.

Exercise.

Verify that ζ/η is an $n - k$ - dimensional vector bundle over B.

Theorem 2.9. *There is a splitting of vector bundles*

$$\zeta \cong \eta \oplus \zeta/\eta.$$

Proof. Give ζ a Euclidean structure. Define $\eta^{\perp} \subset \zeta$ to be the subbundle whose fiber at x is the orthogonal complement

$$E_x^{\eta^{\perp}} = \{v \in E_x^{\zeta} : v \cdot w = 0 \text{ for all } w \in E_x^{\eta}\}$$

Then clearly there is an isomorphism of bundles

$$\eta \oplus \eta^{\perp} \cong \zeta.$$

Moreover the composition

$$\eta^{\perp} \subset \zeta \to \zeta/\eta$$

is also an isomorphism. The theorem follows. □

Corollary 2.10. *Let ζ be a Euclidean n - dimensional vector bundle. Then ζ has a $O(k) \times O(n-k)$ - structure if and only if ζ admits a k - dimensional subbundle $\eta \subset \zeta$.*

We now describe the dual of a vector bundle. So let $\zeta : E^{\zeta} \to B$ be an n - dimensional bundle. Its dual, $\zeta^* : E^{\zeta^*} \to B$ is the bundle whose fiber at $x \in B$ is the dual vector space $E_x^{\zeta^*} = Hom(E^{\zeta}, \mathbb{R})$. If

$$\{\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$$

are clutching functions for ζ, then

$$\{\phi^*_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$$

form the clutching functions for ζ^*, where $\phi^*_{\alpha,\beta}(x)$ is the adjoint (transpose) of $\phi_{\alpha,\beta}(x)$. The dual of a complex bundle is defined similarly.

Exercise.

Prove that ζ and ζ^* are isomorphic vector bundles. *Hint*. Give ζ a Euclidean structure.

Now let $\eta : E^{\eta} \to B$ be a k - dimensional, and as above, $\zeta : E^{\zeta} \to B$ an n - dimensional bundle. We define the tensor product bundle $\eta \otimes \zeta$ to be the bundle whose fiber at $x \in B$ is the tensor product of vector spaces, $E_x^{\eta} \otimes E_x^{\zeta}$. The clutching fucntions can be thought of as compositions of the form

$$\phi^{\eta\otimes\zeta}_{\alpha,\beta} : U_\alpha \cap U_\beta \xrightarrow{\phi^{\eta}_{\alpha,\beta} \times \phi^{\zeta}_{\alpha,\beta}} GL(k, \mathbb{R}) \times GL(n, \mathbb{R}) \xrightarrow{\otimes} GL(kn, \mathbb{R})$$

where the tensor product of two linear transformations $A : V_1 \to V_2$ and $B : W_1 \to W_2$ is the induced linear transformation $A \otimes B : V_1 \otimes W_1 \to V_2 \otimes W_2$.

With these two constructions we are now able to define the "homomorphism bundle", $Hom(\eta, \zeta)$. This will be the bundle whose fiber at $x \in B$ is the $k \cdot m$ - dimensional vector space of linear transformations

$$Hom(E^\eta_x, E^\zeta_x) \cong (E^\eta_x)^* \otimes E^\zeta_x.$$

So as bundles we can define

$$Hom(\eta, \zeta) = \eta^* \otimes \zeta.$$

Observation. A bundle homomorphsim $\theta : \eta \to \zeta$ assigns to every $x \in B$ a linear transformation of the fibers, $\theta_x : E^\eta_x \to E^\zeta_x$. Thus a bundle homomorphism can be thought of as a section of the bundle $Hom(\eta, \zeta)$. That is, there is a bijection between the space of sections, $\Gamma(Hom(\eta, \zeta))$ and the space of bundle homomorphisms, $\{\theta : \eta \to \zeta\}$.

2.3 The homotopy invariance of fiber bundles

The goal of this section is to prove the following theorem, establishing the homotopy invariance of fiber bundles. This will be very important in applications, as we will see in chapter 5.

Theorem 2.11. *Let $p : E \to B$ be a fiber bundle with fiber F, and let $f_0 : X \to B$ and $f_1 : X \to B$ be homotopic maps, where X is a compact space. Then the pull - back bundles are isomorphic,*

$$f_0^*(E) \cong f_1^*(E).$$

The main step in the proof of this theorem is the basic *Covering Homotopy Theorem* for fiber bundles which we now state and prove.

Theorem 2.12. Covering Homotopy theorem. *Let $p_0 : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber, F, where B is normal and locally compact. Let h_0 be a bundle map*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h}_0} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[h_0]{} & Y \end{array}$$

Let $H : B \times I \to Y$ be a homotopy of h_0 (i.e $h_0 = H_{|_{B \times \{0\}}}$.) Then there exists *a covering of the homotopy H by a bundle map*

$$\begin{array}{ccc} E \times I & \xrightarrow{\tilde{H}} & Z \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow[H]{} & Y. \end{array}$$

Proof. We prove the theorem here when the base space B is compact. The natural extension is to when B has the homotopy type of a CW - complex. The proof in full generality can be found in Steenrod's book [143].

The idea of the proof is to decompose the homotopy H into homotopies that take place in local neighborhoods where the bundle is trivial. The theorem is obviously true for trivial bundles, and so the homotopy H can be covered on each local neighborhood. One then must be careful to patch the coverings together so as to obtain a global covering of the homotopy H.

Since the space X is compact, we may assume that the pull - back bundle $H^*(Z) \to B \times I$ has locally trivial neighborhoods of the form $\{U_\alpha \times I_j\}$, where $\{U_\alpha\}$ is a locally trivial covering of B (i.e there are local trivializations $\phi_{\alpha,\beta} : U_\alpha \times F \to p^{-1}(U_\alpha)$), and $I_1, \cdots, I_r$ is a finite sequence of open intervals covering $I = [0,1]$, so that each I_j meets only I_{j-1} and I_{j+1} nontrivially. Choose numbers

$$0 = t_0 < t_1 < \cdots < t_r = 1$$

so that $t_j \in I_j \cap I_{j+1}$. We assume inductively that the covering homotopy $\tilde{H}(x,t)$ has been defined $E \times [0, t_j]$ so as to satisfy the theorem over this part.

For each $x \in B$, there is a pair of neighborhoods (W, W') such that for $x \in W$, $\bar{W} \subset W'$ and $\bar{W}' \subset U_\alpha$ for some U_α. Choose a finite number of such pairs (W_i, W_i'), $(i = 1, \cdots, s)$ covering B. Then the Urysohn lemma implies there is a map $u_i : B \to [t_j, t_{j+1}]$ such that $u_i(\bar{W}_i) = t_{j+1}$ and $u_j(B - W_i') = t_j$. Define $\tau_0(x) = t_j$ for $x \in B$, and

$$\tau_i(x) = max(u_1(x), \cdots, u_i(x)), \quad x \in B, \quad i = 1, \cdots, s.$$

Then

$$t_j = \tau_0(x) \leq \tau_1(x) \leq \cdots \leq t_s(x) = t_{j+1}.$$

Define B_i to be the set of pairs (x,t) such that $t_j \leq t \leq \tau_i(x)$. Let E_i be the part of $E \times I$ lying over B_i. Then we have a sequence of total spaces of bundles

$$E \times t_j = E_0 \subset E_1 \subset \cdots \subset E_s = E \times [t_j, t_{j+1}].$$

We suppose inductively that $\tilde{H}$ has been defined on E_{i-1} and we now define its extension over E_i.

By the definition of the τ's, the set $B_i - B_{i-1}$ is contained in $W_i' \times [t_j, t_{j+1}]$; and by the definition of the W's, $\bar{W'}_i \times [t_j, t_{j+1}] \subset U_\alpha \times I_j$ which maps via H

to a locally trivial neighborhood, say V_k, for $q : Z \to Y$. Say $\phi_k : V_k \times F \to q^{-1}(V_k)$ is a local trivialization. In particular we can define $\rho_k : q^{-1}(V_k) \to F$ to be the inverse of ϕ_k followed by the projection onto F. We now define

$$\tilde{H}(e,t) = \phi_k(H(x,t), \rho(\tilde{H}(e, \tau_{i-1}(x)))$$

where $(e,t) \in E_i - E_{i-1}$ and $x = p(e) \in B$.

It is now a straightforward verification that this extension of $\tilde{H}$ is indeed a bundle map on E_i. This then completes the inductive step. □

We now prove Theorem 2.11 using the covering homotopy theorem.

Proof. Let $p : E \to B$, and $f_0; X \to B$ and $f_1 : X \to B$ be as in the statement of the theorem. Let $H : X \times I \to B$ be a homotopy with $H_0 = f_0$ and $H_1 = f_1$. Now by the covering homotopy theorem there is a covering homotopy $\tilde{H} : f_0^*(E) \times I \to E$ that covers $H : X \times I \to B$. By definition this defines a map of bundles over $X \times I$, that by abuse of notation we also call $\tilde{H}$,

$$\begin{array}{ccc} f_0^*(E) \times I & \xrightarrow{\tilde{H}} & H^*(E) \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow[=]{} & X \times I. \end{array}$$

This is clearly a bundle isomorphism since it induces the identity map on both the base space and on the fibers. Restricting this isomorphism to $X \times \{1\}$, and noting that since $H_1 = f_1$, we get a bundle isomorphism

$$\begin{array}{ccc} f_0^*(E) & \xrightarrow[\cong]{\tilde{H}} & f_1^*(E) \\ \downarrow & & \downarrow \\ X \times \{1\} & \xrightarrow[=]{} & X \times \{1\}. \end{array}$$

This proves Theorem 2.11 □

Exercise. Prove the following corollary of Theorem 2.11.

Corollary 2.13. *Any fiber bundle over a contractible space is trivial.*

3

General Background on Differentiable Manifolds

In geometry one most often studies manifolds that have differentiable structures. They are precisely the types of spaces on which one can do calculus and study differential equations. We begin this chapter by defining these "differentiable manifolds".

Definition 3.1. *An n-dimensional topological manifold M^n is a C^r-differentiable manifold if it admits a C^r-differentiable atlas. This is an atlas $\mathcal{A} = \{U_\alpha, \Psi_{U_\alpha})$ such that every composition of the form*

$$\Psi_{U_\beta} \circ \Psi_{U_\alpha}^{-1} : \Psi_{U_\alpha}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta \to \Psi_{U_\beta}(U_\alpha \cap U_\beta)$$

is a C^r-diffeomorphism of open sets in $\mathbb{R}^n$. We say that each pair of charts $(U_\alpha, \Psi_{U_\alpha})$ and $(U_\beta, \Psi_{U_\beta})$ have a "C^r-overlap".

We note that a C^r-differentiable manifold M^n with atlas $\mathcal{A}$ admits a unique maximal C^r-atlas $\tilde{\mathcal{A}}$ containing $\mathcal{A}$. Namely $\tilde{\mathcal{A}}$ consists of all charts which have C^r-overlap with every chart of $\mathcal{A}$.

Notice that with this definition, it makes sense to say that a continuous map between C^r-differentiable manifolds, $f : M^n \to N^m$ is C^r- differentiable at $x \in M^n$ if there are charts (U, Φ) around $x \in M^n$ and (V, Ψ) around $f(x) \in N$ with $f(U) \subset V$ such that the map

$$\Psi \circ f \circ \Phi^{-1} : \Phi(U) \to \Psi(V)$$

is a differentiable map between open sets $\Phi(U) \subset \mathbb{R}^n$ and $\Psi(V) \subset \mathbb{R}^m$. We say that f is C^r-differentiable if it is C^r-differentiable at every point $x \in M^n$.

For the most part, in this book we will be studying the topology of "smooth", meaning C^∞- differentiable manifolds.

In our definition, we assume that manifolds are always **Hausdorff** topological spaces. Recall that this means that any two points x, $y \in M$ can be separated by disjoint open sets. That is, there are open sets $U_1 \subset M$ containing x and $U_2 \subset M$ containing y with $U_1 \cap U_2 = \emptyset$. Throughout these notes we will also assume our manifolds are **paracompact.** Recall that a space X is paracompact if every open cover $\mathcal{U}$ of X has a locally finite refinement. That is there is another cover $\mathcal{V}$, all of whose open sets are all contained in $\mathcal{U}$, and so that $\mathcal{V}$ is locally finite. That is, each $x \in M$ lies in only finitely many of the open sets in $\mathcal{V}$. Recall that a Hausdorff space is paracompact if and only if it admits a *partition of unity* subordinate to any open cover $\mathcal{U} = \{U_i, i \in \Lambda\}$. Such a partition of unity is a collection of maps $\rho_i : X \to [0,1]$ so that

- The support supp $(\rho_i) \subset U_i$, and
- $\sum_{i\in\Lambda} \rho_i(x) = 1 \quad$ for every $x \in X$.

3.1 History

In this section we give a brief sketch of the history of differentiable manifolds. Our sketch follows that of Hirsch's book [72]. We refer the reader to that reference for more details.

Historically, the notion of a differentiable manifold grew from geometry and function theory in the 19th century. Geometers studied curves and surfaces in $\mathbb{R}^3$, and were mainly interested in local structures, such as *curvature*, introduced by Gauss in the early part of the 19th century. Function theorists were interested in studying "level sets" of differentiable functions $F : \mathbb{R}^n \to \mathbb{R}$, i.e the spaces $F^{-1}(c) \subset \mathbb{R}^n$ for $c \in \mathbb{R}$. They observed that for "most" values of c these level sets are "smooth" and nonsingular. This was part of the analytic study of "Calculus of Variations", which let to "Morse theory" in the 20th century.

In the mid-19th century Riemann broke new ground with the study of what are now called "Riemann surfaces". These were historically the first examples of "abstract manifolds", which is to say they are not defined to be a subspace of some Euclidean space. Riemann surfaces represent the global nature of the analytic continuation process. Riemann also studied topological invariants of these surfaces, such as the *"connectivity"* of a surface, which is defined to be the maximal number of embedded closed curves on a surface whose union does not disconnect the surface plus one. Riemann showed in the 1860's that for compact, orientable surfaces, this number classifies the surface up to homeomorphism. In particular for a surface of genus g, Riemann's connectivity number is $2g + 1$.

In the early 20th century, Poincaré studied 3-dimensional manifolds in his famous treatise, "Analysis Situs". In that work Poincaré introduced some notions in Algebraic Topology such as the fundamental group. The famous *"Poincaré Conjecture"* which was proved by Perelman nearly a hundred years later in 2003, states that every simply connected compact 3-dimensional manifold is homeomorphic, and indeed diffeomorphic to the sphere S^3.

Poincaré's conjecture was a statement about the *classification of manifolds.* Such a classification has been a key problem in differential topology for the past hundred years. Currently there is great interest and work on the classification of symmetries ("diffeomorphisms") of manifolds.

Herman Weyl defined abstract differentiable manifolds in 1912. But it was not until the work of H. Whitney (1936-1940) when basic geometric and topological properties of manifolds, such as existence of embeddings into Euclidean space, were proved. At that time the modern notion of differentiable manifold became firmly established as a fundamental object in mathematics.

3.2 Examples and Basic Notions

3.2.1 Examples

Consider the following standard examples of differentiable manifolds:

1. Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. It has an atlas consisting of two charts. Let $\epsilon > 0$ be small. Then define

$$U_1 = \{(x_1, \cdots, x_{n+1}) \,:\, x_{n+1} > -\epsilon\}$$
$$U_2 = \{(x_1, \cdots, x_{n+1}) \,:\, x_{n+1} < \epsilon\}$$

   There are natural projections of U_1 and U_2 onto $B_1(0)$ with C^∞-overlaps, thus defining a smooth structure on S^n.

2. Let $\mathbb{RP}^n = S^n/\sim$ where $x \sim -x$. This is the (real) projective space. This is a C^∞-n-dimensional manifold. To see a smooth atlas we use "projective coordinates". These are obtained by viewing $\mathbb{RP}^n$ as the quotient of the nonzero elements of Euclidean space, $\mathbb{R}^{n+1}$ by the group action of the nonzero real numbers, $\mathbb{R}^\times$ given by scalar multiplication:

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} - \{0\}\right)/\mathbb{R}^\times.$$

   We describe a point in $\mathbb{RP}^n$ as the equivalence class of a point in $\mathbb{R}^{n+1}-\{0\}$, which we denote using square brackets: $[x_0, x_1, \cdots, x_n] \in \mathbb{RP}^n$. For $0 \le i \le n$ define

$$U_i = \{[x_0, \cdots, x_n] \in \mathbb{RP}^n \,:\, x_i \neq 0\}.$$

   Notice that $\mathbb{RP}^n = U_0 \cup \cdots \cup U_n$ and that the map

$$\Psi_i : U_i \to \mathbb{R}^n$$
$$[x_0, \cdots, x_n] \to (\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i})$$

   defines a homeomorphism of U_i onto $\mathbb{R}^n$. Moreover its easily checked that these homeomorphisms have C^∞-overlaps. Thus $\{(U_i, \Psi_i), \,:\, i = 0, \cdots, n\}$ is a smooth (C^∞) atlas for $\mathbb{RP}^n$.

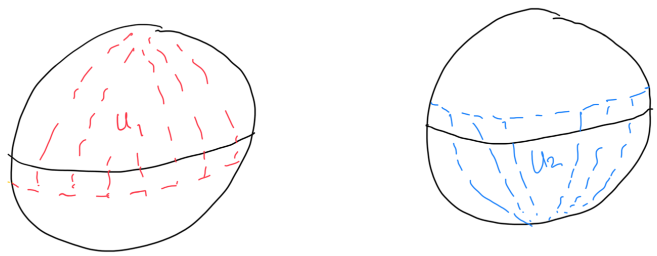


FIGURE 3.1
Charts for S^n

Exercise

Describe atlases for complex projective space $\mathbb{CP}^n$ and quaternionic projective space $\mathbb{HP}^n$, constructed similarly to the atlas described for $\mathbb{RP}^n$ described above, using the complex numbers and the quaternions respectively. Show that $\mathbb{CP}^n$ is a differentiable $2n$-dimensional manifold, and $\mathbb{HP}^n$ is a differentiable $4n$-dimensional manifold.

3.2.2 The tangent bundle

An important concept in the study of differentiable manifolds is that of a **tangent bundle.**

Definition 3.2. *Let M^n be a differentiable (C^1) n-dimensional manifold with an atlas $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$. A tangent vector* to M at $x \in M$ is an equivalence class of triples $(x, \alpha, v) \in M \times \Lambda \times \mathbb{R}^n$ under the equivalence relation

$$(x, \alpha, v) \sim (x, \beta, u)$$

if $D(\phi_\beta \phi_\alpha^{-1})(\phi_\alpha(x))(v) = u$. The *tangent space* of M at x, denoted $T_x M$ is defined to be the set of all tangent vectors at x.

Notice that the functions we are differentiating in this definition are defined on open subspaces of Euclidean space. More specifically, they are defined on open sets of the form $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ and take values in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$.

We leave it to the reader to verify that $T_x M$ is an n-dimensional real vector space. One can also verify that this definition does not depend on the choice of atlas or charts. The *tangent bundle* is defined to be the union of all tangent spaces

$$TM = \bigcup_{x \in M} T_x M.$$

So far TM is defined only set-theoretically. We have yet to discuss its topology. We do so now as follows:

Definition 3.3. *Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha) : \alpha \in \Lambda\}$ be an atlas for a differentiable n-dimensional manifold M^n. Define the tangent bundle*

$$TM = \coprod_{\alpha \in \Lambda} U_\alpha \times \mathbb{R}^n / \sim$$

where $(x, v) \in U_\alpha \times \mathbb{R}^n$ is identified with $(x, u) \in U_\beta \times \mathbb{R}^n$ if $x \in U_\alpha \cap U_\beta$ and $D(\phi_\beta \phi_\alpha^{-1})(\phi_\alpha(x))(v) = u$. TM is given the quotient topology under this identification.

We can give the tangent bundle a more concrete definition in the setting where M^n is a subset of $\mathbb{R}^L$ for some L. (We will later prove that every manifold can be appropriately viewed as a subset of Euclidean space of sufficiently high dimension.)

Assume $M^n \subset \mathbb{R}^L$. Given $x \in M^n \subset \mathbb{R}^L$, we say that a vector $v \in \mathbb{R}^L$ is tangent to M^n at $x \in M$ if there exists an $\epsilon > 0$ and differentiable curve

$$\gamma : (-\epsilon, \epsilon) \to M^n \subset \mathbb{R}^L$$

such that $\frac{d\gamma}{dt}(0) = v$.

We define the tangent space $T_x M^n$ to be the set of all vectors tangent to X. Clearly this is an n-dimensional real vector space. Moreover we can now topologize the tangent bundle as a subspace of $\mathbb{R}^L \times \mathbb{R}^L$:

$$\begin{aligned} TM^n = \bigcup_{x \in M} T_x M^n &\subset \mathbb{R}^L \times \mathbb{R}^L \\ v \in T_x M^n &\to (x, v). \end{aligned}$$

There is a natural continuous projection map

$$\begin{aligned} p : TM &\to M \\ v \in T_x M &\to M. \end{aligned} \tag{3.1}$$

Exercise Prove that the two definitions of tangent bundle given above are equivalent, when the manifold M^n is a submanifold of $\mathbb{R}^L$. By "equivalent" we mean that each of the definitions define vector bundles $TM \to M$ which are isomorphic (as vector bundles).

A differentiable section of the tangent bundle $\sigma : M^n \to TM^n$ is called a *vector field.* At every point of the manifold, a section picks out a tangent vector. The question of which manifolds admit a nowhere zero vector field, and if so, how many linearly independent vector fields are possible, has long been a fundamental question in differential topology. (A collection of vector fields are linearly independent if they pick out linearly independent tangent vectors at every point.) A manifold is called *parallelizable* if its tangent bundle is trivial. Notice that a parallelizable manifold of dimension n admits n linearly independent vector fields.

Exercises

1. Show that a manifold M^n is parallelizable if and only if it admits n linearly independent vector fields.

2. Show that the unit sphere S^n admits a nowhere zero vector field if n is odd.

3. If S^n admits a nowhere zero vector field show that the identity map of S^n is homotopic to the antipodal map. For n even show that the antipodal map of S^n is homotopic to the reflection

$$r(x_1, \cdots, x_{n+1}) = (-x_1, x_2, \cdots, x_{n+1});$$

and therefore has degree -1. Combining these facts, show that S^n is not parallelizable for n even, $n \geq 2$.

3.2.3 The implicit and inverse function theorems, embeddings and immersions

We assume the reader is familiar with the following basic theorems from the analysis of differentiable maps on Euclidean space. We observe that they are local theorems, and so can be used to study differentiable manifolds and maps between them.

Theorem 3.1. *(The Implicit Function Theorem - the surjective version) Let $U \subset \mathbb{R}^m$ be an open subspace and $f : U \to \mathbb{R}^n$ a C^r-map, where $r \geq 1$. For $p \in U$, assume $f(p) = 0$. Suppose the derivative at p,*

$$Df_p : \mathbb{R}^m \to \mathbb{R}^n$$

is surjective. Then there is a local diffeomorphism ϕ of $\mathbb{R}^m$ at 0 such that $\phi(0) = p$ and

$$f \circ \phi(x_1, \cdots, x_n, \cdots, x_m) = (x_1, \cdots, x_n).$$

That is, $f \circ \phi$ is the projection onto the first n-coordinates.

There is another version of the implicit function theorem when the derivative is *injective.*

Theorem 3.2. *(The Implicit Function Theorem - the injective version) Let $U \subset \mathbb{R}^m$ be an open set and $f : U \to \mathbb{R}^n$ a C^r-map, where $r \geq 1$. Let $q \in \mathbb{R}^n$ be such that $0 \in f^{-1}(q)$. Suppose that*

$$Df_0 : \mathbb{R}^m \to \mathbb{R}^n$$

is injective. Then there is a local diffeomorphism ψ of $\mathbb{R}^n$ such that $\psi(q) = 0$ and

$$\psi \circ f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, 0, \cdots, 0) \in \mathbb{R}^n.$$

That is $\psi \circ f$ is the inclusion of the first m-coordinate axes.

Finally, consider the following theorem, which is equivalent to the implicit function theorems.

Theorem 3.3. *(Inverse Function Theorem) Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a C^r-map where $r \geq 1$. If $p \in U$ is such that $Df_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then f is a C^r-local diffeomorphism at p. That is there is an open set $V \subset U \subset \mathbb{R}^n$ such that $f : V \to f(V)$ is a diffeomorphism.*

We end with the definition of immersion and embedding.

Definition 3.4. *Suppose $f : M^m \to N^n$ is C^r, for $r \geq 1$, where M^m and N^n are C^r manifolds of dimensions m and n, respectively. We say that f is* **immersive** at $x \in M$ if the linear map

$$Df_x : T_xM \to T_{f(x)}N$$

is injective. f is an **immersion** if f is immersive at every point $x \in M$. We use the symbol $f : M^m \looparrowright N^n$ to mean that f is an immersion.

Definition 3.5. *Suppose $f : M^m \to N^m$ is C^r, for $r \geq 1$, where M^m and N^n are C^r manifolds of dimensions m and n, respectively. We say that f is* **submersive** at $x \in M$ if the linear map

$$Df_x : T_xM \to T_{f(x)}N$$

is surjective. f is an **submersion** if f is submersive at every point $x \in M$.

Definition 3.6. *A C^r-map $f : M \to N$ is an* **embedding** if it is an immersion and f maps M homeomorphically onto its image. In this case we write $f : M \hookrightarrow N$.

Finally we have the following definition.

Definition 3.7. *Suppose N is a C^r-manifold, $r \geq 1$. A subspace $A \subset N$ is a C^r-submanifold if and only if A is the image of a C^r-embedding of some manifold into N.*

Exercises. 1. Prove that the following are C^∞- submanifolds of the space of $n \times n$ matrices, $Mat_{n,n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Compute their dimensions.

1. $GL_n(\mathbb{R})$
2. $SL_n(\mathbb{R})$
3. $SO(n)$.

2. (a) Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \to \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.
(b) Use the above to show that $\mathbb{RP}^n$ is compact.

The following is an immediate corollary of Implicit Function Theorem (the injective version).

Proposition 3.4. *If $f : M \to N$ is an immersion, then it is a local embedding. That is, around every $x \in M$ there is an open neighborhood U of x so that the restriction $f : U \to N$ is an embedding.*

Exercise Let $\pi : \tilde{X} \to X$ be a covering space. Let Φ be a smooth structure on X. Prove that there is a smooth structure $\tilde{\Phi}$ on $\tilde{X}$ so that $\pi : (\tilde{X}, \tilde{\Phi}) \to (X, \Phi)$ is an immersion.

3.2.4 Manifolds with boundary

In many areas of mathematics one often confronts spaces whose interiors are manifolds. A closed disk in $\mathbb{R}^n$ is a basic example. It is an example of what is called a *"manifold with boundary"*. In this section we define this concept and describe how the constructions and theorems developed above for smooth manifolds, can be generalized to "smooth manifolds with boundary".

Definition 3.8. *The "upper half space" $\mathbb{H}^n \subset \mathbb{R}^n$ is the subspace*

$$\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \text{ such that } x_n \geq 0\}.$$

The boundary points of $\mathbb{H}^n$ are those $(x_1, \cdots, x_n)$ with $x_n = 0$.

An n-dimensional topological manifold with boundary is then a space that that has charts homeomorphic to open sets in $\mathbb{H}^n$ rather than $\mathbb{R}^n$. That is, we have the following definition, which is completely analogous to Definition 1.1 above.

Definition 3.9. *An n-dimensional topological manifold with boundary is a second countable Hausdorff space M^n with the property that for every $x \in M$, there is an open neighborhood U containing x and a homeomorphism,*

$$\psi_U : U \xrightarrow{\cong} V \subset \mathbb{H}^n$$

where V is an open subspace of $\mathbb{H}^n$. The boundary of M^n, written ∂M^n consists of those points $p \in M^n$ for which there is an open neighborhood $p \in U$ and a chart $\psi_U : U \xrightarrow{\cong} V \subset \mathbb{H}^n$ where $\psi_U(p)$ is a boundary point of $\mathbb{H}^n$. Observe that the condition of $p \in M^n$ being a boundary point is independent of the particular chart used.

We leave it for the reader to check that if M^n is a topological n-manifold with boundary, then the boundary ∂M^n is a topological $(n-1)$-dimensional manifold (without boundary).

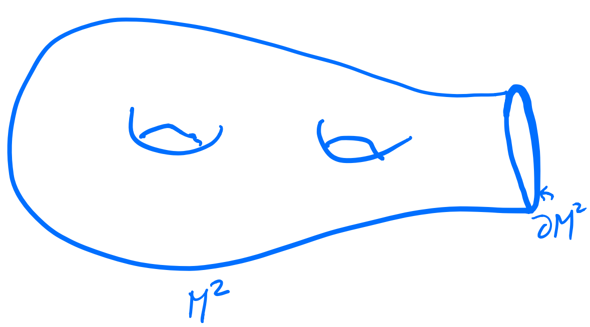


FIGURE 3.2
A 2-dimensional manifold with boundary

We need to be careful about the definition of submanifolds in the setting of manifolds with boundary. First, for $k \leq n$, consider a standard inclusion $\mathbb{H}^k \hookrightarrow \mathbb{R}^n$ mapping $(x_1, \cdots, x_k)$ to $(x_1, \cdots, x_k, 0, \cdots 0)$. A subspace $V \subset \mathbb{R}^n$ is a C^r-dimensional submanifold if each $x \in V$ belongs to the domain of a chart $\phi : U \to \mathbb{R}^n$ of $\mathbb{R}^n$ such that $V \cap U = \phi^{-1}(\mathbb{H}^k)$.

A general definition of a submanifold (with boundary) can be taken to be the following:

Definition 3.10. *Let M be a C^r-manifold, with or without boundary. A subset $N \subset M$ is a C^r-submanifold if each $x \in N$ there is an open set subset U of M containing x, a C^r embedding $g : U \hookrightarrow \mathbb{R}^n$, such that*

$$N \cap U = g^{-1}(\mathbb{H}^k),$$

A particularly important type of embedding of one manifold into another is when one restricts to the boundary of the submanifold, the image of the embedding lies in the boundary of the ambient manifold. This is called a *neat* embedding,

Definition 3.11. *An embedding $e : N \hookrightarrow M$ of C^r-manifolds is neat* if $\partial N = N \cap \partial M$ and N is covered by charts (ϕ, U) of M such that $N \cap U = \phi^{-1}(\mathbb{H}^k)$.

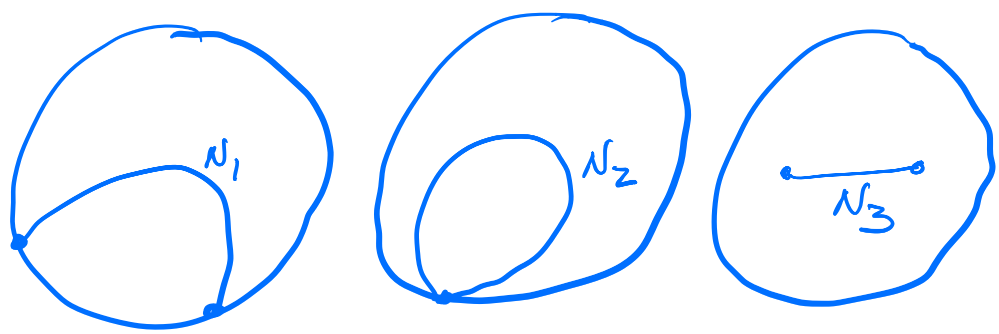


FIGURE 3.3
N_1 is neat, N_2 and N_3 are not.

3.2.5 Regular Values and transversality

We begin this section with the notion of *regular points and values* as well as *critical points and values*.

Definition 3.12. *Suppose $f : M \to N$ is a C^r map between C^r manifolds, where $r \geq 1$. A point $x \in M$ is called a regular point* if f is submersive at x. If $u \in M$ is not a regular point it is called a *critical point*. $f(u) \in N$ is then called a *critical value*. If $y \in N$ is not a critical value it is called a regular value. In particular every point $y \in N$ that is not in the image of f is a regular value. If $y \in N$ is a regular value, its inverse image $f^{-1}(y) \subset M$ is called a *regular level set*.

The following is one of the most fundamental theorems in differential topology:

Theorem 3.5. *(The Regular Value Theorem) Suppose $f : M^n \to N^k$ is a C^r-map between C^r manifolds of dimension n and k respectively. Here $r \geq 1$. If $y \in M$ is a regular value, then the regular level set $f^{-1}(y) \subset M^n$ is a C^r-submanifold of dimension $n - k$.*

Proof. Since being a manifold is a local property, it suffices to prove this theorem in the case when $M^n \subset \mathbb{R}^n$ is an open set, and $N = \mathbb{R}^m$. The theorem now follows from the surjective version of the Implicit Function Theorem. □

The Regular Value Theorem for manifolds with boundary has the following formulation.

Theorem 3.6. *Let M ba a C^r manifold with boundary, and N a C^r manifold (with or without boundary). Here we are assuming $r \geq 1$. Let $f : M \to N$ be a C^r map. If $y \in N - \partial N$ is a regular value for both f and $f_{|\partial M}$, then $f^{-1}(y)$ is a neat C^r submanifold of M.*

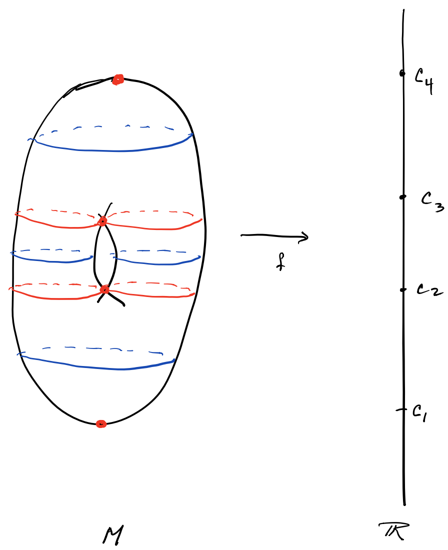


FIGURE 3.4
f is the height function from the torus to the real line. It has 4 critical values. The level sets of the critical values are shown in red, and regular sets of regular values, which are all one-dimensional submanifolds, are shown in blue.

We now want to discuss an important generalization of the concepts involved in the Regular Value Theorem. This is the concept of *transversality*. The following is probably the most conceptual setting for transversality.

Let N^n be an n-dimensional manifold, and let $A \subset N$ and $B \subset N$ be submanifolds of dimensional p and q respectively.

$$\begin{array}{ccc} B & \xrightarrow{\subset} & N \\ & & \uparrow \cup \\ & & A \end{array}$$

We say that A and B have a *transverse intersection* in N if for every $x \in A \cap B$, the tangent spaces of the submanifolds A and B at x, together span the entire tangent space of the ambient manifold N. That is,

$$T_x A + T_x B = T_x N. \tag{3.2}$$

When A and B have transverse intersection we write $A \pitchfork B$. We will see that such transversal intersections are, in an appropriate sense, generic. We begin, though, with the following theorem.

Theorem 3.7. *Let A and B be submanifolds of the n-dimensional manifold N, where $Dim\, A = p$ and $Dim\, B = q$. Suppose furthermore that $A \pitchfork B$. The $A \cap B \subset N$ is a submanifold of dimension $p + q - n$.*

We will actually prove the following generalization of Theorem 3.7.

Let A^p be a p-dimensional manifold and N^n an n-dimensional manifold with a q-dimensional submanifold $B^q \subset N^n$. Let $f : A \to N$ be a smooth map. We say that f is transverse to B, and write $f \pitchfork B$ if whenever $b \in B$ is such that $f^{-1}(b)$ is nonempty, then for any $x \in f^{-1}(b)$

$$Df_x(T_x A) + T_b B = T_b N. \tag{3.3}$$

Notice that if $f : A \to N$ is an embedding, then $f \pitchfork B$ if and only if the submanifold given by the image of f has transverse intersection with B. Notice furthermore that if $B = y \in N$ is a point, viewed as a zero dimensional submanifold, then $f \pitchfork B$ if and only if y is a regular value of f. This is the sense in which the notion of transversality is a generalization of the notion of regular value.

The following is a strengthening of both transversality Theorem 3.7 and of the Regular Value Theorem 3.5:

Theorem 3.8. *Let $f : A^p \to N^n$ and $B^q \subset N^n$ be as above. Then if $f \pitchfork B$, then the inverse image $f^{-1}(B) \subset A$ is a submanifold of codimension $n - q$, which is the same as the codimension of B in N. That is, $f^{-1}(B)$ has dimension $p + q - n$.*

Notice that this theorem is precisely the statement of the Regular Value Theorem when B a point.

Proof. It suffices to prove this theorem locally. By the Implicit Function Theorem, we can locally replace $B^q \subset N^n$ by $U \times \{0\} \subset U \times V$, where $U \subset \mathbb{R}^q$ and $V \subset \mathbb{R}^{n-q}$ are open sets. Notice that

$$f : A^p \to U \times V$$

is transverse to $U \times \{0\}$ if and only if the composition

$$g : A^p \xrightarrow{f} U \times V \xrightarrow{project} V$$

has $0 \in V \subset \mathbb{R}^{n-q}$ as a regular value. Since$f^{-1}(U \times \{0\}) = g^{-1}(0)$, the theorem follows from the Regular Value Theorem (Theorem 3.5). □

A generalization of this theorem to the setting of manifolds with boundary is the following. The above proof applies to this situation with only minor modifications.

Theorem 3.9. *Suppose $B^q \subset N^n$ is a C^r submanifold with boundary. Suppose that either B^q is neat or $B^q \subset N^n - \partial N^n$, or $B^q \subset \partial N^n$. If $f : A^p \to N^n$ is a C^r map between manifolds with boundary with both f and $f_{|\partial A^p}$ transverse to B^q, the $f^{-1}(B^q)$ is a C^r submanifold and $\partial f^{-1}(B^q) = f^{-1}(\partial B^q)$. The dimension of $f^{-1}(B^q)$ is $p + q - n$.*

3.3 Bundles and Manifolds

3.3.1 The tangent bundle of Projective Space

We now use these constructions to identify the tangent bundle of projective spaces, $\tau\mathbb{RP}^n$ and $\tau\mathbb{CP}^n$. We study the real case first.

Recall the canonical line bundle, $\gamma_1 : E^{\gamma_1} \to \mathbb{RP}^n$. If $[x] \in \mathbb{RP}^n$ is viewed as a line in $\mathbb{R}^{n+1}$, then the fiber $E^{\gamma_1}_{[x]}$ is the one dimensional space of vectors in the line $[x]$. Thus γ_1 has a natural embedding into the trivial $n+1$ - dimensional bundle $\epsilon : \mathbb{RP}^n \times \mathbb{R}^{n+1} \to \mathbb{RP}^n$ via

$$E^{\gamma_1} = \{([x], u) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : u \in [x]\} \hookrightarrow \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$

Let $\gamma_1^\perp$ be the n - dimensional orthogonal complement bundle of this embedding.

Theorem 3.10. *There is an isomorphism of the tangent bundle with the homomorphism bundle*
$$\tau\mathbb{RP}^n \cong Hom(\gamma_1, \gamma_1^\perp)$$

Proof. Let $p : S^n \to \mathbb{RP}^n$ be the natural projection. For $x \in S^n$, recall that the tangent space of S^n can be described as

$$T_x S^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \cdot v = 0\}.$$

Notice that $(x, v) \in T_x S^n$ and $(-x, -v) \in T_{-x} S^n$ have the same image in $T_{[x]}\mathbb{RP}^n$ under the derivative $Dp : \tau S^n \to \tau\mathbb{RP}^n$. Since p is a local diffeomorphism, $Dp(x) : T_x S^n \to T_{[x]}\mathbb{RP}^n$ is an isomorphism for every $x \in S^n$. Thus $T_{[x]}\mathbb{RP}^n$ can be identified with the space of pairs

$$T_{[x]}\mathbb{RP}^n = \{(x, v), (-x, -v) : x, v \in \mathbb{R}^{n+1},\ |x| = 1,\ x \cdot v = 0\}.$$

If $x \in S^n$, let $L_x = [x]$ denote the line through $\pm x$ in $\mathbb{R}^{n+1}$. Then a pair $(x, v), (-x, -v) \in T_{[x]}\mathbb{RP}^n$ is uniquely determined by a linear transformation

$$\ell : L_x \to L^\perp$$
$$\ell(tx) = tv.$$

Thus $T_{[x]}\mathbb{RP}^n$ is canonically isomorphic to $Hom(E_x^{\gamma_1}, E_x^{\gamma_1^\perp})$, and so

$$\tau\mathbb{RP}^n \cong Hom(\gamma_1, \gamma_1^\perp),$$

as claimed. □

The following description of the $\tau\mathbb{RP}^n \oplus \epsilon_1$ will be quite helpful to us in future calculations of characteristic classes.

Theorem 3.11. *The Whiney sum of the tangent bundle and a trivial line bundle, $\tau\mathbb{RP}^n \oplus \epsilon_1$ is isomorphic to the Whitney sum of $n+1$ copies of the canonical line bundle γ_1,*

$$\tau\mathbb{RP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1.$$

Proof. Consider the line bundle $Hom(\gamma_1, \gamma_1)$ over $\mathbb{RP}^n$. This line bundle is trivial since it has a canonical nowhere zero section

$$\iota(x) = 1 : E_{[x]}^{\gamma_1} \to E_{[x]}^{\gamma_1}.$$

We therefore have

$$\begin{aligned}\tau\mathbb{RP}^n \oplus \epsilon_1 &\cong \tau\mathbb{RP}^n \oplus Hom(\gamma_1, \gamma_1) \\ &\cong Hom(\gamma_1, \gamma_1^\perp) \oplus Hom(\gamma_1, \gamma_1) \\ &\cong Hom(\gamma_1, \gamma_1^\perp \oplus \gamma_1) \\ &\cong Hom(\gamma_1, \epsilon_{n+1}) \\ &\cong \oplus_{n+1}\gamma_1^* \\ &\cong \oplus_{n+1}\gamma_1\end{aligned}$$

as claimed. □

The following are complex analogues of the above theorems and are proved in the same way.

Theorem 3.12.

$$\tau\mathbb{CP}^n \cong_{\mathbb{C}} Hom_{\mathbb{C}}(\gamma_1, \gamma_1^\perp)$$

and

$$\tau\mathbb{CP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1^*,$$

where $\cong_{\mathbb{C}}$ *and* $Hom_{\mathbb{C}}$ *denote isomorphisms and homomorphisms of complex bundles, respectively.*

Note. γ^* is not isomorphic as *complex* vector bundles to γ_1. It is isomorphic to γ_1 with the *conjugate* complex structure. We will discuss this phenomenon more later.

3.3.2 K - theory

Let $Vect^*(X) = \oplus_{n\geq 0} Vect^n(X)$ where, as above, $Vect^n(X)$ denotes the set of isomorphism classes of n - dimensional complex bundles over X. $Vect^*_{\mathbb{R}}(X)$ denotes the analogous set of real vector bundles. In both these cases $Vect^0(X)$ denotes, by convention, the one point set, representing the unique zero dimensional vector bundle.

Now the Whitney sum operation induces pairings

$$Vect^n(X) \times Vect^m(X) \xrightarrow{\oplus} Vect^{n+m}(X)$$

which in turn give $Vect^*(X)$ the structure of an abelian monoid. Notice that it is indeed abelian because given vector bundles η and ζ we have an obvious isomorphism

$$\eta \oplus \zeta \cong \zeta \oplus \eta.$$

The "zero" in this monoid structure is the unique element of $Vect^0(X)$.

Given an abelian monoid, A, there is a construction due to Grothendieck of its *group completion* $K(A)$. Formally, $K(A)$ is the smallest abelian group equipped with a homomorphism of monoids, $\iota : A \to K(A)$. It is smallest in the sense if G is any abelian group and $\phi : A \to G$ is any homomorphism of monoids, then there is a unique extension of ϕ to a map of abelian groups $\bar{\phi} : K(A) \to G$ making the diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & K(A) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\bar{\phi}} \\ G & = & G \end{array}$$

This formal property, called the *universal property*, characterizes $K(A)$, and can be taken to be the definition. However there is a much more explicit description. Basically the group completion $K(A)$ is obtained by formally adjoining inverses to the elements of A. That is, an element of $K(A)$ can be thought of as a formal difference $\alpha - \beta$, where $\alpha, \beta \in A$. Strictly speaking we have the following definition.

Definition 3.13. *Let $F(A)$ be the free abelian group generated by the elements of A, and let $R(A)$ denote the subgroup of $F(A)$ generated by elements of the form $a \oplus b - (a+b)$ where $a, b \in A$. Here "$\oplus$" is the group operation in the free abelian group and "$+$" is the addition in the monoid structure of A. We then define the Grothendieck group completion $K(A)$ to be the quotient group*

$$K(A) = F(A)/R(A).$$

Notice that an element of $K(A)$ is of the form

$$\theta = \sum_i n_i a_i - \sum_j m_j b_j$$

where the n_i's and m_j's are positive integers, and each a_i and $b_j \in A$. That is, by the relations in $R(A)$, we may write

$$\theta = \alpha - \beta$$

where $\alpha = \sum_i n_i a_i \in A$, and $\beta = \sum_j m_j b_j \in A$.

Notice also that the composition $\iota : A \subset F(A) \to F(A)/R(A) = K(A)$ is a homomorphism of monoids, and clearly has the universal property described above. We can now make the following definition.

Definition 3.14. *Given a compact space X, its complex and real (or orthogonal) K - theories are defined to be the Grothendieck group completions of the abelian monoids of isomorphism classes of vector bundles:*

$$K(X) = K(Vect^*(X))$$
$$KO(X) = K(Vect^*_{\mathbb{R}}(X)$$

An element $\alpha = \zeta - \eta \in K(X)$ is often referred to as a "virtual vector bundle" over X.

When X is not compact, $K(X)$ and $KO(X)$ will have somewhat different definitions as we will see when we talk about generalized cohomology theories in Chapter 10. For the rest of this subsection we will assume that the spaces we discuss will be compact.

Notice that the discusion of the tangent bundles of projective spaces above (section 2.2) can be interpreted in K -theoretic language as follows:

Proposition 3.13. *As elements of $K(\mathbb{CP}^n)$, we have the equation*

$$[\tau\mathbb{CP}^n] = (n+1)[\gamma_1^*] - [1]$$

where $[m] \in K(X)$ refers to the class represented by the trivial bundle of dimension m. Similarly, in the orthogonal K - theory $KO(\mathbb{RP}^n)$ we have the equation

$$[\tau\mathbb{RP}^n] = (n+1)[\gamma_1] - [1].$$

Notice that for a point, $Vect^*(pt) = \mathbb{Z}^+$, the nonnegative integers, since there is precisely one vector bundle over a point (i.e vector space) of each dimension. Thus

$$K(pt) \cong KO(pt) \cong \mathbb{Z}.$$

Notice furthermore that by taking tensor products there are pairings

$$Vect^m(X) \times Vect^n(X) \xrightarrow{\otimes} Vect^{mn}(X).$$

The following is verified by a simple check of definitions.

Proposition 3.14. *The tensor product pairing of vector bundles gives $K(X)$ and $KO(X)$ the structure of commutative rings.*

Now given a bundle ζ over Y, and a map $f : X \to Y$, we saw in the previous section how to define the pull-back, $f^*(\zeta)$ over X. This defines a homomorphism of abelian monoids

$$f^* : Vect^*(Y) \to Vect^*(X).$$

After group completing we have the following:

Proposition 3.15. *A continuous map $f : X \to Y$ induces ring homomorphisms,*

$$f^* : K(Y) \to K(X)$$

and

$$f^* : KO(Y) \to KO(X).$$

In particular, consider the inclusion of a basepoint $x_0 \hookrightarrow X$. This induces a map of rings, called the augmentation,

$$\epsilon : K(X) \to K(x_0) \cong \mathbb{Z}.$$

This map is a split surjection of rings, because the constant map $c : X \to x_0$ induces a right inverse of ϵ, $c^* : \mathbb{Z} = K(x_0) \to K(X)$. Notice that the augmentation can be viewed as the "dimension" map in that when restricted to the monoid $Vect^*(X)$, then $\epsilon : Vect^m(X) \to \{m\} \subset \mathbb{Z}$. Equivalently, on an element $\zeta - \eta \in K(X)$, $\epsilon(\zeta - \eta) = dim(\zeta) - dim(\eta)$. We then define the reduced K -theory as follows.

Definition 3.15. *The reduced K - theory of X, denoted $\tilde{K}(X)$ is defined to be the kernel of the augmentation map*

$$\tilde{K}(X) = ker\{\epsilon : K(X) \to \mathbb{Z}\}$$

and so consists of classes $\zeta - \eta \in K(X)$ such that $dim(\zeta) = dim(\eta)$. The reduced orthogonal K - theory, $\tilde{KO}(X)$ is defined similarly.

The following is an immediate consequence of the above observations:

Proposition 3.16. *There are natural splittings of rings*

$$K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$$
$$KO(X) \cong \tilde{KO}(X) \oplus \mathbb{Z}.$$

Clearly then the reduced K - theory is the interesting part of K - theory. Notice that a bundle $\zeta \in Vect^n(X)$ determines the element $[\zeta] - [n] \in \tilde{K}(X)$, where $[n]$ is the K - theory class of the trivial n - dimensional bundle.

The definitions of K - theory are somewhat abstract. The following discussion makes it clear precisely what K - theory measures in the case of compact spaces.

Definition 3.16. *Let ζ and η be vector bundles over a space X. ζ and η are said to be stably isomorphic* if for some m and n, there is an isomorphism

$$\zeta \oplus \epsilon_n \cong \eta \oplus \epsilon_m$$

where, as above, ϵ_k denotes the trivial bundle of dimension k. We let $\mathcal{S}Vect(X)$ denote the set of stable isomorphism classes of vector bundles over X.

Notice that $\mathcal{S}Vect(X)$ is also an abelian monoid under Whitney sum, and that since any two trivial bundles are stably isomorphic, and that adding a trivial bundle to a bundle does not change the stable isomorphic class, then any trivial bundle represents the zero element of $\mathcal{S}Vect(X)$.

Theorem 3.17. *Let X be a compact space, then $\mathcal{S}Vect(X)$ is an abelian group and is isomorphic to the reduced K -theory,*

$$\mathcal{S}Vect(X) \cong \tilde{K}(X).$$

Proof. A main component of the proof is the following result, which we will prove in the next chapter when we study the classification of vector bundles.

Theorem 3.18. *Every vector bundle over a compact space can be embedded in a trivial bundle. That is, if ζ is a bundle over a compact space X , then for sufficiently large $N > 0$, there is bundle embedding*

$$\zeta \hookrightarrow \epsilon_N.$$

We use this result in the following way in order to prove the above theorem. Let ζ be a bundle over a compact space X. Then by this result we can find an embedding $\zeta \hookrightarrow \epsilon_N$. Let $\zeta^\perp$ be the orthogonal complement bundle to this embedding. So that

$$\zeta \oplus \zeta^\perp = \epsilon_N.$$

Since ϵ_N represents the zero element in $\mathcal{S}Vect(X)$, then as an equation in $\mathcal{S}Vect(X)$ this becomes

$$[\zeta] + [\zeta^\perp] = 0.$$

Thus every element in $\mathcal{S}Vect(X)$ is invertible in the monoid structure, and hence $\mathcal{S}Vect(X)$ is an abelian group.

To prove that $\mathcal{S}Vect(X)$ is isomorphic to $\tilde{K}(X)$, notice that the natural surjection of $Vect^*(X)$ onto $\mathcal{S}Vect(X)$ is a morphism of abelian monoids, and since $\mathcal{S}Vect(X)$ is an abelian group, this surjection extends linearly to a surjective homomorphism of abelian groups,

$$\rho : K(X) \to \mathcal{S}Vect(X).$$

Since $[\epsilon_n] = [n] \in K(X)$ maps to zero in $\mathcal{S}Vect(X)$ under ρ, this map factors through a surjective homomorphism from reduced K - theory, which by abuse of notation we also call ρ,

$$\rho : \tilde{K}(X) \to \mathcal{S}Vect(X).$$

To prove that ρ is a injective (and hence an isomorphism), we will construct a left inverse to ρ. This is done by considering the composition

$$Vect^*(X) \xrightarrow{\iota} K(X) \to \tilde{K}(X)$$

which is given by mapping an n - dimensional bundle ζ to $[\zeta] - [n]$. This map clearly sends two bundles which are stably isomorphic to the same class in $\tilde{K}(X)$, and hence factors through a homomorphism

$$j : \mathcal{S}Vect(X) \to \tilde{K}(X).$$

By checking its values on bundles, it becomes clear that the composition $j \circ \rho : \tilde{K}(X) \to \mathcal{S}Vect(X) \to \tilde{K}(X)$ is the identity map. This proves the theorem. □

We end this section with the following observation. As we said above, in the next chapter we will study the classification of bundles. In the process we will show that homotopic maps induce isomorphic pull - back bundles, and therefore homotopy equivalences induce bijections, via pulling back, on the sets of isomorphism classes of bundles. This tells us that K -theory is a "homotopy invariant" of topological spaces and continuous maps between them. More precisely, the results of the next chapter will imply the following important properties of K - theory.

Theorem 3.19. *Let X and Y be compact spaces of the homotopy type of CW complexes. Let $f : X \to Y$ and $g : X \to Y$ be homotopic maps. then the pull back homomorphisms are equal*

$$f^* = g^* : K(Y) \to K(X)$$

and

$$f^* = g^* : KO(Y) \to KO(X).$$

This can be expressed in categorical language as follows: (Notice the similarity of role K - theory plays to cohomology theory in the following theorem.)

Theorem 3.20. *The assignments $X \to K(X)$ and $X \to KO^*(X)$ are contravariant functors from the category of topological spaces and homotopy classes of continuous maps to the category of rings and ring homomorphisms.*

3.3.3 Differential Forms

In the next two sections we describe certain differentiable constructions on bundles over smooth manifolds that are basic in geometric analysis. We begin by recalling some "multilinear algebra".

Let V be a vector space over a field k. Let $T(V)$ be the associated tensor algebra

$$T(V) = \oplus_{n \geq 0} V^{\otimes n}$$

where $V^0 = k$. The algebra structure is comes from the natural pairings

$$V^{\otimes n} \otimes V^{\otimes m} \xrightarrow{=} V^{\otimes (n+m)}.$$

Recall that the *exterior algebra*

$$\Lambda(V) = T(V)/\mathcal{A}$$

where $\mathcal{A} \subset T(V)$ is the two sided ideal generated by $\{a \otimes b + b \otimes a : a, b \in V\}$.

The algebra $\Lambda(V)$ inherits the grading from the tensor algebra, $\Lambda(V) = \oplus_{n \geq 0} \Lambda^n(V)$, and the induced multiplication is called the "wedge product", $u \wedge v$. Recall that if V is an n - dimensional vector space, $\Lambda^k(V)$ is an $\binom{n}{k}$ - dimensional vector space.

Assume now that V is a real vector space. An element of the dual space, $(V^{\otimes n})^* = Hom(V^{\otimes n}, \mathbb{R})$ is a multilinear form $V \times \cdots \times V \to \mathbb{R}$. An element of the dual space $(\Lambda^k(V))^*$ is an alternating form, i.e a multilinear function θ so that

$$\theta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = sgn(\sigma)\theta(v_1, \cdots, v_k)$$

where $\sigma \in \Sigma_k$ is any permutation.

Let $\mathcal{A}^k(V) = (\Lambda^k(V))^*$ be the space of alternating k - forms. Let $U \subset \mathbb{R}^n$ be an open set. Recall the following definition.

Definition 3.17. *A differential k - form on the open set $U \subset \mathbb{R}^n$ is a smooth function*

$$\omega : U \to \mathcal{A}^k(\mathbb{R}^n).$$

By convention, 0 -forms are just smooth functions, $f : U \to \mathbb{R}$. Notice that given such a smooth function, its differential, df assigns to a point $x \in U \subset \mathbb{R}^n$ a linear map on tangent spaces, $df(x) : \mathbb{R}^n = T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R} = \mathbb{R}$. That is, $df : U \to (\mathbb{R}^n)^*$, and hence is a one form on U.

Let $\Omega^k(U)$ denote the space of k - forms on the open set U. Recall that any k -form $\omega \in \Omega^k(U)$ can be written in the form

$$\omega(x) = \sum_I f_I(x) dx_I \tag{3.4}$$

where the sum is taken over all sequences of length k of integers from 1 to n, $I = (i_1, \cdots, i_k)$, $f_I : U \to \mathbb{R}$ is a smooth function, and where

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Here dx_i denotes the differential of the function $x_i : U \subset \mathbb{R}^n \to \mathbb{R}$ which is the projection onto the i^{th} - coordinate.

Recall also that there is an exterior derivative,

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

defined by

$$d(f dx_I) = df \wedge dx_I == \sum_{j=1}^{k} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

A simple calculation shows that $d^2(\omega) = d(d\omega) = 0$, using the symmetry of second order partial derivatives.

These constructions can be extended to arbitrary differentiable manifolds in the following way. Given an n - dimensional smooth (C^∞) manifold M, let $\Lambda^k(T(M))$ be the $\binom{n}{k}$ - dimensional vector bundle whose fiber at $x \in M$ is the k - fold exterior product, of the tangent space, $\Lambda^k(T_xM)$.

Exercise.

Define clutching functions of $\Lambda^k(T(M))$ in terms of clutching functions of the tangent bundle, $T(M)$

Definition 3.18. *A differential k-form on M is a section of the dual bundle,*

$$\Lambda^k(T(M))^* \cong \Lambda^k(T^*(M)) \cong Hom(\Lambda^k(T(M)), \epsilon_1).$$

That is, the space of k -forms is given by the space of sections,

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*(M))).$$

So a k -form $\omega \in \Omega^k(M)$ assigns to $x \in M$ an alternating k form on its tangent space,

$$\omega(x) : T_xM \times \cdots \times T_xM \to \mathbb{R}.$$

and hence given a local chart with a local coordinate system, then locally ω can be written in the form (3.4).

Since differentiation is a local operation, we may extend the definition of the exterior derivative of forms on open sets in $\mathbb{R}^n$ to all n - manifolds,

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

In particular, the zero forms are the space of functions, $\Omega^0(M) = C^\infty(M; \mathbb{R})$, and for $f \in \Omega^0(M)$, then $df \in \Omega^1(M) = \Gamma(T(M)^*)$ is the 1 -form defined by the differential,

$$df(x) : T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}.$$

Now as above, $d^2(\omega) = 0$ for any form ω. Thus we have a cochain complex, called the *deRham* complex,

$$\begin{array}{l}\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \\ \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.\end{array} \tag{3.5}$$

Recall that a k - form ω with $d\omega = 0$ is called a *closed* form. A k - form ω in the image of d, i.e $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$ is called an *exact* form. The quotient vector space of closed forms modulo exact forms defined the "deRham cohomology" group:

Definition 3.19.

$$H^k_{deRham}(M) = \{closed\ k\text{ - }forms\}/\{exact\ k\text{ - }forms\}.$$

The famous *de Rham theorem* asserts that these cohomology groups are isomorphic to singular cohomology with $\mathbb{R}$ - coefficients. To see the relationship, let $C_k(M)$ be the space of k - dimensional singular chains on M, (i.e the free abelian group generated by smooth singular simplices $\sigma : \Delta^k \to M$), and let

$$C^k(M;\mathbb{R}) = Hom(C_k(M), \mathbb{R})$$

be the space of real valued singular cochains. Notice that a k -form ω gives rise to a k - dimensional singular cochain in that it acts on a singular simplex $\sigma : \Delta^k \to M$ by

$$\langle \omega, \sigma \rangle = \int_\sigma \omega.$$

This defines a homomorphism

$$\gamma : \Omega^k(M) \to C^k(M;\mathbb{R})$$

for each k.

Exercise. Prove that γ is a map of cochain complexes. That is,

$$\gamma(d\omega) = \delta\gamma(\omega)$$

where $\delta : C^k(M;\mathbb{R}) \to C^{k+1}(M;\mathbb{R})$ is the singular coboundary operator.
Hint. Use Stokes' theorem.

We refer the reader to [15] for a proof of the *deRham Theorem:*

Theorem 3.21. *The map of cochain complexes,*

$$\gamma : \Omega^*(M) \to C^*(M;\mathbb{R})$$

is a chain homotopy equivalence. Therefore it induces an isomorphism in cohomology

$$H^*_{deRham}(M) \xrightarrow{\cong} H^*(M;\mathbb{R}).$$

3.3.4 Lie Groups

Lie groups play a central role in bundle theory and in differential topology and geometry. In this section we give a basic description of Lie groups, their actions on manifolds (and other spaces), as well as their associated principal bundles.

Definition 3.20. *A Lie group is a topological group G which has the structure of a differentiable manifold. Moreover the multiplication map*

$$G \times G \to G$$

and the inverse map

$$\begin{aligned} G &\to G \\ g &\to g^{-1} \end{aligned}$$

are required to be differentiable maps.

The following is an important basic property of the differential topology of Lie groups.

Theorem 3.22. *Let G be a Lie group. Then G is parallelizable. That is, its tangent bundle TG is trivial.*

Proof. Let $1 \in G$ denote the identity element, and T_1G the tangent space of G at 1. If G is an n - dimensional manifold, T_1G is an n- dimensional vector space. We define a bundle isomorphism of the tangent bundle TG with the trivial bundle $G \times T_1(G)$, which, on the total space level is given by a map

$$\phi : G \times T_1G \longrightarrow TG$$

defined as follows. Let $g \in G$. Then multiplication by g on the right is a diffeomorphism

$$\begin{aligned} r_g : G &\to G \\ x &\to xg \end{aligned}$$

Left multiplication $\ell_g : G \to G$ is defined similarly. Since r_g is a diffeomorphism, its derivative is a linear isomorphism at every point:

$$Dr_g(x) : T_xG \xrightarrow{\cong} T_{xg}G.$$

We can now define

$$\phi : G \times T_1G \to TG$$

by

$$\phi(g, v) = Dr_g(1)(v) \in T_gG.$$

Clearly ϕ is a bundle isomorphism. □

If G is a Lie group and M is a smooth manifold with a right G - action. We say that the action is *smooth* if the homomorphism $\mu : G \to Homeo(G)$ defined by the action factors through a homomorphism

$$\mu : G \to Diffeo(M)$$

where $Diffeo(M)$ is the group of diffeomorphisms of M.

The following result is originally due to A. Gleason [60], and its proof can be found in Steenrod's book [143]. It is quite helpful in studying free group actions.

Theorem 3.23. *Let E be a smooth manifold, having a free, smooth G - action, where G is a compact Lie group. Then the projection map*

$$p : E \to E/G$$

is a principal G - bundle.

The following was one of the early theorems in fiber bundle theory, appearing originally in H. Samelson's thesis. [133]

Corollary 3.24. *Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space*

$$p : G \to G/H$$

is a principal H - bundle.

3.3.5 Connections and Curvature

In modern geometry, differential topology, and geometric analysis, one often needs to study not only smooth functions on a manifold, but more generally, spaces of smooth sections of a vector bundle $\Gamma(\zeta)$. (Notice that sections of bundles are indeed a generalization of smooth functions in that the space of sections of the n - dimensional trivial bundle over a manifold M, $\Gamma(\epsilon_n) = C^\infty(M;\mathbb{R}^n) = \oplus_n C^\infty(M;\mathbb{R})$.) Similarly, one needs to study differential forms that take values in vector bundles. These will be defined in this subsection.

Definition 3.21. *Let ζ be a smooth finite dimensional vector bundle over a manifold M. A differential k - form with values in ζ is defined to be a smooth section of the bundle of homomorphisms, $Hom(\Lambda^k(T(M)), \zeta) = \Lambda^k(T(M)^*) \otimes \zeta$.*

We write the space of k -forms with values in ζ as

$$\Omega^k(M;\zeta) = \Gamma(\Lambda^k(T(M)^* \otimes \zeta)).$$

The zero forms are simply the space of sections, $\Omega^0(M;\zeta) = \Gamma(\zeta)$. Notice that if ζ is the trivial bundle $\zeta = \epsilon_n$, then one gets standard forms,

$$\Omega^k(M;\epsilon_n) = \Omega^k(M) \otimes \mathbb{R}^n = \oplus_n \Omega^k(M).$$

Even though spaces of forms with values in a bundle are easy to define, there is no canonical analogue of the exterior derivative. There do however exist differential operators

$$D : \Omega^k(M;\zeta) \to \Omega^{k+1}(M;\zeta)$$

that satisfy familiar product formulas. These operators are called *covariant derivatives* (or *connections*) and are related to the notion of a connection on a principal bundle, which we now define and study.

Let G be a compact Lie group. Recall that the tangent bundle TG has a canonical trivialization

$$\begin{aligned} \psi : G \times T_1G &\to TG \\ (g,v) &\to D(\ell_g)(v) \end{aligned}$$

where for any $g \in G$, $\ell_g : G \to G$ is the map given by left multiplication by g, and $D(\ell_g) : T_hG \to T_{gh}G$ is its derivative. r_g and $D(r_g)$ will denote the analogous maps corresponding to right multiplication.

The differential of right multiplication on G defines a right action of G on the tangent bundle TG. We claim that the trivialization ψ is equivariant with respect to this action, if we take as the right action of G on T_1G to be the *adjoint action:*

$$\begin{aligned} T_1G \times G &\to T_1G \\ (v,g) &\to D(\ell_{g^{-1}})(v)D(r_g). \end{aligned}$$

Exercise. Verify this claim.

Multiplication in the group $G \times G \to G$, defines upon differentiation at the identity element $1 \in G$, an algebra structure on T_1G:

$$T_1G \times T_1G \to T_1G.$$

This is called the *"Lie algebra"* of the group G, and we denote it by $\mathfrak{g}$.

The action G on $\mathfrak{g}$ described above is referred to as the *adjoint representation* of the Lie group G on its Lie algebra $\mathfrak{g}$. Now let

$$p : P \to M$$

be a smooth principal G -bundle over a manifold M. This adjoint representation induces a vector bundle $ad(P)$,

$$ad(P) : P \times_G \mathfrak{g} \to M. \tag{3.6}$$

This bundle has the following relevance. Let $p^*(TM) : p^*(TM) \to P$ be the pull - back over the total space P of the tangent bundle of M. We have a surjective map of bundles

$$TP \to p^*(TM).$$

Define T_FP to be the kernel bundle of this map. So the fiber of T_FP at a point $y \in P$ is the kernel of the surjective linear transformation $Dp(y) : T_yP \to T_{p(y)}M$. Notice that the right action of G on the total space of the principal bundle P defines an action of G on the tangent bundle TP, which restricts to an action of G on T_FP. Furthermore, by recognizing that the fibers are equivariantly homeomorphic to the Lie group G, the following is a direct consequence of the above considerations:

Proposition 3.25. *T_FP is naturally isomorphic to the pull - back of the adjoint bundle,*

$$T_FP \cong p^*(ad(P)).$$

Thus we have an exact sequence of G - equivariant vector bundles over P:

$$0 \to p^*(ad(P)) \to TP \xrightarrow{Dp} p^*(TM) \to 0. \tag{3.7}$$

Recall that short exact sequences of bundles split as Whitney sums. A connection is a G - equivariant splitting of this sequence:

Definition 3.22. *A* **connection** on the principal bundle P is a G - equivariant splitting

$$\omega_A : TP \to p^*(ad(P))$$

of the above sequence of vector bundles. That is, ω_A defines a G - equivariant isomorphism

$$\omega_A \oplus Dp : TP \to p^*(ad(P)) \oplus p^*(TM).$$

The following is an important description of the space of connections on P, which we call $\mathcal{A}(P)$.

Proposition 3.26. *The space of connections on the principal bundle P, $\mathcal{A}(P)$, is an affine space modeled on the infinite dimensional vector space of one forms on M with values in the bundle $ad(P)$, $\Omega^1(M; ad(P))$.*

Proof. Consider two connections ω_A and ω_B,

$$\omega_A, \omega_B : TP \to p^*(ad(P).$$

Since these are splittings of the exact sequence 3.7, they are both the identity when restricted to $p^*(ad(P)) \hookrightarrow TP$. Thus their difference, $\omega_A - \omega_B$ is zero when restricted to $p^*(ad(P))$. By the exact sequence it therefore factors as a composition

$$\omega_A - \omega_B : TP \to p^*(TM) \xrightarrow{\alpha} p^*(ad(P))$$

for some bundle homomorphism $\alpha : p^*(TM) \to p^*(ad(P))$. That is, for every $y \in P$, α defines a linear transformation

$$\alpha_y : p^*(TM)_y \to p^*(ad(P))_y.$$

Hence for every $y \in P$, α defines (and is defined by) a linear transformation

$$\alpha_y : T_{p(y)}M \to ad(P)_{p(y)}.$$

Furthermore, the fact that both ω_A and ω_B are *equivariant* splittings says that $\omega_A - \omega_B$ is equivariant, which translates to the fact that α_y only depends on the orbit of y under the G - action. That is,

$$\alpha_y = \alpha_{yg} : T_{p(y)}M \to ad(P)_{p(y)}$$

for every $g \in G$. Thus α_y only depends on $p(y) \in M$. Hence for every $x \in M$, α defnes, and is defined by, a linear transformation

$$\alpha_x : T_xM \to ad(P)_x.$$

Thus α may be viewed as a section of the bundle of homormorphisms, $Hom(TM, ad(P))$, and hence is a one form,

$$\alpha \in \Omega^1(M; ad(P)).$$

Thus any two connections on P differ by an element in $\Omega^1(M; ad(P))$ in this sense.

Now reversing the procedure, an element $\beta \in \Omega^1(M; ad(P))$ defines an equivariant homomorphism of bundles over P,

$$\beta : p^*(TM) \to p^*(ad(P)).$$

By adding the composition

$$TP \xrightarrow{Dp} p^*(TM) \xrightarrow{\beta} p^*(ad(P))$$

to any connection (equivariant splitting)

$$\omega_A : TP \to p^*(ad(P))$$

one produces a new equivariant splitting of TP, and hence a new connection. The proposition follows. □

Remark. Even though the space of connections $\mathcal{A}(P)$ is affine, it is not, in general a vector space. There is no "zero" in $\mathcal{A}(P)$ since there is no pre-chosen, canonical connection. The one exception to this, of course, is when P is the trivial G - bundle,

$$P = M \times G \to M.$$

In this case there is an obvious equvariant splitting of TP, which serves as the "zero" in $\mathcal{A}(P)$. Moreover in this case the adjoint bundle $ad(P)$ is also trivial,

$$ad(P) = M \times \mathfrak{g} \to M.$$

Hence there is a canonical identification of the space of connections on the trivial bundle with $\Omega^1(M; \mathfrak{g}) = \Omega^1(M) \otimes \mathfrak{g}$.

Let $p : P \to M$ be a principal G - bundle and let $\omega_A \in \mathcal{A}(P)$ be a connection.

The curvature F_A of ω_A is a two form

$$F_A \in \Omega^2(M; ad(P))$$

which measures to what extent the splitting ω_A commutes with the braket operation on vector fields. More precisely, let X and Y be vector fields on M. The connection ω_A defines an equivariant splitting of TP and hence defines a "horizontal" lifting of these vector fields, which we denote by $\tilde{X}$ and $\tilde{Y}$ respectively.

Definition 3.23. *The curvature $F_A \in \Omega^2(M; ad(P))$ is defined by*

$$F_A(X, Y) = \omega_A[\tilde{X}, \tilde{Y}].$$

For those unfamiliar with the bracket operation on vector fields, we refer you to [142].

Another important construction with connections is the associated *covariant derivative* which is defined as follows.

Definition 3.24. *The covariant derivative induced by the connection ω_A*

$$D_A : \Omega^0(M; ad(P)) \to \Omega^1(M; ad(P))$$

is defined by

$$D_A(\sigma)(X) = [\tilde{X}, \sigma].$$

where X is a vector field on M.

The notion of covariant derivative, and hence connection, extends to vector bundles as well. Let $\zeta : p : E^\zeta \to M$ be a finite dimensional vector bundle over M.

Definition 3.25. *A connection on ζ (or a covariant derivative) is a linear transformation*

$$D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$$

that satisfies the Leibnitz rule

$$D_A(f\phi) = df \otimes \phi + f D_A(\phi) \tag{3.8}$$

for any $f \in C^\infty(M;\mathbb{R})$ and any $\phi \in \Omega^0(M;\zeta)$.

Now we can model the space of connections on a vector bundle, $\mathcal{A}(\zeta)$ similarly to how we modeled the space of connections on a principal bundle $\mathcal{A}(P)$. Namely, given any two connections D_A and D_B on ζ and a function $f \in C^\infty(M;\mathbb{R})$, one can take the convex combination

$$f \cdot D_A + (1-f) \cdot D_B$$

and obtain a new connection. From this it is not difficult to see the following. We leave the proof as an exercise to the reader.

Proposition 3.27. *The space of connections on the vector bundle ζ, $\mathcal{A}(\zeta)$ is an affine space modeled on the vector space of one forms $\Omega^1(M;End(\zeta))$, where $End(\zeta)$ is the bundle of endomorphisms of ζ.*

Let X be a vector field on M and D_A a connection on the vector bundle ζ. The covariant derivative in the direction of X, which we denote by $(D_A)_X$ is an operator on the space of sections of ζ,

$$(D_A)_X : \Omega^0(M;\zeta) \to \Omega^0(M;\zeta)$$

defined by

$$(D_A)_X(\sigma) = \langle D_A(\phi); X\rangle.$$

One can then define the curvature $F_A \in \Omega^2(M;End(\zeta))$ by defining its action on a pair of vector fields X and Y to be

$$F_A(X,Y) = (D_A)_X(D_A)_Y - (D_A)_Y(D_A)_X - (D_A)_{[X,Y]}. \tag{3.9}$$

To interpret this formula notice that a - priori $F_A(X,Y)$ is a second order differential operator on the space of sections of ζ. However a direct calculation shows that for $f \in C^\infty(M;\mathbb{R})$ and $\sigma \in \Omega^0(M;\zeta)$, then

$$F_A(X,Y)(f\sigma) = fF_A(X,Y)(\sigma)$$

and hence $F_A(X,Y)$ is in fact a zero - order operator on $\Omega^0(M;\zeta)$. But a zero order operator on the space of sections of ζ is a section of the endomorphism bundle $End(\zeta)$. Thus F_A assigns to any pair of vector fields X and Y a section of $End(\zeta)$. Moreover it is straightforward to check that this assignment is tensorial in X and Y (i.e $F_A(fX,Y) = F_A(X,fY) = fF_A(X,Y)$). Thus F_A is an element of $\Omega^2(M; End(\zeta))$. The curvature measures the lack of commutativity in second order partial covariant derivatives.

Given a connection on a bundle ζ the linear mapping $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ extends to a *deRham* type sequence,

$$\Omega^0(M;\zeta) \xrightarrow{D_A} \Omega^1(M;\zeta) \xrightarrow{D_A} \Omega^2(M;\zeta) \xrightarrow{D_A} \cdots$$

where for $\sigma \in \Omega^p(M;\zeta)$, $D_A(\sigma)$ is the $p+1$ -form defined by the formula

$$D_A(\sigma)(X_0,\cdots,X_p) = \sum_{j=0}^{p}(-1)^j (D_A)_{X_j}(\sigma(X_0,\cdots,\hat{X}_j,\cdots,X_p)) \tag{3.10}$$
$$+ \sum_{i<j}(-1)^{i+j}\sigma([X_i,X_j],X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_p).$$

We observe that unlike with the standard deRham exterior derivative (which can be viewed as a connection on the trivial line bundle), it is not generally true that $D_A \circ D_A = 0$. In fact we have the following, whose proof is a direct calculation that we leave to the reader.

Proposition 3.28.

$$D_A \circ D_A = F_A : \Omega^0(M;\zeta) \to \Omega^2(M;\zeta)$$

where in this context the curvature F_A is interpreted as a assigning to a section $\sigma \in \Omega^0(M;\zeta)$ the 2 - form $F_A(\sigma)$ which associates to vector fields X and Y the section $F_A(X,Y)(\sigma)$ as defined in (3.9).

Thus the curvature of a connection F_A can also be viewed as measuring the extent to which the covariant derivative D_A fails to form a cochain complex on the space of differential forms with values in the bundle ζ. However it is always true that the covariant derivative of the curvature tensor is zero. This is the well known *Bianchi identity* (see [142] for a complete discussion).

Theorem 3.29. *Let A be a connection on a vector bundle ζ. Then*

$$D_A F_A = 0.$$

We end this section by observing that if P is a principal G - bundle with a connection ω_A, then any representation of G on a finite dimensional vector space V induces a connection on the corresponding vector bundle

$$P \times_G V \to M.$$

We refer the reader to [69] and [142] for thorough discussions of the various ways of viewing connections. [9] has a nice, brief discussion of connections on principal bundles, and [49] and [90] have similarly concise discussions of connections on vector bundles.

3.3.6 The Levi - Civita Connection

Let M be a manifold equipped with a Riemannian structure. Recall that this is a Euclidean structure on its tangent bundle. In this section we will show how this structure induces a connection, or covariant derivative, on the tangent bundle. This connection is called the *Levi - Civita* connection associated to the Riemannian structure. Our treatment of this topic follows that of Milnor and Stasheff [121].

Let $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ be a connection (or covariant derivative) on an n - dimensional vector bundle ζ. Its curvature is a two- form with values in the endomorphism bundle

$$F_A \in \Omega^2(M; End(\zeta)).$$

The endomorphism bundle can be described alternatively as follows. Let E_ζ be the principal $GL(n,\mathbb{R})$ bundle associated to ζ. Then of course $\zeta = E_\zeta \otimes_{GL(n,\mathbb{R})} \mathbb{R}^n$. The endomorphism bundle can then be described as follows. The proof is an easy exercise that we leave to the reader.

Proposition 3.30.

$$End(\zeta) \cong ad(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R})$$

where $GL(n,\mathbb{R})$ acts on $M_n(\mathbb{R})$ by conjugation,

$$A \cdot B = ABA^{-1}.$$

Let ω be a differential p - form on M with values in $End(\zeta)$,

$$\omega \in \Omega^p(M; End(\zeta)) \cong \Omega^p(M; ad(\zeta)) = \Omega^p(M; E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R})).$$

Then on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ for ζ, (and hence the induced coordinate chart and local trivialization for $ad(\zeta)$), ω can be viewed as an $n \times n$ matrix of p -forms on M. We write

$$\omega = (\omega_{i,j}).$$

Of course this description depends on the coordinate chart and local trivialization chosen, but at any $x \in U$, by the above proposition, two trivializations yield conjugate matrices. That is, if $(\omega_{i,j}(x))$ and $(\omega'_{i,j}(x))$ are two matrix descriptions of $\omega(x)$ defined by two different local trivializations of $\zeta_{|_U}$, then there exists an $A \in GL(n,\mathbb{C})$ with

$$A(\omega_{i,j}(x))A^{-1} = (\omega'_{i,j}(x)).$$

Now suppose the bundle ζ is equipped with a Euclidean structure. As seen earlier in this chapter this is equivalent to its associated principal $GL(n, \mathbb{R})$ - bundle E_ζ having a reduction to the structure group $O(n)$. We let $E_{O(n)} \to M$ denote this principal $O(n)$ - bundle.

Now the Lie algebra $\mathfrak{o}(n)$ of $O(n)$ (i.e the tangent space $T_1(O(n))$) is a subspace of the Lie algebra of $GL(n, \mathbb{R})$, i.e

$$\mathfrak{o}(n) \subset M_n(\mathbb{R}).$$

The following is well known (see, for example[134]).

Proposition 3.31. *The Lie algebra $\mathfrak{o}(n) \subset M_n(\mathbb{R})$ is the subspace consisting of skew symmetric $n \times n$ - matrices. That is, $A \in \mathfrak{o}(n)$ if and only if*

$$A^t = -A$$

where A^t denotes the transpose of A.

So if ζ has a Euclidean structure, we can form the adjoint bundle

$$ad^O(\zeta) = E_{O(n)} \times_{O(n)} \mathfrak{o}(n) \subset E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R}) = ad(\zeta)$$

where, again $O(n)$ acts on $\mathfrak{o}(n)$ by conjugation.

Now suppose D_A is an orthogonal connection on ζ. That is, it is induced by a connection on the principal $O(n)$ - bundle $E_{O(n)} \to M$. The following is fairly clear, and we leave its proof as an exercise.

Corollary 3.32. *If D_A is an orthogonal connection on a Euclidean bundle ζ, then the curvature F_A lies in the space of $\mathfrak{o}(n)$ valued two forms*

$$F_A \in \Omega^2(M; ad^O(\zeta)) \subset \Omega^2(M; ad(\zeta)) = \Omega^2(M; End(\zeta)).$$

Furthermore, on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ that preserves the Euclidean structure, we may write the form F_A as a skew - symmetric matrix of two forms,

$$F_{A_{|_U}} = (\omega_{i,j}) \quad i, j = 1, \cdots, n$$

where each $\omega_{i,j} \in \Omega^2(M)$ and $\omega_{i,j} = -\omega_{j,i}$. In fact the connection D_A itself can be written as skew symmetric matrix of one forms

$$D_{A_{|_U}} = (\alpha_{i,j})$$

where each $\alpha_{i,j} \in \Omega^1(M)$.

We now describe the notion of a "symmetric" connection on the cotangent bundle of a manifold, and then show that if the manifold is equipped with a Riemannian structure (i.e a Euclidean structure on the (co) - tangent bundle), then there is a unique symmetric, orthogonal connection on the cotangent bundle.

Definition 3.26. *A connection D_A on the cotangent bundle T^*M is symmetric* (or *torsion free*) if the composition

$$\Gamma(T^*) = \Omega^0(M;T^*) \xrightarrow{D_A} \Omega^1(M;T^*) = \Gamma(T^* \otimes T^*) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*)$$

is equal to the exterior derivative d.

In terms of local coordinates $x_1, \cdots, x_n$, if we write

$$D_A(dx_k) = \sum_{i,j} \Gamma^k_{i,j} dx_i \otimes dx_j \tag{3.11}$$

(the functions $\Gamma^k_{i,j}$ are called the "Christoffel symbols"), then the requirement that D_A is symmetric is that the image $\sum_{i,j} \Gamma^k_{i,j} dx_i \otimes dx_j$ be equal to the exterior derivative $d(dx_k) = 0$. This implies that the Christoffel symbols $\Gamma^k_{i,j}$ must be symmetric in i and j. The following is straightforward to verify.

Lemma 3.33. *A connection D_A on T^* is symmetric if and only if the covariant derivative of the differential of any smooth function*

$$D_A(df) \in \Gamma(T^* \otimes T^*)$$

is a symmetric tensor. That is, if $\psi_1, \cdots, \psi_n$ form a local basis of sections of T^*, and we write the corresponding local expression

$$D_A(df) = \sum_{i,j} a_{i,j}\, \psi_i \otimes \psi_j$$

then $a_{i,j} = a_{j,i}$.

We now show that the (co)-tangent bundle of a Riemannian metric has a preferred connection.

Theorem 3.34. *The cotangent bundle T^*M of a Riemannian manifold has a unique orthogonal, symmetric connection. (It is orthogonal with respect to the Euclidean structure defined by the Riemannian metric.)*

Proof. Let U be an open neighborhood in M with a trivialization

$$\psi : U \times \mathbb{R}^n :\to T^*_{|U}$$

which preserves the Euclidean structure. ψ defines n orthonormal sections of $T^*_{|U}$, $\psi_1, \cdots, \psi_n$. The ψ_j's constitute an orthonormal basis of one forms on M. We will show that there is one and only one skew-symmetric matrix $(\alpha_{i,j})$ of one forms such that

$$d\psi_k = \sum \alpha_{k,j} \wedge \psi_j.$$

We can then define a connection D_A on $T^*_{|U}$ by requiring that

$$D_A(\psi_k) = \sum \alpha_{k,j} \otimes \psi_j.$$

It is then clear that D_A is the unique symmetric connection which is compatible with the metric. Since the local connections are unique, they glue together to yield a unique global connection with this property.

In order to prove the existence and uniqueness of the skew symmetric matrix of one forms $(\alpha_{i,j})$ we need the following combinatorial observation.

Any $n \times n \times n$ array of real valued functions $A_{i,j,k}$ can be written uniquely as the sum of an array $B_{i,j,k}$ which is symmetric in i, j, and an array $C_{i,j,k}$ which is skew symmetric in j, k. To see this, consider the formulas

$$B_{i,j,k} = \frac{1}{2}(A_{i,j,k} + A_{j,i,k} - A_{k,i,j} - A_{k,j,i} + A_{j,k,i} + A_{i,k,j})$$
$$C_{i,j,k} = \frac{1}{2}(A_{i,j,k} - A_{j,i,k} + A_{k,i,j} + A_{k,j,i} - A_{j,k,i} - A_{i,k,j})$$

Uniqueness would follow since if an array $D_{i,j,k}$ were both symmetric in i, j and skew symmetric in j, k, then one would have

$$D_{i,j,k} = D_{j,i,k} = -D_{j,k,i} = -D_{k,j,i} = D_{k,i,j} = D_{i,k,j} = -D_{i,j,k}$$

and hence all the entries are zero.

Now choose functions $A_{i,j,k}$ such that

$$d\psi_k = \sum A_{i,j,k}\, \psi_i \wedge \psi_j$$

and set $A_{i,j,k} = B_{i,j,k} + C_{i,j,k}$ as above. It then follows that

$$d\psi_k = \sum C_{i,j,k}\, \psi_i \wedge \psi_j$$

by the symmetry of the $B_{i,j,k}$'s. Then we define the one forms

$$\alpha_{k,j} = \sum C_{i,j,k}\, \psi_i.$$

They clearly form the unique skew symmetric matrix of one forms with $d\psi_k = \sum \alpha_{k,j} \wedge \psi_j$. This proves the lemma. □

This preferred connection on the (co)tangent bundle of a Riemannian metric is called the Levi - Civita connection. Statements about the curvature of a metric on a manifold are actually statements about the curvature form of the Levi - Civita connection associated to the Riemannian metric. For example, a "flat metric" on a manifold is a Riemannian structure whose corresponding Levi-Civita connection has zero curvature form. As is fairly clear, these connections form a central object of study in Riemannian geometry.

4

Homotopy Theory of Fibrations

In this chapter we study the basic algebraic topological properties of fiber bundles, and their generalizations, "Serre fibrations". We begin with a discussion of homotopy groups and their basic properties. We then show that fibrations yield long exact sequences in homotopy groups and use it to show that the loop space of the classifying space of a group is homotopy equivalent to the group. We then develop basic obstruction theory for liftings in fibrations, use it to interpret characteristic classes as obstructions, and apply them in several geometric contexts, including vector fields, Spin structures, and classification of $SU(2)$ - bundles over four dimensional manifolds. We also use obstruction theory to prove the existence of Eilenberg - MacLane spaces, and to prove their basic property of classifying cohomology. We then develop the theory of spectral sequences and then discuss the famous Leray - Serre spectral sequence of a fibration. We use it in several applications, including a proof of the theorem relating homotopy groups and homology groups, a calculation of the homology of the loop space ΩS^n, and a calculation of the homology of the Lie groups $U(n)$ and $O(n)$.

4.1 Homotopy Groups

We begin by adopting some conventions and notation. In this chapter, unless otherwise specified, we will assume that all spaces are of the homotopy type of CW complexes, are connected, and come equipped with a basepoint. When we write $[X, Y]$ we mean homotopy classes of basepoint preserving maps $X \to Y$. Suppose $x_0 \in X$ and $y_0 \in Y$ are the basepoints. Then a basepoint preserving homotopy between basepoint preserving maps f_0 and $f_1 : X \to Y$ is a map

$$F : X \times I \to Y$$

such that each $F_t : X \times \{t\} \to Y$ is a basepoint preserving map and $F_0 = f_0$ and $F_1 = f_1$. If $A \subset X$ and $B \subset Y$, are subspaces that contain the basepoints, ($x_0 \in A$, and $y_0 \in B$), we write $[X, A; Y, B]$ to mean homotopy classes of maps $f : X \to Y$ so that the restriction $f_{|_A}$ maps A to B. Moreover homotopies are assumed to preserve these subsets as well. That is, a homotopy defining this equivalence relation is a map $F : X \times I \to Y$ that restricts to a basepoint preserving homotopy $F : A \times I \to B$. We can now give a careful strict definition of homotopy groups.

Definition 4.1. *The n^{th} homotopy group of a space X with basepoint $x_0 \in X$ is defined to be the set*

$$\pi_n(X) = \pi_n(X, x_0) = [S^n, X].$$

Equivalently, this is the set

$$\pi_n(X) = [D^n, S^{n-1}; X, x_0]$$

where $S^{n-1} = \partial D^n$ is the boundary sphere.

Exercise. Prove that these two definitions are in fact equivalent.

Remarks. 1. It will often helpful to us to use as our model of the disk D^n the n - cube $I^n = [0,1]^n$. Notice that in this model the boundary ∂I^n consists of n - tuples $(t_1, \cdots, t_n)$ with $t_i \in [0,1]$ where at least one of the coordinates is either 0 or 1.
2. Notice that for $n = 1$, this definition of the first homotopy group is the usual definition of the fundamental group.

So far the homotopy "groups" have only been defined as sets. We now examine the group structure. To do this, we will define our homotopy groups via the cube I^n, which we give the basepoint $(0, \cdots, 0)$. Let

$$f \quad \text{and} \quad g : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

be two maps representing elements $[f]$ and $[g] \in \pi_n(X, x_0)$. Define

$$f \cdot g : I^n \longrightarrow X$$

by

$$f \cdot g(t_1, t_2, \cdots, t_n) = \begin{cases} f(2t_1, t_2, \cdots, t_n) & \text{for } t_1 \in [0, 1/2] \\ g(2t-1, t_2, \cdots, t_n) & \text{for } t_1 \in [1/2, 1] \end{cases}$$

The map $f \cdot g : (I^n, \partial I^n) \to (X, x_0)$ represents the product of the classes

$$[f \cdot g] = [f] \cdot [g] \in \pi_n(X, x_0).$$

Notice that in the case $n = 1$ this is precisely the definition of the product structure on the fundamental group $\pi_1(X, x_0)$. The same proof that this product structure is well defined and gives the fundamental group the structure of an associative group extends to prove that all of the homotopy groups are in fact groups under this product structure. We leave the details of checking this to the reader. We refer the reader to any introductory textbook on algebraic topology for the details. A good reference is [67].

As we know the fundamental group of a space can be quite complicated. Indeed any group can be the fundamental group of a space. In particular fundamental groups can be very much noncommutative. However we recall the relation of the fundamental group to the first homology group, for which we again refer the reader to any introductory textbook:

Theorem 4.1. *Let X be a connected space. Then the abelianization of the fundamental group is isomorphic to the first homology group,*

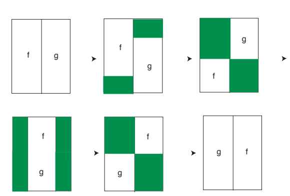
$$\pi_1(X)/[\pi_1, \pi_1] \cong H_1(X)$$

where $[\pi_1, \pi_1]$ is the commutator subgroup of $\pi_1(X)$.

We also have the following basic result about higher homotopy groups.

Proposition 4.2. *For $n \geq 2$, the homotopy group $\pi_n(X)$ is abelian.*

Proof. Let $[f]$ and $[g]$ be elements of $\pi_n(X)$ represented by basepoint preserving maps $f : (I^n, \partial I^n) \to (X, x_0)$ and $g : (I^n, \partial I^n) \to (X, x_0)$, respectively. We need to find a homotopy between the product maps $f \cdot g$ and $g \cdot f$ defined above. The following schematic diagram suggests such a homotopy. We leave it to the reader to make this into a well defined homotopy.



□

Now assume $A \subset X$ is a subspace containing the basepoint $x_0 \in A$.

Definition 4.2. *For $n \geq 1$ we define the relative homotopy group $\pi_n(X, A) = \pi_n(X, A, x_0)$ to be homotopy classes of maps of pairs*

$$\pi_n(X, A) = [(D^n, \partial D^n, t_0); (X, A, x_0)].$$

where $t_0 \in \partial D^n = S^{n-1}$ and $x_0 \in A$ are the basepoints.

Exercise. Show that for $n > 1$ the relative homotopy group $\pi_n(X, A)$ is in fact a group. Notice here that the zero element is represented by any basepoint preserving map of pairs$f : (D^n, \partial I^n) \to (X, A)$ that is homotopic (through maps of pairs) to one whose image lies entirely in $A \subset X$.

Again, let $A \subset X$ be a subset containing the basepoint $x_0 \in A$, and let $i : A \hookrightarrow X$ be the inclusion. This induces a homomorphism of homotopy groups

$$i_* : \pi_n(A, x_0) \to \pi_n(X, x_0).$$

Also, by ignoring the subsets, a basepoint preserving map $f : (D^n, \partial D^n) \to (X, x_0)$ defines a map of pairs $f : (D^n, \partial D^n, t_0) \to (X, A, x_0)$ which defines a homomorphism

$$j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0).$$

Notice furthermore, that by construction, the composition

$$j_* \circ i_* : \pi_n(A) \to \pi_n(X) \to \pi_n(X, A)$$

is zero. Finally, if given a map of pairs $g : (D^n, S^{n-1}, t_0) \to (X, A, x_0)$, then we can restrict g to the boundary sphere S^{n-1} to produce a basepoint preserving map

$$\partial g : (S^{n-1}, t_0) \to (A, x_0).$$

This defines a homomorphism

$$\partial_* : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

Notice here that the composition

$$\partial_* \circ j_* : \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A)$$

is also zero, since the application of this composition to any representing map $f : (D^n, S^{n-1}) \to (X, x_0)$ yields the constant map $S^{n-1} \to x_0 \in A$. We now have the following fundamental property of homotopy groups. Compare with the analogous theorem in homology.

Theorem 4.3. *Let $A \subset X$ be a subspace containing the basepoint $x_0 \in A$. Then we have a long exact sequence in homotopy groups*

$$\cdots \xrightarrow{\partial_*} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial_*} \pi_{n-1}(A) \to \cdots$$
$$\xrightarrow{\partial_*} \pi_1(A) \xrightarrow{i_*} \pi_1(X) \to$$

Proof. We've already observed that $j_* \circ i_*$ and $\partial_* \circ j_*$ are zero. Similarly, $i_* \circ \partial_*$ is zero because an element in the image of ∂_* is represented by a basepoint preserving map $S^{n-1} \to A$ that extends to a map $D^n \to X$. Thus the image under i_*, namely the composition $S^{n-1} \to A \hookrightarrow X$ has an extension to D^n and is therefore null homotopic. We therefore have

$$\begin{aligned} image(\partial_*) &\subset kernel(i_*) \\ image(i_*) &\subset kernel(j_*) \\ image(j_*) &\subset kernel(\partial_*). \end{aligned}$$

To finish the proof we need to show that all of these inclusions are actually equalities. Consider the kernel of (i_*). An element $[f] \in \pi_n(A)$ is in $ker(i_*)$ if and only if the basepoint preserving composition $f : S^n \to A \subset X$ is null homotopic. Such a null - homotopy gives an extension of this map to the disk $F : D^{n+1} \to X$. The induced map of pairs $F : (D^{n+1}, S^n) \to (X, A)$ represents an element in $\pi_{n+1}(X, A)$ whose image under ∂_* is $[f]$. This proves that $image(\partial_*) = kernel(i_*)$. The other equalities are proved similarly, and we leave their verification to the reader. □

Remark. Even though this theorem is analogous to the existence of exact sequences for pairs in homology, notice that its proof is much easier.

Notice that $\pi_0(X)$ is the set of path components of X. So a space is (path) - connected if and only if $\pi_0(X) = 0$ (i.e the set with one element). We generalize this notion as follows.

Definition 4.3. *A space X is said to be m - connected if $\pi_q(X) = 0$ for $0 \leq q \leq m$.*

We now do our first calculation.

Proposition 4.4. *An n - sphere is $n - 1$ connected.*

Proof. We need to show that any map $S^k \to S^n$, where $k < n$ is null homotopic. Now since spheres can be given the structure of simplicial complexes, the simplicial approximation theorem says that any map $f : S^k \to S^n$ is homotopic to a simplicial map (after suitable subdivisions). So we assume without loss of generality that f is simplicial. But since $k < n$, the image of f lies in the k - skeleton of the n - dimensional simplicial complex S^n. In particular this means that $f : S^k \to S^n$ is not surjective. Let $y_0 \in S^n$ be a point that is not in the image of f. Then f has image in $S^n - y_0$ which is homeomorphic to the open disk D^n, and is therefore contractible. This implies that f is null homotopic. □

4.2 Fibrations

In chapter 2, in our discussion of the homotopy invariance of fiber bundles, we proved that locally trivial fiber bundles satisfy the Covering Homotopy Theorem 2.11. A generalization of the notion of a fiber bundle, due to Serre, is simply a map that satisfies this type of property.

Definition 4.4. *A Serre fibration is a surjective, continuous map $p : E \to B$ that satisfies the* Homotopy Lifting Property for CW - complexes. That is, if X is any CW - complex and $F : X \times I \to B$ is any continuous homotopy so that $F_0 : X \times \{0\} \to B$ factors through a map $f_0 : X \to E$, then there exists a lifting $\bar{F} : X \times I \to E$ that extends f_0 on $X \times \{0\}$, and makes the following diagram commute:

$$\begin{array}{ccc} X \times I & \xrightarrow{\bar{F}} & E \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B. \end{array}$$

A Hurewicz fibration is a surjective, continuous map $p : E \to B$ that satisfies the homotopy lifting property for all spaces.

Remarks. 1. Obviously every Hurewicz fibration is a Serre fibration. The converse is false. In these notes, unless otherwise stated, we will deal with Serre fibrations, which we will simply refer to as fibrations.
2. The Covering Homotopy Theorem implies that a fiber bundle is a fibration in this sense.

The following is an important example of a fibration.

Proposition 4.5. *Let X be any connected space with basepoint $x_0 \in X$. Let PX denote the space of based paths in X. That is,*

$$PX = \{\alpha : I \to X : \alpha(0) = x_0\}.$$

The path space PX is topologized using the compact - open function space topology. Define

$$p : PX \to X$$

by $p(\alpha) = \alpha(1)$. Then PX is a contractible space, and the map $p : PX \to X$ is a fibration, whose fiber at x_0, $p^{-1}(x_0)$ is the loop space ΩX.

Proof. The fact that PX is contractible is straightforward. For a null homotopy of the identity map one can take the map $H : PX \times I \to PX$, defined by $H(\alpha, s)(t) = \alpha((1-s)t)$.

To prove that $p : PX \to X$ is a fibration, we need to show it satisfies the Homotopy Lifting Property. So let $F : Y \times I \to X$ and $f_0 : X \to PX$ be maps making the following diagram commute:

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{f_0} & PX \\ \cap\big\downarrow & & \big\downarrow p \\ Y \times I & \xrightarrow[F]{} & X \end{array}$$

Then we can define a homotopy lifting, $\bar{F} : Y \times I \to PX$ by defining for $(y, s) \in Y \times I$, the path

$$\bar{F}(y, s) : I \to X$$

$$\bar{F}(y, s)(t) = \begin{cases} f_0(y)(\frac{2t}{2-s}) & \text{for } t \in [0, \frac{2-s}{2}] \\ F(y, 2t - 2 + s) & \text{for } t \in [\frac{2-s}{2}, 1] \end{cases}$$

One needs to check that this definition makes $\bar{F}(y, s)(t)$ a well defined continous map and satisfies the boundary conditions

$$\bar{F}(y, 0)(t) = f_0(y, t)$$
$$\bar{F}(y, s)(0) = x_0$$
$$\bar{F}(y, s)(1) = F(y, s)$$

These verifications are all straightforward. □

The following is just the observation that one can pull back the Homotopy Lifting Property.

Proposition 4.6. *Let $p : E \to B$ be a fibration, and $f : X \to B$ a continuous map. Then the pull back, $p_f : f^*(E) \to X$ is a fibration, where*

$$f^*(E) = \{(x, e) \in X \times E \text{ such that } f(x) = p(e)\}$$

and $p_f(x, e) = x$.

The following shows that in the setting of homotopy theory, every map can be viewed as a fibration in this sense.

Theorem 4.7. *Every continuous map $f : X \to Y$ is homotopic to a fibration in the sense that there exists a fibration*

$$\tilde{f} : \tilde{X} \to Y$$

and a homotopy equivalence

$$h : X \xrightarrow{\simeq} \tilde{X}$$

making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow[\simeq]{h} & \tilde{X} \\ f\downarrow & & \downarrow\tilde{f} \\ Y & = & Y. \end{array}$$

Proof. Define $\tilde{X}$ to be the space

$$\tilde{X} = \{(x, \alpha) \in X \times Y^I \text{ such that } \alpha(0) = x.\}$$

where here Y^I denotes the space of continuous maps $\alpha : [0, 1] \to Y$ given the compact open topology. The map $\tilde{f} : \tilde{X} \to Y$ is defined by $\tilde{f}(x, \alpha) = \alpha(1)$. The fact that $\tilde{f} : \tilde{X} \to Y$ is a fibration is proved in the same manner as theorem 4.5, and so we leave it to the reader.

Define the map $h : X \to \tilde{X}$ by $h(x) = (x, \epsilon_x) \in \tilde{X}$, where $\epsilon_x(t) = x$ is the constant path at $x \in X$. Clearly $\tilde{f} \circ h = f$ so the diagram in the statement of the theorem commutes. Now define $g : \tilde{X} \to X$ by $g(x, \alpha) = x$. Clearly $g \circ h$ is the identity map on X. To see that $h \circ g$ is homotopic to the identity on $\tilde{X}$, consider the homotopy $F : \tilde{X} \times I \to \tilde{X}$, defined by $F((x, \alpha), s) = (x, \alpha_s)$, where $\alpha_s : I \to X$ is the path $\alpha_s(t) = \alpha(st)$. So in particular $\alpha_0 = \epsilon_x$ and $\alpha_1 = \alpha$. Thus F is a homotopy between $h \circ g$ and the identity map on $\tilde{X}$. Thus h is a homotopy equivalence, which completes the proof of the theorem. □

The homotopy fiber of a map $f : X \to Y$, F_f, is defined to be the fiber of the fibration $\tilde{f} : \tilde{X} \to Y$ defined in the proof of this theorem. That is,

Definition 4.5. *The homotopy fiber F_f of a basepoint preserving map $f : X \to Y$ is defined to be*

$$F_f = \{(x, \alpha) \in X \times Y^I \text{ such that } \alpha(0) = f(x) \text{ and } \alpha(1) = y_0.\}$$

where $y_0 \in Y$ is the basepoint.

So for example, the homotopy fiber of the inclusion of the basepoint $y_0 \hookrightarrow Y$ is the loop space ΩY. The homotopy fiber of the identity map $id : Y \to Y$ is the path space PY. The homotopy fibers are important invariants of the map $f : X \to Y$.

The following is the basic homotopy theoretic property of fibrations.

Theorem 4.8. *Let $p : E \to B$ be a fibration over a connected space B with fiber F. So we are assuming the basepoint of E, is contained in F, $e_0 \in F$, and that $p(e_0) = b_0$ is the basepoint in B. Let $i : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence of homotopy groups:*

$$\cdots \xrightarrow{\partial_*} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial_*} \pi_{n-1}(F) \to$$
$$\cdots \to \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B).$$

Proof. Notice that the projection map $p : E \to B$ induces a map of pairs

$$p : (E, F) \to (B, b_0).$$

By the exact sequence for the homotopy groups of the pair (E, F), 4.3 it is sufficient to prove that the induced map in homotopy groups

$$p_* : \pi_n(E, F) \to \pi_n(B, b_0)$$

is an isomorphism for all $n \geq 1$. We first show that p_* is surjective. So let $f : (I^n, \partial I^n) \to (B, b_0)$ represent an element of $\pi_n(B)$. We can think of a map from a cube as a homotopy of maps of cubes of one lower dimension. Therefore by induction on n, the homotopy lifting property says that that $f : I^n \to B$ has a basepoint preserving lifting $\bar{f} : I^n \to E$. Since $p \circ \bar{f} = f$, and since the restriction of f to the boundary ∂I^n is constant at b_0, then the image of the restriction of $\bar{f}$ to the boundary ∂I^n has image in the fiber F. That is, $\bar{f}$ induces a map of pairs

$$\bar{f} : (I^n, \partial I^n) \to (E, F)$$

which in turn represents an element $[\bar{f}] \in \pi_n(E, F)$ whose image under p_* is $[f] \in \pi_n(B, b_0)$. This proves that p_* is surjective.

We now prove that $p_* : \pi_n(E, F) \to \pi_n(B, b_0)$ is injective. So let $f : (D^n, \partial D^n) \to (E, F)$ be a map of pairs that represents an element in the kernel of p_*. That means $p \circ f : (D^n, \partial D^n) \to (B, b_0)$ is null homotopic. Let

$F : (D^n, \partial D^n) \times I \to (B, b_0)$ be a null homotopy between $F_0 = f$ and the constant map $\epsilon : D^n \to b_0$. By the Homotopy Lifting Property there exists a basepoint preserving lifting

$$\bar{F} : D^n \times I \to E$$

having the properties that $p \circ \bar{F} = F$ and $\bar{F} : D^n \times \{0\} \to E$ is equal to $f : (D^n, \partial D^n) \to (E, F)$. Since $p \circ \bar{F} = F$ maps $\partial D^n \times I$ to the basepoint b_0, we must have that $\bar{F}$ maps $\partial D^n \times I$ to $p^{-1}(b_0) = F$. Thus $\bar{F}$ determines a homotopy of pairs,

$$\bar{F} : (D^n, \partial D^n) \times I \to (E, F)$$

with $\bar{F}_0 = f$. Now consider $\bar{F}_1 : (D^n, \partial D^n) \times \{1\} \to E$. Now $p \circ \bar{F}_1 = F_1 = \epsilon : D^n \to b_0$. Thus the image of $\bar{F}_1$ lies in $p^{-1}(b_0) = F$. Thus $\bar{F}$ gives a homotopy of the map of pairs $f : (D^n, \partial D^n) \to (E, F)$ to a map of pairs whose image lies entirely in F. Such a map represents the zero element of $\pi_n(E, F)$. This completes the proof that p_* is injective, and hence is an isomorphism. As observed earlier, this is what was needed to prove the theorem. □

We now use this theorem to make several important calculations of homotopy groups. In particular, we prove the following seminal result of Hopf.

Theorem 4.9.

$\pi_2(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$.
$\pi_k(S^3) \cong \pi_k(S^2)$ *for all* $k \geq 3$. *In particular,*
$\pi_3(S^2) \cong \mathbb{Z}$, *generated by the Hopf map* $\eta : S^3 \to S^2$.

Proof. Consider the Hopf fibration $\eta : S^3 \to S^2 = \mathbb{CP}^1$ with fiber S^1. Recall that S^1 is an Eilenberg - MacLane space $K(\mathbb{Z}, 1)$. In other words,

$$\pi_q(S^1) = \begin{cases} \mathbb{Z} & \text{for q = 1} \\ 0 & \text{for all other } q. \end{cases}$$

Using this fact in the exact sequence in homotopy groups for the Hopf fibration $\eta : S^3 \to S^2$, together with the fact that $\pi_q(S^3) = 0 \quad \text{for } q \leq 2$, one is led to the facts that $\pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}$, and that $\eta_* : \pi_k(S^3) \to \pi_k(S^2)$ is an isomorphism for $k \geq 3$. To examine the case $k = 3$, consider the homomorphism (called the *Hurewicz homomorphism*)

$$h : \pi_3(S^3) \to H_3(S^3) = \mathbb{Z}$$

defined by sending a class represented by a self map $f : S^3 \to S^3$, to the image of the fundamental class in homology, $f_*([S^3]) \in H^3(S^3) \cong \mathbb{Z}$. Clearly this is a homomorphism (check this!). Moreover it is surjective since the image of the identity map is the fundamental class, and thus generates, $H_3(S^3)$, $H([id]) = [S^3] \in H_3(S^3)$. Thus $\pi_3(S^3)$ contains an integral summand generated by the identity. In particular, since $\eta_* : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism, this implies that $\pi_3(S^2)$ contains an integral summand generated by the Hopf map $[\eta] \in \pi_3(S^2)$. The fact that these integral summands generate the entire groups $\pi_3(S^3) \cong \pi_3(S^2)$ will follow once we know that the Hurewicz homomorphism is an isomorphism in this case. Later in this chapter we will prove the more general "Hurewicz theorem" that says that for any $k > 1$, and any $(k-1)$ - connected space X, the Hurewicz homomorphism is an isomorphism in dimension k: $h : \pi_k(X) \cong H_k(X)$. □

Remark. As we remarked earlier in these notes. these were the first nontrivial elements found in the higher homotopy groups of spheres, $\pi_{n+k}(S^n)$, and Hopf's proof of their nontriviality is commonly viewed as the beginning of modern Homotopy Theory [160]

Before we continue to apply the notion of fibrations to homotopy theory, we point out that there is a dual notion of a *cofibration* that is also very important. Instead of satisfying a homotopy lifting property, cofibratons satisfy a *homotopy extension property,*

Definition 4.6. *A map $\iota : A \to X$ of topological spaces is called a cofibration if for any map $f : X \to Y$ and any homotopy*

$$H : A \times [0,1] \to Y$$

with $H(a,0) = f(\iota(a))$, then there is an extension of the homotopy H to $X \times I$,

$$\bar{H} : X \times [0,1] \to Y.$$

so that $\bar{H}(x,0) = f(x)$ for all $x \in X$, and $\bar{H}(\iota(a),t) = H(a,t)$ for all $a \in A$ and $t \in [0,1]$.

Exercise. Show that if X is a CW-complex and $A \subset X$ is a subcomplex then the inclusion map $\iota : A \hookrightarrow X$ is a cofibration.

Notice we have the following analogue of Theorem 4.7:

Theorem 4.10. *Every map $g : A \to X$ is homotopic to a cofibration in the sense that there is a space $\bar{X}$ equipped with a deformation retraction $j : X \xrightarrow{\simeq} \bar{X}$ and a cofibration*

$$\bar{g} : A \to \bar{X}$$

that is homotopic to $j \circ g : A \to X \xrightarrow{\simeq} \bar{X}$.

Proof. Define $\bar{X}$ to be the mapping cylinder

$$\bar{X} = X \cup (A \times I)/\sim$$

where $(a,0) \in A \times I$ is identified with $g(a) \in X$. Define $\bar{g} : A \to \bar{X}$ to be the inclusion as $A \times \{0\}$. We leave it to the reader to verify that the pair $(\bar{X}, \bar{g} : A \to \bar{X})$ satisfies the required properties. □

Definition 4.7. *Let $\iota : A \to X$ be a cofibration. The cofiber of ι is the quotient space X/A defined to be*

$$X/A = X/\sim$$

where the equivalence relation is given by $\iota(a) \sim \iota(b)$ for any two points a, $b \in A$. Notice that in the case where ι is the inclusion of a subcomplex $\iota : A \subset X$ of a CW complex, the cofiber is the quotient complex, X/A.

Exercises.

1. Show that if $\iota : A \to X$ is a cofibration, it's cofiber X/A is homotopy equivalent to the mapping cone

$$X \cup_\iota c(A)$$

where $c(A) = A \times [0,1]/A \times \{1\}$, and the notation $X \cup_\iota c(A)$ refers to the disjoint union of X with $c(A)$, modulo the identification $(a,0) \in c(A)$ is identified with $\iota(a) \in X$ for all $a \in A$.

2. Show that if $\iota : A \to X$ is a cofibration, then there is an isomorphism of homology groups,

$$H_*(X, A) \cong \tilde{H}_*(X/A).$$

Remark. . Since any map $f : X \to Y$ is homotopic to a cofibration with cofiber the mapping cone $Y \cup_f c(X)$, the mapping cone is sometimes referred to as the *"homotopy cofiber"* of f. Notice furthermore that the inclusion of Y into the mapping cone,

$$Y \subset Y \cup_f c(X)$$

is a cofibration with cofiber the suspension $\Sigma X = c(X)/X \times \{0\}$.

We end this section with an application to the "homotopy stability" of the orthogonal and unitary groups, as well as their classifying spaces.

Theorem 4.11. *The inclusion maps*

$$\begin{aligned} \iota : O(n) &\hookrightarrow O(n+1) \quad \textit{and} \\ U(n) &\hookrightarrow U(n+1) \end{aligned}$$

induce isomorphisms in homotopy groups through dimensions $n-2$ and $2n-1$ respectively.

Proof. These statements follow from the existence of fiber bundles

$$O(n) \hookrightarrow O(n+1) \to S^n$$

and

$$U(n) \hookrightarrow U(n+1) \to S^{2n+1},$$

the connectivity of spheres 4.4, and by applying the exact sequence in homotopy groups to these fiber bundles. □

4.3 Obstruction Theory

In this section we discuss the obstructions to obtaining a lifting to the total space of a fibration of a map to the base space. As an application we prove the important "Whitehead theorem" in homotopy theory, and we prove general results about the existence of cross sections of principal $O(n)$ or $U(n)$ - bundles. We do not develop a formal theory here - we just develop what we will need for our applications to fibrations. For a full development of obstruction theory we refer the reader to [159].

Let X be a CW - complex. Recall that its cellular k - chains, $C_k(X)$ is the free abelian group generated by the k - dimensional cells in X. The co-chains with coefficients in a group G are defined by

$$C^k(X,G) = Hom(C_k(X),G).$$

Theorem 4.12. *Let $p : E \to B$ be a fibration with fiber F. Let $f : X \to B$ be a continuous map, where X is a CW - complex. Suppose there is a lifting of the $(k-1)$ - skeleton $\tilde{f}_{k-1} : X^{(k-1)} \to E$. That is, the following diagram commutes:*

$$\begin{array}{ccc} X^{(k-1)} & \xrightarrow{\tilde{f}_{k-1}} & E \\ \cap\downarrow & & \downarrow p \\ X & \xrightarrow[f]{} & B. \end{array}$$

Then the obstruction to the existence of a lifting to the k -skeleton, $\tilde{f}_k : X^{(k)} \to E$ that extends $\tilde{f}_{k-1}$, is a cochain $\gamma \in C^k(X;\pi_{k-1}(F))$. That is, $\gamma = 0$ if and only if such a lifting $\tilde{f}_k$ exists.

Proof. We will first consider the special case where $X^{(k)}$ is obtained from $X^{(k-1)}$ by adjoining a single k -dimensional cell. So assume

$$X^{(k)} = X^{(k-1)} \cup_\alpha D^k$$

where $\alpha : \partial D^k = S^{(k-1)} \to X^{k-1)}$ is the attaching map. We therefore have the following commutative diagram:

$$\begin{array}{ccccc}
S^{k-1} & \xrightarrow{\alpha} & X^{(k-1)} & \xrightarrow{\bar{f}_{k-1}} & E \\
\cap\downarrow & & \cap\downarrow & & \downarrow p \\
D^k & \xrightarrow[\subset]{} & X^{(k-1)} \cup_\alpha D^k & \xrightarrow[f]{} & B
\end{array}$$

Notice that $\bar{f}_{k-1}$ has an extension to $X^{(k-1)} \cup_\alpha D^k = X^{(k)}$ that lifts f, if and only if the composition $D^k \subset X^{(k-1)} \cup_\alpha D^k \xrightarrow{f} B$ lifts to E in such a way that it extends $\bar{f}_{k-1} \circ \alpha$.

Now view the composition $D^k \subset X^{(k-1)} \cup_\alpha D^k \xrightarrow{f} B$ as a map from the cone on S^{k-1} to B, or in other words, as a null homotopy$F : S^{k-1} \times I \to B$ from $F_0 = p \circ \bar{f}_{k-1} \circ \alpha : S^{k-1} \to X^{(k-1)} \to E \to B$ to the constant map $F_1 = \epsilon : S^{(k-1)} \to b_0 \in B$. By the Homotopy Lifting Property, F lifts to a homotopy

$$\bar{F} : S^{(k-1)} \times I \to E$$

with $\bar{F}_0 = \bar{f}_{k-1} \circ \alpha$. Thus the extension f_k exists on $X^{(k-1)} \cup_\alpha D^k$ if and only if this lifting $\bar{F}$ can be chosen to be a null homotopy of $\bar{f}_{k-1} \circ \alpha$. But we know $\bar{F}_1 : S^{k-1} \times \{1\} \to E$ lifts F_1 which is the constant map $\epsilon : S^{k-1} \to b_0 \in B$. Thus the image of $\bar{F}_1$ lies in the fiber F, and therefore determines an element $\gamma \in \pi_{k-1}(F)$. The homotopy $\bar{F}_1$ can be chosen to be a null homotopy if and only if $\bar{F}_1 : S^{k-1} \to F$ is null homotopic. (Because combining $\bar{F}$ with a null homotopy of $\bar{F}_1$, i.e an extension of $\bar{F}_1$ to a map $D^k \to F$, is still a lifting of F, since the extension lives in a fiber over a point.) But this is only true if the homotopy class $\gamma = 0 \in \pi_{k-1}(F)$.

This proves the theorem in the case when $X^{(k)} = X^{(k-1)} \cup_\alpha D^k$. In the general case, suppose that $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching a collection of k - dimensional disks, indexed on a set, say J. That is,

$$X^{(k)} = X^{(k-1)} \bigcup_{j \in J} \cup_{\alpha_j} D^k.$$

The above procedure assigns to every $j \in J$ an "obstruction" $\gamma_j \in \pi_{k-1}(F)$. An extension $\bar{f}_k$ exists if and only if all these obstructions are zero. This assignment from the indexing set of the k - cells to the homotopy group can be extended linearly to give a homomorphism γ from the free abelian group generated by the k - cells to the homotopy group $\pi_{k-1}(F)$, which is zero if and only if the extension $\bar{f}_k$ exists. Such a homomorphism γ is a cochain, $\gamma \in C^k(X; \pi_{k-1}(F))$. This completes the proof of the theorem. □

We now discuss several applications of this obstruction theory.

Corollary 4.13. *Any fibration $p : E \to B$ over a CW - complex with a contractible fiber F admits a cross section.*

Proof. Since $\pi_q(F) = 0$ for all q, by the theorem, there are no obstructions to constructing a cross section inductively on the skeleta of B. □

We now use this obstruction theory to prove the well known "Whitehead Theorem", one of the most important foundational theorems in homotopy theory.

Theorem 4.14. *Suppose X and Y are CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homotopy groups,*

$$f_* : \pi_k(X) \xrightarrow{\cong} \pi_k(Y) \quad \textit{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

Note. A map between any two topological spaces $f : X \to Y$ that induces an isomorphism in homotopy groups is called a *"weak homotopy equivalence"*. This theorem says that if X and Y have the homotopy type of CW complexes, a weak homotopy equivalence is a homotopy equivalence.

Proof. By 4.7 we can replace $f : X \to Y$ by a homotopy equivalent fibration

$$\tilde{f} : \tilde{X} \to Y.$$

That is, there is a homotopy equivalence $h : X \to \tilde{X}$ so that $\tilde{f} \circ h = f$. Since f induces an isomorphism in homotopy groups, so does $\tilde{f}$. By the exact sequence in homotopy groups for this fibration, this means that the fiber of the fibration $\tilde{f} : \tilde{X} \to Y$, i.e the homotopy fiber of f, is aspherical. thus by 4.12 there are no obstructions to finding a lifting $\tilde{g} : Y \to \tilde{X}$ of the identity map of Y. Thus $\tilde{g}$ is a section of the fibration, so that $\tilde{f} \circ \tilde{g} = id : Y \to Y$. Now let $h^{-1} : \tilde{X} \to X$ denote a homotopy inverse to the homotopy equivalence h. Then if we define

$$g = h^{-1} \circ \tilde{g} : Y \to X$$

we then have $f \circ g : Y \to Y$ is given by

$$\begin{aligned} f \circ g &= f \circ h^{-1} \circ \tilde{g} \\ &= \tilde{f} \circ h \circ h^{-1} \circ \tilde{g} \\ &\sim \tilde{f} \circ \tilde{g} \\ &= id : Y \to Y. \end{aligned}$$

Thus $f \circ g$ is homotopic to the identity of Y. To show that $g \circ f$ is homotopic to the identity of X, we need to construct a homotopy $X \times I \to X$ that lifts a homotopy $X \times I \to Y$ from $f \circ g \circ f$ to f. This homotopy is constructed inductively on the skeleta of X, and like in the argument proving 4.12, one finds that there are no obstructions in doing so because the homotopy fiber of f is aspherical. We leave the details of this obstruction theory argument to the reader. Thus f and g are homotopy inverse to each other, which proves the theorem. □

The following is an immediate corollary.

A "weakly contractible" space is one in which all of its homotopy groups are zero.

Corollary 4.15. *A weakly contractible CW - complex is contractible.*

Proof. If X is a weakly contractible CW - complex, then the constant map to a point, $\epsilon : X \to pt$ induces an isomorphism on homotopy groups, and is therefore, by the above theorem, is a homotopy equivalence. □

4.4 Eilenberg - MacLane Spaces

In this section we prove a classification theorem for cohomology. We show that there are classifying spaces $K(G, n)$ that classify n - dimensional cohomology with coefficients in G in an appropriate sense. These are Eilenberg - MacLane spaces. In this section we prove their existence and describe their properties.

4.4.1 Obstruction theory and the existence of Eilenberg - MacLane spaces

The main goal in this section is to prove the following.

Theorem 4.16. *Let G be any abelian group and n an integer with $n \geq 2$. Then there exists a space $K(G, n)$ with*

$$\pi_k(K(G,n)) = \begin{cases} G, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

This theorem will basically be proven using obstruction theory. For this we will assume the following famous theorem of Hurewicz, which we will prove later in this chapter. We first recall the Hurewicz homomorphism from homotopy to homology.

Let $f : (D^n, S^{n-1}) \to (X, A)$ represent an element $[f] \in \pi_n(X, A)$. Let $\sigma_n \in H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ be a preferred, fixed generator. Define $h([f]) = f_*(\sigma_n) \in H_n(X, A)$. The following is straightforward, and we leave its verification to the reader.

Lemma 4.17. *The above construction gives a well defined homomorphism*

$$h_* : \pi_n(X, A) \to H_n(X, A)$$

called the "Hurewicz homomorphism".

The following is the "Hurewicz theorem".

Theorem 4.18. *Let X be simply connected, and let $A \subset X$ be a simply connected subspace. Suppose that the pair (X, A) is $(n-1)$ - connected, for $n > 2$. That is,*

$$\pi_k(X, A) = 0 \quad \textit{if} \quad k \leq n-1.$$

Then the Hurewicz homomorphism $h_* : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

We now prove the following basic building block type result concerning how the homotopy groups change as we build a CW - complex cell by cell.

Theorem 4.19. *Let X be a simply connected, CW - complex and let*

$$f : S^k \to X$$

be a map. Let X' be the mapping cone of f. That is,

$$X' = X \cup_f D^{k+1}$$

which denotes the union of X with a disk D^{k+1} glued along the boundary sphere $S^k = \partial D^{k+1}$ via f. That is we identify $t \in S^k$ with $f(t) \in X$. Let

$$\iota : X \hookrightarrow X'$$

be the inclusion. Then

$$\iota_* : \pi_k(X) \to \pi_k(X')$$

is surjective, with kernel equal to the cyclic subgroup generated by $[f] \in \pi_k(X)$.

Proof. Let $g : S^q \to X'$ represent an element in $\pi_q(X')$ with $q \leq k$. By the cellular approximation theorem, g is homotopic to a cellular map, and therefore one whose image lies in the q - skeleton of X'. But for $q \leq k$, the q - skeleton of X' is the q - skeleton of X. This implies that

$$\iota_* : \pi_q(X) \to \pi_q(X')$$

is surjective for $q \leq k$. Now assume $q \leq k-1$, then if $g : S^q \to X \subset X'$ is null homotopic, any null homotopy, i.e extension to the disk $G : D^{q+1} \to X'$ can be assumed to be cellular, and hence has image in X. This implies that for $q \leq k-1$, $\iota_* : \pi_q(X) \to \pi_q(X')$ is an isomorphism. By the exact sequence in homotopy groups of the pair (X', X), this implies that the pair (X', X) is k - connected. By the Hurewicz theorem that says that

$$\pi_{k+1}(X', X) \cong H_{k+1}(X', X) = H_{k+1}(X \cup_f D^{k+1}, X)$$

which, by analyzing the cellular chain complex for computing $H_*(X')$ is $\mathbb{Z}$ if and only if $f : S^k \to X$ is zero in homology, and zero otherwise. In particular, the generator $\gamma \in \pi_{k+1}(X', X)$ is represented by the map of pairs given by the inclusion

$$\gamma : (D^{k+1}, S^k) \hookrightarrow (X \cup_f D^{k+1}, X)$$

and hence in the long exact sequence in homotopy groups of the pair (X', X),

$$\cdots \to \pi_{k+1}(X', X) \xrightarrow{\partial_*} \pi_k(X) \xrightarrow{\iota_*} \pi_k(X') \to \cdots$$

we have $\partial_*(\gamma) = [f] \in \pi_k(X)$. Thus $\iota_* : \pi_k(X) \to \pi_k(X')$ is surjective with kernel generated by $[f]$. This proves the theorem. □

We will now use this basic homotopy theory result to establish the existence of Eilenberg - MacLane spaces.

Proof. of Theorem 4.16.

Fix the group G and the integer $n \geq 2$. Let $\{\gamma_\alpha : \alpha \in \mathcal{A}\}$ be a set of generators of G, where $\mathcal{A}$ denotes the indexing set for these generators. Let $\{\theta_\beta : \beta \in \mathcal{B}\}$ be a corresponding set of relations. In other words G is isomorphic to the free abelian group $F_{\mathcal{A}}$ generated by $\mathcal{A}$, modulo the subgroup $R_{\mathcal{B}}$ generated by $\{\theta_\beta : \beta \in \mathcal{B}\}$.

Consider the wedge of spheres $\bigvee_{\mathcal{A}} S^n$ indexed on the set $\mathcal{A}$. Then by the Hurewicz theorem,

$$\pi_n(\bigvee_{\mathcal{A}} S^n) \cong H_n(\bigvee_{\mathcal{A}} S^n) \cong F_{\mathcal{A}}.$$

Now the group $R_{\mathcal{B}}$ is a subgroup of a free abelian group, and hence is itself free abelian. Let $\bigvee_{\mathcal{B}} S^n$ be a wedge of spheres whose n^{th} - homotopy group (which by the Hurewicz theorem is isomorphic to its homology, which is free abelian) is $R_{\mathcal{B}}$. Moreover there is a natural map

$$j : \bigvee_{\mathcal{B}} S^n \to \bigvee_{\mathcal{A}} S^n$$

which, on the level of the homotopy group π_n is the inclusion $R_{\mathcal{B}} \subset F_{\mathcal{A}}$. Let X_{n+1} be the mapping cone of j:

$$X_{n+1} = \bigvee_{\mathcal{A}} S^n \cup_j \bigcup_{\mathcal{B}} D^{n+1}$$

where the disk D^{n+1} corresponding to a generator in $R_{\mathcal{B}}$ is attached via the map $S^n \to \bigvee_{\mathcal{A}} S^n$ giving the corresponding element in $\pi_n(\bigvee_{\mathcal{A}} S^n) = F_{\mathcal{A}}$. Then by using 4.19 one cell at a time, we see that X_{n+1} is an $n-1$ - connected space and $\pi_n(X_n)$ is generated by $F_{\mathcal{A}}$ modulo the subgroup $R_{\mathcal{B}}$. In other words,

$$\pi_n(X_{n+1}) \cong G.$$

Now inductively assume we have constructed an space X_{n+k} with

$$\pi_q(X_{n+k}) = \begin{cases} 0 & \text{if } q < n, \\ G & \text{if } q = n \quad \text{and} \\ 0 & \text{if } n < q \leq n+k-1 \end{cases}$$

Notice that we have begun the inductive argument with $k = 1$, by the construction of the space X_{n+1} above. So again, assume we have constructed X_{n+k}, and we need to show how to construct X_{n+k+1} with these properties. Once we have done this, by induction we let $k \to \infty$, and clearly X_∞ will be a model for $K(G, n)$.

Now suppose $\pi = \pi_{n+k}(X_{n+k})$ is has a generating set $\{\gamma_u : u \in \mathcal{C}\}$, where $\mathcal{C}$ is the indexing set. Let $F_{\mathcal{C}}$ be the free abelian group generated by the elements in this generating set. Let $\bigvee_{u \in \mathcal{C}} S_u^{n+k}$ denote a wedge of spheres indexed by this indexing set. Then, like above, by applying the Hurewicz theorem we see that

$$\pi_{n+k}(\bigvee_{u \in \mathcal{C}} S_u^{n+k}) \cong H_{n+k}(\bigvee_{\mathcal{C}} S^{n+k}) \cong F_{\mathcal{C}}.$$

Let

$$f : \bigvee_{\mathcal{C}} S^{n+k} \to X_{n+k}$$

be a map which, when restricted to the sphere S_u^{n+k} represents the generator $\gamma_u \in \pi = \pi_{n+k}(X_{n+k})$. We define X_{n+k+1} to be the mapping cone of f:

$$X_{n+k+1} = X_{n+k} \cup_f \bigcup_{u \in \mathcal{C}} D^{n+k+1}.$$

Then by 4.19 we have that $\pi_q(X_{n+k}) \to \pi_q(X_{n+k+1})$ is an isomorphism for $q < n+k$, and

$$\pi_{n+k}(X_{n+k}) \to \pi_{n+k}(X_{n+k+1})$$

is surjective, with kernel the subgroup generated by $\{\gamma_u : u \in \mathcal{C}\}$. But since this subgroup generates $\pi = \pi_{n+k}(X_{n+k})$ we see that this homomorphism is zero. Since it is surjective, that implies $\pi_{n+k}(X_{n+k+1}) = 0$. Hence X_{n+k+1} has the required properties on its homotopy groups, and so we have completed our inductive argument. □

4.4.2 The Hopf - Whitney theorem and the classification theorem for Eilenberg - MacLane spaces

We now know that the Eilenberg - MacLane spaces $K(G,n)$ exist for every n and every abelian group G, and when $n = 1$ for every group G. Furthermore, by their construction in the proof of Theorem refkgnexist they can be chosen to be CW - complexes. In this section we prove their main property, that is they classify cohomology.

In order to state the classification theorem properly, we need to recall the universal coefficient theorem, which says the following.

Theorem 4.20. *(Universal Coefficient Theorem) Let G be an abelian group. Then there is a split short exact sequence*

$$0 \to Ext(H_{n-1}(X); G) \to H^n(X; G) \to Hom(H^n(X), G) \to 0.$$

Corollary 4.21. *If Y is $(n-1)$ - connected for $n > 1$, and $\pi = \pi_n(Y)$, then*

$$H^n(Y; \pi) \cong Hom(\pi, \pi).$$

Proof. Since Y is $(n-1)$ connected, $H_{n-1}(Y) = 0$, so the universal coefficient theorem says that $H^n(Y;\pi) \cong Hom(H_n(Y), \pi)$. But the Hurewicz theorem says that the Hurewicz homomorphism $h_* : \pi = \pi_n(Y) \to H_n(Y)$ is an isomorphism. The corollary follows by combining these two isomorphisms. □

For an $(n-1)$ - connected space Y as above, let $\iota \in H^n(Y;\pi)$ be the class corresponding to the identity map $id \in Hom(\pi, \pi)$ under the isomorphism in this corollary. This is called the fundamental class. Given any other space X, we therefore have a set map

$$\phi : [X, Y] \to H^n(X, \pi)$$

defined by $\phi([f]) = f^*(\iota) \in H^n(X; \pi)$. The classification theorem for Eilenberg - MacLane spaces is the following.

Theorem 4.22. *For $n \geq 2$ and π any abelian group, let $K(\pi, n)$ denote an Eilenberg - MacLane space with $\pi_n(K(\pi, n)) = \pi$, and all other homotopy groups zero. Let $\iota \in H^n(K(\pi, n); \pi)$ be the fundamental class. Then for any CW - complex X, the map*

$$\phi : [X, K(\pi, n)] \to H^n(X; \pi)$$
$$[f] \to f^*(\iota)$$

is a bijective correspondence.

We have the following immediate corollary, giving a uniqueness theorem regarding Eilenberg - MacLane spaces.

Corollary 4.23. *Let $K(\pi, n)_1$ and $K(\pi, n)_2$ be CW - complexes that are both Eilenberg - MacLane spaces with the same homotopy groups. Then there is a natural homotopy equivalence between $K(\pi, n)_1$ and $K(\pi, n)_2$.*

Proof. Let $f : K(\pi, n)_1 \to K(\pi, n)_2$ be a map whose homotopy class is the inverse image of the fundamental class under the bijection

$$\phi : [K(\pi, n)_1, K(\pi, n)_2] \xrightarrow{\cong} H^n(K(\pi, n)_1; \pi) \cong Hom(\pi, \pi).$$

This means that $f : K(\pi, n)_1 \to K(\pi, n)_2$ induces the identity map in $Hom(\pi, \pi)$, and in particular induces an isomorphism on π_n. Since all other homotopy groups are zero in both of these complexes, f induces an isomorphism in homotopy groups in all dimensions. Therefore by the Whitehead theorem 4.14, f is a homotopy equivalence. □

We begin our proof of this classification theorem by proving a special case, known as the Hopf - Whitney theorem. This predates knowledge of the existence of Eilenberg - MacLane spaces.

Theorem 4.24. *(Hopf-Whitney theorem) Let Y be any $(n-1)$ - connected space with $\pi = \pi_n(Y)$. Let X be any n - dimensional CW complex. Then the map*

$$\phi : [X, Y] \to H^n(X; \pi)$$
$$[f] \to f^*(\iota)$$

is a bijective correspondence.

Remark. This theorem is most often used in the context of manifolds, where it implies that if M^n is any closed, orientable manifold the correspondence

$$[M^n, S^n] \to H^n(M^n; \mathbb{Z}) \cong \mathbb{Z}$$

is a bijection.

Exercise. Show that this correspondence can alternatively be described as assigning to a smooth map $f : M^n \to S^n$ its degree, $deg(f) \in \mathbb{Z}$.

Proof. (Hopf - Whitney theorem) We first set some notation. Let Y be $(n-1)$ - connected, and have basepoint $y_0 \in Y$. Let $X^{(m)}$ denote the m - skeleton of the n - dimensional complex X. Let $C_k(X) = H_k(X^{(k)}, X^{(k-1)})$ be the cellular k - chains in X. Alternatively, $C_k(X)$ can be thought of as the free abelian group on the k - dimensional cells in the CW - decomposition of X. Let $Z^k(X)$ and $B^k(X)$ denote the subgroups of cocycles and coboundaries respectively. Let J_k be the indexing set for the set of k - cells in this CW - structure. So that there are attaching maps

$$\alpha_k : \bigvee_{j \in J_k} S^k_j \to X^{(k)}$$

so that the $(k+1)$ - skeleton $X^{(k+1)}$ is the mapping cone

$$X^{(k+1)} = X^{(k)} \cup_{\alpha_k} \bigcup_{j \in J_k} D^{k+1}_j.$$

We prove this theorem in several steps, each translating between cellular cochain complexes or cohomology on the one hand, and homotopy classes of maps on the other hand. The following is the first step.

Step 1. There is a bijective correspondence between the following set of homotopy classes of maps of pairs, and the cochain complex with values in π:

$$\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X; \pi).$$

Proof. A map of pairs $f : (X^{(n)}, X^{(n-1)}) \to (Y, y_0)$ is the same thing as a basepoint preserving map from the quotient,

$$f : X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j \to Y.$$

So the homotopy class of f defines and is defined by an assignment to every $j \in J_n$, an element $[f_j] \in \pi_n(Y) = \pi$. But by extending linearly, this is the same as a homomorphism from the free abelian group generated by J_n, i.e the chain group $C_n(X)$, to π. That is, this is the same thing as a cochain $[f] \in C^n(X; \pi)$. □

Step 2. The map $\phi : [X, Y] \to H^n(X; \pi)$ is surjective.

Proof. Notice that since X is an n - dimensional CW - complex, all n - dimensional cochains are cocycles, $C^n(X;\pi) = Z^n(X;\pi)$. So in particular there is a surjective homomorphism $\mu : C^n(X;\pi) = Z^n(X;\pi) \to Z^n(X;\pi)/B^n(X;\pi) = H^n(X;\pi)$. A check of the definitions of the maps defined so far yields that the following diagram commutes:

$$\begin{array}{ccc} [(X^{(n)}, X^{(n-1)}), (Y, y_0)] & \xrightarrow[\cong]{\phi} & C^n(X;\pi) \\ \rho \downarrow & & \downarrow \mu \\ [X, Y] & \xrightarrow[\phi]{} & H^n(X;\pi) \end{array}$$

where ρ is the obvious restriction map. By the commutativity of this diagram, since μ is surjective and $\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X;\pi)$ is bijective, then we must have that $\phi : [X, Y] \to H^n(X;\pi)$ is surjective, as claimed. $\square$

In order to show that ϕ is injective, we will need to examine the coboundary map

$$\delta : C^{n-1}(X;\pi) \to C^n(X;\pi)$$

from a homotopy point of view. To do this, recall that the boundary map on the chain level, $\partial : C_k(X) \to C_{k-1}(X)$ is given by the connecting homomorphism $H_n(X^{(k)}, X^{(k-1)}) \to H_{k-1}(X^{(k-1)}, X^{(k-2)})$ from the long exact sequence in homology of the triple, $(X^{(k)}, X^{(k-1)}, X^{(k-2)})$. This boundary map can be realized homotopically as follows. Let $c(X^{(k-1)})$ be the cone on the subcomplex $X^{(k-1)}$,

$$c(X^{(k-1)}) = X^{(k-1)} \times I/(X^{(k-1)} \times \{1\} \cup \{x_0\} \times I),$$

which is obviously a contractible space. Consider the mapping cone of the inclusion $X^{(k-1)} \hookrightarrow X^{(k)}$, $X^{(k)} \cup c(X^{(k-1)})$. By projecting the cone to a point, there is a projection map

$$p_k : X^{(k)} \cup c(X^{(k-1)}) \to X^{(k)}/X^{(k-1)} = \bigvee_{j \in J_k} S^k_j$$

which is a homotopy equivalence. (**Note.** The fact that this map induces an isomorphism in homology is straight forward by computing the homology exact sequence of the pair $(X^{(k)} \cup c(X^{(k-1)}), X^{(k)})$. The fact that this map is a homotopy equivalence is a basic point set topological property of CW - complexes coming from the so - called “Homotopy Extension Property”. However it can be proved directly, by hand, in this case. We leave its verification to the reader.) Let

$$u_k : X^{(k)} \to \bigvee_{j \in J_k} S^k_j$$

be the composition

$$X^{(k)} \hookrightarrow X^{(k)} \cup c(X^{(k-1)}) \xrightarrow{p_k} X^{(k)}/X^{(k-1)} = \bigvee_{j \in J_k} S^k_j.$$

Then the composition of u_k with the attaching map

$$\alpha_{k+1} : \bigvee_{j \in J_{k+1}} S^k_j \to X^{(k)}$$

(whose mapping cone defines the $(k+1)$ - skeleton $X^{(k+1)}$), is a map between wedges of k - spheres,

$$d_{k+1} : \bigvee_{j \in J_{k+1}} S^k_j \xrightarrow{\alpha_{k+1}} X^{(k)} \xrightarrow{u_k} \bigvee_{j \in J_k} S^k_j.$$

The following is immediate from the definitions.

Step 3. The induced map in homology,

$$(d_{k+1})_* : H_k(\bigvee_{j \in J_{k+1}} S^k_j) \to H_k(\bigvee_{j \in J_k} S^k_j)$$
$$C_{k+1}(X) \to C_k(X)$$

is the boundary homomorphism in the chain complex $\partial_{k+1} : C_{k+1}(X) \to C_k(X)$.

Now consider the map

$$[(X^{(n)}, X^{(n-1)}), (Y, y_0)] \xrightarrow[\cong]{\phi} C^n(X;\pi) = Z^n(X;\pi) \xrightarrow{\mu} H^n(X;\pi).$$

We then have the following corollary.

Step 4. A map $f : X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j \to Y$ has the property that

$$\mu \circ \phi([f]) = 0 \in H^n(X;\pi)$$

if and only if there is a map

$$f_{n-1} : \bigvee_{j \in J_{(n-1)}} S^n_j \to Y$$

so that f is homotopic to the composition

$$\bigvee_{j \in J_n} S^n_j \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S^n_j \xrightarrow{f_{n-1}} Y.$$

Proof. Since $\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X;\pi) = Z^n(X;\pi)$ is a bijection, $\mu \circ \phi([f]) = 0$ if and only if $\phi([f])$ is in the image of the coboundary map. The result then follows from step 3. $\square$

Step 5. The composition

$$X^{(n)} \xrightarrow{u_n} \bigvee_{j\in J_n} S_j^n \xrightarrow{d_n} \bigvee_{j\in J_{n-1}} S_j^n$$

is null homotopic.

Proof. The map u_n was defined by the composition

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)} \xrightarrow[\simeq]{p_n} \bigvee_{j\in J_n} S_j^n.$$

But notice that if we take the quotient $X^{(n)} \cup c(X^{(n-1)}/X^{(n)}$ we get the suspension

$$X^{(n)} \cup c(X^{(n-1)}/X^{(n)} = \Sigma X^{(n-1)}.$$

Furthermore, the map between the wedges of the spheres, $d_n : \bigvee_{j\in J_n} S_j^n \to \bigvee_{j\in J_{n-1}} S_j^n$ is directly seen to be the composition

$$d_n : \bigvee_{j\in J_n} S_j^n \simeq X^{(n)} \cup c(X^{(n-1)} \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)}/X^{(n)} = \Sigma X^{(n-1)} \xrightarrow{\Sigma u_{n-1}} \bigvee_{j\in J_{n-1}} S_j^n.$$

Thus the composition $d_n \circ u_n : X^{(n)} \to \bigvee_{j\in J_n} S_j^n \to \bigvee_{j\in J_{n-1}} S_j^n$ factors as the composition

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)} \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)}/X^{(n)} = \Sigma X^{(n-1)} \xrightarrow{\Sigma u_{n-1}} \bigvee_{j\in J_{n-1}} S_j^n.$$

But the composite of the first two terms in this composition,

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)} \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)}/X^{(n)}$$

is clearly null homotopic, and hence so is $d_n \circ u_n$. □

We now complete the proof of the theorem by doing the following step.

Step. 6. The correspondence $\phi : [X, Y] \to H^n(X; \pi)$ is injective.

Proof. Let $f, g : X \to Y$ be maps with $\phi([f]) = \phi([g]) \in H^n(X; \pi)$. Since Y - is $(n-1)$ - connected, given any map $h : X \to Y$, the restriction to its $(n-1)$ - skeleton is null homotopic. (**Exercise.** Check this!) Null homotopies define maps

$$\tilde{f}, \tilde{g} : X \cup c(X^{(n-1)}) \to Y$$

given by f and g respectively on X, and by their respective null homotopies on the cones, $c(X^{(n-1)})$. Using the homotopy equivalence $p_n : X^{(n)} \cup c(X^{(n-1)}) \simeq X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j$, we then have maps

$$\bar{f}, \, \bar{g} : X^{(n)}/X^{(n-1)} \to Y$$

which, when composed with the projection $X = X^{(n)} \to X^{(n)}/X^{(n-1)}$ are homotopic to f and g respectively. Now by the commutativity of the diagram in step 2, since $\phi([f]) = \phi([g])$, then $\mu \circ \phi([\bar{f}]) = \mu \circ \phi([\bar{g}])$. Or equivalently,

$$\mu \circ \phi([\bar{f}] - [\bar{g}]) = 0$$

where we are using the fact that

$$[(X^{(n)}, X^{(n-1)}), Y] = [\bigvee_{j \in J_n} S^n_j, Y] = \oplus_{j \in J_n} \pi_n(Y)$$

is a group, and maps to $C^n(X;\pi)$ as a group isomorphism.

Let $\psi : X^{(n)}/X^{(n-1)} \to Y$ represent $[\bar{f}] - [\bar{g}] \in [\bigvee_{j \in J_n} S^n_j, Y]$. Then $\mu \circ \phi(\psi) = 0$. Then by step 4, there is a map $\psi_{n-1} : \bigvee_{j \in J_{n-1}} S^n_j \to Y$ so that $\psi_{n-1} \circ d_n$ is homotopic to ψ. Thus the composition

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\psi} Y$$

is homotopic to the composition

$$X \to X^{(n)}/X^{n-1} = \bigvee_{j \in J_n} S^n_j \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S^n_j \xrightarrow{\psi_{n-1}} Y.$$

But by step 5, this compostion is null homotopic. Now since ψ represents $[\bar{f}] - [\bar{g}]$, a null homotopy of the composition

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\psi} Y$$

defines a homotopy between the compositions

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\bar{f}} Y \quad \text{and} \quad X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\bar{g}} Y.$$

The first of these maps is homotopic to $f : X \to Y$, and the second is homotopic to $g : X \to Y$. Hence $f \simeq g$, which proves that ϕ is injective. □

We now know that the correspondence $\phi : [X,Y] \to H^n(X;\pi)$ is surjective (step 2) and injective (step 6). This completes the proof of this theorem. □

We now proceed with the proof of the main classification theorem for cohomology, using Eilenberg - MacLane spaces, Theorem 4.22.

Proof. The Hopf Whitney theorem proves this theorem when X is an n - dimensional CW - complex. We split the proof for general CW - complexes into two cases.

Case 1. X is $n+1$ - dimensional.

Consider the following commutative diagram

$$\begin{array}{ccc} [X, K(\pi, n)] & \xrightarrow{\phi} & H^n(X;\pi) \\ \rho\downarrow & & \downarrow\rho \\ [X^{(n)}, K(\pi, n)] & \xrightarrow[\cong]{\phi_n} & H^n(X^{(n)};\pi) \end{array} \tag{4.1}$$

where the vertical maps ρ denote the obvious restriction maps, and ϕ_n denotes the restriction of the correspondence ϕ to the n - skeleton, which is an isomorphism by the Hopf - Whitney theorem.

Now by considering the exact sequence for cohomology of the pair $(X, X^{(n)}) = (X^{(n+1)}, X^{(n)})$, one sees that the restriction map $\rho : H^n(X, \pi) \to H^n(X^{(n)}, \pi)$ is injective. Using this together with the fact that ϕ_n is an isomorphism and the commutativity of this diagram, one sees that to show that $\phi : [X, K(\pi, n)] \to H^n(X;\pi)$ is surjective, it suffices to show that for $\gamma \in H^n(X, \pi)$ with $\rho(\gamma) = \phi_n([f_n])$, where $f_n : X^{(n)} \to K(\pi, n)$, then f_n can be extended to a map $f : X \to K(\pi, n)$.

Using the same notation as was used in the proof of the Hopf - Whitney theorem, since $X = X^{(n+1)}$, we can write

$$X = X^{(n)} \cup_{\alpha_{n+1}} \bigcup_{j\in J_{n+1}} D^{(n+1)}$$

where $\alpha_{n+1} : \bigvee_{j\in J_{n+1}} S^n_j \to X^{(n)}$ is the attaching map. Thus the obstruction to finding an extension $f : X \to K(\pi, n)$ of the map $f_n : X^{(n)} \to K(\pi, n)$, is the compostion

$$\bigvee_{j\in J_{n+1}} S^n_j \xrightarrow{\alpha_{n+1}} X^{(n)} \xrightarrow{f_n} K(\pi, n).$$

Now since $\bigvee_{j\in J_{n+1}} S^n_j$ is n - dimensional, the Hopf - Whitney theorem says that this map is determined by its image under ϕ,

$$\phi([f_n \circ \alpha_{n+1}]) \in H^n(\bigvee_{j\in J_{n+1}} S^n_j;\pi).$$

But this class is $\alpha^*_{n+1}(\phi([f_n]))$, which by assumption is $\alpha^*_{n+1}(\rho(\gamma))$. But the composition

$$H^n(X;\pi) \xrightarrow{\rho} H^n(X^{(n)}, \pi) \xrightarrow{\alpha^*_{n+1}} H^n(\bigvee_{j\in J_{n+1}} S^n_j;\pi)$$

are two successive terms in the long exact sequence in cohomology of the pair $(X^{(n+1)}, X^{(n)})$ and is therefore zero. Thus the obstruction to finding the extension $f : X \to K(\pi, n)$ is zero. As observed above this proves that $\phi : [X, K(\pi, n)] \to H^n(X; \pi)$ is surjective.

We now show that ϕ is injective. So suppose $\phi([f]) = \phi([g])$ for f, $g : X \to K(\pi, n)$. To prove that ϕ is injective we need to show that this implies that f is homotopic to g. Let f_n and g_n be the restrictions of f and g to $X^{(n)}$. That is,

$$f_n = \rho([f]) : X^{(n)} \to K(\pi, n) \quad \text{and} \quad g_n = \rho([g]) : X^{(n)} \to K(\pi, n)$$

Now by the commutativity of diagram 4.1 and the fact that ϕ_n is an isomorphism, we have that f_n and g_n are homotopic maps. Let

$$F_n : X^{(n)} \times I \to K(\pi, n)$$

be a homotopy between them. That is, $F_0 = f_n : X^{(n)} \times \{0\} \to K(\pi, n)$ and $F_1 = g_n : X^{(n)} \times \{1\} \to K(\pi, n)$. This homotopy defines a map on the $(n+1)$ -subcomplex of $X \times I$ defined to be

$$\tilde{F} : (X \times \{0\}) \cup (X \times \{1\}) \cup X^{(n)} \times I \to K(\pi, n)$$

where $\tilde{F}$ is defined to be f and g on $X \times \{0\}$ and $X \times \{1\}$ respectively, and F on $X^{(n)} \times I$. But since X is $(n+1)$ - dimensional, $X \times I$ is $(n+2)$ - dimensional, and this subcomplex is its $(n+1)$ - skeleton. So $X \times I$ is the union of this complex with $(n+2)$ - dimensional disks, attached via maps from a wedge of $(n+1)$ - dimensional spheres. Hence the obstruction to extending $\tilde{F}$ to a map $F : X \times I \to K(\pi, n)$ is a cochain in $C^{n+2}(X \times I; \pi_{n+1}(K(\pi, n))$. But this group is zero since $\pi_{n+1}(K(\pi, n)) = 0$. Thus there is no obstruction to extending $\tilde{F}$ to a map $F : X \times I \to K(\pi, n)$, which is a homotopy between f and g. As observed before this proves that ϕ is injective. This completes the proof of the theorem in this case.

General Case. Since, by case 1, we know the theorem for $(n+1)$ - dimensional CW - complexes, we assume that the dimension of X is $\geq n+2$. Now consider the following commutative diagram:

$$\begin{array}{ccc}
[X, K(\pi, n)] & \xrightarrow{\phi} & H^n(X; \pi) \\
\rho \downarrow & & \downarrow \rho \\
[X^{(n+1)}, K(\pi, n)] & \underset{\cong}{\xrightarrow{\phi_{n+1}}} & H^n(X^{(n+1)}; \pi)
\end{array}$$

where, as earlier, the maps ρ denote the obvious restriction maps, and ϕ_{n+1} denotes the restriction of ϕ to the $(n+1)$ skeleton, which we know is an isomorphism, by the result of case 1.

Now in this case the exact sequence for the cohomology of the pair $(X, X^{(n+1)})$ yields that the restriction map $\rho : H^n(X; \pi) \to H^n(X^{(n+1)}, \pi)$

is an isomorphism. Therefore by the commutativity of this diagram, to prove that $\phi : [X, K(\pi, n)] \to H^n(X; \pi)$ is an isomorphism, it suffices to show that the restriction map

$$\rho : [X, K(\pi, n)] \to [X^{(n+1)}, K(\pi, n)]$$

is a bijection. This is done by induction on the skeleta $X^{(K)}$ of X, with $K \geq n + 1$. To complete the inductive step, one needs to analyze the obstructions to extending maps $X^{(K)} \to K(\pi, n)$ to $X^{(K+1)}$ or homotopies $X^{(K)} \times I \to K(\pi, n)$ to $X^{(K+1)} \times I$, like what was done in the proof of case 1. However in these cases the obstructions will always lie is spaces of cochains with coefficients in $\pi_q(K(\pi, n))$ with $q = K \,\text{or}\, K + 1$, and so $q \geq n + 1$. But then $\pi_q(K(\pi, n)) = 0$ and so these obstructions will always vanish. We leave the details of carrying out this argument to the reader. □

4.5 Spectral Sequences

One of the great technical achievements of Algebraic Topology was the development of spectral sequences. They were originally invented by Leray in the late 1940's and since that time have become fundamental calculational tools in many areas of Geometry, Topology, and Algebra. One of the earliest and most important applications of spectral sequences was the work of Serre [136] for the calculation of the homology of a fibration. We divide our discussion of spectral sequences in these notes into three parts. In the first section we develop the notion of a spectral sequence of a filtration. In the next section we discuss the Leray - Serre spectral sequence for a fibration. In the final two sections we discuss applications: we prove the Hurewicz theorem, calculate the cohomology of the Lie groups $U(n)$, and $O(n)$, and of the loop spaces ΩS^n. We refer the reader to [109] for a more complete discussion of spectral sequences.

4.5.1 The spectral sequence of a filtration

A spectral sequence is the algebraic machinery for studying sequences of long exact sequences that are interelated in a particular way. We begin by illustrating this with the example of a filtered complex.

Let C_* be a chain complex, and let $A_* \subset C_*$ be a subcomplex. The short exact sequence of chain complexes

$$0 \longrightarrow A_* \hookrightarrow C_* \longrightarrow C_*/A_* \longrightarrow 0$$

leads to a long exact sequence in homology:

$$\longrightarrow \cdots \longrightarrow H_{q+1}(C_*, A_*) \longrightarrow H_q(A_*) \longrightarrow H_q(C_*) \longrightarrow H_q(C_*, A_*) \longrightarrow H_{q-1}(A_*) \longrightarrow \cdots$$

This is useful in computing the homology of the big chain complex, $H_*(C_*)$ in terms of the homology of the subcomplex $H_*(A_*)$ and the homology of the quotient complex $H_*(C_*, A_*)$. A spectral sequence is the machinery used to study the more general situation when one has a *filtration* of a chain complex C_* by subcomplexes

$$0 = F_0(C_*) \hookrightarrow F_1(C_*) \hookrightarrow \cdots \hookrightarrow F_k(C_*) \hookrightarrow F_{k+1}(C_*) \hookrightarrow \cdots \hookrightarrow C_* = \bigcup_k F_k(C_*).$$

Let D_*^k be the subquotient complex $D_*^k = F_k(C_*)/F_{k-1}(C_*)$ and so for each k there is a long exact sequence in homology

$$\longrightarrow H_{q+1}(D_*^k) \longrightarrow H_q(F_{k-1}(C_*)) \longrightarrow H_q(F_k(C_*)) \longrightarrow H_q(D_*^k) \longrightarrow \cdots$$

By putting these long exact sequences together, in principle one should be able to use information about $\oplus_k H_*(D_*^k)$ in order to obtain information about

$$H_*(C_*) = \varinjlim_k H_*(F_k(C_*)).$$

A spectral sequence is the bookkeeping device that allows one to do this. To be more specific, consider the following diagram.

$$
\begin{array}{ccccccc}
0 & & & & 0 & & \\
\downarrow i & & & & \downarrow i & & \\
H_q(F_1(C_*)) & & & & H_{q-1}(F_1(C_*)) & \xrightarrow{=} & H_{q-1}(D_*^1) \\
\downarrow i & & & & \downarrow i & & \\
\vdots & & & & \vdots & & \\
\downarrow i & & & & \downarrow i & & \\
H_q(F_{k-p}(C_*)) & \xrightarrow{j} & H_q(D_*^{k-p}) & \xrightarrow{\partial} & H_{q-1}(F_{k-p-1}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-p-1}) \\
\downarrow i & & & & \downarrow i & & \\
\vdots & & & & H_{q-1}(F_{k-p}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-p}) \\
\downarrow i & & & & \downarrow i & & \\
\vdots & & & & \vdots & & \\
\downarrow i & & & & \downarrow i & & \\
H_q(F_{k-2}(C_*)) & & & & H_{q-1}(F_{k-3}(C_*)) & & \\
\downarrow i & & & & \downarrow i & & \\
H_q(F_{k-1}(C_*)) & \xrightarrow{j} & H_q(D_*^{k-1}) & \xrightarrow{\partial} & H_{q-1}(F_{k-2}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-2})) \\
\downarrow i & & & & \downarrow i & & \\
H_q(F_k(C_*)) & \xrightarrow{j} & H_q(D_*^{k}) & \xrightarrow{\partial} & H_{q-1}(F_{k-1}(C_*)) & \xrightarrow{j} & H_{q-1}(D_*^{k-1})) \\
\downarrow i & & & & \downarrow i & & \\
\vdots & & & & \vdots & & \\
\downarrow i & & & & \downarrow i & & \\
H_q(C_*) & & & & H_{q-1}(C_*) & &
\end{array}
\tag{4.2}
$$

The columns represent the homology filtration of $H_*(C_*)$ and the three maps ∂, j, and i combine to give long exact seqences at every level.

Let $\alpha \in H_q(C_*)$. We say that α has *algebraic filtration* k, if α is in the image of a class $\alpha_k \in H_q(F_k(C_*))$ but is not in the image of $H_q(F_{k-1}(C_*))$. In such a case we say that the image $j(\alpha_k) \in H_q(D_*^k)$ is a *representative* of α. Notice that this representative is not unique. In particular we can add any class in the image of

$$d_1 = j \circ \partial : H_{q+1}(D_*^{k+1}) \longrightarrow H_q(D_*^k)$$

to $j(\alpha_k)$ and we would still have a representative of $\alpha \in H_q(C_*)$ under the above definition.

Conversely, let us consider when an arbitrary class $\beta \in H_q(D_*^k)$ represents a class in $H_q(C_*)$. By the exact sequence this occurs if and only if the image $\partial(\beta) = 0$, for this is the obstruction to β being in the image of $j : H_q(F_k(C_*)) \to H_q(D_*^k)$. Furthermore if $j(\tilde{\beta}) = \beta$ then β represents the image

$$i \circ \cdots \circ i(\tilde{\beta}) \in H_q(C_*).$$

Now $\partial(\beta) = 0$ if and only if it lifts all the way up the second vertical tower in diagram 4.2. The first obstruction to this lifting, (i.e the obstruction to lifting $\partial(\beta)$ to $H_{q-1}(F_{k-2}(C_*))$ is that the composition

$$d_1 = j \circ \partial : H_q(D_*^k) \longrightarrow H_{q-1}(D_*^{k-1})$$

maps β to zero. That is elements of $H_q(C_*)$ are represented by elements in the subquotient

$$ker(d_1)/Im(d_1)$$

of $H_q(D_*^k)$. We use the following notation to express this. We define

$$E_1^{r,s} = H_{r+s}(D_*^r)$$

and define

$$d_1 = j \circ \partial : E_1^{r,s} \longrightarrow E_1^{r-1,s}.$$

r is said to be the *algebraic filtration* of elements in $E_1^{r,s}$ and $r+s$ is the *total degree* of elements in $E_1^{r,s}$. Since $\partial \circ j = 0$, we have that

$$d_1 \circ d_1 = 0$$

and we let

$$E_2^{r,s} = Ker(d_1 : E_1^{r,s} \to E_1^{r-1,s})/Im(d_1 : E_1^{r+1,s} \to E_1^{r,s})$$

be the resulting homology group. We can then say that the class $\alpha \in H_q(C_*)$ has as its representative, the class $\alpha_k \in E_2^{k,q-k}$.

Now let us go back and consider further obstructions to an arbitrary class $\beta \in E_2^{k,q-k}$ representing a class in $H_q(C_*)$. Represent β as a cycle in E_1: $\beta \in Ker(d_1 = j \circ \partial \in H_q(D_*^k))$. Again, β represents a class in $H_q(C_*)$ if and only if $\partial(\beta) = 0$. Now since $j \circ \partial(\beta) = 0$, $\partial(\beta) \in H_{q-1}(F_{k-1}(C_*))$ lifts to a class, say $\tilde{\beta} \in H_{q-1}F_{k-2}(C_*)$. Remember that the goal was to lift $\partial(\beta)$ all the way up the vertical tower (so that it is zero). The obstruction to lifting it the next stage, i.e to $H_{q-1}(F_{k-3}(C_*))$ is that $j(\tilde{\beta}) \in H_{q-1}(D_*^{k-2})$. That is, we can find such a lifting if and only if $j(\tilde{\beta}) = 0$. Now the fact that a d_1 cycle β has the property that $\partial(\beta)$ lifts to $H_{q-1}F_{k-2}(C_*))$ allows to define a map

$$d_2 : E_2^{k,q-k} \longrightarrow E_2^{k-2,q-k+1}$$

and more generally,

$$d_2 : E_2^{r,s} \longrightarrow E_2^{r-2,s+1}$$

by composing this lifting with

$$j : H_{s+r-1}(F_{r-2}(C_*)) \longrightarrow H_{s+r-1}(D_*^{r-2}).$$

That is, $d_2 = j \circ i^{-1} \circ \partial$. It is straightforward to check that $d_2 : E_2^{r,s} \longrightarrow E_2^{r-2,s+1}$ is well defined, and that elements of $H_q(C_*)$ are actually represented by elements in the subquotient homology groups of $E_2^{*,*}$:

$$E_3^{r,s} = Ker(d_2 : E_2^{r,s} \to E_2^{r-2,s+1})/Im(d_2 : E_2^{r+2,s-1} \to E_1^{r,s})$$

Inductively, assume the subquotient homology groups $E_j^{r,s}$ have been defined for $j \leq p-1$ and differentials

$$d_j : E_j^{r,s} \longrightarrow E_j^{r-j,s+j-1}$$

defined on representative classes in $H_{r+s}(D_*^r)$ to be the composition

$$d_j = j \circ (i^{j-1} = i \circ \cdots \circ i)^{-1} \circ \partial$$

so that $E_{j+1}^{*,*}$ is the homology $Ker(d_j)/Im(d_j)$. We then define

$$E_p^{r,s} = Ker(d_{p-1} : E_{p-1}^{r,s} \to E_{p-1}^{r-p+1,s+p-2})/Im(d_{p-1} : E_{p-1}^{r+p-1,s-p+2} \to E_{p-1}^{r,s}).$$

Thus $E_p^{k,q-k}$ is a subquotient of $H_q(D_*^k)$, represented by elements β so that $\partial(\beta)$ lifts to $H_q(F_{k-p}(C_*))$. That is, there is an element $\tilde{\beta} \in H_q(F_{k-p}(C_*))$ so that

$$i^{p-1}(\tilde{\beta}) = \partial(\beta) \in H_{q-1}(F_{k-1}(C_*)).$$

The obstruction to $\tilde{\beta}$ lifting to $H_{q-1}(F_{k-p-1}(C_*))$ is $j(\beta) \in H_q(D_*^{k-p})$. This procedure yields a well defined map

$$d_p : E_p^{r,s} \longrightarrow E_p^{r-p,s+p-1}$$

given by $j \circ (i^{p-1})^{-1} \circ \partial$ on representative classes in $H_q(D_*^k)$. This completes the inductive step. Notice that if we let

$$E_\infty^{r,s} = \varinjlim_p E_p^{r,s}$$

then $E_\infty^{k,q-k}$ is a subquotient of $H_q(D_*^k)$ consisting of precisely those classes represented by elements $\beta \in H_q(D_*^k)$ so that $\partial(\beta)$ lifts all the way up the vertical tower i.e $\partial(\beta)$ is in the image of i^p for all p. This is equivalent to the condition that $\partial(\beta) = 0$ which as observed above is precisely the condition necessary for β to represent a class in $H_q(C_*)$. These observations can be made more precise as follows.

Theorem 4.25. *Let* $I^{r,s} = Image(H_{r+s}(F_r(C_*)) \longrightarrow H_{r+s}(C_*))$. *Then* $E_\infty^{r,s}$ *is isomorphic to the quotient group*

$$E_\infty^{r,s} \cong I^{r,s}/I^{r-1,s+1}.$$

Thus the $E_\infty^{*,*}$ *determines* $H_*(C_*)$ *up to extensions. In particular, if all homology groups are taken with field coefficients we have*

$$H_q(C_*) \cong \bigoplus_{r+s=q} E_\infty^{r,s}.$$

In this case we say that $\{E_p^{r,s}, d_p\}$ is a *spectral sequence* starting at $E_1^{r,s} = H_{r+s}(D_*^r)$, and converging to $H_{r+s}(C_*)$.

Often times a filtration of this type occurs when one has a topological space X filtered by subspaces,

$$* = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \cdots \hookrightarrow X.$$

An important example is the filtration of a CW - complex X by its skeleta, $X_k = X^{(k)}$. We get a spectral sequence as above by applying the homology of the chain complexes to this topological filtration. This spectral sequence converges to $H_*(X)$ with E_1 term $E_1^{r,s} = H_{r+s}(X_r, X_{r-1})$. From the construction of this spectral sequence one notices that chain complexes are irrelevant in this case; indeed all one needs is the fact that each inclusion $X_{k-1} \hookrightarrow X_k$ induces a long exact sequence in homology.

Exercise. Show that in the case of the filtration of a CW - complex X by its skeleta, that the E_1 -term of the corresponding spectral sequence is the cellular chain complex, and the E_2 - term is the homology of X,

$$E_2^{r,s} = \begin{cases} H_r(X), & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, show that this spectral sequence "collapses" at the E_2 level, in the sense that

$$E_p^{r,s} = E_2^{r,s} \quad \text{for all } p \geq 2$$

and hence

$$E_\infty^{r,s} = E_2^{r,s}.$$

In Chapter 10 we will discuss the notion of a *generalized homology theory*. Roughly speaking, a functor $h_*(-)$ is a *generalized homology theory.* if it satisfies all the Eilenberg - Steenrod axioms but dimension. In this case the inclusions of a filtration as above $X_{k-1} \hookrightarrow X_k$ induce long exact sequences in $h_*(-)$, and one gets, by a procedure completely analogous to the above, a spectral sequence converging to $h_*(X)$ with E_1 term

$$E_1^{r,s} = h_{r+s}(X_r, X_{r-1}).$$

Again, for the skeletal filtration of a CW complex, this spectral sequence is called the Atiyah - Hirzebruch spectral sequence for the generalized homology h_*. We will discuss this spectral sequence in more detail in Chapter 10.

4.5.2 The Leray - Serre spectral sequence for a fibration

One of the most important examples of a spectral sequence is the Leray - Serre spectral sequence of a fibration. Given a fibration $F \to E \to B$, the goal is to understand how the homology of the three spaces (fiber, total space, base space) are related. In the case of a trivial fibration, $E = B \times F \to B$, the answer to this question is given by the Kunneth formula, which says, that when taken with field coefficients,

$$H_*(B \times F; k) \cong H_*(B; k) \otimes_k H_*(F : k),$$

where k is the field.

When $p : E \to B$ is a nontrivial fibration with fiber F, one needs a spectral sequence to study the relation between the homology of E and those of F and B. This is the Leray-Serre spectral sequence. The idea is to construct a filtration on a chain complex $C_*(E)$ in terms of the skeletal filtration of a CW - decomposition of the base space B.

Assume for the moment that $p : E \to B$ is a fiber bundle with fiber F. For the purposes of our discussion we will assume that the base space B is simply connected. Let $B^{(k)}$ be the k - skeleton of B, and define

$$E(k) = p^{-1}(B^{(k)}) \subset E.$$

We then have a filtration of the total space E by subspaces

$$* \hookrightarrow E(0) \hookrightarrow E(1) \hookrightarrow \cdots \hookrightarrow E(k) \hookrightarrow E(k+1) \hookrightarrow \cdots \hookrightarrow E.$$

To analyze the E_1 - term of the associated homology spectral spectral sequence we need to compute the E_1 - term, $E_1^{r,s} = H_{r+s}(E(r), E(r-1))$. To do this, write the skeleta of B in the form

$$B^{(r)} = B^{(r-1)} \cup \bigcup_{j \in J_r} D_j^r.$$

Now since each cell D_r is contractible, the restriction of the fibration E to the cells is trivial, and so

$$E(r) - E(r-1) \cong \bigcup_{j \in J_r} D^r \times F.$$

Moreover the attaching maps are via the maps

$$\tilde{\alpha}_r : \bigvee_{j \in J_r} S_j^{r-1} \times F \to E(r-1)$$

induced by the cellular attaching maps $\alpha_k : \bigvee_{j \in J_k} S_j^{k-1} \to B^{(k-1)}$. Using the Mayer - Vietoris sequence, one then computes that

$$\begin{aligned} E_1^{r,s} = H_{r+s}(E(r), E(r-1)) &= H_{r+s}(\bigcup_{j \in J_r} D^r \times F, \bigcup_{j \in J_r} S^{r-1} \times F) \\ &= H_{r+s}(\bigvee_{j \in J_r} S^r \times F, F) \\ &= H_r(\bigvee_{j \in J_r} S^r) \otimes H_s(F) \\ &= C_r(B; H_s(F)). \end{aligned}$$

These calculations indicate the following result, due to Serre in his thesis [136]. We refer the reader to that paper for details. It is one of the great pieces of mathematics literature from the twentieth century.

Theorem 4.26. *Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there are chain complexes $C_*(E)$ and $C_*(B)$ computing the homology of E and B respectively, and a filtation of $C_*(E)$ leading to a spectral sequence converging to $H_*(E)$ with the following properties:*

1. $E_1^{r,s} = C_r(B) \otimes H_s(F)$

2. $E_2^{r,s} = H_r(B; H_s(F))$

3. *The differential d_j has bidegree $(-j, j-1)$:*

$$d_j : E_j^{r,s} \to E_j^{r-j, s+j-1}.$$

4. *The inclusion of the fiber into the total space induces a homomorphism*

$$i_* : H_n(F) \to H_n(E)$$

which can be computed as follows:

$$i_* : H_n(F) = E_2^{0,n} \to E_\infty^{0,n} \subset H_n(E)$$

where $E_2^{0,n} \to E_\infty^{0,n}$ is the projection map which exists because all the differentials d_j are zero on $E_j^{0,n}$.

5. *The projection map induces a homomorphism*

$$p_* : H_n(E) \to H_n(B)$$

which can be computed as follows:

$$H_n(E) \to E_\infty^{n,0} \subset E_2^{n,0} = H_n(B)$$

where $E_\infty^{n,0}$ *includes into* $E_2^{n,0}$ *as the subspace consisting of those classes on which all differentials are zero. This is well defined because no class in* $E_j^{n,0}$ *can be a boundary for any* j*.*

Remark. The theorem holds when the base space is not simply connected also. However in that case the E_2 -term is homology with "twisted coefficients". This has important applications in many situations, however we will not consider this issue in this book. Again, we refer the reader to Serre's thesis [136] or McCleary's text [109] for details.

We will finish this chapter by describing several applications of this important spectal sequence. The first, due to Serre himself [136], is the use of this spectral sequence to prove that even though fibrations do not, in general, admit long exact sequences in homology, they do admit exact sequences in homology through a range of dimensions depending on the connectivity of the base space and fiber.

Theorem 4.27. *Let* $p : E \to B$ *be a fibration with connected fiber* F*, where* B *is simply connected and* $H_i(B) = 0$ *for* $0 < i < n$*, and* $H_i(F) = 0$ *for* $0 < i < m$*. Then there is an exact sequence*

$$\begin{aligned} H_{n+m-1}(F) \xrightarrow{i_*} H_{n+m-1}(E) \xrightarrow{p_*} H_{n+m-1}(B) \xrightarrow{\tau} H_{n+m-2}(F) \\ \to \cdots \to H_1(E) \to 0. \end{aligned}$$

Proof. The E_2 -term of the Serre spectral sequence is given by

$$E_2^{r,s} = H_r(B; H_s(F))$$

which, by hypothesis is zero for $0 < r < n$ or $0 < j < m$. Let $q < n + m$. Then this implies that the composition series for $H_q(E)$, given by the filtration defining the spectral sequence, reduces to the short exact sequence

$$0 \to E_\infty^{0,q} \to H_q(E) \to E_\infty^{q,0} \to 0.$$

Now in general, for these "edge terms", we have

$$\begin{aligned} E_\infty^{q,0} &= kernel\{d_q : E_q^{q,0} \to E_q^{0,q-1}\} \quad \text{and} \\ E_\infty^{0,q} &= coker\{d_q : E_q^{q,0} \to E_q^{0,q-1}\}. \end{aligned}$$

But when $q < n+m$, we have $E_q^{q,0} = E_2^{q,0} = H_q(B)$ and $E_q^{0,q-1} = E_2^{0,q-1} = H_{q-1}(F)$ because there can be no other differentials in this range. Thus if we define

$$\tau : H_q(B) \to H_{q-1}(F)$$

to be $d_q : E_q^{q,0} \to E_q^{0,q-1}$, for $q < n+m$, we then have that $p_* : H_q(E) \to H_q(B)$ maps surjectively onto the kernel of τ, and if $q < n+m-1$, then the kernel of p_* is the cokernel of $\tau : H_{q+1}(B) \to H_q(F)$. This establishes the existence of the long exact sequence in homology in this range. □

Remark. The homomorphism $\tau : H_q(B) \to H_{q-1}(F)$ for $q < n+m$ in the proof of this theorem is called the "transgression" homomorphism.

4.5.3 Applications I: The Hurewicz theorem

As promised earlier in this chapter, we now use the Serre spectral sequence to prove the Hurewicz theorem. The general theorem is a theorem comparing relative homotopy groups with relative homology groups. We begin by proving the theorem comparing homotopy groups and homology of a single space.

Theorem 4.28. *Let X be an $n-1$ - connected space, $n \geq 2$. That is, we assume $\pi_q(X) = 0$ for $q \leq n-1$. Then $H_q(X) = 0$ for $q \leq n-1$ and the previously defined "Hurewicz homomorphism"*

$$h : \pi_n(X) \to H_n(X)$$

is an isomorphism.

Proof. We assume the reader is familiar with the analogue of the theorem when $n = 1$, which says that for X connected, the first homology group $H_1(X)$ is given by the abelianization of the fundamental group

$$h : \pi_1(X)/[\pi_1, \pi_1] \cong H_1(X)$$

where $[\pi_1, \pi_1] \subset \pi_1(X)$ is the commutator subgroup. We use this preliminary result to begin an induction argument to prove this theorem. Namely we assume that the theorem is true for $n-1$ replacing n in the statement of the theorem. We now complete the inductive step. By our inductive hypotheses, $H_i(X) = 0$ for $i \leq n-2$ and $\pi_{n-1}(X) \cong H_{n-1}(X)$. But we are assuming that $\pi_{n-1}(X) = 0$. Thus we need only show that $h : \pi_n(X) \to H_n(X)$ is an isomorphism.

Consider the path fibration $p : PX \to X$ with fiber the loop space ΩX. Now $\pi_i(\Omega X) \cong \pi_{i+1}(X)$, and so $\pi_i(\Omega X) = 0$ for $i \leq n-2$. So our inductive assumption applied to the loop space says that

$$h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$$

is an isomorphism. But $\pi_{n-1}(\Omega X) = \pi_n(X)$. Also, by the Serre exact sequence applied to this fibration, using the facts that

1. the total space PX is contractible, and

2. the fiber ΩX is $n-2$ - connected and the base space X is $(n-1)$ - connected

we then conclude that the transgression,

$$\tau : H_n(X) \to H_{n-1}(\Omega X)$$

is an isomorphism. Hence the Hurewicz map $h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ is the same as the Hurewicz map $h : \pi_n(X) \to H_n(X)$, which is therefore an isomorphism. □

We are now ready to prove the more general relative version of this theorem 4.18

Theorem 4.29. *Let X be simply connected, and let $A \subset X$ be a simply connected subspace. Suppose that the pair (X, A) is $(n-1)$ - connected, for $n > 2$. That is,*

$$\pi_k(X, A) = 0 \quad \textit{if} \quad k \leq n-1.$$

Then the Hurewicz homomorphism $h_* : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

Proof. . Replace the inclusion

$$\iota : A \hookrightarrow X.$$

by a homotopy equivalent fibration $\tilde{\iota} : \tilde{A} \to X$ as in 4.7. Let F_ι be the fiber. Then $\pi_i(F_\iota) \cong \pi_{i+1}(X, A)$, by comparing the long exact sequences of the pair (X, A) to the long exact sequence in homotopy groups for the fibration $\tilde{A} \to X$. So by the Hurewicz theorem 4.28 we know that $\pi_i(F) = H_i(F) = 0$ for $i \leq n-2$ and

$$h : \pi_{n-1}(F) \to H_{n-1}(F)$$

is an isomorphism. But as mentioned, $\pi_{n-1}(F) \cong \pi_n(X, A)$ and by comparing the homology long exact sequence of the pair (X, A) to the Serre exact sequence for the fibration $F \to \tilde{A} \to B$, one has that $H_{n-1}(F) \cong H_n(X, A)$. The theorem follows. □

As a corollary, we obtain the following strengthening of the Whitehead Theorem 4.14 which is quite useful in calculations.

Corollary 4.30. *Suppose X and Y are simply connected CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homology groups,*

$$f_* : H_k(X) \xrightarrow{\cong} H_k(Y) \quad \text{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

Proof. Replace $f : X \to Y$ by the inclusion into the mapping cylinder

$$\bar{f} : X \hookrightarrow \bar{Y}$$

where $\bar{Y} = Y \cup_f X \times I$ which is homotopy equivalent to Y, and $\bar{f}$ includes X into $\bar{Y}$ as $X \times \{1\}$.

Since X and Y are simply connected, we have that $\pi_2(X) \cong H_2(X)$ and $\pi_2(Y) \cong H_2(Y)$. Thus $f_* : \pi_2(X) \to \pi_2(Y)$ is an isomorphism. Again, since X and Y are simply connected, this implies that $\pi_q(\bar{Y}, X) = 0$ for $q = 1, 2$. Thus we can apply the relative Hurewicz theorem. However since $f_* : H_k(X) \cong H_k(Y) \quad$ for all $k \geq 0$, we have that $H_k(\bar{Y}, X) = 0$ for all $k \geq 0$. But then the Hurewicz theorem implies that $\pi_k(\bar{Y}, X) = 0$ for all k, which in turn implies that $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all k. The theorem follows from the Whitehead Theorem 4.14. □

4.5.4 Applications II: $H_*(\Omega S^n)$ and $H^*(U(n))$

In this section we will use the Serre spectral sequence to compute the homology of the loop space ΩS^n and the cohomology ring of the Lie groups, $H^*(U(n))$.

Theorem 4.31.

$$H_q(\Omega S^n) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is a multiple of } n-1, \text{ i.e } q = k(n-1) \\ 0 & \text{otherwise} \end{cases}$$

Proof. ΩS^n is the fiber of the path fibration $p : PS^n \to S^n$. Since the total space of this fibration is contractible, the Serre spectral sequence converges to 0 in positive dimensions. That is,

$$E_\infty^{r,s} = 0$$

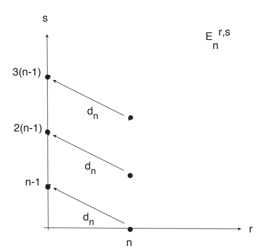
for all r, s, except that $E_\infty^{0,0} = \mathbb{Z}$. Now since the base space, S^n has nonzero homology only in dimensions 0 and n (when it is $\mathbb{Z}$), then

$$E_2^{r,s} = H_r(S^n; H_s(\Omega S^n))$$

is zero unless $r = 0$ or n. In particular, since $d_q : E_q^{r,s} \to E_q^{r-q,s+q-1}$, we must have that for $q < n$, $d_q = 0$. Thus $E_2^{r,s} = E_n^{r,s}$ and the only possible nonzero differential d_n occurs in dimensions

$$d_n : E_n^{n,s} \to E_n^{0,s+n-1}.$$

It is helpful to picture this spectral sequence as in the following diagram, where a dot in the (r, s) - entry denotes a copy of the integers in $E_n^{r,s} = H_r(S^n; H_s(\Omega S^n))$.



Notice that if the generator $\sigma_{n,0} \in E_n^{n,0}$ is in the kernel of d_n, then it would represent a nonzero class in $E_{n+1}^{n,0}$. But d_{n+1} and all higher differentials on $E_{n+1}^{n,0}$ must be zero, for dimensional reasons. That is, $E_{n+1}^{n,0} = E_\infty^{n,0}$. But we saw that $E_\infty^{n,0} = 0$. Thus we must conclude that $d_n(\sigma_{n,0}) \neq 0$. For the same reasoning, (i.e the fact that $E_{n+1}^{n,0} = 0$) we must have that $d_n(k\sigma_{n,0}) \neq 0$ for all integers k. This means that the image of

$$d_n : E_n^{n,0} \to E_n^{0,n-1}$$

is $\mathbb{Z} \subset E_n^{0,n-1} = H_{n-1}(\Omega S^n)$. On the other hand, we claim that $d_n : E_n^{n,0} \to E_n^{0,n-1}$ must be surjective. For if $\alpha \in E_n^{0,n-1}$ is not in the image of d_n, then it represents a nonzero class in $E_{n+1}^{0,n-1} = E_\infty^{0,n-1}$. But as mentioned earlier $E_\infty^{0,n-1} = 0$. So d_n is surjective as well. In fact we have proven that

$$d_n : \mathbb{Z} = H_n(S^n) = E_n^{n,0} \to E_n^{0,n-1} = E_2^{0,n-1} = H_{n-1}(\Omega S^n)$$

is an isomorphism. Hence $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, as claimed. Now notice this calculation implies a calculation of $E_2^{n,n-1}$, namely,

$$E_2^{n,n-1} = H_n(S^n; H_{n-1}(\Omega S^n) = \mathbb{Z}.$$

Repeating the above argument shows that $E_2^{n,n-1} = E_n^{n,n-1}$ and that

$$d_n : E_n^{n,n-1} \to E_n^{0,2(n-1)}$$

must be an isomorphism. This yields that

$$\mathbb{Z} = E_2^{0,2(n-1)} = H_{2(n-1)}(\Omega S^n).$$

Repeating this argument shows that for every q, $\mathbb{Z} = E_2^{n,q(n-1)} = E_n^{n,q(n-1)}$ and that

$$d_n : E_n^{n,q(n-1)} \to E_n 0, (q+1)(n-1) \cong H_{(q+1)(n-1)}(\Omega S^n)$$

is an isomorphism. And so $H_{k(n-1)}(\Omega S^n) = \mathbb{Z}$ for all k.

We can also conclude that in dimensions j not a multiple of $n-1$, then $H_j(\Omega S^n)$ must be zero. This is true by the following argument. Assume the contrary, so that there is a smallest $j > 0$ not a multiple of $n-1$ with $H_j(\Omega S^n) = E_2^{0,j} \neq 0$. But for dimensional reasons, this group cannot be in the image of any differential, because the only $E_q^{r,s}$ that can be nonzero with $r > 0$ is when $r = n$. So the only possibility for a class $\alpha \in E_2^{0,j}$ to represent a class which is in the image of a differential is $d_n : E_n^{n,s} \to E_n^{0,s+n-1}$. So $j = s+n-1$. But since j is the smallest positive integer not of the form a multiple of $n-1$ with $H_j(\Omega S^n)$ nonzero, then for $s < j$, $E_n^{n,s} = H_n(S^n, H_s(\Omega S^n)) = H_s(\Omega S^n)$ can only be nonzero if s is a multiple of $(n-1)$, and therefore so is $s+n-1 = j$. This contradiction implies that if j is not a multiple of $n-1$, then $H_j((\Omega S^n))$ is zero. This completes our calculation of$H_*(\Omega S^n)$. □

We now use the cohomology version of the Serre spectral sequence to compute the cohomology of the unitary groups. We first give the cohomological analogue of 4.26. Again, the reader should consult [136] for details.

Theorem 4.32. *Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there is a cohomology spectral sequence converging to $H^*(E)$, with $E_2^{r,s} = H^r(B; H^s(F))$, having the following properties.*

1. *The differential d_j has bidegree $(j, -j+1)$:*

$$d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}.$$

2. *For each j, $E_j^{*,*}$ is a bigraded ring. The ring multiplication consists of pairings*

$$E_j^{p,q} \otimes E_j^{i,j} \to E_j^{p+i,q+j}.$$

3. *The differential* $d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}$. *is an antiderivation in the sense that it satisfies the product rule:*

$$d_j(ab) = d_j(a) \cdot b + (-1)^{u+v} a \cdot d_j(b)$$

where $a \in E_j^{u,v}$.

4. *The product in the ring* E_{j+1} *is induced by the product in the ring* E_j, *and the product in* E_∞ *is induced by the cup product in* $H^*(E)$.

We apply this to the following calculation.

Theorem 4.33. *There is an isomorphism of graded rings,*

$$H^*(U(n)) \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}],$$

the graded exterior algebra on one generator σ_{2k-1} *in every odd dimension* $2k-1$ *for* $1 \leq k \leq n$.

Proof. We prove this by induction on n. For $n = 1$, $U(1) = S^1$ and we know the assertion is correct. Now assume that $H^*(U(n-1)) \cong \Lambda[\sigma_1, \cdots \sigma_{2n-3}]$. Consider the Serre cohomology spectral sequence for the fibration

$$U(n-1) \subset U(n) \to U(n)/U(n-1) \cong S^{2n-1}.$$

Then the E_2 - term is given by

$$E_2^{*,*} \cong H^*(S^{2n-1}; H^*(U(n-1)) = H^*(S^{2n-1}) \otimes H^*(U(n-1))$$

and this isomorphism is an isomorphism of graded rings. But by our inductive assumption we have that

$$\begin{aligned} H^*(S^{2n-1}) \otimes H^*(U(n-1)) &\cong \Lambda[\sigma_{2n-1}] \otimes \Lambda[\sigma_1, \cdots \sigma_{2n-3}] \\ &\cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]. \end{aligned}$$

Thus

$$E_2^{**} \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]$$

as graded algebras. Now since all the nonzero classes in $E_2^{*,*}$ have odd total degree (where the total degree of a class $\alpha \in E_2^{r,s}$ is $r+s$), and all differentials increase the total degree by one, we must have that all differentials in this spectral sequence are zero. Thus

$$E_\infty^{*,*} = E_2^{*,*} \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}].$$

We then conclude that $H^*(U(n)) =\cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]$ which completes the inductive step in our proof. □

4.5.5 Applications III: $H_*(K(\mathbb{Q},n))$

We will use the Serre spectral sequence to compute the homology of the rational Eilenberg-MacLane spaces, $K(\mathbb{Q};n)$.

Theorem 4.34. *The homology of the Eilenberg-MacLane spaces $K(\mathbb{Q},n)$ is given as follows:*

$$\tilde{H}_q(K(\mathbb{Q},2m);\mathbb{Z}) = \begin{cases} \mathbb{Q}, & \text{if } q \text{ is a positive multiple of } 2m\ , \\ 0 & \text{otherwise.} \end{cases}$$
$$\tilde{H}_q(K(\mathbb{Q},2m+1);\mathbb{Z}) = \begin{cases} \mathbb{Q}, & \text{if } q = 2m+1, \\ 0 & \text{otherwise.} \end{cases} \quad .$$

Proof. Consider the path-loop fibration, $\Omega K(\mathbb{Q},n) \to PK(\mathbb{Q},n) \to K(\mathbb{Q},n)$. Notice that the based loop space, $\Omega K(\mathbb{Q},n)$ is an Eilenberg-MacLane space of type $K(\mathbb{Q},n-1)$. We now prove the theorem by induction on n. For $n=0$, the statement is obvious. Inductively assume the theorem is true for $n-1$ and we want to prove it for n. We consider the Serre spectral sequence for this fibration. Since the path space $PK(\mathbb{Q},n)$ is contractible, the spectral sequence must converge to zero in positive dimensions. For this to happen, the spectral sequences must have the following form, depending on whether n is even or odd. The argument is very similar to that which was carried out in the calculation of $H^*(\Omega S^n)$ (Theorem 4.31). We leave the verification of these descriptions as an exercise for the reader.

The result follows from these spectral sequences.

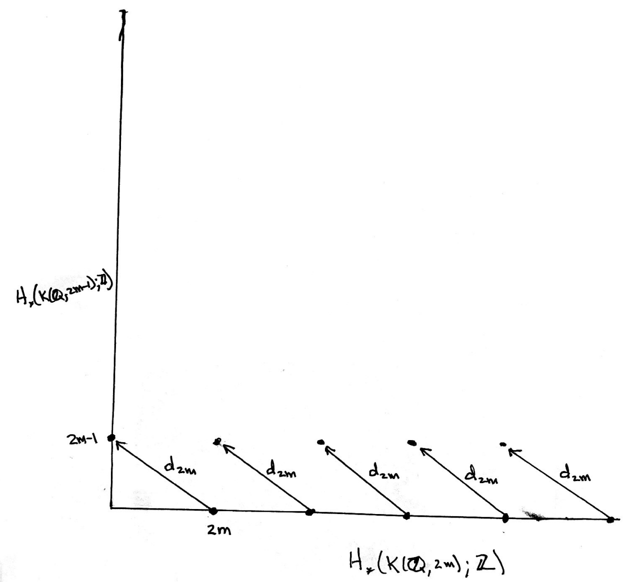


FIGURE 4.1
The Serre spectral sequence for the homology of the fibration $K(\mathbb{Q}, 2m-1) \to PK(\mathbb{Q}, 2m) \to K(\mathbb{Q}, 2m)$.

□

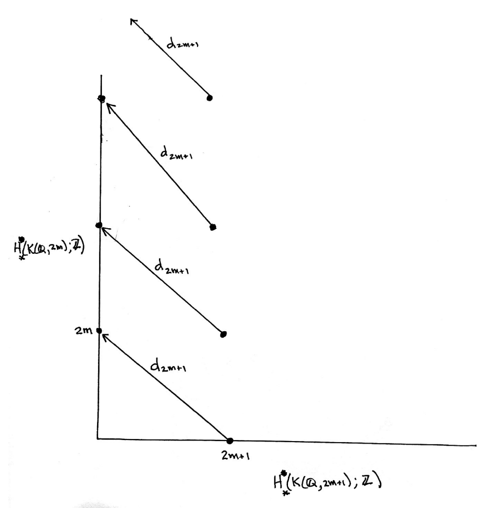


FIGURE 4.2
The Serre spectral sequence for the homology of the fibration $K(\mathbb{Q}, 2m) \to PK(\mathbb{Q}, 2m+1) \to K(\mathbb{Q}, 2m+1)$.

5

Classification of Bundles

In this chapter we prove Steenrod's classification theorem of principal G - bundles, and the corresponding classification theorem of vector bundles. This theorem states that for every group G, there is a "classifying space" BG with a well defined homotopy type so that the homotopy classes of maps from a space X, $[X, BG]$, is in bijective correspondence with the set of isomorphism classes of principal G - bundles, $Prin_G(X)$. We then describe various examples and constructions of these classifying spaces, and use them to study structures on principal bundles, vector bundles, and manifolds.

5.1 Consequences of the homotopy invariance of fiber bundles

The goal of this section is to examine certain applications of the homotopy invariance of fiber bundles (Theorem 2.11), such as the classification of principal bundles over spheres in terms of the homotopy groups of Lie groups.

The following is a direct corollary of the homotopy invariance of fiber bundles, Theorem 2.11.

Corollary 5.1. *Let $p : E \to B$ be a principal G - bundle over a connected space B. Then for any space X the pull back construction gives a well defined map from the set of homotopy classes of maps from X to B to the set of isomorphism classes of principal G - bundles,*

$$\rho_E : [X, B] \to Prin_G(X).$$

Definition 5.1. *A principal G - bundle $p : EG \to BG$ is called* universal if the pull back construction

$$\rho_{EG} : [X, BG] \to Prin_G(X)$$

is a bijection for every space X of the homotopy type of a CW complex. The base space of the universal bundle BG is called a *classifying space* for G (or for principal G - bundles).

The main goal of this chapter is to show that universal bundles exist for every group G, and that the classifying spaces are unique up to homotopy type.

Applying Theorem 2.11 to vector bundles gives the following,

Corollary 5.2. *If $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic, they induce the same homomorphism of abelian monoids,*

$$f_0^* = f_1^* : Vect^*(Y) \to Vect^*(X)$$
$$Vect^*_{\mathbb{R}}(Y) \to Vect^*_{\mathbb{R}}(X)$$

and hence of K theories

$$f_0^* = f_1^* : K(Y) \to K(X)$$
$$KO(Y) \to KO(X)$$

Corollary 5.3. *If $f : X \to Y$ is a homotopy equivalence, then it induces isomorphisms*

$$f^* : Prin_G(Y) \xrightarrow{\cong} Prin_G(X)$$
$$Vect^*(Y) \xrightarrow{\cong} Vect^*(X)$$
$$K(Y) \xrightarrow{\cong} K(X)$$

Note. In the above statements regarding K-theory, the spaces involved are assumed to be compact.

The following result is a classification theorem for bundles over spheres. It begins to describe why understanding the homotopy type of Lie groups is so important in Topology.

Theorem 5.4. *There is a bijective correspondence between principal bundles and homotopy groups*

$$Prin_G(S^n) \cong \pi_{n-1}(G)$$

where as a set $\pi_{n-1}G = [S^{n-1}, x_0; G, \{1\}]$, which refers to (based) homotopy classes of basepoint preserving maps from the sphere S^{n-1} with basepoint $x_0 \in S^{n-1}$, to the group G with basepoint the identity $1 \in G$.

Proof. Let $p : E \to S^n$ be a G - bundle. Write S^n as the union of its upper and lower hemispheres,

$$S^n = D^n_+ \cup_{S^{n-1}} D^n_-.$$

Since D^n_+ and D^n_- are both contractible, the above corollary says that E restricted to each of these hemispheres is trivial. Morever if we fix a trivialization of the fiber of E at the basepoint $x_0 \in S^{n-1} \subset S^n$, then we can extend this trivialization to both the upper and lower hemispheres. We may therefore write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

where θ is a clutching function defined on the equator, $\theta : S^{n-1} \to G$. That is, E consists of the two trivial components, $(D^n_+ \times G)$ and $(D^n_- \times G)$ where if $x \in S^{n-1}$, then $(x, g) \in (D^n_+ \times G)$ is identified with $(x, \theta(x)g) \in (D^n_- \times G)$. Notice that since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}G.$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow[\cong]{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

By restricting to the hemispheres, the isomorphism Ψ defines maps

$$\Psi_+ : D^n_+ \to G$$

and

$$\Psi_- : D^n_- \to G$$

which both map the basepoint $x_0 \in S^{n-1}$ to the identity $1 \in G$, and furthermore have the property that for $x \in S^{n-1}$,

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x),$$

or, $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. Now by considering the linear homotopy $\Psi_+(tx)\theta_1(x)\Psi_-(tx)^{-1}$ for $t \in [0, 1]$, we see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, where the two zeros in this description refer to the origins of D^n_+ and D^n_- respectively, i.e the north and south poles of the sphere S^n. Now since Ψ_+ and Ψ_- are defined on connected spaces, their images lie in a connected component of the group G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t = 0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t = 1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\Theta : Prin_G(S^n) \to \pi_{n-1}G$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

as seen in our discussion of clutching functions in chapter 1.

We now show that Θ is injective. That is, suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2 As above we write

$$E_1 = (D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G)$$

and

$$E_2 = (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

Let $H : S^{n-1} \times [-1, 1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator S^{n-1} in S^n with $S^{n-1} \times [-1, 1]$ Write $\mathcal{D}_+ = D^2_+ \cup \bar{\mathcal{N}}$ and $\mathcal{D}_- = D^2_- \cup \bar{\mathcal{N}}$ Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \bar{\mathcal{N}} \cong S^{n-1} \times [-1, 1].$$

Thus we may form the principal G - bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_- \times G$$

where by abuse of notation, H refers to the composition

$$\bar{\mathcal{N}} \cong S^{n-1} \times [-1, 1] \xrightarrow{H} G\cdot$$

We leave it to the interested reader to verify that E is isomorphic to both E_1 and E_2. This completes the proof of the theorem. □

5.2 Universal bundles and classifying spaces

The goal of this section is to study universal principal G - bundles, the resulting classification theorem, and the corresponding classifying spaces. We will discuss several examples including the universal bundle for any subgroup of the general linear group. We postpone the proof of the existence of universal bundles for all groups until the next section.

In order to identify universal bundles, we need to recall the following definition from homotopy theory. Recall that all spaces we are considering have the homotopy type of CW-complexes.

The following is the main result of this section. It identifies when a principal bundle is universal.

Theorem 5.5. *Let $p : E \to B$ be a principal G - bundle, where the total space E is contractible. Then this bundle is universal* in the sense that if X is any space of the homotopy type of a CW-complex, the induced pull-back map

$$\psi : [X, B] \to Prin_G(X)$$
$$f \to f^*(E)$$

is a bijective correspondence.

For the purposes of this book we will prove the theorem in the setting where the action of G on the total space E is *cellular.* That is, there is a CW - decomposition of the space E which, in an appropriate sense, is respected by the group action. In practical terms there is not much loss in making these assumptions, since the actions of compact Lie groups on manifolds, and algebraic actions on projective varieties satisfy this property. For the proof of the theorem in its full generality we refer the reader to Steenrod's book [143], and for a full reference on equivariant CW - complexes and how they approximate a wide range of group actions, we refer the reader to [91].

In order to make the notion of cellular action precise, we need to define the notion of an *equivariant CW - complex,* or a G - CW - complex. The idea is the following. Recall that a CW - complex is a space that is made up out of disks of various dimensions whose interiors are disjoint. In particular it can be built up skeleton by skeleton, and the $(k+1)^{st}$ skeleton $X^{(k+1)}$ is constructed out of the k^{th} skeleton $X^{(k)}$ by attaching $(k+1)$ - dimensional disks via "attaching maps", $S^k \to X^{(k)}$.

A "G - CW - complex" is one that has a group action so that the orbits of the points on the interior of a cell are uniform in the sense that each point in a cell D^k has the same isotropy subgroup, say H, and the orbit of a cell itself is of the form $G/H \times D^k$. This leads to the following definition.

Definition 5.2. *A G - CW - complex is a space with G -action X which is topologically the direct limit of G - invariant subspaces $\{X^{(k)}\}$ called the equivariant skeleta,*

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(k-1)} \subset X^{(k)} \subset \cdots X$$

where for each $k \geq 0$ there is a countable collection of k dimensional disks, subgroups of G, and maps of boundary spheres

$$\{D^k_j,\, H_j < G,\, \phi_j : \partial D^k_j \times G/H_j = S^{k-1}_j \times G/H_j \to X^{(k-1)} \quad j \in I_k\}$$

so that

1. *Each "attaching map" $\phi_j : S^{k-1}_j \times G/H_j \to X^{(k-1)}$ is G -equivariant, and*

2.

$$X^{(k)} = X^{k-1)} \bigcup_{\phi_j \; j \in I_j} (D^k_j \times G/H_j).$$

This notation means that each " disk orbit " $D^k_j \times G/H_j$ *is attached to* $X^{(k-1)}$ *via the map* $\phi_j : S^{k-1}_j \times G/H_j \to X^{(k-1)}$.

We leave the following as an exercise to the reader.

Exercise. Prove that when X is a G - CW complex the orbit space X/G has the an induced structure of a (non-equivariant) CW - complex.

Note. Observe that in a G -CW complex X with a *free* G action, all disk orbits are of the form $D^k \times G$, since all isotropy subgroups are trivial.

We now prove Theorem 5.5 under the assumption that the principal bundle $p : E \to B$ has the property that with respect to group action of G on E, then E has the structure of a G - CW - complex. The basespace is then given the induced CW - structure. The spaces X in the statement of the theorem are assumed to be of the homotopy type of CW - complexes.

Proof. We first prove that the pull - back map

$$\psi : [X, B] \to Prin_G(X)$$

is surjective. So let $q : P \to X$ be a principal G - bundle, with P a G - CW - complex. We prove there is a G - equivariant map $h : P \to E$ that maps each orbit pG homeomorphically onto its image, $h(y)G$. We prove this by induction on the equivariant skeleta of P. So assume inductively that the map h has been constructed on the $(k-1)$ - skeleton,

$$h_{k-1} : P^{(k-1)} \to E.$$

Since the action of G on P is free, all the k - dimensional disk orbits are of the form $D^k \times G$. Let $D^k_j \times G$ be a disk orbit in the G-CW - structure of the k - skeleton $P^{(k)}$. Consider the disk $D^k_j \times \{1\} \subset D^k_j \times G$. Then the map h_{k-1} extends to $D^k_j \times \{1\}$ if and only if the composition

$$S^{k-1}_j \times \{1\} \subset S^{k-1}_j \times G \xrightarrow{\phi_j} P^{(k-1)} \xrightarrow{h_{k-1}} E$$

is null homotopic. But since E is contractible, any such map is null homotopic and extends to a map of the disk, $\gamma : D^k_j \times \{1\} \to E$. Now extend γ equivariantly to a map $h_{k,j} : D^k_j \times G \to E$. By construction $h_{k,j}$ maps the orbit of each point $x \in D^k_j$ equivariantly to the orbit of $\gamma(x)$ in E. Since both orbits are isomorphic to G (because the action of G on both P and E are free), this map is a homeomorphism on orbits. Taking the collection of the extensions $h_{k,j}$ together then gives an extension

$$h_k : P^{(k)} \to E$$

with the required properties. This completes the inductive step. Thus we may conclude we have a G - equivariant map $h : P \to E$ that is a homeomorphism on the orbits. Hence it induces a map on the orbit space $f : P/G = X \to E/G = B$ making the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{h} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

Since h induces a homeomorphism on each orbit, the maps h and f determine a homeomorphism of principal G - bundles which induces an equivariant isomorphism on each fiber. This implies that h induces an isomorphism of principal bundles to the pull - back

$$\begin{array}{ccc} P & \xrightarrow[\cong]{h} & f^*(E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[=]{} & X. \end{array}$$

Thus the isomorphism class $[P] \in Prin_G(X)$ is given by $f^*(E)$. That is, $[P] = \psi(f)$, and hence

$$\psi : [X, B] \to Prin_G(X)$$

is surjective.

We now prove ψ is injective. To do this, assume $f_0 : X \to B$ and $f_1 : X \to B$ are maps so that there is an isomorphism

$$\Phi : f_0^*(E) \xrightarrow{\cong} f_1^*(E).$$

We need to prove that f_0 and f_1 are homotopic maps. Now by the cellular approximation theorem (see [141]) we can find cellular maps homotopic to f_0 and f_1 respectively. We therefore assume without loss of generality that f_0 and f_1 are cellular. This, together with the assumption that E is a G - CW complex, gives the pull back bundles $f_0^*(E)$ and $f_1^*(E)$ the structure of G -CW complexes.

Define a principal G - bundle $\mathcal{E} \to X \times I$ by

$$\mathcal{E} = f_0^*(E) \times [0, 1/2] \cup_\Phi f_1^*(E) \times [1/2, 1]$$

where $v \in f_0^*(E) \times \{1/2\}$ is identified with $\Phi(v) \in f_1^*(E) \times \{1/2\}$. $\mathcal{E}$ also has the structure of a G - CW - complex.

Now by the same kind of inductive argument that was used in the surjectivity argument above, we can find an equivariant map $H : \mathcal{E} \to E$ that induces a homeomorphism on each orbit, and that extends the obvious maps $f_0^*(E) \times \{0\} \to E$ and $f_1^*(E) \times \{1\} \to E$. The induced map on orbit spaces

$$F : \mathcal{E}/G = X \times I \to E/G = B$$

is a homotopy between f_0 and f_1. This proves the correspondence Ψ is injective, and completes the proof of the theorem. □

The following result establishes the homotopy uniqueness of universal bundles.

Theorem 5.6. *Let $E_1 \to B_1$ and $E_2 \to B_2$ be universal principal G - bundles. Then there is a bundle map*

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{h}} & E_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow[h]{} & B_2 \end{array}$$

so that h is a homotopy equivalence.

Proof. The fact that $E_2 \to B_2$ is a universal bundle means, by Theorem 5.5 that there is a "classifying map" $h : B_1 \to B_2$ and an isomorphism $\tilde{h} : E_1 \to h^*(E_2)$. Equivalently, $\tilde{h}$ can be thought of as a bundle map $\tilde{h} : E_! \to E_2$ lying over $h : B_1 \to B_2$. Similarly, using the universal property of $E_1 \to B_1$, we get a classifying map $g : B_2 \to B_1$ and an isomorphism $\tilde{g} : E_2 \to g^*(E_1)$, or equivalently, a bundle map $\tilde{g} : E_2 \to E_1$. Notice that the composition

$$g \circ f : B_1 \to B_2 \to B_1$$

is a map whose pull back,

$$\begin{aligned} (g \circ f)^*(E_1) &= g^*(f^*(E_1)) \\ &\cong g^*(E_2) \\ &\cong E_1. \end{aligned}$$

That is, $(g \circ f)^*(E_1) \cong id^*(E_1)$, and hence by Theorem 5.5 we have $g \circ f \simeq id : B_1 \to B_1$. Similarly, $f \circ g \simeq id : B_2 \to B_2$. Thus f and g are homotopy inverses of each other. □

Because of this theorem, the basespace of a universal principal G - bundle has a well defined homotopy type. We denote this homotopy type by BG, and refer to it as the *classifying space* of the group G. We also use the notation EG to denote the total space of a universal G - bundle.

We have the following immediate result about the homotopy groups of the classifying space BG.

Corollary 5.7. *For any group G, there is an isomorphism of homotopy groups,*

$$\pi_{n-1}G \cong \pi_n(BG).$$

Proof. By considering Theorems 5.4 and 5.5 we see that both of these homotopy groups are in bijective correspondence with the set of principal bundles $Prin_G(S^n)$. To realize this bijection by a group homomorphism, consider the "suspension" of the group G, ΣG obtained by attaching two cones on G along the equator. That is,

$$\Sigma G = G \times [-1,1]/\sim$$

where all points of the form $(g,1)$, $(h,-1)$, or $(1,t)$ are identified to a single point.

Notice that this suspension construction can be applied to any space with a basepoint, and in particular $\Sigma S^{n-1} \cong S^n$.

Consider the principal G bundle E over ΣG defined to be trivial on both cones with clutching function $id : G \times \{0\} \xrightarrow{=} G$ on the equator. That is, if $C_+ = G \times [0,1]/\sim \subset \Sigma G$ and $C_- = G \times [-1,0] \subset \Sigma E$ are the upper and lower cones, respectively, then

$$E = (C_+ \times G) \cup_{id} (C_- \times G)$$

where $((g,0),h) \in C_+ \times G$ is identified with $((g,0)gh \in C_- \times G$. Then by Theorem 5.5 there is a classifying map

$$f : \Sigma G \to BG$$

such that $f^*(EG) \cong E$.

Now for any space X, let ΩX be the *loop space* of X,

$$\Omega X = \{\gamma : [-1,1] \to X \text{ such that } \gamma(-1) = \gamma(1) = x_0 \in X\}$$

where $x_0 \in X$ is a fixed basepoint. Then the map $f : \Sigma G \to BG$ determines a map (its adjoint)

$$\bar{f} : G \to \Omega BG$$

defined by $\bar{f}(g)(t) = f(g,t)$. But now the loop space ΩX of any connected space X has the property that $\pi_{n-1}(\Omega X) = \pi_n(X)$ (see the exercise below). We then have the induced group homomorphism

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

which induces the bijective correspondence described above. □

Exercises. 1. Let X and Y be connected spaces equipped with basepoints. Prove that there is a bijection

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Here the notation $[-,-]$ denotes the set of homotopy classes of basepoint preserving maps. As a special case, conclude that $\pi_n(Y, y_o) \cong \pi_{n-1}(\Omega Y, \epsilon_0)$, where $\epsilon_0 : S^1 \to Y$ is the constant map at the basepoint y_0.

2. Let G be a topological group, and consider the map $f : G \to \Omega BG$ defined in the above proof of Corollary 4.10. Prove that f induces an isomorphism in homotopy groups (in all degrees). Such a map is called a "weak homotopy equivalence".

3. Prove that the composition

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

yields the bijection associated with identifying both $\pi_{n-1}(G)$ and $\pi_n(BG)$ with $Prin_G(S^n)$.

We recall the following definition from homotopy theory.

Definition 5.3. *An Eilenberg - MacLane space of type (G, n) is a space X such that*

$$\pi_k(X) = \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

We write $K(G, n)$ for an Eilenberg - MacLane space of type (G, n). Recall that for $n \geq 2$, the homotopy groups $\pi_n(X)$ are abelian groups, so in this case $K(G, n)$ can only exist if G is abelian.

Corollary 5.8. *Let π be a discrete group. Then the classifying space $B\pi$ is an Eilenberg - MacLane space $K(\pi, 1)$.*

Examples.

- $\mathbb{R}$ has a free, cellular action of the integers $\mathbb{Z}$ by

  $$(t, n) \to t + n \quad t \in \mathbb{R},\, n \in \mathbb{Z}.$$

  Since $\mathbb{R}$ is contractible, $\mathbb{R}/\mathbb{Z} = S^1 = B\mathbb{Z} = K(\mathbb{Z}, 1)$.

- The inclusion $S^n \subset S^{n+1}$ as the equator is clearly null homotopic since the inclusion obviously extends to a map of the disk. Hence the direct limit space

  $$\varinjlim_n S^n = \cup_n S^n = S^\infty$$

is contractible. Now $\mathbb{Z}_2$ acts freely on each S^n by the antipodal map, and the inclusions $S^n \subset S^{n+1}$ are equivariant with respect to these actions. Hence there is an induced free action of $\mathbb{Z}_2$ on S^∞. Thus the projection map

$$S^\infty \to S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$$

is a universal principal $\mathbb{Z}_2 = O(1)$ - bundle, and so

$$\mathbb{RP}^\infty = BO(1) = B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1)$$

.

- Similarly, the inclusion of the unit sphere in $\mathbb{C}^n$ into the unit sphere in $\mathbb{C}^{n+1}$ gives an the inclusion $S^{2n-1} \subset S^{2n+1}$ which is null homotopic. It is also equivariant with respect to the free $S^1 = U(1)$ - action given by (complex) scalar multiplication. Then the limit $S^\infty = \cup_n S^{2n+1}$ is aspherical with a free S^1 action. We therefore have that the projection

$$S^\infty \to S^\infty/S^1 = \mathbb{CP}^\infty$$

is a principal $S^1 = U(1)$ bundle. Hence we have

$$\mathbb{CP}^\infty = BS^1 = BU(1).$$

Moreover since S^1 is a $K(\mathbb{Z}, 1)$, then we have that

$$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

- The cyclic groups $\mathbb{Z}_n$ are subgroups of $U(1) = SO(2)$ given by rotation by $2\pi/n$. These groups therefore act freely on S^∞ as well. Thus the projection maps

$$S^\infty \to S^\infty/\mathbb{Z}_n$$

is a universal principal $\mathbb{Z}_n$ bundle. The quotient space $S^\infty/\mathbb{Z}_n$ is denoted $L^\infty(n)$ and is referred to as the infinite $\mathbb{Z}_n$ - lens space.

These examples allow us to give the following description of line bundles and their relation to cohomology. We first recall a result that is a special case of Corollary 4.21 above.

Theorem 5.9. *Let G be an abelian group. Then there is a natural isomorphism*

$$\phi : H^n(K(G, n); G) \xrightarrow{\cong} Hom(G, G).$$

Let $\iota \in H^n(K(G, n); G)$ be $\phi^{-1}(id)$. This is called the fundamental class.

Then as was seen in Chapter 4, if X has the homotopy type of a CW - complex, the mapping

$$[X, K(G,n)] \to H^n(X;G)$$
$$f \to f^*(\iota)$$

is a bijective correspondence.

With this we can now prove the following:

Theorem 5.10. *There are bijective correspondences which allow us to classify complex line bundles,*

$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{CP}^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z})$$

where the last correspondence takes a map $f : X \to \mathbb{CP}^\infty$ *to the class*

$$c_1 = f^*(c) \in H^2(X),$$

where $c \in H^2(\mathbb{CP}^\infty)$ *is the generator. In the composition of these correspondences, the class* $c_1 \in H^2(X)$ *corresponding to a line bundle* $\zeta \in Vect^1(X)$ *is called the first Chern class of* ζ *(or of the corresponding principal* $U(1)$ *- bundle).*

Proof. These correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X) \cong [X, BGL(1,\mathbb{C})]$, and that $\mathbb{CP}^\infty$ is a model for $BGL(1,\mathbb{C})$ as well as $BU(1)$. This is because, we can express $\mathbb{CP}^\infty$ in its homogeneous form as

$$\mathbb{CP}^\infty = \varinjlim_n (\mathbb{C}^{n+1} - \{0\})/GL(1,\mathbb{C}),$$

and that $\varinjlim_n (\mathbb{C}^{n+1} - \{0\})$ is a contractible space with a free action of $GL(1,\mathbb{C}) = \mathbb{C}^*$. □

There is a similar theorem classifying real line bundles:

Theorem 5.11. *There are bijective correspondences*

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{RP}^\infty] \cong [X, K(\mathbb{Z}_2,1)] \cong H^1(X;\mathbb{Z}_2)$$

where the last correspondence takes a map $f : X \to \mathbb{RP}^\infty$ *to the class*

$$w_1 = f^*(w) \in H^1(X;\mathbb{Z}_2),$$

where $w \in H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$ *is the generator. In the composition of these correspondences, the class* $w_1 \in H^1(X;\mathbb{Z}_2)$ *corresponding to a line bundle* $\zeta \in Vect^1_{\mathbb{R}}(X)$ *is called the first Stiefel - Whitney class of* ζ *(or of the corresponding principal* $O(1)$ *- bundle).*

More Examples.

- Let $V_n(\mathbb{C}^N)$ be the Stiefel - manifold described in Chapter 2. We claim that the inclusion of vector spaces $\mathbb{C}^N \subset \mathbb{C}^{2N}$ as the first N - coordinates induces an embedding $V_n(\mathbb{C}^N) \hookrightarrow V_n(\mathbb{C}^{2N})$ which is null homotopic. To see this, let $\iota : \mathbb{C}^n \to \mathbb{C}^{2N}$ be a fixed linear embedding, whose image lies in the last N - coordinates in $\mathbb{C}^{2N}$. Then given any $\rho \in V_n(\mathbb{C}^N) \subset V_n(\mathbb{C}^{2N})$, then $t \cdot \iota + (1-t) \cdot \rho$ for $t \in [0,1]$ defines a one parameter family of linear embeddings of $\mathbb{C}^n$ in $\mathbb{C}^{2N}$, and hence a contraction of the image of $V_n(\mathbb{C}^N)$ onto the element ι. Hence the limiting space $V_n(\mathbb{C}^\infty)$ is contractible with a free $GL(n,\mathbb{C})$ - action. Therefore the projection

  $$V_n(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/GL(n,\mathbb{C}) = Gr_n(\mathbb{C}^\infty)$$

  is a universal $GL(n,\mathbb{C})$ - bundle. Hence the infinite Grassmannian is the classifying space

  $$Gr_n(\mathbb{C}^\infty) = BGL(n,\mathbb{C})$$

  and so we have a classification

  $$Vect^n(X) \cong Prin_{GL(n,\mathbb{C})}(X) \cong [X, BGL(n,\mathbb{C})] \cong [X, Gr_n(\mathbb{C}^\infty)]. \quad (5.1)$$

- A simlar argument shows that the infinite unitary Stiefel manifold, $V_n^U(\mathbb{C}^\infty)$ is aspherical with a free $U(n)$ - action. Thus the projection

  $$V_n^U(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/U(n) = Gr_n(\mathbb{C}^\infty)$$

  is a universal principal $U(n)$ - bundle. Hence the infinite Grassmanian $Gr_n(\mathbb{C}^\infty)$ is the classifying space for $U(n)$ bundles as well,

  $$Gr_n(\mathbb{C}^\infty) = BU(n).$$

  The fact that this Grassmannian is both $BGL(n,\mathbb{C})$ and $BU(n)$ reflects the fact that every n - dimensional complex vector bundle has a $U(n)$ - structure, and that structure is unique up to homotopy.

- We have similar universal $GL(n,\mathbb{R})$ and $O(n)$ - bundles:

  $$V_n(\mathbb{R}^\infty) \to V_n(\mathbb{R}^\infty)/GL(n,\mathbb{R}) = Gr_n(\mathbb{R}^\infty)$$

  and

  $$V_n^O(\mathbb{R}^\infty) \to V_n^O(\mathbb{R}^\infty)/O(n) = Gr_n(\mathbb{R}^\infty).$$

  Thus we have

  $$Gr_n(\mathbb{R}^\infty) = BGL(n,\mathbb{R}) = BO(n)$$

  and so this infinite dimensional Grassmannian classifies real n - dimensional vector bundles as well as principal $O(n)$ - bundles.

Now suppose $p : EG \to EG/G = BG$ is a universal G - bundle. Suppose further that $H < G$ is a subgroup. Then H acts freely on EG as well, and hence the projection

$$EG \to EG/H$$

is a universal H - bundle. Hence $EG/H = BH$. Using the infinite dimensional Stiefel manifolds described above, this observation gives us models for the classifying spaces for any subgroup of a general linear group. So for example if we have a subgroup (i.e a faithful representation) $H \subset GL(n, \mathbb{C})$, then

$$BH = V_n(\mathbb{C}^\infty)/H.$$

This observation also leads to the following useful fact.

Proposition 5.12. . *Let $p : EG \to BG$ be a universal principal G - bundle, and let $H < G$. Then there is a fiber bundle*

$$BH \to BG$$

with fiber the orbit space G/H.

Proof. This bundle is given by

$$G/H \to EG \times_G G/H \to EG/G = BG$$

together with the observation that $EG \times_G G/H = EG/H = BH$. □

The Whitehead Theorem 4.14 will now allow us to prove the following important relationship between the homotopy type of a topological group and its classifying space.

Theorem 5.13. *Let G be a topological group with the homotopy type of a CW complex., and BG its classiftying space. Then there is a homotopy equivalence between G and the loop space,*

$$G \simeq \Omega BG.$$

Proof. Let $p : EG \to BG$ be a universal G bundle with EG a G - equivariant CW - complex. In particular, EG is contractible. So let and

$$H : EG \times I \to EG$$

be a contraction. That is, $H_0 : EG \times \{0\} \to EG$ is the constant map at the basepoint $e_0 \in EG$, , and $H_1 : EG \times \{1\} \to EG$ is the identity. Composing with the projection map,

$$\Phi = p \circ H : EG \times I \to BG$$

is a homotopy between the constant map to the basepoint $\Phi_0 : EG \times \{0\} \to b_0 \in BG$ and the projection map $\Phi_1 = p : EG \times \{1\} \to BG$. Consider the adjoint of Φ,

$$\bar{\Phi} : EG \to P(BG) = \{\alpha : I \to BG \text{ such that } \alpha(0) = b_0.\}$$

defined by $\bar{\Phi}(e)(t) = \Phi(e,t) \in BG$. Then by definition, the following diagram commutes:

$$\begin{array}{ccc} EG & \xrightarrow{\bar{\Phi}} & P(BG) \\ p\big\downarrow & & \big\downarrow q \\ BG & = & BG \end{array}$$

where $q(\alpha) = \alpha(1)$, for $\alpha \in P(BG)$. Thus Φ is a map of fibrations that induces a map on fibers

$$\phi : G \to \Omega BG.$$

Comparing the exact sequences in homotopy groups of these two fibrations, we see that ϕ induces an isomorphism in homotopy groups. A result of Milnor [114] that we will not prove says that if X is a CW complex, then the loop space ΩX has the homotopy type of a CW - complex. Then the Whitehead theorem implies that $\phi : G \to \Omega BG$ is a homotopy equivalence. □

We end this section with the following result whose proof is fairly easy using the theory of classifying spaces.

Proposition 5.14. *Let X be an n - dimensional CW - complex, and let ζ be an m - dimensional vector bundle over X, with $m \geq n$. Then ζ has $m-n$ linearly independent cross sections. If ξ is a d - dimensional complex bundle over X, then ξ admits $d - [n/2]$ linearly independent cross sections, where $[n/2]$ is the integral part of $n/2$.*

.

Proof. Let ζ be classified by a map $f_m : X \to BO(m)$. To prove the theorem we need to prove that f_m lifts (up to homotopy) to a map $f_n X \to BO(n)$. We would then have that

$$\zeta \cong f_m^*(\gamma_m) \cong f_n^*(\gamma_n) \oplus \epsilon_{m-n}$$

where γ_k is the universal k - dimensional vector bundle over $BO(k)$, and ϵ_j represents the j - dimensional trivial bundle. These isomorphisms would then produce the $m-n$ linearly independent cross sections of ζ. over X.

We first observe the following lemma.

Lemma 5.15. *The homotopy fiber of the map $BO(n) \to BO(m)$ is homotopy equivalent to the space of orbits, $O(m)/O(n)$.*

Proof. Notice that if $EO(m) \to BO(m)$ is a universal $O(m)$-bundle, then by viewing $O(n)$ as a subgroup of $O(m)$ we may consider the projection onto the orbit space

$$EO(m) \to EO(m)/O(n).$$

Notice that this is a principal $O(n)$ bundle, and since the total space $EO(m)$ is contractible, it is a model for a universal principal $O(n)$-bundle. In particular we have that the base space is a model for the classifying space,

$$EO(m)/O(n) \simeq BO(n).$$

Moreover this says that the projection map $EO(m)/O(n) \to EO(m)/O(m)$, which is obviously a fiber bundle, is a model for the map $BO(n) \to BO(m)$. But the fiber of this bundle is clearly the orbit space, $O(m)/O(n)$. The lemma follows. □

To complete the proof of the proposition, notice that by a simple induction argument using Theorem 4.33 shows that the fiber $O(m)/O(n)$ is $(n-1)$-connected. That is, $\pi_q(O(m)/O(n)) = 0$ for $q \leq n-1$. This means that all obstructions vanish for lifting the n - skeleton of X to the total space $BO(n)$. Since we are assuming X is n - dimensional, this completes the proof. The complex case is proved similarly. □

Corollary 5.16. *Let X be a compact, n - dimensional CW complex. Then every element of the reduced real K - theory, $\tilde{K}O(X)$ can be represented by a n - dimensional vector bundle. Every element of the complex K - theory, $\tilde{K}(X)$ can be represented by an $[n/2]$ - dimensional complex vector bundle.*

5.3 Classifying gauge groups

In this section we describe the classifying space of the group of automorphisms of a principal G - bundle, or the *gauge group* of the bundle. We describe the classifying space in two different ways: in terms of the space of connections on the bundle, and in terms of the mapping space of the base manifold to the classifying space BG. These constructions are important in Yang - Mills theory, and we refer the reader to [9] and [42] for more details.

Let A be a connection on a principal bundle $P \longrightarrow M$ where M is a closed manifold equipped with a Riemannian metric. The Yang - Mills functional applied to A, $\mathcal{YM}(A)$ is the square of the L^2 norm of the curvature,

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \|F_A\|^2 \, d(vol).$$

We view $\mathcal{YM}$ as a mapping $\mathcal{YM} : \mathcal{A}(P) \longrightarrow \mathbb{R}$. The relevance of the gauge group in Yang - Mills theory is that $\mathcal{YM}$ preserves this group of symmetries.

Definition 5.4. *The gauge group $\mathcal{G}(P)$ of the principal bundle P is the group of bundle automorphisms of $P \longrightarrow M$. That is, an element $\phi \in \mathcal{G}(P)$ is a bundle isomorphism of P with itself lying over the identity:*

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\phi} & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M. \end{array}$$

Equivalently, $\mathcal{G}(P)$ is the group $\mathcal{G}(P) = Aut_G(P)$ of G - equivariant diffeomorphisms of the space P, inducing the identity map on the orbit space $P/G = M$.

The gauge group $\mathcal{G}(P)$ can be thought of in several equivalent ways. The following one is particularly useful.

Consider the conjugation action of the Lie group G on itself,

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longrightarrow ghg^{-1}. \end{aligned}$$

This left action defines a fiber bundle

$$Ad(P) = P \times_G G \longrightarrow P/G = M$$

with fiber G. We leave the following as an exercise for the reader.

Proposition 5.17. *The gauge group of a principal bundle $P \longrightarrow M$ is naturally isomorphic (as topological groups) to the group of sections of $Ad(P)$, $C^\infty(M; Ad(P))$.*

The gauge group $\mathcal{G}(P)$ acts on the space of connections $\mathcal{A}(P)$ by the pull-back construction. More specifically, if $f : P \to Q$ is any smooth map of principal G - bundles and A is a connection on Q, then there is a natural pull back connection $f^*(A)$ on Q, defined by pulling back the equivariant splitting of the tangent bundle TQ to an equivariant splitting of TP in the obvious way. The pull - back construction for automorphisms $\phi : P \longrightarrow P$ defines an action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$.

We leave the proof of the following is an exercise for the reader.

Proposition 5.18. *Let P be the trivial bundle $M \times G \to M$. Then the gauge group $\mathcal{G}(P)$ is given by the function space from M to G,*

$$\mathcal{G}(P) \cong C^\infty(M; G).$$

Furthermore if $\phi : M \to G$ *is identified with an element of* $\mathcal{G}(P)$*, and* $A \in \Omega^1(M; \mathfrak{g})$ *is identified with an element of* $\mathcal{A}(G)$*, then the induced action of* ϕ *on* G *is given by*

$$\phi^*(A) = \phi^{-1}A\phi + \phi^{-1}d\phi.$$

It is not difficult to see that in general the gauge group $\mathcal{G}(P)$ does not act freely on the space of connections $\mathcal{A}(P)$. However there is an important subgroup $\mathcal{G}_0(P) < \mathcal{G}(P)$ that does. This is the group of *based* gauge transformations. To define this group, let $x_0 \in M$ be a fixed basepoint, and let P_{x_0} be the fiber of P at x_0.

Definition 5.5. *The based gauge group* $\mathcal{G}_0(P)$ *is a subgroup of the group of bundle automorphisms* $\mathcal{G}(P)$ *which pointwise fix the fiber* P_{x_0}*. That is,*

$$\mathcal{G}_0(P) = \{\phi \in \mathcal{G}(P) : \textit{if} \quad v \in P_{x_0} \textit{ then} \quad \phi(v) = v\}.$$

Theorem 5.19. *If* M *is connected the based gauge group* $\mathcal{G}_0(P)$ *acts freely on the space of connections* $\mathcal{A}(P)$*.*

Proof. (Sketch) Suppose that $A \in \mathcal{A}(P)$ is a fixed point of $\phi \in \mathcal{G}_0(P)$. That is, $\phi^*(A) = A$. We need to show that $\phi = 1$.

The equivariant splitting ω_A given by a connection A defines a notion of parallel transport in P along curves in M (see [69]). It is not difficult to see that the statement $\phi^*(A) = A$ implies that application of the automorphism ϕ commutes with parallel transport. Now let $w \in P_x$ be a point in the fiber of an element $x \in M$. Given curve γ in M between the basepoint x_0 and x this means that

$$\phi(w) = T_\gamma(\phi(T_{\gamma^{-1}}(w))$$

where T_γ is parallel transport along γ. But since $T_{\gamma^{-1}}(w) \in P_{x_0}$ and $\phi \in \mathcal{G}_0(P)$,

$$\phi(T_{\gamma^{-1}}(w)) = T_{\gamma^{-1}}(w).$$

Hence $\phi(w) = T_\gamma(T_\gamma^{-1}(w)) = w$. That is, $\phi = 1$. □

Remark. Notice that this argument actually says that if $A \in \mathcal{A}(P)$ is the fixed point of *any* gauge transformation $\phi \in \mathcal{G}(P)$, then ϕ is determined by its action on a single fiber.

Let $\mathcal{B}(P)$ and $\mathcal{B}_0(P)$ be the orbit spaces of connections on P up to guage and based gauge equivalence respectively,

$$\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P) \quad \mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P).$$

Now it is straightforward to check directly that the Yang - Mills functional in invariant under gauge transformations. Thus it yields maps

$$\mathcal{YM} : \mathcal{B}(P) \to \mathbb{R} \quad \text{and} \quad \mathcal{YM} : \mathcal{B}_0(P) \to \mathbb{R}.$$

It is therefore important to understand the homotopy types of these orbit spaces. Because of the freeness of the action of $\mathcal{G}_0(P)$, the homotopy type of the orbit space $\mathcal{G}_0(P)$ is easier to understand.

We end this section with a discussion of its homotopy type. Since the space of connections $\mathcal{A}(P)$ is affine, it is contractible. Moreover it is possible to show that the free action of the based gauge group $\mathcal{G}_0(P)$ defines a principal bundle $\mathcal{A}(P) \to \mathcal{A}(P)/\mathcal{G}_0(P) = \mathcal{B}_0(P)$ (See [42]). Thus $\mathcal{B}_0(P)$ is a model for the classifying space of the based gauge group,

$$\mathcal{B}_0(P) = B\mathcal{G}_0(P).$$

But the classifying spaces of the gauge groups are relatively easy to understand. (see [9].)

Theorem 5.20. *Let $G \longrightarrow EG \longrightarrow BG$ be a universal principal bundle for the Lie group G (so that EG is aspherical). Let $y_0 \in BG$ be a fixed basepoint. Then there are homotopy equivalences*

$$B\mathcal{G}(P) \simeq Map^P(M, BG) \quad \text{and} \quad \mathcal{B}_0(P) \simeq B\mathcal{G}_0(P) \simeq Map_0^P(M, BG)$$

where $Map(M, BG)$ is the space of all continuous maps from M to BG and $Map_0(M, BG)$ is the space of those maps that preserve the basepoints. The superscript P denotes the path component of these mapping spaces consisting of the homotopy class of maps that classify the principal G - bundle P.

Proof. (Sketch) Consider the space of all G - equivariant maps from P to EG, $Map^G(P, EG)$. The gauge group $\mathcal{G}(P) \cong Aut^G(P)$ acts freely on the left of this space by composition. It is easy to see that $Map^G(P, EG)$ is contractible, and its orbit space is given by the space of maps from the G - orbit space of P $(= M)$ to the G - orbit space of EG (= BG),

$$Map^G(P, EG)/\mathcal{G}(P) \cong Map^P(M, BG).$$

Furthermore the projection map to the orbit space is known to be a locally trivial fiber bundle (see [9]). This proves that $Map(M, BG) = B\mathcal{G}(P)$. Similarly $Map_0^G(P, EG)$, the space of G - equivariant maps that send the fiber P_{x_0} to the fiber EG_{y_0}, is an contractible space with a free $\mathcal{G}_0(P)$ action, whose orbit space is $Map_0^P(M, BG)$. As before we can conclude that $Map_0^P(M, BG) = B\mathcal{G}_0(P)$. □

5.4 Existence of universal bundles: the Milnor join construction and the simplicial classifying space

In the last section we proved a "recognition principle" for universal principal G bundles. Namely, if the total space of a principal G - bundle $p : E \to B$ is contractible, then it is universal. We also proved a homotopy uniqueness theorem, stating among other things that the homotopy type of the base space of a universal bundle, i.e the classifying space BG, is well defined. We also described many examples of universal bundles, and in particular have a model for the classifying space BG, using Stiefel manifolds, for every subgroup of a general linear group.

The goal of this section is to prove the general existence theorem. Namely, for every group G, there is a universal principal G - bundle $p : EG \to BG$. We will give two constructions of the universal bundle and the corresponding classifying space. One, due to Milnor [115] involves taking the "infinite join" of a group with itself. The other is an example of a simplicial space, called the simplicial bar construction. It is originally due to Eilenberg and MacLane [44]. These constructions are essentially equivalent when G has a CW-structure, and they both yield G - CW - complexes. Since they are so useful in algebraic topology and combinatorics, we will also take this opportunity to introduce the notion of a general simplicial space and show how these classifying spaces are important examples.

5.4.1 The join construction

One can think of the "join" of two spaces X and Y, written $X * Y$ as the space consisting of points that lie on a line that connects a point in X to a point in Y. The following is a more precise definition:

Definition 5.6. *The join $X * Y$ is defined by*

$$X * Y = X \times I \times Y / \sim$$

where $I = [0, 1]$ is the unit interval and the equivalence relation is given by $(x, 0, y_1) \sim (x, 0, y_2)$ for any two points y_1, $y_2 \in Y$, and similarly $(x_1, 1, y) \sim (x_2, 1, y)$ for any two points x_1, $x_2 \in X$.

A point $(x, t, y) \in X * Y$ can be viewed as a point on the line connecting x to y. Here are some examples.

Examples.

- Let y be a single point. Then $X * y$ is the cone $CX = X \times I / X \times \{1\}$.

- Let $Y = \{y_1, y_2\}$ be the space consisting of two distinct points. Then $X * Y$ is the suspension ΣX discussed earlier. Notice that the suspension can be viewed as the union of two cones, with vertices y_1 and y_2 respectively, attached along the equator.

- **Exercise.** Prove that the join of two spheres, is another sphere,

$$S^n * S^m \cong S^{n+m+1}.$$

- Let $\{x_0, \cdots, x_k\}$ be a collection of $k+1$ - distinct points. Then the k - fold join $x_0 * x_1 * \cdots * x_k$ is the convex hull of these points and hence is the k - dimensional simplex Δ^k with vertices $\{x_0, \cdots, x_k\}$.

Observe that the space X sits naturally as a subspace of the join $X * Y$ as endpoints of line segments,

$$\begin{aligned} \iota : X &\hookrightarrow X * Y \\ x &\to (x, 0, y). \end{aligned}$$

Notice that this formula for the inclusion makes sense and does not depend on the choice of $y \in Y$. There is a similar embedding

$$\begin{aligned} j : Y &\hookrightarrow X * Y \\ y &\to (x, 1, y). \end{aligned}$$

Lemma 5.21. *The inclusions $\iota : X \hookrightarrow X * Y$ and $j : Y \hookrightarrow X * Y$ are null homotopic.*

Proof. Pick a point $y_0 \in Y$. By definition, the embedding $\iota : X \to X * Y$ factors as the composition

$$\begin{aligned} \iota : X &\hookrightarrow X * y_0 \subset X * Y \\ x &\to (x, 0, y_0). \end{aligned}$$

But as observed above, the join $X * y_0$ is the cone on X and hence contractible. This means that ι is null homotopic, as claimed. The fact that $j : Y \hookrightarrow X * Y$ is null homotopic is proved in the same way. □

Now let G be a group and consider the iterated join

$$G^{*(k+1)} = G * G * \cdots * G$$

where there are $k+1$ copies of the group element. This space has a free G action given by the diagonal action

$$g \cdot (g_0, t_1, g_1, \cdots, t_k, g_k) = (gg_0, t_1, gg_1, \cdots, t_k, gg_k).$$

Exercise. 1. Prove that there is a natural G - equivariant map

$$\Delta^k \times G^{k+1} \to G^{*(k+1)}$$

which is a homeomorphism when restricted to $\tilde{\Delta}^k \times G^{k+1}$ where $\tilde{\Delta}^k \subset \Delta^k$ is the interior. Here G acts on $\Delta^k \times G^{k+1}$ trivially on the simplex Δ^k and diagonally on G^{k+1}.
2. Use exercise 1 to prove that if G is a CW complex, the iterated join $G^{*(k+1)}$ has the structure of a G - CW - complex.

Define $\mathcal{J}(G)$ to be the infinite join

$$\mathcal{J}(G) = \lim_{k \to \infty} G^{*(k+1)}$$

where the limit is taken over the embeddings $\iota : G^{*(k+1)} \hookrightarrow G^{*(k+2)}$ Since these embedding maps are G -equivariant, we have an induced G - action on $\mathcal{J}(G)$.

Theorem 5.22. *If G is a CW-complex, the projection map*

$$p : \mathcal{J}(G) \to \mathcal{J}(G)/G$$

is a universal principal G - bundle.

Proof. By the above exercise the space $\mathcal{J}(G)$ has the structure of a G - CW - complex with a free G - action. Therefore by the results of the last section the projection $p : \mathcal{J}(G) \to \mathcal{J}(G)/G$ is a principal G - bundle. To see that $\mathcal{J}(G)$ has trivial homotopy groups, and therefore since it is a CW complex it is contractible, notice that since S^n is compact, any map $\alpha : S^n \to \mathcal{J}(G)$ is homotopic to one that factors through a finite join (that by abuse of notation we still call α), $\alpha : S^n \to G^{*(n+1)} \hookrightarrow \mathcal{J}(G)$. But by the above lemma the inclusion $G^{*(n+1)} \subset \mathcal{J}(G)$ is null homotopic, and hence so is α. Thus $\mathcal{J}(G)$ is contractible. By the results of last section, this means that the projection $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is a universal G - bundle. □

5.4.2 Simplicial spaces and classifying spaces

We therefore now have a universal bundle for every topological group G with a CW-structure. We actually know a fair amount about the geometry of the total space $EG = \mathcal{J}(G)$ which, by the above exercise can be described as the union of simplices, where the k - simplices are parameterized by $k+1$ -tuples of elements of G,

$$EG = \mathcal{J}(G) = \bigcup_k \Delta^k \times G^{k+1} / \sim$$

and so the classifying space can be described by

$$BG = \mathcal{J}(G)/G \cong \bigcup_k \Delta^k \times G^k / \sim$$

It turns out that in these constructions, the simplices are glued together along faces, and these gluings are parameterized by the $k+1$ - product maps $\partial_i : G^{k+2} \to G^{k+1}$ given by multiplying the i^{th} and $(i+1)^{st}$ coordinates.

Having this type of data (parameterizing spaces of simplices as well as gluing maps) is an example of an object known as a *"simplicial set"* which is an important combinatorial object in topology. We now describe this notion in more detail and show how these universal G - bundles and classifying spaces can be viewed in these terms.

Good references for this theory are [38], [103].

The idea of simplicial sets is to provide a combinatorial technique to study cell complexes built out of simplices; i.e simplicial complexes. A simplicial complex X is built out of a union of simplices, glued along faces. Thus if X_n denotes the indexing set for the n - dimensional simplices of X, then we can write

$$X = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

where Δ^n is the standard n - simplex in $\mathbb{R}^n$;

$$\Delta^n = \{(t_1, \cdots, t_n) \in \mathbb{R}^n : 0 \leq t_j \leq 1, \text{ and } \sum_{i=1}^n t_i \leq 1\}.$$

The gluing relation in this union can be encoded by set maps among the X_n's that would tell us for example how to identify an $n-1$ simplex indexed by an element of X_{n-1} with a particular face of an n - simplex indexed by an element of X_n. Thus in principal simplicial complexes can be studied purely combinatorially in terms of the sets X_n and set maps between them. The notion of a *simplicial set* is a generalization of simplicial complex that makes this idea precise.

Definition 5.7. *A simplicial set* X_* is a collection of sets

$$X_n, \quad n \geq 0$$

together with set maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

for $0 \leq i, j \leq n$ called **face** and **degeneracy** maps respectively. These maps are required to satisfy the following compatibility conditions

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{for } i < j$$
$$s_i s_j = s_{j+1} s_i \quad \text{for } i < j$$

and

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{for } i < j \\ 1 & \text{for } i = j,\, j+1 \\ s_j \partial_{i-1} & \text{for } i > j+1 \end{cases}$$

As mentioned above, the maps ∂_i and s_j encode the combinatorial information necessary for gluing the simplices together. To say precisely how this works, consider the following maps between the standard simplices:

$$\delta_i : \Delta^{n-1} \longrightarrow \Delta^n \quad \text{and} \quad \sigma_j : \Delta^{n+1} \longrightarrow \Delta^n$$

for $0 \leq i, j \leq n$ defined by the formulae

$$\delta_i(t_1, \cdots, t_{n-1}) = \begin{cases} (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) & \text{for } i \geq 1 \\ (1 - \sum_{q=1}^{n-1} t_q, t_1, \cdots, t_{n-1}) & \text{for } i = 0 \end{cases}$$

and

$$\sigma_j(t_1, \cdots, t_{n+1}) = \begin{cases} (t_1, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}) & \text{for } i \geq 1 \\ (t_2, \cdots, t_{n+1}) & \text{for } i = 0\ . \end{cases}$$

δ_i includes Δ^{n-1} in Δ^n as the i^{th} face, and σ_j projects, in a linear fashion, Δ^{n+1} onto its j^{th} face.

We can now define the space associated to the simplicial set X_* as follows.

Definition 5.8. *The geometric realization* of a simplicial set X_* is the space

$$\|X_*\| = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

where if $t \in \Delta^{n-1}$ and $x \in X_n$, then

$$(t, \partial_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{n+1}$ and $x \in X_n$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

In the topology of $\|X_*\|$, each X_n is assumed to have the discrete topology, so that $\Delta^n \times X_n$ is a discrete set of n - simplices.

Thus $\|X_*\|$ has one n - simplex for every element of X_n, glued together in a way determined by the face and degeneracy maps.

Example. Consider the simplicial set $\mathbf{S}_*$ defined as follows. The set of n - simplices is given by

$$\mathbf{S}_n = \mathbb{Z}/(n+1), \text{ generated by an element } \tau_n.$$

The face maps are given by

$$\partial_i(\tau_n^r) = \begin{cases} \tau_{n-1}^r & \text{if } r \leq i \leq n \\ \tau_{n-1}^{r-1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

The degeneracies are given by

$$s_i(\tau_n^r) = \begin{cases} \tau_{n+1}^r & \text{if } r \leq i \leq n \\ \tau_{n+1}^{r+1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

Notice that there is one zero simplex, two one simplices, one of them the image of the degeneracy $s_0 : \mathbf{S}_0 \longrightarrow \mathbf{S}_1$, and the other nondegenerate (i.e not in the image of a degeneracy map). Notice also that all simplices in dimensions larger than one are in the image of degeneracy maps. Hence we have that the geometric realization

$$\|\mathbf{S}_*\| = \Delta^1/0 \sim 1 = S^1.$$

Let X_* be any simplicial set. There is a particularly nice and explicit way for computing the homology of the geometric realization, $H_*(\|X_*\|)$.

Consider the following chain complex. Define $C_n(X_*)$ to be the free abelian group generated by the set of n - simplices X_n. Define the homomorphism

$$d_n : C_n(X_*) \longrightarrow C_{n-1}(X_*)$$

by the formula

$$d_n([x]) = \sum_{i=0}^{n} (-1)^i \partial_i([x])$$

where $x \in X_n$.

Proposition 5.23. *The homology of the geometric realization $H_*(\|X_*\|)$ is the homology of the chain complex*

$$\longrightarrow \cdots \xrightarrow{d_{n+1}} C_n(X_*) \xrightarrow{d_n} C_{n-1}(X_*) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} C_0(X_*).$$

Proof. It is straightforward to check that the geometric realization $\|X_*\|$ is a CW - complex and that this is the associated cellular chain complex. $\square$

Besides being useful computationally, the following result establishes the fact that all CW complexes can be studied simplicially.

Theorem 5.24. *Every CW complex has the homotopy type of the geometric realization of a simplicial set.*

Proof. Let X be a CW complex. Define the singular simplicial set of X , $\mathcal{S}(X)_*$ as follows. The n simplices $\mathcal{S}(X)_n$ is the set of singular n - simplices,

$$\mathcal{S}(X)_n = \{c : \Delta^n \longrightarrow X\}.$$

The face and degeneracy maps are defined by

$$\partial_i(c) = c \circ \delta_i : \Delta^{n-1} \longrightarrow \Delta^n \longrightarrow X$$

and

$$s_j(c) = c \circ \sigma_i : \Delta^{n+1} \longrightarrow \Delta^n \longrightarrow X.$$

Notice that the associated chain complex to $\mathcal{S}(X)_*$ as in 5.23 is the singular chain complex of the space X. Hence by 5.23 we have that

$$H_*(\|\mathcal{S}(X)\|) \cong H_*(X).$$

This isomorphism is actually realized by a map of spaces

$$E : \|\mathcal{S}(X)_*\| \longrightarrow X$$

defined by the natural evaluation maps

$$\Delta^n \times \mathcal{S}(X)_n \longrightarrow X$$

given by

$$(t, c) \longrightarrow c(t).$$

It is straightforward to check that the map E does induce an isomorphism in homology. In fact it induces an isomorphism in homotopy groups. We will not prove this here; it is more technical and we refer the reader to [103] for details. Note that it follows from the homological isomorphism by the Hurewicz theorem if we knew that X was simply connected. As we've mentioned before, a map between spaces that induces an isomorphism in homotopy groups is called a *weak homotopy equivalence*. Thus *any* space is weakly homotopy equivalent to a CW - complex (i.e the geometric realization of its singular simplicial set). But by the Whitehead theorem, two CW complexes that are weakly homotopy equivalent are homotopy equivalent. Hence X and $\|\mathcal{S}(X)_*\|$ are homotopy equivalent. □

We next observe that the notion of simplicial set can be generalized as follows. We say that X_* is a **simplicial space** if it is a simplicial set (i.e it satisfies definition 5.7) with the extra data that the sets X_n have the structure of a compactly-generated topological space, and the face and degeneracy maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

are continuous maps. The definition of the geometric realization of a simplicial space X_*, $\|X_*\|$, is the same as in 5.8 with the proviso that the topology of each $\Delta^n \times X_n$ is the product topology. Notice that since the "set of n - simplices" X_n is actually a space, it is not necessarily true that $\|X_*\|$ is a CW complex. However if in fact each X_n is a CW complex and the face and degeneracy maps are cellular, then $\|X_*\|$ does have a natural CW structure induced by the product CW - structures on $\Delta^n \times X_n$.

Notice that this simplicial notion generalizes even further. For example a **simplicial group** would be defined similarly, where each X_n would be a group and the face and degeneracy maps are group homomorphisms. Simplicial vector spaces, modules, etc. are defined similarly. The categorical nature of these definitions should by now be coming clear. Indeed more generally one can define a **simplicial object in a category** $\mathcal{C}$ using the above definition where now the X_n's are assumed to be objects in the category and the face and degenarcies are assumed to be morphisms. If the category $\mathcal{C}$ is a subcategory of the category of compactly-generated topological spaces, then geometric realizations can be defined as in Definition 5.8. For example the geometric realization of a simplicial (abelian) group turns out to be a topological (abelian) group. (Try to verify this for yourself!)

The notion of a simplicial object in a category $\mathcal{C}$ can be formalized somewhat in the following way.

Let Δ denote the *simplex category*. The objects of Δ are nonempty, linearly ordered sets of the form $[n] = \{0, 1, \cdots, n\}$. A morphism $\phi : [n] \to [m]$ is a non-strictly order-preserving set map. Important examples of such morphisms are "*coface* maps" δ_i, $i = 0, \cdots, n : [n-1] \to [n]$, where δ_i is defined to be the unique injective, order preserving set map from $[n-1]$ to $[n]$ whose image does not contain i. There are also "*codegeneracy* maps" σ_j, $j = 0, \cdots, n : [n+1] \to [n]$, where σ_j is the unique surjective order preserving set map $[n+1] \to [n]$ such that j is in the image of two elements.

We can then define a simplicial object $\mathbf{X}$ in a category $\mathcal{C}$ to be a contravariant functor

$$\mathbf{X} : \Delta \to \mathcal{C}.$$

Given such a simplicial object $\mathbf{X}$, the p simplicies are given by $X_p = \mathbf{X}([\mathrm{p}])$, and the face and degeneracy maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

are given by $\mathbf{X}(\delta_{\mathrm{i}})$ and $\mathbf{X}(\sigma_{\mathrm{j}})$, respectively.

Exercise: Show that he two definitions of a simplicial object in a category $\mathcal{C}$ given above are equivalent.

If $\mathcal{C}$ is a subcategory of the category of compactly-generated topological spaces, then the *geometric realization*, $\|\mathbf{X}\|$ of a simplicial object in $\mathcal{C}$, can be defined as in Definition 5.8. Observe that $\|\mathbf{X}\|$ is then object in $\mathcal{C}$.

We now use this simplicial theory to construct universal principal G - bundles and classifying spaces.

Let G be a topological group and let $\mathcal{E}G_*$ be the simplicial space defined as follows. The space of n - simplices is given by the $n+1$ - fold cartesian product

$$\mathcal{E}G_n = G^{n+1}.$$

The face maps $\partial_i : G^{n+1} \longrightarrow G^n$ are given by the formula

$$\partial_i(g_0, \cdots, g_n) = (g_0, \cdots, \hat{g}_i, \cdots, g_n).$$

The degeneracy maps $s_j : G^{n+1} \longrightarrow G^{n+2}$ are given by the formula

$$s_j(g_0, \cdots, g_n) = (g_0, \cdots, g_j, g_j, \cdots, g_n).$$

Exercise. Show that the geometric realization $\|\mathcal{E}G_*\|$ is weakly contractible (i.e has the weak homotopy type of a point).

Hint. Let $\|\mathcal{E}G_*\|^{(n)}$ be the n^{th} - skeleton,

$$\|\mathcal{E}G_*\|^{(n)} = \bigcup_{p=0}^{n} \Delta^p \times G^{p+1}.$$

Then show that the inclusion of one skeleton in the next
$\|\mathcal{E}G_*\|^{(n)} \hookrightarrow \|\mathcal{E}G_*\|^{(n+1)}$ is null - homotopic. One way of doing this is to establish a homeomorphism between $\|\mathcal{E}G_*\|^{(n)}$ and n - fold join $G * \cdots * G$.

Notice that the group G acts freely on the right of $\|\mathcal{E}G_*\|$ by the rule

$$\begin{aligned} \|\mathcal{E}G_*\| \times G = \left(\bigcup_{p \geq 0} \Delta^p \times G^{p+1} \right) \times G &\longrightarrow \|\mathcal{E}G_*\| \\ (t; (g_0, \cdots, g_p)) \times g &\longrightarrow (t; (g_0 g, \cdots, g_p g)) \,. \end{aligned} \tag{5.2}$$

Thus we can define $EG = \|\mathcal{E}G_*\|$. The projection map

$$p : EG \to EG/G = BG$$

is principal G-bundles whose total space is weakly contractible. Therefore it is universal principal G - bundle.

This description gives the classfiying space BG an induced simplicial structure described as follows.

Let $\mathcal{B}G_*$ be the simplicial space whose n - simplices are the cartesian product

$$\mathcal{B}G_n = G^n. \tag{5.3}$$

The face and degeneracy maps are given by

$$\partial_i(g_1, \cdots, g_n) = \begin{cases} (g_2, \cdots, g_n) & \text{for } i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots g_n) & \text{for } 1 \leq i \leq n-1 \\ (g_1, \cdots, g_{n-1}) & \text{for } i = n. \end{cases}$$

The degeneracy maps are given by

$$s_j(g_1, \cdots, g_n) = \begin{cases} (1, g_1, \cdots, g_n) & \text{for } j = 0 \\ (g_1, \cdots g_j, 1, g_{j+1}, \cdots, g_n) & \text{for } j \geq 1. \end{cases}$$

The simplicial projection map

$$p : \mathcal{E}G_* \longrightarrow \mathcal{B}G_*$$

defined on the level of n - simplicies by

$$p(g_0, \cdots, g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \cdots, g_{n-1} g_n^{-1})$$

is easily checked to commute with face and degeneracy maps and so induces a map on the level of geometric realizations

$$p : EG = \|\mathcal{E}G_*\| \longrightarrow \|\mathcal{B}G_*\|$$

which induces a homeomorphism

$$BG = EG/G \xrightarrow{\cong} \|\mathcal{B}G_*\|.$$

Thus for any topological group this construction gives a simplicial space model for its classifying space. This is referred to as the **simplicial bar construction.** Notice that when G is discrete the bar construction is a CW complex for the classifying space $BG = K(G, 1)$ and 5.23 gives a particularly nice complex for computing its homology. (The homology of a $K(G, 1)$ is referred to as the homology of the group G.)

The n - chains are the group ring

$$C_n(\mathcal{B}G_*) = \mathbb{Z}[G^n] \cong \mathbb{Z}[G]^{\otimes n}$$

and the boundary homomorphisms

$$d_n : \mathbb{Z}[G]^{\otimes n} \longrightarrow \mathbb{Z}[G]^{\otimes n-1}$$

are given by

$$\begin{aligned} d_n(a_1 \otimes \cdots \otimes a_n) = (a_2 \otimes \cdots \otimes a_n) + &\sum_{i=1}^{n-1} (-1)^i (a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^n (a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

This complex is called the **bar complex** for computing the homology of a group and was discovered by Eilenberg and MacLane in the mid 1950's.

5.4.3 Topological categories, their classifying spaces, and Quillen's Theorems A and B

We end this chapter by discussing a generalization of the notion of the classifying space of a topological group. Notice that the bar construction of the classifying space of a group did not use the full group structure. It only used the existence of an associative multiplication with unit. In particular it did not use the existence of inverse. So one can study the classifying space BA of a monoid A. Indeed one can define the classifying space $B\mathcal{C}$ of any "small category" $\mathcal{C}$ in a similar way. (A "small" category is one whose objects and morphisms are sets.) These are important construction in algebraic - K - theory as well as homotopy theory, and we describe some basic properties of this construction here. The main reference for this material is Quillen's groundbreaking work [129].

Throughout this section $\mathcal{C}$ will be a topological category, meaning it is a small category, and the sets of objects and morphisms are topologized. Furthermore, the source and target maps,

$$\sigma_{\mathcal{C}} : Mor_{\mathcal{C}} \to Ob_{\mathcal{C}}. \quad \text{and} \quad \tau_{\mathcal{C}} : Mor_{\mathcal{C}} \to Ob_{\mathcal{C}}$$

are continuous. Of course important examples of topological categories are *discrete* categories where the topologies on the objects and morphisms are all discrete.

As suggested above, one can associate to a topological category $\mathcal{C}$ a simplicial space, called its *nerve*, denoted $\mathcal{N}\mathcal{C}$. This is a generalization of the bar construction of a group. The space of 0-simplices of $\mathcal{N}\mathcal{C}$ is the space of objects, $\mathcal{N}_0\mathcal{C} = Ob_{\mathcal{C}}$. For $p \geq 1$ the space of p-simplices, $\mathcal{N}_p\mathcal{C}$ is given by the space of p-tuples of composable morphisms

$$X_0 \xrightarrow{\mu_1} X_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_p} X_p \tag{5.4}$$

topologized as a subspace of the p-fold cartesian product of $Mor_{\mathcal{C}}$. The face and degeneracy maps are defined as they are for the simplicial classifying space of a group (5.3). Namely, the i^{th} face of this simplex is obtained by deleting the object X_i and composing the morphisms $\mu_{i+1} \circ \mu_i : X_{i-1} \to X_{i+1}$. The i^{th} degeneracy of this simplex is obtained by inserting the identity morphism $id : X_i \to X_i$ to obtain a $(p+1)$-simplex.

Definition 5.9. *The classifying space of the topological category $\mathcal{C}$, $B\mathcal{C}$, is the geometric realizaiton of the nerve*

$$B\mathcal{C} = |\mathcal{N}\mathcal{C}|.$$

Notice that from this perspective, a topological monoid can be thought of as a topological category in which there is only one object, and a topological group is a topological monoid in which every morphism is invertible. To complete this collection of ideas, a topological *groupoid* is a topological category in which every morphism is invertible.

Given a continuous functor between topological categories (i.e a functor that is continuous on both object and morphism spaces), $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$, one gets an induced map of classifying spaces,

$$B\mathcal{F} : B\mathcal{C}_1 \to B\mathcal{C}_2.$$

Definition 5.10. *We say that a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is a fibration if the induced map of classifying spaces*

$$B\mathcal{F} : B\mathcal{C}_1 \to B\mathcal{C}_2$$

is a fibration. Similarly, we say that such a functor is an "equivalence" (sometimes called a "Quillen equivalence") if the induced map classifying spaces, $B\mathcal{F}$ is a weak homotopy equivalence.

It is natural to ask what properties of a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ assure that it will be a fibration or an equivalence. Quillen began this study in [129]. In this section we will describe a few of these results. These results are quite useful in modern homotopy theory.

One observation, due to Milnor [120] is that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two topological categories, then the canonical map

$$B(\mathcal{C}_1 \times \mathcal{C}_2) \to B\mathcal{C}_1 \times B\mathcal{C}_2$$

is a homeomorphism, assuming the product is given the compactly generated topology. (Try to prove this yourself!) This quickly implies the following, which was originally observed by Segal [135].

Proposition 5.25. *A natural transformation $\Theta : \mathcal{F} \to \mathcal{G}$ of functors between topological categories $\mathcal{C}_1$ and $\mathcal{C}_2$ induces a homotopy*

$$B\Theta : B\mathcal{C}_1 \times I \to B\mathcal{C}_2$$

between $B\mathcal{F}$ and $B\mathcal{G}$.

Proof. Consider the category $\mathcal{I}$ defined by the ordered set $\{0 < 1\}$. In otherwords, $\mathcal{I}$ has two objects, 0 and 1, and a unique nonidentity morphism $0 \to 1$. Observe that the classifying space $B\mathcal{I}$ is homeomorphic to the interval, $I = [0,1]$. The triple $(\mathcal{F}, \mathcal{G}, \Theta)$ can be viewed as a functor

$$\mathcal{C}_1 \times \mathcal{I} \to \mathcal{C}_2$$

and therefore gives, upon passage to classifying spaces, a homotopy

$$B\mathcal{C}_1 \times I \to B\mathcal{C}_2.$$

□

Now recall that a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ has a left adjoint functor $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_1$ if there is a natural transformation

$$\Theta : \mathcal{G} \circ \mathcal{F} \to id : \mathcal{C}_1 \to \mathcal{C}_1 \quad \text{and} \quad \Psi : id \to \mathcal{F} \circ \mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_2.$$

A functor $\mathcal{F}$ having a right adjoint functor $\mathcal{G}$ is defined in terms of the existence of a natural transformations $\mathcal{F} \circ \mathcal{G} \to id : \mathcal{C}_2 \to \mathcal{C}_2$ and $id \to \mathcal{G} \circ \mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_1$. The following is then an immediate corollary.

Corollary 5.26. *If a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ has either a left or right adjoint, then $\mathcal{F}$ is an equivalence.*

An object X_0 of a category $\mathcal{C}$ is said to be *initial*, if given any other object X, there is a unique morphism $X_0 \to X$. Similarly, an object X_1 is *final* if, given any other object X there is a unique morphism $X \to X_1$. We now have the following very useful corollary that allows us to recognize contractible classifying spaces.

Corollary 5.27. *A topological category having either an initial or final object has a contractible classifying space. (In this case we say the category is contractible.)*

Proof. In these cases the unique functor from the category to the category $\{0\}$ consisting of one object and no nonidentity morphisms, has an adjoint. The result follows. □

As an example of a topological category with an initial object we define the "based path category".

Definition 5.11. *Let X be a connected space with basepoint $x_0 \in X$, Let $\mathcal{P}_0 X$ be the "based path category" of X. The objects of $\mathcal{P}X$ are paths starting at x_0, $\alpha : [0,r] \to X$ for some $r \geq 0$ such that $\alpha(0) = x_0$. A morphism from $\alpha_1 : [0,r] \to X$ to $\alpha_2 : [0,s] \to X$ is a path $\phi : [r,s] \to X$ such that the concantenation of paths, $\phi \cdot \alpha_1 : [0,s] \to X$ is equal to the path α_2. The concantenation $\phi \cdot \alpha_1$ is defined by*

$$\phi \cdot \alpha_1(t) = \begin{cases} \alpha_1(t) \text{ if } t \in [0,r], \\ \phi(t) \text{ if } t \in [r,s]. \end{cases}$$

This category is topologized in the obvious way.

Exercise. Show that the constant path $\epsilon : [0,0] \to X$ at x_0 is an initial object of $\mathcal{P}_0 X$. We therefore may conclude that the classifying space, $B(\mathcal{P}X)$ is contractible. The reader may want to compare this path category and its classifying space to the path space PX defined in Proposition 4.5.

Now let $\mathcal{C}$ and $\mathcal{D}$ be topological categories, and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a continuous functor between them. We will now describe a slight generalization of a result of Quillen [129] that can identify the homotopy type of the homotopy fiber of the induced map on classifying spaces $B\mathcal{F} : B\mathcal{C} \to B\mathcal{D}$.

Definition 5.12. *(1). Let $Y \in \mathcal{D}$ be an object. The "under category" $\mathcal{F} \searrow Y$ is the topological category whose objects are pairs (X, v) where X is an object of $\mathcal{C}$ and v is a morphism in $\mathcal{D}$ from Y to $\mathcal{F}(X)$. A morphism from (X, v) to (X', v') is a morphism $\alpha : X \to X'$ in $\mathcal{C}$ such that $\mathcal{F}(\alpha) \circ v = v'$. Notice that a morphism in $\mathcal{D}$ from Y to Y' naturally induces a functor $\mathcal{F} \searrow Y' \to \mathcal{F} \searrow Y$.*

(2) The "over category" $\mathcal{F} \swarrow Y$ is defined similarly. Its objects are pairs (X, v) where X is an object in $\mathcal{C}$ and v is a morphism in $\mathcal{D}$ from $\mathcal{F}(X)$ to Y. A morphism from (X, v) to (X', v') is a morphism $\alpha : X \to X'$ in $\mathcal{C}$ such that $v' \circ \mathcal{F}(\alpha) = v$. Notice that a morphism in $\mathcal{D}$ from Y to Y' induces a functor $\mathcal{F} \swarrow Y \to \mathcal{F} \swarrow Y'$.

We have the following immediate result.

Lemma 5.28. *Consider the identity functor of a topological category, $id : \mathcal{C} \to \mathcal{C}$. Then for any object Y in $\mathcal{C}$, (Y, id_Y) is an initial object in the undercategory $id \searrow Y$. Here $id_Y : Y \to Y$ is the identity morphism of Y. Similarly, for any object Y in $\mathcal{C}$, then in the overcategory $id \swarrow Y$ the object (Y, id_Y) is a final object. Therefore the classifying spaces $B(id \searrow Y)$ and $B(id \swarrow Y)$ are contractible.*

Now given a continuous functor between topological categories $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, and $Y \in \mathcal{D}$ an object, notice that we have natural functors $j : \mathcal{F} \swarrow Y \to \mathcal{C}$, $\mathcal{F}' : \mathcal{F} \swarrow Y \to id_{\mathcal{D}} \swarrow Y$ and $j' : id_{\mathcal{D}} \swarrow \mathcal{D} \to \mathcal{D}$ defined on objects as follows.

- $j(X, v) = X$
- $\mathcal{F}'(X, v) = \mathcal{F}(X)$
- $j'(Y, v) = Y$

To describe the functors on the level of morphisms suppose $\alpha : (X, v) \to (X', v')$ is a morphism in $\mathcal{F} \swarrow Y$, then $j(\alpha) : j(X, v) = X \to j(X', v') = X'$ is simply given by $\alpha : X \to X'$. $\mathcal{F}'$ and j' are defined similarly on morphisms. Now notice that the following square of categories and functors commutes for every object Y in $\mathcal{D}$:

$$\begin{array}{ccc} \mathcal{F} \swarrow Y & \xrightarrow{j} & \mathcal{C} \\ {\scriptstyle \mathcal{F}'}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\ id_{\mathcal{D}} \swarrow Y & \xrightarrow[j']{} & \mathcal{D} \end{array}$$

The following is a statement of Quillen's famous "Theorem B" in [129]. For it we need the following definition.

Definition 5.13. *Consider a commutative square of maps between spaces,*

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & .Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow[f]{} & W. \end{array}$$

Let $hF(p)$ and $hF(q)$ be the homotopy fibers of the maps p and q respectively. There is an induced map $\tilde{f} : hF(p) \to hF(q)$.

The above square is said to be "homotopy cartesian" if the induced map of homotopy fibers

$$\tilde{f} : hF(p) \to hF(q)$$

is a weak homotopy equivalence.

Theorem 5.29. *(Quillen [129]) Suppose $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a continuous functor of topological categories. Suppose that the following conditions are met:*

1. *The target maps $\tau_{\mathcal{C}} : Mor_{\mathcal{C}} \to Ob_{\mathcal{C}}$ and $\tau_{\mathcal{D}} : Mor_{\mathcal{D}} \to Ob_{\mathcal{D}}$ are fibrations.*
2. *The restriction of the functor to object spaces, $\mathcal{F} : Ob_{\mathcal{C}} \to Ob_{\mathcal{D}}$ is a fibration.*

3. For any morphism α in $\mathcal{D}$ between objects Y_1 and Y_2, the induced functor

$$\mathcal{F} \swarrow Y_1 \to \mathcal{F} \swarrow Y_2$$

is an equivalence (i.e induces a weak homotopy equivalence of classifying spaces).

Then the induced square of maps classifying spaces,

$$\begin{array}{ccc} B(\mathcal{F} \swarrow Y) & \xrightarrow{Bj} & B\mathcal{C} \\ {\scriptstyle B\mathcal{F}'}\downarrow & & \downarrow{\scriptstyle B\mathcal{F}} \\ B(id_{\mathcal{D}} \swarrow Y) & \xrightarrow[j']{} & B\mathcal{D} \end{array}$$

is homotopy cartesian.

Remark.

Quillen's Theorem B assumes that the categories are discrete, and it therefore does not need the first and second hypotheses stated above. His "Theorem A" is a special case, when one assumes that the undercategories are all contractible. In this case one can conclude that the functor $\mathcal{F}$ is a (Quillen) equivalence in that it induces a weak homotopy equivalence of classifying spaces. The reader is encouraged to read Quillen's beautiful proofs of his Theorems A and B.

The result as stated here in the setting of topological categories has been known for many years, but has been difficult to find explicitly in the literature. A writeup in this form can be found in [58], [59].

We now have the following immediate corollary to Quillen's Theorem B.

Corollary 5.30. *Assuming the hypotheses of Theorem 5.29, then the homotopy fiber of*

$$B\mathcal{F} : B\mathcal{C} \to B\mathcal{D}$$

is weakly homotopy equivalent to $B(\mathcal{F} \swarrow Y)$ for any object Y of $\mathcal{D}$.

Proof. This follows from Theorem 5.29 immediately because by Lemma 5.28 above, $B(id_{\mathcal{D}} \swarrow Y)$ is contractible. □

We now consider the following very useful application of this theorem.

Definition 5.14. *Let X be a connected topological space with basepoint $x_0 \in X$. Let X^I be the (unbased) path category of X. It is a topological category whose objects are points $x \in X$, topologized using the topology of X. The space of morphisms from x_1 to x_2 is the space of paths $\alpha : [0, r] \to X$ for some $r > 0$, with $\alpha(0) = x_1$ and $\alpha(1) = x_2$. Composition of morphisms is concantenation of paths.*

Corollary 5.31. *There is a weak homotopy equivalence, $B(X^I) \simeq X$.*

Proof. Let $\mathcal{P}_0 X$ be the based path category considered above (5.11). We consider the "evaluation functor"

$$\mathcal{E}v : \mathcal{P}_0 X \to X^I$$

defined on an object $\alpha : [0, r] \to X$ to be $\mathcal{E}v(\alpha) = \alpha(r) \in X$. Recall that a morphism between objects $\alpha_1 : [0, r] \to X$ and $\alpha_2 : [0, s] \to X$ is a path $\gamma : [r, s] \to X$ such that the concantenation $\gamma \cdot \alpha_1 = \alpha_2 : [0, s] \to X$. We define the value of the functor $\mathcal{E}v$ on the morphism γ to equal γ, viewed as a path between $\alpha_1(r)$ and $\alpha_2(s)$. Clearly $\mathcal{E}v$ is a well-defined continuous functor.

We leave it to the reader to check that the functor $\mathcal{E}v$ satisfies the first two hypotheses of Theorem 5.29. We now check that it satisfies the third hypothesis. Namely, we need to check that for any morphism γ between objects x_1 and x_2 in the category X^I, the induced functor on over categories, which by abuse of notation we also call γ,

$$\gamma : \mathcal{E}v \swarrow x_1 \to \mathcal{E}v \swarrow x_2$$

is a Quillen equivalence. Now γ is a path $\gamma : [r, s] \to X$ between x_1 and x_2. Consider the "inverse" path $\gamma^{-1} : [s, 2s - r] \to X$ given by

$$\gamma^{-1}(t) = \gamma(2s - t).$$

Since γ^{-1} is a morphism between $\gamma^{-1}(s) = \gamma(s) = x_2$ and $\gamma^{-1}(2s - r) = \gamma(r) = x_1$, it induces a functor of over categories,

$$\gamma^{-1} : \mathcal{E}v \swarrow x_2 \to \mathcal{E}v \swarrow x_1.$$

We leave it to the reader to check that the composition functors

$$\gamma^{-1} \circ \gamma : \mathcal{E}v \swarrow x_1 \to \mathcal{E}v \swarrow x_1 \quad \text{and} \quad \gamma \circ \gamma^{-1} : \mathcal{E}v \swarrow x_2 \to \mathcal{E}v \swarrow x_2$$

induce maps between classifying spaces that are homotopic to the identity.

Thus $\mathcal{E}v : \mathcal{P}_0 X \to X^I$ satisfies the hypotheses of Theorem 5.29, and we conclude that the induced map of classifying spaces

$$B\mathcal{E}v : B\mathcal{P}_0 X \to BX^I$$

has homotopy fiber weakly homotopy equivaent to $B(\mathcal{E}v \swarrow x)$ for any $x \in X$.

We consider the homotopy type of $B(\mathcal{E}v \swarrow x_0)$ where $x_0 \in X$ is the basepoint. Notice that an object in $\mathcal{E}v \swarrow x_0$ is a pair (α, β), where $\alpha : [0, r] \to X$ is a path that starts at x_0 (i.e $\alpha(0) = x_0$), and β is a path $\beta : [r, s] \to X$ with $\beta(r) = \alpha(r)$ and $\beta(s) = x_0$. By concantinating these two paths, we may view this object as a loop $\gamma : [0, s] \to X$ starting and ending at x_0. The only extra data in this object is the number r with $0 \leq r \leq s$. So an equivalent formulation of the category $\mathcal{E}v \swarrow x_0$ is the category whose objects are triples (γ, r, s), where $\gamma : [0, s] \to X$ is a loop at x_0 (i.e $\gamma(0) = \gamma(s) = x_0$) and r is a number with $0 \leq r \leq s$.

From this point of view, the morphisms can be described as follows. There is a unique morphism $(\gamma_1, r_1, s_1) \to (\gamma_2, r_2, s_2)$ if and only if

1. $r_1 \leq r_2$, and
2. $s_1 = s_2$ and $\gamma_2 = \gamma_1 : [0, s_1] \to X$.

There are no other morphisms. In other words, the category $\mathcal{E}v \swarrow x_0$ is a partially ordered topological space, with

$$(\gamma_1, r_1, s_1) \leq (\gamma_2, r_2, s_2)$$

if and only if the above conditions are satisfied.

From this description of the over category $\mathcal{E}v \swarrow x_0$ it is straightforward to complete the following exercise.

Exercise. Use this description of the category $\mathcal{E}v \swarrow x_0$ to show that the classifying space $B(\mathcal{E}v \swarrow x_0)$ has the weak homotopy type of the based loop space ΩX.

With this exercise we now know that the homotopy fiber of the map $B\mathcal{E}v : B\mathcal{P}_0 X \to BX^I$ is the based loop space ΩX. Since the total space of this fibration, $B\mathcal{P}_0 X$ is contractible (see the exercise after Definition 5.11), this implies the base space BX^I must be weakly homotopy equivalent to X. □

5.5 Some Applications

In a sense, much of what we will study in the next chapter are applications of the classification theorem for principal bundles. In this section we describe a few immediate applications.

5.5.1 Line bundles over projective spaces

By the classification theorem we know that the set of isomorphism classes of complex line bundles over the projective space $\mathbb{CP}^n$ is given by

$$\begin{aligned} Vect^1(\mathbb{CP}^n) \cong Prin_{GL(1,\mathbb{C})}(\mathbb{CP}^n) \cong Prin_{U(1)}(\mathbb{CP}^n) &\cong [\mathbb{CP}^n, BU(1)] = [\mathbb{CP}^n, \mathbb{CP}^\infty] \\ &= [\mathbb{CP}^n, K(\mathbb{Z}, 2)] \cong H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z} \end{aligned}$$

Theorem 5.32. *Under the above isomorphism,*

$$Vect^1(\mathbb{CP}^n) \cong \mathbb{Z}$$

the n - fold tensor product of the universal line bundle $\gamma_1^{\otimes n}$ corresponds to the integer $n \geq 0$.

Proof. The classification theorem says that every line bundle ζ over $\mathbb{CP}^n$ is the pull back of the universal line bundle via a map $f_\zeta : \mathbb{CP}^n \to \mathbb{CP}^\infty$. That is,

$$\zeta \cong f_\zeta^*(\gamma_1).$$

The cohomology class corresponding to ζ, *the first Chern class* $c_1(\zeta)$, is given by

$$c_1(\zeta) = f_\zeta^*(c) \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$$

where $c \in H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ is the generator. Clearly $\iota^*(c) \in H^2(\mathbb{CP}^n)$ is the generator, where $\iota : \mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ is natural inclusion. But $\iota^*(\gamma_1) = \gamma_1 \in Vect^1(\mathbb{CP}^n)$. Thus $\gamma_1 \in Vect^1(\mathbb{CP}^n) \cong \mathbb{Z}$ corresponds to the generator.

To see the effect of taking tensor products, consider the following "tensor product map"

$$BU(1) \times \cdots \times BU(1) \xrightarrow{\otimes} BU(1)$$

defined to be the unique map (up to homotopy) that classifies the external tensor product $\gamma_1 \otimes \cdots \otimes \gamma_1$ over $BU(1) \times \cdots \times BU(1)$. Using $\mathbb{CP}^\infty \cong Gr_1(\mathbb{C}^\infty)$ as our model for $BU(1)$, this tensor product map is given by taking k lines $\ell_1, \cdots, \ell_k$ in $\mathbb{C}^\infty$ and considering the tensor product line

$$\ell_1 \otimes \cdots \otimes \ell_k \subset \mathbb{C}^\infty \otimes \cdots \otimes \mathbb{C}^\infty \xrightarrow[\psi]{\cong} \mathbb{C}^\infty$$

where $\psi : \mathbb{C}^\infty \otimes \cdots \otimes \mathbb{C}^\infty \cong \mathbb{C}^\infty$ is a fixed isomorphism. The induced map

$$\tau : \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \cong K(\mathbb{Z}, 2)$$

is determined up to homotopy by its effect on H^2. Clearly the restriction to each factor is the identity map and so

$$\tau^*(c) = c_1 + \cdots + c_k \in H^2(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty) = H^2(\mathbb{CP}^\infty) \oplus \cdots \oplus H^2(\mathbb{CP}^\infty) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

where c_i denotes the generator of H^2 of the i^{th} factor in the product. Therefore the composition

$$t_k : \mathbb{CP}^\infty \xrightarrow{\Delta} \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \xrightarrow{\tau} \mathbb{CP}^\infty$$

has the property that $t_k^*(c) = kc \in H^2(\mathbb{CP}^\infty)$. But also we have that on the bundle level,

$$t_k^*(\gamma_1) = \gamma_1^{\otimes k} \in Vect^1(\mathbb{CP}^\infty).$$

The theorem now follows. □

We have a similar result for real line vector bundles over real projective spaces.

Theorem 5.33. *The only nontrivial real line bundle over* $\mathbb{RP}^n$ *is the canonical line bundle* γ_1.

Proof. We know that γ_1 is nontrivial because its restriction to $S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^n$ is the Moebeus strip line bundle, which is nonorientable, and hence nontrivial. On the other hand, by the classification theorem,

$$Vect^1_{\mathbb{R}}(\mathbb{RP}^n) \cong [\mathbb{RP}^n, BGL(1,\mathbb{R})] = [\mathbb{RP}^n, \mathbb{RP}^\infty] = [\mathbb{RP}^n, K(\mathbb{Z}_2, 1)] \\ \cong H^1(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Hence there is only one nontrivial line bundle over $\mathbb{RP}^n$. □

.

5.5.2 Structures on bundles and homotopy liftings

The following theorem is a direct consequence of the classification theorem. We leave its proof as an exercise.

Theorem 5.34. . *Let $p : E \to B$ be a principal G - bundle classified by a map $f : B \to BG$. Let $H < G$ be a subgroup. By the naturality of the construction of classifying spaces, this inclusion induces a map (well defined up to homotopy) $\iota : BH \to BG$. Then the bundle $p : E \to B$ has an H - structure (i.e a reduction of its structure group to H) if and only if there is a map*

$$\tilde{f} : B \to BH$$

so that the composition

$$B \xrightarrow{\tilde{f}} BH \xrightarrow{\iota} BG$$

is homotopic to $f : B \to BG$. In particular if $\tilde{p} : \tilde{E} \to B$ is the principal H - bundle classified by $\tilde{f}$, then there is an isomorphism of principal G bundles,

$$\tilde{E} \times_H G \cong E.$$

The map $\tilde{f} : B \to BH$ is called a “lifting” of the classifying map $f : B \to BG$. It is called a lifting because, as we saw at the end of the last section, the map $\iota : BH \to BG$ can be viewed as a fiber bundle, by taking our model for BH to be $BH = EG/H$. Then ι is the projection for the fiber bundle

$$G/H \to EG/H = BH \xrightarrow{\iota} EG/G = BG.$$

This bundle structure will allow us to analyze in detail what the obstructions are to obtaining a lift $\tilde{f}$ of a classifying map $f : B \to BG$. We will study this is Chapter 6.

Examples.

- An orientation of a bundle classified by a map $f : B \to BO(k)$ is a lifting $\tilde{f} : B \to BSO(k)$. Notice that the map $\iota : BSO(k) \to BO(k)$ can be viewed as a two - fold covering map
$$\mathbb{Z}_2 = O(k)/SO(k) \to BSO(k) \xrightarrow{\iota} BO(k).$$
- An almost complex structure on a bundle classified by a map $f : B \to BO(2n)$ is a lifting $\tilde{f} : B \to BU(n)$. Notice we have a bundle
$$O(2n)/U(n) \to BU(n) \to BO(2n).$$

The following example will be particularly useful in the next chapter when we define characteristic classes and do calculations with them.

Theorem 5.35. *A complex bundle vector bundle ζ classified by a map $f : B \to BU(n)$ has a nowhere zero section if and only if f has a lifting $\tilde{f} : B \to BU(n-1)$. Similarly a real vector bundle η classified by a map $f : B \to BO(n)$ has a nowhere zero section if and only if f has a lifting $\tilde{f} : B \to BO(n-1)$. Notice we have the following bundles:*
$$S^{2n-1} = U(n)/U(n-1) \to BU(n-1) \to BU(n)$$
and
$$S^{n-1} = O(n)/O(n-1) \to BO(n-1) \to BO(n).$$

This theorem says that $BU(n-1)$ forms a sphere bundle (S^{2n-1}) over $BU(n)$, and similarly, $BO(n-1)$ forms a S^{n-1} - bundle over $BO(n)$. We identify these sphere bundles as follows.

Corollary 5.36. *The sphere bundles*
$$S^{2n-1} \to BU(n-1) \to BU(n)$$
and
$$S^{n-1} \to BO(n-1) \to BO(n)$$
are isomorphic to the unit sphere bundles of the universal vector bundles γ_n over $BU(n)$ and $BO(n)$ respectively.

Proof. We consider the complex case. The real case is proved in the same way. Notice that the model for the sphere bundle in the above theorem is the projection map
$$p : BU(n-1) = EU(n)/U(n-1) \to EU(n)/U(n) = BU(n).$$

But γ_n is the vector bundle $EU(n) \times_{U(n)} \mathbb{C}^n \to BU(n)$ which therefore has unit sphere bundle

$$S(\gamma_n) = EU(n) \times_{U(n)} S^{2n-1} \to BU(n) \tag{5.5}$$

where $S^{2n-1} \subset \mathbb{C}^n$ is the unit sphere with the induced $U(n)$ - action. But $S^{2n-1} \cong U(n)/U(n-1)$ and this diffeomorphism is equivariant with respect to this action. Thus the unit sphere bundle is given by

$$S(\gamma_n) = EU(n) \times_{U(n)} U(n)/U(n-1) \cong EU(n)/U(n-1) = BU(n-1)$$

as claimed. □

We observe that by using the Grassmannian models for $BU(n)$ and $BO(n)$, then their relation to the sphere bundles can be seen explicitly in the following way. This time we work in the real case.

Consider the embedding

$$\iota : Gr_{n-1}(\mathbb{R}^N) \hookrightarrow Gr_n(\mathbb{R}^N \times \mathbb{R}) = Gr_n(\mathbb{R}^{N+1})$$

defined by

$$(V \subset \mathbb{R}^N) \to (V \times \mathbb{R} \subset \mathbb{R}^N \times \mathbb{R}).$$

Clearly as $N \to \infty$ this map becomes a model for the inclusion $BO(n-1) \hookrightarrow BO(n)$. Now for $V \in Gr_{n-1}(\mathbb{R}^N)$ consider the vector $(0,1) \in V \times \mathbb{R} \subset \mathbb{R}^N \times \mathbb{R}$. This is a unit vector, and so is an element of the fiber of the unit sphere bundle $S(\gamma_n)$ over $V \times \mathbb{R}$. Hence this association defines a map

$$j : Gr_{n-1}(\mathbb{R}^N) \to S(\gamma_n)$$

which lifts $\iota : Gr_{n-1}(\mathbb{R}^N) \hookrightarrow Gr_n(\mathbb{R}^{N+1})$. By taking a limit over N we get a map $j : BO(n-1) \to S(\gamma_n)$.

There is a homotopy inverse to the map j which we will call $\rho : S(\gamma_n) \to BO(n-1)$. To define ρ we again first work on the finite Grassmannian level.

Let $(W, w) \in S(\gamma_n)$, the unit sphere bundle over $Gr_n(\mathbb{R}^K)$. Thus $W \subset \mathbb{R}^K$ is an n -dimensional subspace and $w \in W$ is a unit vector. Let $W_w \subset W$ denote the orthogonal complement to the vector w in W. Thus $W_w \subset W \subset \mathbb{R}^K$ is an $n-1$ - dimensional subspace. This association defines a map

$$\rho : S(\gamma_n) \to Gr_{n-1}(\mathbb{R}^K)$$

and by taking the limit over K, defines a map $\rho : S(\gamma_n) \to BO(n-1)$. We leave it to the reader to verify that $j : BO(n-1) \to S(\gamma_n)$ and $\rho : S(\gamma_n) \to BO(n-1)$ are homotopy inverse to each other.

5.5.3 Embedded bundles and K -theory

The classification theorem for vector bundles says that for every n - dimensional complex vector bundle ζ over X, there is a classifying map $f_\zeta : X \to BU(n)$ so that ζ is isomorphic to pull back, $f^*(\gamma_n)$ of the universal vector bundle. A similar statement holds for real vector bundles. Using the Grassmannian models for these classifying spaces, we obtain the following as a corollary.

Theorem 5.37. *Every n - dimensional complex bundle ζ over a space X can be embedded in a trivial infinite dimensional bundle, $X \times \mathbb{C}^\infty$. Similarly, every n - dimensional real bundle η over X can be embedded in the trivial bundle $X \times \mathbb{R}^\infty$.*

Proof. Let $f_\zeta : X \to Gr_n(\mathbb{C}^\infty) = BU(n)$ classify ζ. So $\zeta \cong f^*(\gamma_n)$. But recall that

$$\gamma_n = \{(V, v) \in Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty \quad \text{such that}\, v \in V.\}$$

Hence γ_n is naturally embedded in the trivial bundle $Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty$. Thus $\zeta \cong f^*(\gamma_n)$ is naturally embedded in $X \times \mathbb{C}^\infty$. The real case is proved similarly. □

Notice that because of the direct limit topology on $Gr_n(\mathbb{C}^\infty) = \varinjlim Gr_n(\mathbb{C}^N)$, if X is a compact space, any map $f : X \to Gr_n(\mathbb{C}^\infty)$ has image that lies in $Gr_n(\mathbb{C}^N)$ for some finite N. But notice that over this finite Grassmannian, $\gamma_n \subset Gr_n(\mathbb{C}^N) \times \mathbb{C}^N$. The following is then an immediate corollary. This result was used in Chapter 3 in our discussion about K -theory.

Corollary 5.38. *If X is a compact space of the homotopy type of a CW-complex, then every n - dimensional complex bundle zeta can be embedded in a trivial bundle $X \times \mathbb{C}^N$ for some N. The analogous result also holds for real vector bundles.*

Let $f : X \to BU(n)$ classify the n - dimensional complex vector bundle ζ. Then clearly the composition $f : X \to BU(n) \hookrightarrow BU(n+1)$ classifies the $n+1$ dimensional vector bundle $\zeta \oplus \epsilon_1$, where as before, ϵ_1 is the one dimensional trivial line bundle. This observation leads to the following.

Proposition 5.39. *Let ζ_1 and ζ_2 be two n -dimensional vector bundles over X classified by f_1 and $f_2 : X \to BU(n)$ respectively. Then if we add trivial bundles, we get an isomorphism*

$$\zeta_1 \oplus \epsilon_k \cong \zeta_2 \oplus \epsilon_k$$

if and only if the compositions,

$$f_1,\, f_2 : X \to BU(n) \hookrightarrow BU(n+k)$$

are homotopic.

Now recall from the discussion of K - theory in the Chapter 3 that the set of *stable* isomorphism classes of vector bundles $\mathcal{S}Vect(X)$ is isomorphic to the reduced K - theory, $\tilde{K}(X)$, when X is compact. This proposition then implies the following important result, which displays how in the case of compact spaces, computing K -theory reduces to a specific homotopy theory calculation.

Definition 5.15. *Let BU be the limit of the spaces*

$$BU = \varinjlim_n BU(n).$$

Similarly,

$$BO = \varinjlim_n BO(n).$$

Theorem 5.40. *For X compact there are isomorphisms (bijective correspondences)*

$$\tilde{K}(X) \cong \mathcal{S}Vect(X) \cong [X, BU]$$

and

$$\tilde{K}O(X) \cong \mathcal{S}Vect_{\mathbb{R}}(X) \cong [X, BO].$$

5.5.4 Representations and flat connections

Recall the following classification theorem for covering spaces.

Theorem 5.41. . *Let X be a connected space. Then the set of isomorphism classes of connected covering spaces, $p : E \to X$ is in bijective correspondence with conjugacy classes of normal subgroups of $\pi_1(X)$. This correspondence sends a covering $p : E \to B$ to the image $p_*(\pi_1(E)) \subset \pi_1(X)$.*

Let $\pi = \pi_1(X)$ and let $p : E \to X$ be a connected covering space with $\pi_1(E) = N \lhd \pi$. Then the group of deck transformations of E is the quotient group π/N, and so can be thought of as a principal π/N - bundle. Viewed this way it is classified by a map $f_E : X \to B(\pi/N)$, which on the level of fundamental groups,

$$f_* : \pi = \pi_1(X) \to \pi_1(B(\pi/N)) = \pi/N \tag{5.6}$$

is just the projection on to the quotient space. In particular the universal cover $\tilde{X} \to X$ is the unique simply connected covering space. It is classified by a map

$$\gamma_X : X \to B\pi$$

which induces an isomorphism on the fundamental group.

Now let $\theta : \pi \to G$ be any group homomorphism. By the naturality of classifying spaces this induces a map on classifying spaces,

$$B\theta : B\pi \to BG.$$

This induces a principal G - bundle over X classified by the composition

$$X \xrightarrow{\gamma_X} B\pi \xrightarrow{B\theta} BG.$$

The bundle this map classifies is given by

$$\tilde{X} \times_\pi G \to X$$

where π acts on G via the homomorphism $\theta : \pi \to G$.

This construction defines a map

$$\rho : Hom(\pi_1(X), G) \to Prin_G(X).$$

Now if X is a smooth manifold then its universal cover $p : \tilde{X} \to X$ induces an isomorphism on tangent spaces,

$$Dp(x) : T_x\tilde{X} \to T_{p(x)}X$$

for every $x \in \tilde{X}$. Thus, viewed as a principal π - bundle, it has a canonical connection. Notice furthermore that this connection is *flat,* i.e its curvature is zero.

Exercise.
Check this claim! That is, show that the canonical connection on a covering space is flat.

Now notice that any bundle of the form $\tilde{X} \times_\pi G \to X$ has an induced flat connection. This says that the image of $\rho : Hom(\pi_1(X), G) \to Prin_G(X)$ consists of principal bundles equipped with flat connections.

Notice furthermore that by taking $G = GL(n, \mathbb{C})$ the map ρ assigns to an n - dimensional representation an n - dimensional vector bundle with flat connection

$$\rho : Rep_n(\pi_1(X)) \to Vect_n(X).$$

By taking the sum over all n and passing to the Grothendieck group completion,we get a homomorphism of rings from the representation ring to K - theory,

$$\rho : R(\pi_1(X)) \to K(X).$$

An important question is what is the image of this map of rings. We end this section by describing a famous theorem of Atiyah and Segal describing this image. As we said, we know the image is contained in the classes represented by bundles that have flat connections. Let $X = B\pi$, where π a finite group. Here $K(B\pi)$ will denote $[B\pi, \mathbb{Z} \times BO]$ (which may not be isomorphic to the Grothendieck group of vector bundles since $B\pi$ is not necessarily compact).

Let

$$\epsilon : R(\pi) \to \mathbb{Z} \quad \text{and} \quad \epsilon : K(B\pi) \to \mathbb{Z}$$

be the augmentation maps induced by sending a representation or a vector bundle to its dimension. Let $I \subset R(\pi)$ and $I \subset K(B\pi)$ denote the kernels of these augmentations, i.e the "augmentation ideals". Finally let $\bar{R}(\pi)$ and $\bar{K}(B\pi)$ denote the completions of these rings with respect to these ideals. That is,

$$\bar{R}(\pi) = \varprojlim_n R(\pi)/I^n \quad \text{and} \quad \bar{K}(B\pi) = \varprojlim_n K(B\pi)/I^n$$

where I^n is the product of the ideal I with itself n - times.

Theorem 5.42. *(Atiyah and Segal) [11] For π a finite group, the induced map on the completions of the rings with respect to the augmentation ideals,*

$$\rho : \bar{R}(\pi) \to \bar{K}(B\pi)$$

is an isomorphism.

6

Characteristic Classes

In this chapter we define and calculate characteristic classes for principal bundles and vector bundles. Characteristic classes are the basic cohomological invariants of bundles and have a wide variety of applications throughout topology and geometry. Characteristic classes were introduced originally by E. Stiefel in Switzerland and H. Whitney in the United States in the mid 1930's. Stiefel, who was a student of H. Hopf intoduced in his thesis certain "characteristic homology classes" determined by the tangent bundle of a manifold. At about the same time Whitney studied general sphere bundles, and later introduced the general notion of a characteristic cohomology class coming from a vector bundle, and proved the product formula for their calculation.

In the early 1940's, L. Pontrjagin, in Moscow, introduced new characteristic classes by studying the Grassmannian manifolds, using work of C. Ehresmann from Switzerland. In the mid 1940's, after just arriving in Princeton from China, S.S Chern defined characteristic classes for complex vector bundles using differential forms and his calculations resulted in a great clarification of the theory.

Much of the modern view of characteristic classes has been greatly influenced by the highly influential book of Milnor and Stasheff [121]. This book was originally circulated as lecture notes written in 1957 and finally published in 1974. This book is one of the great textbooks in modern mathematics. These notes follow, in large part, their treatment of the subject. The reader is encouraged to consult their book for further details.

6.1 Preliminaries

Definition 6.1. *Let G be a topological group (possibly with the discrete topology). Then a "characteristic class" for principal G - bundles is an assignment to each principal G - bundle $p : P \to B$ a cohomology class*

$$c(P) \in H^*(B)$$

satisfying the following naturality condition. If

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{f}} & P_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array}$$

is a map of principal G - bundles inducing an equivariant homeomorphism on fibers, then

$$f^*(c(P_2)) = c(P_1) \in H^*(B_1).$$

Remarks. 1. In this definition cohomology could be taken with any coefficients, including, for example, DeRham cohomology, which has coefficients in the real numbers $\mathbb{R}$. The particular cohomology theory used is referred to as the "values" of the characteristic classes.

2. The same definition of characteristic classes applies to real or complex vector bundles as well as principal bundles.

The following is an easy consequence of the definition.

Lemma 6.1. *Let c be a characteristic class for principal G - bundles so that c takes values in $H^q(-)$, for $q \geq 1$. Then if ϵ is the trivial G bundle,*

$$\epsilon = X \times G \to X$$

then $c(\epsilon) = 0$.

Proof. The trivial bundle ϵ is the pull - back of the constant map to the one point space $e : X \to pt$ of the bundle $\nu = G \to pt$. Thus $c(\epsilon) = e^*(c(\nu))$. But $c(\nu) \in H^q(pt) = 0$ when $q > 0$. □

The following observation is also immediate from the definition.

Lemma 6.2. *Characteristic classes are invariant under isomorphism. More specifically, Let c be a characteristic class for principal G - bundles. Also let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be isomorphic principal G - bundles. Then*

$$c(E_1) = c(E_2) \in H^*(X).$$

Thus for a given space X, a characteristic class c can be viewed as a map

$$c : Prin_G(X) \to H^*(X).$$

3. The naturality property in the definition can be stated in more functorial terms in the following way.

Cohomology (with any coefficients) $H^*(-)$ is a contravariant functor from the category $ho\mathcal{T}op$ of topological spaces and homotopy classes of maps, to the category $\mathcal{A}b$ of abelian groups. By the results of chapter 2, the set of principal G - bundles $Prin_G(-)$ can be viewed as a contravariant functor from the category $ho\mathcal{T}op$ to the category of sets $\mathcal{S}ets$.

Definition 6.2. (Alternative) A characteristic class is a natural transformation c between the functors $Prin_G(-)$ and $H^*(-)$:

$$c : Prin_G(-) \rightsquigarrow H^*(-)$$

Examples.

1. The *first Chern class* $c_(\zeta)$ is a characteristic class on principal $U(1)$ - bundles, or equivalently, complex line bundles. If ζ is a line bundle over X, then $c_1(\zeta) \in H^2(X;\mathbb{Z})$. As we saw in the last chapter, c_1 is a complete invariant of line bundles. That is to say, the map

   $$c_1 : Prin_{U(1)}(X) \to H^2(X;\mathbb{Z})$$

   is an isomorphism.

2. The *first Stiefel - Whitney class* $w_1(\eta)$ is a characteristic class of two fold covering spaces (i.e a principal $\mathbb{Z}_2 = O(1)$ - bundles) or of real line bundles. If η is a real line bundle over a space X, then $w_1(\eta) \in H^1(X;\mathbb{Z}_2)$. Moreover, as we saw in the last chapter, the first Stiefel - Whitney class is a complete invariant of line bundles. That is, the map

   $$w_1 : Prin_{O(1)}(X) \to H^1(X;\mathbb{Z}_2)$$

   is an isomorphism.

We remark that the first Stiefel - Whitney class can be extended to be a characteristic class of real n - dimensional vector bundles (or principal $O(n)$ - bundles) for any n. To see this, consider the subgroup $SO(n) < O(n)$. As we saw in the last chapter, a bundle has an $SO(n)$ structure if and only if it is orientable. Moreover the induced map of classifying spaces gives a 2 - fold covering space or principal $O(1)$ - bundle,

$$\mathbb{Z}_2 = O(1) = O(n)/SO(n) \to BSO(n) \to BO(n).$$

This covering space defines, via its classifying map $w_1 : BO(n) \to BO(1) = \mathbb{RP}^\infty$ an element $w_1 \in H^1(BO(n);\mathbb{Z}_2)$ which is the first Stiefel - Whitney class of this covering space.

Now let η be any n - dimensional real vector bundle over X, and let

$$f_\eta : X \to BO(n)$$

be its classifying map.

Definition 6.3. *The first Stiefel - Whitney class* $w_1(\eta) \in H^1(X; \mathbb{Z}_2)$ *is defined to be*

$$w_1(\eta) = f_\eta^*(w_1) \in H^1(X; \mathbb{Z}_2)$$

The first Chern class c_1 *of an* n *- dimensional complex vector bundle* ζ *over* X *is defined similarly, by pulling back the first Chern class of the principal* $U(1)$ *- bundle*

$$U(1) \cong U(n)/SU(n) \to BSU(n) \to BU(n)$$

via the classifying map $f_\zeta : X \to BU(n)$.

The following is an immediate consequence of the above lemma and the meaning of $SO(n)$ and $SU(n)$ - structures.

Theorem 6.3. *Given a complex* n *- dimensional vector bundle* ζ *over* X, *then* $c_1(\zeta) \in H^2(X)$ *is zero if and only if* ζ *has an* $SU(n)$ *-structure.*

Furthermore, given a real n *- dimensional vector bundle* η *over* X, *then* $w_1(\eta) \in H^1(X; \mathbb{Z}_2)$ *is zero if and only if the bundle* η *has an* $SO(n)$ *- structure, which is equivalent to* η *being orientable.*

We now use the classification theorem for bundles to describe the set of characteristic classes for principal G - bundles.

Let R be a commutative ring and let $Char_G(R)$ be the set of all characteristic classes for principal G bundles that take values in $H^*(-; R)$. Notice that the sum (in cohomology) and the cup product of characteristic classes is again a characteristic class. This gives $Char_G$ the structure of a ring. (Notice that the unit in this ring is the constant characteristic class $c(\zeta) = 1 \in H^0(X)$.

Theorem 6.4. *There is an isomorphism of rings*

$$\rho : Char_G(R) \xrightarrow{\cong} H^*(BG; R)$$

Proof. Let $c \in Char_G(R)$. Define

$$\rho(c) = c(EG) \in H^*(BG; R)$$

where $EG \to BG$ is the universal G - bundle over BG. By definition of the ring structure of $Char_G(R)$, ρ is a ring homomorphism.

Now let $\gamma \in H^q(BG; R)$. Define the characteristic class c_γ as follows. Let $p : E \to X$ be a principal G - bundle classified by a map $f_E : X \to BG$. Define

$$c_\gamma(E) = f_E^*(\gamma) \in H^q(X; R)$$

where $f_E^* : H^*(BG : R) \to H^*(X; R)$ is the cohomology ring homomorphism induced by f_E. This association defines a map

$$c : H^*(BG; R) \to Char_G(R)$$

which immediately seen to be inverse to ρ. □

6.2 Chern Classes and Stiefel - Whitney Classes

In this section we compute the rings of unitary characteristic classes $Char_{U(n)}(\mathbb{Z})$ and $\mathbb{Z}_2$ - valued orthogonal characteristic classes $Char_{O(n)}(\mathbb{Z}_2)$. These are the characteristic classes of complex and real vector bundles and as such have a great number of applications. By Theorem 6.4 computing these rings of characteristic classes reduces to computing the cohomology rings $H^*(BU(n);\mathbb{Z})$ and $H^*(BO(n);\mathbb{Z}_2)$. The following is the main theorem of this section.

Theorem 6.5. *a. The ring of $U(n)$ characteristic classes is a polynomial algebra on n - generators,*

$$Char_{U(n)}(\mathbb{Z}) \cong H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \cdots, c_n]$$

where $c_i \in H^{2i}(BU(n);\mathbb{Z})$ is known as the i^{th} - Chern class.

b. The ring of $\mathbb{Z}_2$ - valued $O(n)$ characteristic classes is a polynomial algebra on n - generators,

$$Char_{O(n)}(\mathbb{Z}_2) \cong H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$$

where $w_i \in H^i(BO(n);\mathbb{Z}_2)$ is known as the i^{th} - Stiefel - Whitney class.

This theorem will be proven by induction on n. For $n = 1$, $BU(1) = \mathbb{CP}^\infty$ and $BO(1) = \mathbb{RP}^\infty$ and so the theorem describes the ring structure in the cohomology of these projective spaces. To complete the inductive step we will study the sphere bundles

$$S^{n-1} \to BO(n-1) \to BO(n)$$

and

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

described in the last chapter. In particular recall from Corollary 5.36 that in these fibrations, $BO(n-1)$ and $BU(n-1)$ are the unit sphere bundles $S(\gamma_n)$ of the universal bundle γ_n over $BO(n)$ and $BU(n)$ respectively. Let $D(\gamma_n)$ be the unit disk bundles of the universal bundles. That is, in the complex case,

$$D(\gamma_n) = EU(n) \times_{U(n)} D^{2n} \to BU(n)$$

and in the real case,

$$D(\gamma_n) = EO(n) \times_{O(n)} D^n \to BO(n)$$

where $D^{2n} \subset \mathbb{C}^n$ and $D^n \subset \mathbb{R}^n$ are the unit disks, and therefore have the induced unitary and orthogonal group actions.

Here is one easy observation about these disk bundles.

Proposition 6.6. *The projection maps*

$$p : D(\gamma_n) = EU(n) \times_{U(n)} D^{2n} \to BU(n)$$

and

$$D(\gamma_n) = EO(n) \times_{O(n)} D^n \to BO(n)$$

are homotopy equivalences.

Proof. Both of these bundles have zero sections $\mathcal{Z} : BU(n) \to D(\gamma_n)$ and $\mathcal{Z} : BO(n) \to D(\gamma_n)$. In both the complex and real cases, we have $p \circ \mathcal{Z} = 1$. To see that $\mathcal{Z} \circ p \simeq 1$ consider the homotopy $H : D(\gamma_n) \times I \to D(\gamma_n)$ defined by $H(v, t) = tv$. □

We will use this result when studying the cohomology exact sequence of the pair $(D(\gamma_n), S(\gamma_n))$:

$$\begin{aligned} \cdots \to H^{q-1}(S(\gamma_n)) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n)) \to H^q(D(\gamma_n)) \to H^q(S(\gamma_n)) \\ &\xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n)) \to H^{q+1}(D(\gamma_n)) \to \cdots \end{aligned} \tag{6.1}$$

Using the above proposition and Corollary 5.36 we can substitute $H^*(BU(n))$ for $H^*(D(\gamma_n))$, and $H^*(BU(n-1))$ for $H^*(S(\gamma_n))$ in this sequence to get the following exact sequence

$$\cdots \to H^{q-1}(BU(n-1)) \xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n)) \to H^q(BU(n)) \tag{6.2}$$
$$\xrightarrow{\iota} H^q(BU(n-1)) \xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n)) \to H^{q+1}(BU(n)) \to \cdots .$$

and we get a similar exact sequence in the real case

$$\begin{aligned} \cdots \to H^{q-1}(BO(n-1); \mathbb{Z}_2) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n); \mathbb{Z}_2) \to H^q(BO(n); \mathbb{Z}_2) \\ &\xrightarrow{\iota} H^q(BO(n-1); \mathbb{Z}_2) \xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n); \mathbb{Z}_2) \to H^{q+1}(BO(n); \mathbb{Z}_2) \to \cdots \end{aligned} \tag{6.3}$$

These exact sequences will be quite useful for inductively computing the cohomology of these classifying spaces, but to do so we need a method for computing $H^*(D(\gamma_n), S(\gamma_n))$, or more generally, $H^*(D(\zeta), S(\zeta))$, where ζ is any Euclidean vector bundle and $D(\zeta)$ and $S(\zeta)$ are the associated unit disk bundles and sphere bundles respectively. The quotient space,

$$T(\zeta) = D(\zeta)/S(\zeta) \tag{6.4}$$

is called the *Thom space* of the bundle ζ. As the name suggests, this construction was first studied by R. Thom [150], and has been quite useful in both bundle theory and cobordism theory. Notice that on each fiber (say at $x \in X$) of the n - dimensional disk bundle ζ, the Thom space construction takes the unit n - dimensional disk modulo its boundary $(n-1)$ - dimensional sphere which therefore yields an n - dimensional sphere, with marked basepoint, say $\infty_x \in S^n(\zeta_x) = D^n(\zeta_x)/S^{n-1}(\zeta_x)$. The Thom space construction then identifies all the basepoints ∞_x to a single point. Notice that for a bundle over a point $\mathbb{R}^n \to pt$, the Thom space $T(\mathbb{R}^n) = D^n/S^{n-1} = S^n \cong \mathbb{R}^n \cup \infty$. More generally, notice that when the basespace X is compact, then the Thom space is simply the one point compactification of the total space of the vector bundle ζ,

$$T(\zeta) \cong \zeta^+ = \zeta \cup \infty \tag{6.5}$$

where we think of the extra point in this compactification as the common point at infinity assigned to each fiber. In order to compute with the above exact sequences, we will need to study the cohomology of Thom spaces. But before we do we examine the topology of the Thom spaces of product bundles. For this we introduce the "smash product" construction.

Let X and Y be spaces with basepoints $x_0 \in X$ and $y_0 \in Y$.

Definition 6.4. *The wedge* $X \vee Y$ is the "one point union",

$$X \vee Y = X \times y_0 \cup x_0 \times Y \subset X \times Y.$$

The *smash product* $X \wedge Y$ is given by

$$X \wedge Y = X \times Y / X \vee Y.$$

Observations. 1. The k be a field. Then the Kunneth formula gives

$$\tilde{H}^*(X \wedge Y; k) \cong \tilde{H}^*(X; k) \otimes \tilde{H}^*(Y; k).$$

2. Let V and W be vector spaces, and let V^+ and W^+ be their one point compactifications. These are spheres of the same dimension as the respective vector spaces. Then

$$V^+ \wedge W^+ = (V \times W)^+.$$

So in particular,

$$S^n \wedge S^m = S^{n+m}.$$

Proposition 6.7. *Let ζ be an n - dimensional vector bundle over a space X, and let η be an m - dimensional bundle over X. Let $\zeta \times \eta$ be the product $n+m$ - dimensional vector bundle over $X \wedge Y$. Then the Thom space of $\zeta \times \eta$ is given by*

$$T(\zeta \times \eta) \cong T(\zeta) \wedge T(\eta).$$

Proof. Notice that the disk bundle is given by

$$D(\zeta \times \eta) \cong D(\zeta) \times D(\eta)$$

and its boundary sphere bundle is given by

$$S(\zeta \times \eta) \cong S(\zeta) \times D(\eta) \cup D(\zeta) \times S(\eta).$$

Thus

$$\begin{aligned} T(\zeta \times \eta) = D(\zeta \times \eta)/S(\zeta \times \eta) &\cong (D(\zeta) \times D(\eta))/\left(S(\zeta) \times D(\eta) \cup D(\zeta) \times S(\eta)\right) \\ &\cong (D(\zeta)/S(\zeta)) \wedge (D(\eta)/S(\eta)) \\ &\cong T(\zeta) \wedge T(\eta). \end{aligned}$$

□

We now proceed to study the cohomology of Thom spaces.

6.2.1 The Thom Isomorphism Theorem

We begin by describing a cohomological notion of orientability of an vector bundle ζ over a space X. Give ζ a Euclidean structure.

Consider the 2 - fold cover over X defined as follows. Let E_ζ be the principal $O(n)$ bundle associated to ζ. Also let Gen_n be the set of generators of $H^n(S^n) \cong \mathbb{Z}$. So Gen_n is a set with two elements. Moreover the orthogonal group $O(n)$ acts on $S^n = \mathbb{R}^n \cup \infty$ by the usual linear action on $\mathbb{R}^n$ extended to have a fixed point at $\infty \in S^n$. By looking at the induced map on cohomology, there is an action of $O(n)$ on Gen_n. We can then define the double cover

$$\mathcal{G}(\zeta) = E_\zeta \times_{O(n)} Gen_n \longrightarrow E_\zeta/O(n) = X.$$

Lemma 6.8. *The double covering $\mathcal{G}(\zeta)$ is isomorphic to the orientation double cover $Or(\zeta)$.*

Proof. Recall from chapter 1 that the orientation double cover $Or(\zeta)$ is given by

$$Or(\zeta) = E_\zeta \times_{O(n)} Or(\mathbb{R}^n)$$

where $Or(\mathbb{R}^n)$ is the two point set consisting of orientations of the vector space $\mathbb{R}^n$. A matrix $A \in O(n)$ acts on this set trivially if and only if the determinant $detA$ is positive. It acts nontrivially (i.e permutes the two elements) if and only if $detA$ is negative. Now the same is true of the action of $O(n)$ on Gen_n. This is because $A \in O(n)$ induces multiplication by $detA$ on $H^n(S^n)$.

Exercise. Verify this claim. That is, prove that $A \in GL(n, \mathbb{R})$ induces multiplication by the sign of $detA$ on $H^n(S^n)$.

Since $Or(\mathbb{R}^n)$ and Gen_n are both two point sets with the same action of $GL(n, \mathbb{R})$, the corresponding two fold covering spaces $Or(\zeta)$ and $\mathcal{G}(\zeta)$ are isomorphic. □

Corollary 6.9. *An orientation of an n - dimensional vector bundle ζ is equivalent to a section of $\mathcal{G}(\zeta)$ and hence defines a continuous family of generators*

$$u_x \in H^n(S^n(\zeta_x)) \cong \mathbb{Z}$$

for every $x \in X$. Here $S^n(\zeta_x)$ is the unit disk of the fiber ζ_x modulo its boundary sphere. $S^n(\zeta_x)$ is called the sphere at x.

Now recall that given a pair of spaces $A \subset Y$, there is a relative cup product in cohomology,

$$H^q(Y) \otimes H^r(Y, A) \xrightarrow{\cup} H^{q+r}(Y, A).$$

So in particular the relative cohomology $H^*((Y, A)$ is a (graded) module over the (graded) ring $H^*(Y)$.

In the case of a vector bundle ζ over a space X, we then have that $H^*(D(\zeta), S(\zeta)) = \tilde{H}^*(T(\zeta))$ is a module over $H^*(D(\zeta)) \cong H^*(X)$. So in particular, given any cohomology class in the Thom space, $\alpha \in H^r(T(\zeta))$ we get an induced homomorphism

$$H^q(X) \xrightarrow{\cup\alpha} H^{q+r}(T(\zeta)).$$

Our next goal is to prove the famous *Thom Isomorphism Theorem* which can be stated as follows.

Theorem 6.10. *Let ζ be an oriented n - dimensional real vector bundle over a connected space X. Let R be any commutative ring with unit. The orientation gives generators $u_x \in H^n(S^n(\zeta_x); R) \cong R$. Then there is a unique class (called the Thom class) in the cohomology of the Thom space*

$$u \in H^n(T(\zeta); R)$$

so that for every $x \in X$, if

$$j_x : S^n(\zeta_x) \hookrightarrow D(\zeta)/S(\zeta) = T(\zeta)$$

is the natural inclusion of the sphere at x in the Thom space, then under the induced homomorphism in cohomology,

$$j_x^* : H^n(T(\zeta); R) \to H^n(S^n(\zeta_x); R) \cong R$$

$j_x^*(u) = u_x$.

Furthermore The induced cup product map

$$\gamma : H^q(X; R) \xrightarrow{\cup u} \tilde{H}^{q+n}(T(\zeta); R)$$

is an isomorphism for every $q \in \mathbb{Z}$. So in particular $\tilde{H}^r(T(\zeta); R) = 0$ for $r < n$.

If ζ is not an orientable bundle over X, then the theorem remains true if we take $\mathbb{Z}_2$ coefficients, $R = \mathbb{Z}_2$.

Proof. We prove the theorem for oriented bundles. We leave the nonorientable case (when $R = \mathbb{Z}_2$) to the reader. We also restrict our attention to the case $R = \mathbb{Z}$, since the theorem for general coefficients will follow immediately from this case using the universal coefficient theorem.

Case 1: ζ is the trivial bundle $X \times \mathbb{R}^n$.

In this case the Thom space $T(\zeta)$ is given by

$$T(\zeta) = X \times D^n / X \times S^{n-1}.$$

The projection of X to a point, $X \to pt$ defines a map

$$\pi : T(\zeta) = X \times D^n / X \times S^{n-1} \to D^n / S^{n-1} = S^n.$$

Let $u \in H^n(T(\zeta))$ be the image in cohomology of a generator,

$$\mathbb{Z} \cong H^n(S^n) \xrightarrow{\pi^*} H^n(T(\zeta)).$$

The fact that taking the cup product with this class

$$H^q(X) \xrightarrow{\cup u} H^{q+n}(T(\zeta)) = H^{q+n}(X \times D^n, X \times S^{n-1}) = H^{q+n}(X \times S^n, X \times pt)$$

is an isomorphism for every $q \in \mathbb{Z}$ follows from the universal coefficient theorem.

Case 2: X is the union of two open sets $X = X_1 \cup X_2$, where we know the Thom isomorphism theorem holds for the restrictions $\zeta_i = \zeta_{|X_i}$ for $i = 1, 2$ and for $\zeta_{1,2} = \zeta_{|X_1 \cap X_2}$.

We prove the theorem for X using the Mayer - Vietoris sequence for cohomology. Let $X_{1,2} = X_1 \cap X_2$.

$$\to H^{q-1}(T(\zeta_{1,2})) \to H^q(T(\zeta)) \to H^q(T(\zeta_1)) \oplus H^q(T(\zeta_2)) \to H^q((T(\zeta_{1,2})) \to \cdots$$

Looking at this sequence when $q < n$, we see that since

$$H^q(T(\zeta_{1,2})) = H^q(T(\zeta_1)) = H^q(T(\zeta_2)) = 0,$$

then by exactness we must have that $H^q(T(\zeta)) = 0$.

We now let $q = n$, and we see that by assumption, $H^n(T(\zeta_1)) \cong H^n(T(\zeta_2)) \cong H^n(T(\zeta_{1,2})) \cong \mathbb{Z}$, and that the Thom classes of each of the restriction maps $H^n(T(\zeta_1)) \to H^n(T(\zeta_{1,2}))$ and $H^n(T(\zeta_2)) \to H^n(T(\zeta_{1,2}))$ correspond. Moreover $H^{n-1}(T(\zeta_{1,2})) = 0$. Hence by the exact sequence, $H^n(T(\zeta)) \cong \mathbb{Z}$ and there is a class $u \in H^n(T(\zeta))$ that maps to the direct sum of the Thom classes in $H^n(T(\zeta_1)) \oplus H^n(T(\zeta_2))$.

Now for $q \geq n$ we compare the above Mayer - Vietoris sequence with the one of base spaces,

$$\to H^{q-1}(X_{1,2}) \to H^q(X) \to H^q(X_1) \oplus H^q(X_2) \to H^q(X_{1,2}) \to \cdots$$

This sequence maps to the one for Thom spaces by taking the cup product with the Thom classes. By assumption this map is an isomorphism on $H^*(X_i)$, $i = 1, 2$ and on $H^*(X_{1,2})$. Thus by the Five Lemma it is an isomorphism on $H^*(X)$. This proves the theorem in this case.

Case 3. X is covered by finitely many open sets X_i, $i = 1, \cdots, k$ so that the restrictions of the bundle to each X_i, ζ_i is trivial.

The proof in this case is an easy inductive argument (on the number of open sets in the cover), where the inductive step is completed using cases 1 and 2.

Notice that this case includes the situation when the basespace X is compact.

Case 4. General Case. We now know the theorem for compact spaces. However it is not necessarily true that the cohomology of a general space (i.e homotopy type of a C.W complex) is determined by the cohomology of its compact subspaces. However it is true that the homology of a space X is given by

$$H_*(X) \cong \varinjlim_K H_*(K)$$

where the limit is taken over the partially ordered set of compact subspaces $K \subset X$. Thus we want to first work in homology and then try to transfer our observations to cohomology.

To do this, recall that the construction of the cup product pairing actually comes from a map on the level of cochains,

$$C^q(Y) \otimes C^r(Y, A) \xrightarrow{\cup} C^{q+r}(Y, A)$$

and therefore has a dual map on the chain level

$$C_*(Y, A) \xrightarrow{\psi} C_*(Y) \otimes C_*(Y, A).$$

and thus induces a map in homology

$$\psi : H_k(Y, A) \to \oplus_{r\geq 0} H_{k-r}(Y) \otimes H_r(Y, A).$$

Hence given $\alpha \in H^r(Y, A)$ we have an induced map in homology (the "slant product")

$$/\alpha : H_k(Y, A) \to H_{k-r}(Y)$$

defined as follows. If $\theta \in H_k(Y, A)$ and

$$\psi(\theta) = \sum_j a_j \otimes b_j \in H_*(Y) \otimes H_*(Y, A)$$

then

$$/\alpha(\theta) = \sum_j \alpha(b_j) \cdot a_j$$

where by convention, if the degree of a homology class b_j is not equal to the degree of α, then $\alpha(b_j) = 0$.

Notice that this slant product is dual to the cup product map

$$H^q(Y) \xrightarrow{\cup\alpha} H^{q+r}(Y, A).$$

Again, by considering the pair $(D(\zeta), S(\zeta))$, and identifying $H_*(D(\zeta)) \cong H_*(X)$, we can apply the slant product operation to the Thom class, to define a map

$$/u : H_k(T(\zeta)) \to H_{k-n}(X).$$

which is dual to the Thom map $\gamma : H^q(X) \xrightarrow{\cup u} H^{q+n}(T(\zeta))$. Now since γ is an isomorphism in all dimensions when restricted to compact sets, then by the universal coefficient theorem, $/u : H_q(T(\zeta_{|K})) \to H_{q-n}(K)$ is an isomorphism for all q and for every compact subset $K \subset X$. By taking the limit over the partially ordered set of compact subsets of X, we get that

$$/u : H_q(T(\zeta)) \to H_{q-n}(X)$$

is an isomorphism for all q. Applying the universal coefficient theorem again, we can now conclude that

$$\gamma : H^k(X) \xrightarrow{\cup u} H^{k+n}(T(\zeta))$$

is an isomorphism for all k. This completes the proof of the theorem. □

We now observe that the Thom class of a product of two bundles is the appropriately defined product of the Thom classes.

Lemma 6.11. *Let ζ and η be an n and m dimensional oriented vector bundles over X and Y respectively. Then the Thom class $u(\zeta\times\eta)$ is given by the tensor product: $u(\zeta \times \eta) \in H^{n+m}(T(\zeta \times \eta))$ is equal to*

$$\begin{aligned} u(\zeta) \otimes u(\eta) &\in H^n(T(\zeta)) \otimes H^m(T(\eta)) \\ &\cong H^{n+m}(T(\zeta) \wedge T(\eta)) \\ &= H^{n+m}(T(\zeta \times \eta)). \end{aligned}$$

In this description, cohomology is meant to be taken with $\mathbb{Z}_2$ - coefficients if the bundles are not orientable.

Proof. $u(\zeta) \otimes u(\eta)$ restricts on each fiber $(x, y) \in X \times Y$ to

$$\begin{aligned} u_x \otimes u_y &\in H^n(S^n(\zeta_x)) \otimes H^m(S^m(\eta_y)) \\ &\cong H^{n+m}(S^n(\zeta_x) \wedge S^m(\eta_y)) \\ &= H^{n+m}(S^{n+m}(\zeta \times \eta)_{(x,y)})) \end{aligned}$$

which is the generator determined by the product orientation of $\zeta_x \times \eta_y$. The result follows by the uniqueness of the Thom class. □

We now use the Thom isomorphism theorem to define a characteristic class for oriented vector bundles, called the *Euler class*.

Definition 6.5. *The Euler class of an oriented, n dimensional bundle ζ, over a connected space X, is the n - dimensional cohomology class*

$$\chi(\zeta) \in H^n(X)$$

defined to be the image of the Thom class $u(\zeta) \in H^n(T(\zeta))$ under the composition

$$H^n(T(\zeta)) = H^n(D(\zeta), S(\zeta)) \to H^n(D(\zeta)) \cong H^n(X).$$

Again, if ζ is not orientable, cohomology is taken with $\mathbb{Z}_2$ - coefficients.

Exercise. Verify that the Euler class is a characteristic class according to our definition.

The following is then a direct consequence of Lemma 6.11.

Corollary 6.12. *Let ζ and η be as in 6.11. Then the Euler class of the product is given by*

$$\chi(\zeta \times \eta) = \chi(\zeta) \otimes \chi(\eta) \in H^n(X) \otimes H^m(Y) \hookrightarrow H^{n+m}(X \times Y).$$

We will also need the following observation.

Proposition 6.13. *Let η be an odd dimensional oriented vector bundle over a space X. Say dim $(\eta) = 2n+1$. Then its Euler class has order two:*

$$2\chi(\eta) = 0 \in H^{2n+1}(X).$$

Proof. Consider the bundle map

$$\begin{aligned} \nu : \eta &\to \eta \\ v &\to -v. \end{aligned}$$

Since η is odd dimensional, this bundle map is an orientation reversing automorphism of η. This means that $\nu^*(u) = -u$, where $u \in H^{2n+1}(T(\eta))$ is the Thom class. By the definition of the Euler class this in turn implies that $\nu^*(\chi(\eta)) = -\chi(\eta)$. But since the Euler class is a characteristic class and ν is a bundle map, we must have $\nu^*(\chi(\eta)) = \chi(\eta)$. Thus $\chi(\eta) = -\chi(\eta)$. □

6.2.2 The Gysin sequence

We now input the Thom isomorphism theorem into the cohomology exact sequence of the pair $D(\zeta), S(\zeta))$ in order to obtain an important calculational tool for computing the (co)homology of vector bundles and sphere bundles.

Namely, let ζ be an oriented n - dimensional oriented vector bundle over a space X, and consider the exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^q(D(\zeta), S(\zeta)) \to H^q(D(\zeta)) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q+1}(D(\zeta), S(\zeta)) \to H^{q+1}(D(\zeta)) \to \cdots \end{aligned}$$

By identifying $H^*(D(\zeta), S(\zeta)) = \tilde{H}^*(T(\zeta))$ and $H^*(D(\zeta)) \cong H^*(X)$, this exact sequence becomes

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^q(T(\zeta)) \to H^q(X) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q+1}(T(\zeta)) \to H^{q+1}(X) \to \cdots \end{aligned}$$

Finally, by inputting the Thom isomorphism, $H^{q-n}(X) \xrightarrow[\cong]{\cup u} H^q(T(\zeta))$ we get the following exact sequence known as the **Gysin sequence:**

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^{q-n}(X) \xrightarrow{\chi} H^q(X) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q-n+1}(X) \xrightarrow{\chi} H^{q+1}(X) \to \cdots \end{aligned} \tag{6.6}$$

We now make the following observation about the homomorphism $\chi : H^q(X) \to H^{q+n}(X)$ in the Gysin sequence.

Proposition 6.14. *The homomorphism $\chi : H^q(X) \to H^{q+n}(X)$ is given by taking the cup product with the Euler class,*

$$\chi : H^q(X) \xrightarrow{\cup\chi} H^{q+n}(X).$$

Proof. The theorem is true for $q = 0$, by definition. Now in general, the map χ was defined in terms of the Thom isomorphism $\gamma : H^r(X) \xrightarrow{\cup u} H^{r+n}(T(\zeta))$, which, by definition is a homomorphism of graded $H^*(X)$ - modules. This will then imply that

$$\chi : H^q(X) \to H^{q+n}(X)$$

is a homomorphism of graded $H^*(X)$ - modules. Thus

$$\begin{aligned} \chi(\alpha) &= \chi(1 \cdot \alpha) \\ &= \chi(1) \cup \alpha \quad \text{since } \chi \text{ is an } H^*(X) \text{ - module homomorphism} \\ &= \chi(\zeta) \cup \alpha \end{aligned}$$

as claimed. □

6.2.3 Proof of Theorem 6.5

the goal of this section is to use the Gysin sequence to prove Theorem 6.5, which we begin by restating:

Theorem 6.15. *a. The ring of $U(n)$ characteristic classes is a polynomial algebra on n - generators,*

$$Char_{U(n)}(\mathbb{Z}) \cong H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \cdots, c_n]$$

where $c_i \in H^{2i}(BU(n);\mathbb{Z})$ is known as the ith - Chern class.

b. The ring of $\mathbb{Z}_2$ - valued $O(n)$ characteristic classes is a polynomial algebra on n - generators,

$$Char_{O(n)}(\mathbb{Z}_2) \cong H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$$

where $w_i \in H^i(BO(n);\mathbb{Z}_2)$ is known as the ith - Stiefel - Whitney class.

Proof. We start by considering the Gysin sequence, applied to the universal bundle γ_n over $BU(n)$. We input the fact that the sphere bundle $S(\gamma_n)$ is given by $BU(n-1)$ see 6.2:

$$\begin{aligned} \cdots \to H^{q-1}(BU(n-1)) \xrightarrow{\delta} H^{q-2n}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^q(BU(n)) \\ \xrightarrow{\iota^*} H^q(BU(n-1)) \xrightarrow{\delta} H^{q-2n+1}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^{q+1}(BU(n)) \to \cdots \end{aligned} \tag{6.7}$$

and we get a similar exact sequence in the real case

$$\begin{aligned} \cdots \to H^{q-1}(BO(n-1);\mathbb{Z}_2) \xrightarrow{\delta} H^{q-n}(BO(n);\mathbb{Z}_2) \xrightarrow{\cup\chi(\gamma_n)} H^q(BO(n);\mathbb{Z}_2) \\ \xrightarrow{\iota^*} H^q(BO(n-1);\mathbb{Z}_2) \xrightarrow{\delta} H^{q-n+1}(BO(n));\mathbb{Z}_2) \xrightarrow{\cup\chi(\gamma_n)} H^{q+1}(BO(n);\mathbb{Z}_2) \to \cdots \end{aligned} \tag{6.8}$$

We use these exact sequences to prove the above theorem by induction on n. For $n=1$ then sequence 6.7 reduces to the short exact sequences,

$$0 \to H^{q-2}(BU(1)) \xrightarrow[\cong]{\cup\chi(\gamma_1)} H^q(BU(1)) \to 0$$

for each $q \geq 2$. We let $c_1 \in H^2(BU(1)) = H^2(\mathbb{CP}^\infty)$ be the Euler class $\chi(\gamma_1)$. These isomorphisms imply that the ring structure of $H^*(BU(1))$ is that of a polynomial algebra on this single generator,

$$H^*(BU(1)) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c_1]$$

which is the statement of the theorem in this case.

In the real case when $n=1$ the Gysin sequence 6.8 reduces to the short exact sequences,

$$0 \to H^{q-1}(BO(1);\mathbb{Z}_2) \xrightarrow[\cong]{\cup\chi(\gamma_1)} H^q(BO(1);\mathbb{Z}_2) \to 0$$

for each $q \geq 1$. We let $w_1 \in H^1(BO(1);\mathbb{Z}_2) = H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$ be the Euler class $\chi(\gamma_1)$. These isomorphisms imply that the ring structure of $H^*(BO(1);\mathbb{Z}_2)$ is that of a polynomial algebra on this single generator,

$$H^*(BO(1);\mathbb{Z}_2) = H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$$

which is the statement of the theorem in this case.

We now inductively assume the theorem is true for $n-1$. That is,

$$H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \cdots, c_{n-1}] \quad \text{and} \quad H^*(BO(n-1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_{n-1}].$$

We first consider the Gysin sequence 6.7, and observe that by exactness, for $q \leq 2(n-1)$, the homomorphism

$$\iota^* : H^q(BU(n)) \to H^q(BU(n-1))$$

is an isomorphism. That means there are unique classes, $c_1, \cdots, c_{n-1} \in H^*(BU(n))$ that map via ι^* to the classes of the same name in $H^*(BU(n-1))$. Furthermore, since ι^* is a ring homomorphism, every polynomial in $c_1, \cdots, c_{n-1}$ in $H^*(BU(n-1))$ is in the image under ι^* of the corresponding polynomial in the these classes in $H^*(BU(n))$. Hence by our inductive assumption,

$$\iota^* : H^*(BU(n)) \to H^*(BU(n-1)) = \mathbb{Z}[c_1, \cdots, c_{n-1}]$$

is a split surjection of rings. But by the exactness of the Gysin sequence 6.7 this implies that this long exact splits into short exact sequences,

$$0 \to H^{*-2n}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^*(BU(n)) \xrightarrow{\iota^*} H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \cdots c_{n-1}] \to 0$$

Define $c_n \in H^{2n}(BU(n))$ to be the Euler class $\chi(\gamma_n)$. Then this sequence becomes

$$0 \to H^{*-2n}(BU(n)) \xrightarrow{\cup c_n} H^*(BU(n)) \xrightarrow{\iota^*} \mathbb{Z}[c_1, \cdots c_{n-1}] \to 0$$

which implies that $H^*(BU(n)) \cong \mathbb{Z}[c_1, \cdots, c_n]$. This completes the inductive step in this case.

In the real case now consider the Gysin sequence 6.8, and observe that by exactness, for $q < n-1$, the homomorphism

$$\iota^* : H^q(BO(n); \mathbb{Z}_2) \to H^q(BO(n-1); \mathbb{Z}_2)$$

is an isomorphism. That means there are unique classes, $w_1, \cdots, w_{n-2} \in H^*(BO(n); \mathbb{Z}_2)$ that map via ι^* to the classes of the same name in $H^*(BO(n-1); \mathbb{Z}_2)$.

In dimension $q = n-1$, the exactness of the Gysin sequence tells us that the homomorphism $\iota^* H^{n-1}(BO(n); \mathbb{Z}_2) \to H^{n-1}(BO(n-1); \mathbb{Z}_2)$ is injective. Also by exactness we see that ι^* is surjective if and only if $\chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2)$ is nonzero. But to see this, by the universal property of γ_n, it suffices to prove that there exists some n -dimensional bundle ζ with Euler class $\chi(\zeta) \neq 0$. Now by 6.12, the Euler class of the product

$$\begin{aligned}\chi(\gamma_k \times \gamma_{n-k}) &= \chi(\gamma_k) \otimes \chi(\gamma_{n-k}) \in H^k(BO(k) \times BO(n-k); \mathbb{Z}_2) \\ &= w_k \otimes w_{n-k} \in H^*(BO(k); \mathbb{Z}_2) \otimes H^{n-k}(BO(n-k); \mathbb{Z}_2)\end{aligned}$$

which, by the inductive assumption is nonzero for $k \geq 1$. Thus $\chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2)$ is nonzero, and we define it to be the n^{th} Stiefel - Whitney class

$$w_n = \chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2).$$

As observed above, the nontriviality of $\chi(\gamma_n)$ implies that $\iota^* H^{n-1}(BO(n);\mathbb{Z}_2) \to H^{n-1}(BO(n-1);\mathbb{Z}_2)$ is an isomorphism, and hence there is a unique class $w_{n-1} \in H^{n-1}(BO(n-1);\mathbb{Z}_2)$ (as well as $w_1, \cdots w_{n-2}$) restricting to the inductively defined classes of the same names in $H^*(BO(n-1);\mathbb{Z}_2)$.

Furthermore, since ι^* is a ring homomorphism, every polynomial in $w_1, \cdots, w_{n-1}$ in $H^*(BO(n-1);\mathbb{Z}_2)$ is in the image under ι^* of the corresponding polynomial in the these classes in $H^*(BO(n);\mathbb{Z}_2)$. Hence by our inductive assumption,

$$\iota^* : H^*(BO(n);\mathbb{Z}_2) \to H^*(BO(n-1);\mathbb{Z}_2) = \mathbb{Z}_2[w_1, \cdots, w_{n-1}]$$

is a split surjection of rings. But by the exactness of the Gysin sequence 6.8 this implies that this long exact splits into short exact sequences,

$$0 \to H^{*-n}(BO(n);\mathbb{Z}_2) \xrightarrow{\cup w_n} H^*(BO(n);\mathbb{Z}_2) \xrightarrow{\iota^*} H^*(BO(n-1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots w_{n-1}] \to 0$$

which implies that $H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_n]$. This completes the inductive step and therefore the proof of the theorem. □

6.3 The product formula and the splitting principle

Perhaps the most important calculational tool for characteristic classes is the Whitney sum formula, which we now state and prove.

Theorem 6.16. *a. Let ζ and η be vector bundles over a space X. Then the Stiefel - Whitney classes of the Whitney sum bundle $\zeta \oplus \eta$ are given by*

$$w_k(\zeta \oplus \eta) = \sum_{j=0}^{k} w_j(\zeta) \cup w_{k-j}(\eta) \in H^k(X;\mathbb{Z}_2).$$

where by convention, $w_0 = 1 \in H^0(X;\mathbb{Z}_2)$.

b. If ζ and η are complex vector bundles, then the Chern classes of the Whitney sum bundle $\zeta \oplus \eta$ are given by

$$c_k(\zeta \oplus \eta) = \sum_{j=0}^{k} c_j(\zeta) \cup c_{k-j}(\eta) \in H^{2k}(X).$$

Again, by convention, $c_0 = 1 \in H^0(X)$.

Proof. We prove the formula in the real case. The complex case is done the same way.

Let ζ be an n - dimensional vector bundle over X, and let η be an m - dimensional bundle. Let $N = n + m$. Since we are computing $w_k(\zeta \oplus \eta)$, we may assume that $k \leq N$, otherwise this characteristic class is zero.

We prove the Whitney sum formula by induction on $N \geq k$. We begin with the case $N = k$. Since $\zeta \oplus \eta$ is a k - dimensional bundle, the k^{th} Stiefel - Whitney class, $w_k(\zeta \oplus \eta)$ is equal to the Euler class $\chi(\zeta \oplus \eta)$. We then have

$$\begin{aligned} w_k(\zeta \oplus \eta) &= \chi(\zeta \oplus \eta) \\ &= \chi(\zeta) \cup \chi(\eta) \quad \text{by} \quad 6.12 \\ &= w_n(\zeta) \cup w_m(\eta). \end{aligned}$$

This is the Whitney sum formula in this case as one sees by inputting the fact that for a bundle ρ with $j > dim\,(\rho)$, $w_j(\rho) = 0$.

Now inductively assume that the Whitney sum formula holds for computing w_k for any sum of bundles whose sum of dimensions is less that or equal to $N - 1$ which is greater than or equal to k. Let ζ have dimension n and η have dimension m with $n + m = N$. To complete the inductive step we need to compute $w_k(\zeta \oplus \eta)$.

Suppose ζ is classified by a map $f_\zeta : X \to BO(n)$, and η is classified by a map $f_\eta : X \to BO(m)$. Then $\zeta \oplus \eta$ is classified by the composition

$$f_{\zeta \oplus \eta} : X \xrightarrow{f_\zeta \times f_\eta} BO(n) \times BO(m) \xrightarrow{\quad \mu \quad} BO(n+m)$$

where μ is the map that classifies the product of the universal bundles $\gamma_n \times \gamma_m$ over $BO(n) \times BO(m)$. Equivalently, μ is the map on classifying spaces induced by the inclusion homomorphism of the subgroup $O(n) \times O(m) \hookrightarrow O(n + m)$. Thus to prove the theorem we must show that the map $\mu : BO(n) \times BO(m) \to BO(n + m)$ has the property that

$$\mu^*(w_k) = \sum_{j=0}^{k} w_j \otimes w_{k-j} \in H^*(BO(n); \mathbb{Z}_2) \otimes H^*(BO(m); \mathbb{Z}_2). \tag{6.9}$$

For a fixed $j \leq k$, let

$$p_j : H^k(BO(n) \times BO(m); \mathbb{Z}_2) \to H^j(BO(n); \mathbb{Z}_2) \otimes H^{k-j}(BO(m); \mathbb{Z}_2)$$

be the projection onto the summand. So we need to show that $p_j(\mu^*(w_k)) = w_j \otimes w_{k-j}$. Now since $n + m = N > k$, then either $j < n$ or $k - j < m$ (or both). We assume without loss of generality that $j < n$. Now by the proof of Theorem 6.5

$$\iota^* : H^j(BO(n); \mathbb{Z}_2) \to H^j(BO(j); \mathbb{Z}_2)$$

is an isomorophism. Moreover we have a commutative diagram:

$$\begin{array}{ccccc}
H^k(BO(N);\mathbb{Z}_2) & \xrightarrow{\mu^*} & H^k(BO(n)\times BO(m);\mathbb{Z}_2) & \xrightarrow{p_j} & H^j(BO(n);\mathbb{Z}_2)\otimes H^{k-j}(BO(m);\mathbb{Z}_2) \\
\iota^* \downarrow & & & & \downarrow \iota^*\otimes 1 \\
H^k(BO(j+m);\mathbb{Z}_2) & \xrightarrow[\mu^*]{} & H^k(BO(j)\times BO(m);\mathbb{Z}_2) & \xrightarrow[p_j]{} & H^j(BO(j);\mathbb{Z}_2)\otimes H^{k-j}(BO(m);\mathbb{Z}_2).
\end{array}$$

Since $j < n$, $j+m < n+m = N$ and $\iota^*(w_k) = w_k \in H^k(BO(j+m);\mathbb{Z}_2)$. This fact and the commutativity of this diagram give,

$$\begin{aligned}
(\iota^*\otimes 1)\circ p_j\circ\mu^*(w_k) &= p_j\circ\mu^*\circ\iota^*(w_k) \\
&= p_j\circ\mu^*(w_k) \\
&= w_j\otimes w_{k-j} \quad \text{by the inductive assumption.}
\end{aligned}$$

Since $\iota^*\otimes 1$ is an isomorphism in this dimension, and since $\iota^*(w_j\otimes w_{k-j}) = w_j\otimes w_{k-j}$ we have that

$$p_j\circ\mu^*(w_k) = w_j\otimes w_{k-j}.$$

As remarked above, this suffices to complete the inductive step in the proof of the theorem. □

We can restate the Whitney sum formula in the following convenient way. For an n - dimensional bundle ζ, let

$$w(\zeta) = 1 + w_1(\zeta) + w_2(\zeta) + \cdots + w_n(\zeta) \in H^*(X;\mathbb{Z}_2)$$

This is called the *total Stiefel - Whitney class.* The total Chern class of a complex bundle is defined similarly.

The Whitney sum formula can be interpreted as saying these total characteristic classes have the "exponential property" that they take sums to products. That is, we have the following:

Corollary 6.17.

$$w(\zeta\oplus\eta) = w(\zeta)\cup w(\eta)$$

and

$$c(\zeta\oplus\eta) = c(\zeta)\cup c(\eta).$$

Our next observation implies that these characteristic classes are invariants of the stable isomorphism types of bundles:

Corollary 6.18. *If ζ and η are stably equivalent real vector bundles over a space X, then*

$$w(\zeta) = w(\eta) \in H^*(X;\mathbb{Z}_2),$$

Similarly if they are complex bundles,

$$c(\zeta) = c(\eta) \in H^*(X).$$

Proof. If ζ and η are stably equivalent, then

$$\zeta \oplus \epsilon^m \cong \eta \oplus \epsilon^r$$

for some m and r. So

$$w(\zeta \oplus \epsilon^m) = w(\eta \oplus \epsilon^r).$$

But by 6.17

$$w(\zeta \oplus \epsilon^m) = w(\zeta)w(\epsilon) = w(\zeta) \cdot 1 = w(\zeta).$$

Similarly $w(\eta \oplus \epsilon^r) = w(\eta)$. The statement follows. The complex case is proved in the same way. □

By our description of K - theory in chapter 3, we have that these characteristic classes define invariants of K - theory.

Theorem 6.19. *The Chern classes c_i and the Stiefel - Whitney classes w_i define natural transformations*

$$c_i : K(X) \to H^{2i}(X)$$

and

$$w_i : KO(X) \to H^i(X; \mathbb{Z}_2).$$

The total characteristic classes

$$c : K(X) \to \bar{H}^*(X)$$

and

$$w : KO(X) \to \bar{H}^*(X; \mathbb{Z}_2)$$

are exponential in the sense that

$$c(\alpha + \beta) = c(\alpha)c(\beta) \quad \textit{and} \quad w(\alpha + \beta) = w(\alpha)w(\beta).$$

Here $\bar{H}^*(X)$ is the direct product $\bar{H}^*(X) = \prod_q H^q(X)$.

As an immediate application of these product formulas, we can deduce a "splitting principle" for characteristic classes. We now explain this principle.

Recall that an n - dimensional bundle ζ over X splits as a sum of n line bundles if and only if its associated principal bundle has an $O(1) \times \cdots \times O(1)$ - structure. That is, the classifying map $f_\zeta : X \to BO(n)$ lifts to the n -fold product, $BO(1)^n$. The analogous observation also holds for complex vector bundles. If we have such a lifting, then in cohomology, $f_\zeta^* : H^*(BO(n); \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ factors through $\otimes_n H^*(BO(1); \mathbb{Z}_2)$.

The "splitting principle" for characteristic classes says that this cohomological property always happens.

To state this more carefully, recall that $H^*(BO(1);\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$. Thus

$$H^*(BO(1)^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \cdots, x_n]$$

where $x_j \in H^1$ is the generator of the cohomology of the j^{th} factor in this product. Similarly,

$$H^*(BU(1)^n) \cong \mathbb{Z}[y_1, \cdots, y_n]$$

where $y_j \in H^2$ is the generator of the cohomology of the j^{th} factor in this product.

Notice that the symmetric group Σ_n acts on these polynomial algebras by permuting the generators. The subalgebra consisting of polynomials fixed under this symmetric group action is called the algebra of symmetric polynomials, $Sym[x_1, \cdots, x_n]$ or $Sym[y_1, \cdots, y_n]$.

Theorem 6.20. *(***Splitting Principle.)** The maps

$$\mu : BU(1)^n \to BU(n) \quad \text{and} \quad \mu : BO(1)^n \to BO(n)$$

induce injections in cohomology

$$\mu^* : H^*(BU(n)) \to H^*(BU(1)^n) \quad \text{and} \quad \mu^* : H^*(BO(n);\mathbb{Z}_2) \to H^*(BO(1)^n;\mathbb{Z}_2).$$

Furthermore the images of these monomorphisms are the symmetric polynomials

$$H^*(BU(n)) \cong Sym[y_1, \cdots, y_n] \quad \text{and} \quad H^*(BO(n);\mathbb{Z}_2) \cong Sym[x_1, \cdots, x_n].$$

Proof. By the Whitney sum formula,

$$\mu^*(w_j) = \sum_{j_1+\cdots+j_n=j} w_{j_1} \otimes \cdots \otimes w_{j_n} \in H^*(BO(1);\mathbb{Z}_2) \otimes \cdots \otimes H^*(BO(1);\mathbb{Z}_2).$$

But $w_i(\gamma_1) = 0$ unless $i = 0, 1$. So

$$\mu^*(w_j) = \sum_{1 \leq i_1 < \cdots < i_j \leq n} x_{i_1} \cdots x_{i_j} \in \mathbb{Z}_2[x_1, \cdots, x_n].$$

This is the j^{th} - elementary symmetric polynomial, $\sigma_j(x_1, \cdots, x_n)$. Thus the image of $\mathbb{Z}_2[w_1, \cdots, w_n] = H^*(BO(n);\mathbb{Z}_2)$ is the subalgebra of $\mathbb{Z}_2[x_1, \cdots, x_n]$ generated by the elementary symmetric polynomials, $\mathbb{Z}_2[\sigma_1, \cdots, \sigma_n]$. But it is well known that the elementary symmetric polynomials generate $Sym[x_1, \cdots, x_n]$ (see [88]). The complex case is proved similarly. □

This result gives another way of producing characteristic classes which is particularly useful in index theory.

Let $p(x)$ be a power series in one variable, which is assumed to have a grading equal to one. Say

$$p(x) = \sum_i a_i x^i.$$

Consider the corresponding symmetric power series in n -variables,

$$p(x_1, \cdots, x_n) = p(x_1) \cdots p(x_n).$$

Let $p_j(x_1, \cdots, x_n)$ be the homogeneous component of $p(x_1, \cdots, x_n)$ of grading j. So

$$p_j(x_1, \cdots, x_n) = \sum_{i_1 + \cdots + i_n = j} a_{i_1} \cdots a_{i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

Since p_j is symmetric, by the splitting principle we can think of

$$p_j \in H^j(BO(n); \mathbb{Z}_2)$$

and hence determines a characteristic class (i.e a polynomial in the Stiefel - Whitney classes).

Similarly if we give x grading 2, we can think of $p_j \in H^{2j}(BU(n))$ and so determines a polynomial in the Chern classes.

In particular, given a real valued smooth function $y = f(x)$, its Taylor series $p_f(x) = \sum_k \frac{f^{(k)}(0)}{k!} x^k$ determines characteristic classes $f_i \in H^i(BO(n); \mathbb{Z}_2)$ or $f_i \in H^{2i}(BU(n); \mathbb{Z}_2)$.

Exercise. Consider the examples $f(x) = e^x$, and $f(x) = tanh(x)$. Write the low dimensional characteristic classes f_i in $H^*(BU(n))$ for $i = 1, 2, 3$, as explicit polynomials in the Chern classes.

6.4 Applications

In this section all cohomology will be taken with $\mathbb{Z}_2$ - coefficients, even if not explicitly written.

6.4.1 Characteristic classes of manifolds

We have seen that the characteristic classes of trivial bundles are trivial. However the converse is not true, as we will now see, by examining the characteristic classes of manifolds.

Definition 6.6. *The characteristic classes of a manifold M, $w_j(M)$, $c_i(M)$, are defined to be the characteristic classes of the tangent bundle, τM.*

Theorem 6.21. *$w_j(S^n) = 0$ for all $j, n > 0$.*

Proof. As we saw in chapter 1, the normal bundle of the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a trivial line bundle. Thus

$$\tau S^n \oplus \epsilon_1 \cong \epsilon_{n+1}$$

and so τS^n is stably trivial. The theorem follows. □

Of course we know τS^2 is nontrivial since it has no nowhere zero cross sections. Thus the Stiefel- Whitney classes do not form a complete invariant of the bundle. However they do constitute a very important class of invariants, as we will see below.

Write $a \in H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ as the generator. Then the total Stiefel - Whitney class of the canonical line bundle γ_1 is

$$w(\gamma_1) = 1 + a \in H^*(\mathbb{RP}^n).$$

This allows us to compute the Stiefel - Whitney classes of $\mathbb{RP}^n$ (i.e of the tangent bundle $T(\mathbb{RP}^n)$).

Theorem 6.22. *$w(\mathbb{RP}^n) = (1 + a)^{n+1} \in H^*(\mathbb{RP}^n; \mathbb{Z}_2)$. So $w_j(\mathbb{RP}^n) = \binom{n+1}{j} a^j \in H^j(\mathbb{RP}^n)$.*

Note: Even though the polynomial $(1 + a)^{n+1}$ has highest degree term a^{n+1}, this class is zero in $H^*(\mathbb{RP}^n)$ since $H^{n+1}(\mathbb{RP}^n) = 0$.

Proof. As seen in Chapter 3,

$$T\mathbb{RP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1.$$

Thus

$$\begin{aligned} w(T\mathbb{RP}^n) &= w(T(\mathbb{RP}^n) \oplus \epsilon_1) \\ &= w(\oplus_{n+1}\gamma_1) \\ &= w(\gamma_1)^{n+1}, \quad \text{by the Whitney sum formula} \\ &= (1 + a)^{n+1}. \end{aligned}$$

□

Observation. The same argument shows that the total Chern class of $\mathbb{CP}^n$ is

$$c(\mathbb{CP}^n) = (1 + b)^{n+1} \tag{6.10}$$

where $b \in H^2(\mathbb{CP}^n \mathbb{Z})$ is the generator.

This calculation of the Stiefel - Whitney classes of $\mathbb{RP}^n$ allows us to rule out the possibility that many of these projective spaces are parallelizable.

Corollary 6.23. *If $\mathbb{RP}^n$ is parallelizable, then n is of the form $n = 2^k - 1$ for some k.*

Proof. We show that if $n \neq 2^k - 1$ then there is some $j > 0$ such that $w_j(\mathbb{RP}^n) \neq 0$. But $w_j(\mathbb{RP}^n) = \binom{n+1}{j} a^j$, so we are reduced to verifying that if m is not a power of 2, then there is a $j \in \{1, \cdots, m-1\}$ such that $\binom{m}{j} \equiv 1 \bmod 2$. This follows immediately from the following combinatorial lemma, whose proof we leave to the reader.

Lemma 6.24. *Let $j \in \{1, \cdots, m-1\}$. Write j and m in their binary representations,*

$$m = \sum_{i=0}^{k} a_i 2^i$$

$$j = \sum_{i=0}^{k} b_i 2^i$$

where the a_i's and b_i's are either 0 or 1. Then

$$\binom{m}{j} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \, mod\, 2.$$

Note. Here we are adopting the usual conventions that $\binom{0}{0} = 1$, $\binom{1}{0} = 1$, and $\binom{0}{1} = 0$.

□

Since we know that Lie groups are parallelizable, this result says that $\mathbb{RP}^n$ can only have a Lie group structure if n is of the form $2^k - 1$. However a famous theorem of Adams [3] says that the only $\mathbb{RP}^n$'s that are parallelizable are $\mathbb{RP}^1$, $\mathbb{RP}^3$, and $\mathbb{RP}^7$.

Now as seen in chapter 2 an n - dimensional vector bundle ζ^n has k - linearly independent cross sections if and only if

$$\zeta^n \cong \rho^{n-k} \oplus \epsilon_k$$

for some $n-k$ dimensional bundle ρ. Moreover, having this structure is equivalent to the classifying map

$$f_\zeta : X \to BO(n)$$

having a lift (up to homotopy) to a map $f_\rho : X \to BO(n-k)$.

Now the Stiefel - Whitney classes give natural obstructions to the existence of such a lift because the map $\iota : BO(n-k) \to BO(n)$ induces the map of rings

$$\iota^* : \mathbb{Z}_2[w_1, \cdots, w_n] \to \mathbb{Z}_2[w_1, \cdots w_{n-k}]$$

that maps w_j to w_j for $j \leq n-k$, and w_j to 0 for $n \geq j > n-k$. We therefore have the following result.

Theorem 6.25. *Let ζ be an n -dimensional bundle over X. Suppose $w_k(\zeta)$ is nonzero in $H^k(X;\mathbb{Z}_2)$. Then ζ has no more than $n-k$ linearly independent cross sections. In particular, if $w_n(\zeta) \neq 0$, then ζ does not have a nowhere zero cross section.*

This result has applications to the existence of linearly independent vector fields on a manifold. The following is an example.

Theorem 6.26. *If m is even, $\mathbb{RP}^m$ does not have a nowhere zero vector field.*

Proof. By 6.22

$$\begin{aligned} w_m(\mathbb{RP}^m) &= \binom{m+1}{m} a^m \\ &= (m+1)a^m \in H^m(\mathbb{RP}^m; \mathbb{Z}_2). \end{aligned}$$

For m even this is nonzero. Hence $w_m(\mathbb{RP}^m) \neq 0$. □

6.4.2 *Spin* and *Spin*$_{\mathbb{C}}$ structures

In this section we describe the notions of *Spin* and *Spin*$_{\mathbb{C}}$ structures on vector bundles. We then use the Serre spectral sequence to identify characteristic class conditions for the existence of these structures. These structures are particularly important in geometry, geometric analysis, and geometric topology.

Recall from Chapter 5 that an n - dimensional vector bundle ζ over a space X is orientable if and only if it has a $SO(n)$ - structure, which exists if and only if the classifying map $f_\zeta : X \to BO(n)$ has a homotopy lifting to $BSO(n)$. In Chapter 5 we proved the following property as well.

Proposition 6.27. *The n - dimensional bundle ζ is orientable if and only if its first Stiefel - Whitney class is zero,*

$$w_1(\zeta) = 0 \in H^1(X; \mathbb{Z}_2).$$

A *Spin* structure on ζ is a refinement of an orientation. To define it we need to further study the topology of $SO(n)$.

The group $O(n)$ has two path components, i.e $\pi_0(O(n)) \cong \mathbb{Z}_2$ and $SO(n)$ is the path component of the identity map. In particular $SO(n)$ is connected, so $\pi_0(SO(n)) = 0$. We have the following information about $\pi_1(SO(n))$.

Proposition 6.28. *$\pi_1(SO(2)) = \mathbb{Z}$. For $n \geq 3$, we have*

$$\pi_1(SO(n)) = \mathbb{Z}_2.$$

Proof. $SO(2)$ is topologically a circle, so the first part of the theorem follows. $SO(3)$ is topologically the projective space

$$SO(3) \cong \mathbb{RP}^3$$

which has a double cover $\mathbb{Z}_2 \to S^3 \to \mathbb{RP}^3$. Since S^3 is simply connected, this is the universal cover of $\mathbb{RP}^3$ and hence $\mathbb{Z}_2 = \pi_1(\mathbb{RP}^3) = \pi_1(SO(3))$.

Now for $n \geq 3$, consider the fiber bundle $SO(n) \to SO(n+1) \to SO(n+1)/SO(n) = S^n$. By the long exact sequence in homotopy groups for this fibration we see that $\pi_1(SO(n)) \to \pi_1(SO(n+1))$ is an isomorphism for $n \geq 3$. The result follows by induction on n. □

Since $\pi_1(SO(n)) = \mathbb{Z}_2$, the universal cover of $SO(n)$ is a double covering. The group $Spin(n)$ is defined to be this universal double cover:

$$\mathbb{Z}_2 \to Spin(n) \to SO(n).$$

Exercise. Show that $Spin(n)$ is a group and that the projection map $p : Spin(n) \to SO(n)$ is a group homomorphism with kernel $\mathbb{Z}_2$.

Now the group $Spin(n)$ has a natural $\mathbb{Z}_2$ action, since it is the double cover of $SO(n)$. Define the group $Spin_{\mathbb{C}}(n)$ using this $\mathbb{Z}_2$ - action in the following way.

Definition 6.7. *The group $Spin_{\mathbb{C}}(n)$ is defined to be*

$$Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1).$$

where $\mathbb{Z}_2$ acts on $U(1)$ by $z \to -z$ for $z \in U(1) \subset \mathbb{C}$.

Notice that there is a principal $U(1)$ - bundle,

$$U(1) \to Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \to Spin(n)/\mathbb{Z}_2 = SO(n).$$

$Spin_{\mathbb{C}}$ - structures have been shown to be quite important in the Seiberg - Witten theory approach to the study of smooth structures on four dimensional manifolds [87].

The main theorem of this subsection is the following:

Theorem 6.29. *Let ζ be an oriented n - dimensional vector bundle over a CW - complex X. Let $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$ be the second Stiefel - Whitney class of ζ. Then*

1. *ζ has a $Spin(n)$ structure if and only if $w_2(\zeta) = 0$.*

2. *ζ has a $Spin_{\mathbb{C}}(n)$ - structure if and only if $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$ comes from an integral cohomology class. That is, if and only if there is a class $c \in H^2(X;\mathbb{Z})$ which maps to $w_2(\zeta)$ under the projection map*
$$H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2).$$

Proof. The question of the existence of a $Spin$ or $Spin_{\mathbb{C}}$ structure is equivalent to the existence of a homotopy lifting of the classifying map $f_\zeta : X \to BSO(n)$ to $BSpin(n)$ or $BSpin_{\mathbb{C}}(n)$. To examine the obstructions to obtaining such liftings we first make some observations about the homotopy type of $BSO(n)$.

We know that $BSO(n) \to BO(n)$ is a double covering (the orientation double cover of the universal bundle). Furthermore $\pi_1(BO(n)) = \pi_0(O(n)) = \mathbb{Z}_2$, so this is the universal cover of $BO(n)$. In particular this says that $BSO(n)$ is simply connected and
$$\pi_i(BSO(n)) \to \pi_i(BO(n))$$
is an isomorphism for $i \geq 2$.

Recall that for n odd, say $n = 2m+1$, then there is an isomorphism of groups
$$SO(2m+1) \times \mathbb{Z}_2 \cong O(2m+1).$$

Exercise. Prove this!

This establishes a homotopy equivalence
$$BSO(2m+1) \times B\mathbb{Z}_2 \cong BO(2m+1).$$

The following is then immediate from our knowledge of $H^*(BO(2m+1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_{2m+1}]$ and $H^*(B\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1]$.

Lemma 6.30.
$$H^*(BSO(2m+1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, \cdots, w_{2m+1}]$$
where $w_i \in H^i(BSO(2m+1);\mathbb{Z}_2)$ is the i^{th} Stiefel - Whitney class of the universal oriented $(2m+1)$ - dimensional bundle classified by the natural map $BSO(2m+1) \to BO(2m+1)$.

Corollary 6.31. *For $n \geq 3$, $H^2(BSO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$, with nonzero class w_2.*

Proof. This follows from the lemma and the fact that for $n \geq 3$ the inclusion $BSO(n) \to BSO(n+1)$ induces an isomorphism in H^2, which can be seen by looking at the Serre exact sequence for the fibration $S^n \to BSO(n) \to BSO(n+1)$. □

This allows us to prove the following.

Lemma 6.32. *The classifying space $BSpin(n)$ is homotopy equivalent to the homotopy fiber F_{w_2} of the map*

$$w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$$

classifying the second Stiefel - Whitney class $w_2 \in H^2(BSO(n);\mathbb{Z}_2)$.

Proof. The group $Spin(n)$ is the universal cover of $SO(n)$, and hence is simply connected. This means that $BSpin(n)$ is 2 - connected. By the Hurewicz theorem this implies that $H^2(BSpin(n);\mathbb{Z}_2) = 0$. Thus the composition

$$BSpin(n) \xrightarrow{p} BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$$

is null homotopic. Convert the map w_2 to a homotopy equivalent fibration, $\tilde{w}_2 : \tilde{B}SO(n) \to K(\mathbb{Z}_2, 2)$. The map p defines a map (up to homotopy) $\tilde{p} : BSpin(n) \to \tilde{B}SO(n)$, and the composition $\tilde{p} \circ w_2$ is still null homotopic. A null homotopy $\Phi : BSpin(n) \times I \to K(\mathbb{Z}_2, 2)$ between $\tilde{p} \circ w_2$ and the constant map at the basepoint, lifts, due to the homotopy lifting property, to a homotopy $\tilde{\Phi} : BSpin(n) \times I \to \tilde{B}SO(n)$ between $\tilde{p}$ and a map $\bar{p}$ whose image lies entirely in the fiber over the basepont, F_{w_2},

$$\bar{p} : BSpin(n) \to F_{w_2}.$$

We claim that $\bar{p}$ induces an isomorphism in homotopy groups. To see this, observe that the homomorphism $p_q : \pi_q(BSpin(n)) \to \pi_q(BSO(n))$ is equal to the homomorphism $\pi_{q-1}(Spin(n)) \to \pi_{q-1}(SO(n))$ which is an isomorphism for $q \geq 3$ because $Spin(n) \to SO(n)$ is the universal cover. But similarly $\pi_q(F_{w_2}) \to \pi_q(BSO(n))$ is also an isomorphism for $q \geq 3$ by the exact sequence in homotopy groups of the fibration $F_{w_2} \to BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$, since w_2 induces an isomorphism on π_2. $BSpin(n)$ and F_{w_2} are also both 2 - connected. Thus they have the same homotopy groups, and we have a commutative square for $q \geq 3$,

$$\begin{array}{ccc} \pi_q(BSpin(n)) & \xrightarrow{\bar{p}_*} & \pi_q(F_{w_2}) \\ p\downarrow\cong & & \downarrow\cong \\ \pi_q(BSO(n)) & \xrightarrow[=]{} & \pi_q(BSO(n)). \end{array}$$

Thus $\bar{p} : BSpin(n) \to F_{w_2}$ induces an isomorphism in homotopy groups, and by the Whitehead theorem is a homotopy equivalence. □

Notice that we are now able to complete the proof of the first part of the theorem. If ζ is any oriented, n - dimensional bundle with $Spin(n)$ structure, its classifying map$f_\zeta : X \to BSO(n)$ lifts to a map $\tilde{f}_\zeta : X \to BSpin(n)$, and hence by this lemma, $w_2(\zeta) = f_\zeta^*(w_2) = \tilde{f}_\zeta^* \circ p^*(w_2) = 0$. Conversely, if $w_2(\zeta) = 0$, then the classifying map $f_\zeta : X \to BSO(n)$ has the property that $f_\zeta^*(w_2) = 0$. This implies that the composition

$$X \xrightarrow{f_\zeta} BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$$

is null homotopic. A null homotopy lifts to give a homotopy between f_ζ and a map whose image lies in the homotopy fiber F_{w_2}, which, by the above lemma is homotopy equivalent to $BSpin(n)$. Thus $f_\zeta : X \to BSO(n)$ has a homotopy lift $\tilde{f}_\zeta : X \to BSpin(n)$, which implies that ζ has a $Spin(n)$ - structure.

We now turn our attention to $Spin_{\mathbb{C}}$ - structures.

Consider the projection map

$$p : Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \to U(1)/\mathbb{Z}_2 = U(1).$$

p is a group homomorpism with kernel $Spin(n)$. p therefore induces a map on classifying spaces, which we call c,

$$c : BSpin_{\mathbb{C}}(n) \to BU(1) = K(\mathbb{Z}, 2)$$

which has homotopy fiber $BSpin(n)$. But clearly we have the following homotopy commutative diagram

$$\begin{array}{ccccc} BSpin(n) & \xrightarrow{\subset} & B(Spin(n) \times_{bz_2} U(1)) & \xrightarrow{=} & BSpin_{\mathbb{C}}(n) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ BSpin(n) & \longrightarrow & B(Spin(n)/\mathbb{Z}_2) & \xrightarrow[=]{} & BSO(n) \end{array}$$

Therefore we have the following diagram between homotopy fibrations

$$\begin{array}{ccccc} BSpin(n) & \longrightarrow & BSpin_{\mathbb{C}}(n) & \xrightarrow{c} & K(\mathbb{Z}, 2) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ BSpin(n) & \longrightarrow & BSO(n) & \xrightarrow{w_2} & K(\mathbb{Z}_2, 2) \end{array}$$

where $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ is induced by the projection $\mathbb{Z} \to \mathbb{Z}_2$. As we've done before we can assume that $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ and $w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$ have been modified to be fibrations. Then this means that $BSpin_{\mathbb{C}}(n)$ is homotopy equivalent to the pull - back along w_2 of the fibration $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$:

$$BSpin_{\mathbb{C}}(n) \simeq w_2^*(K(\mathbb{Z}, 2)).$$

But this implies that the map $f_\zeta : X \to BSO(n)$ homotopy lifts to $BSpin_{\mathbb{C}}(n)$ if and only if there is a map $u : X \to K(\mathbb{Z}, 2)$ such that $p \circ u : X \to K(\mathbb{Z}_2, 2)$ is homotopic to $w_2 \circ f_\zeta : X \to K(\mathbb{Z}_2, 2)$. Interpreting these as cohomology classes, this says that f_ζ lifts to $BSpin_{\mathbb{C}}(n)$ (i.e ζ has a $Spin_{\mathbb{C}}(n)$ - structure) if and only if there is a class $u \in H^2(X; \mathbb{Z})$ so that the $\mathbb{Z}_2$ reduction of u, $p_*(u)$ is equal to $w_2(\zeta) \in H^2(X; \mathbb{Z}_2)$. This is the statement of the theorem. □

6.4.3 Normal bundles and immersions

Theorem 6.25 has important applications to the existence of immersions of a manifold M in Euclidean space, which we now discuss.

Let $e : M^n \looparrowright \mathbb{R}^{n+k}$ be an immersion. Recall that this means that the derivative at each point,

$$De(x) : T_x M^n \to T_{e(x)} \mathbb{R}^{n+k} = \mathbb{R}^{n+k}$$

is injective. Recall also that the Inverse Function Theorem implies that an immersion is a local embedding.

The immersion e defines a k - dimensional normal bundle ν_e^k whose fiber at $x \in M$ is the orthogonal complement of the image of $T_x M^n$ in $\mathbb{R}^{n+k}$ under $De(x)$. In particular we have

$$TM^n \oplus \nu_e^k \cong e^*(T\mathbb{R}^{n_k}) \cong \epsilon_{n+k}.$$

Thus we have the Whitney sum relation among the Stiefel - Whitney classes

$$w(M^n) \cdot w(\nu_e^k) = 1. \tag{6.11}$$

So we can compute the Stiefel - Whitney clases of the normal bundle formally as the power series

$$w(\nu_e^k) = 1/w(M) \in \bar{H}^*(M; \mathbb{Z}_2).$$

This proves the following:

Proposition 6.33. *The Stiefel - Whitney classes of the normal bundle to an immersion $e : M^n \looparrowright \mathbb{R}^{n+k}$ are independent of the immersion. They are called the normal Stiefel - Whitney classes, and are written $\bar{w}_i(M)$. These classes are determined by the formula*

$$w(M) \cdot \bar{w}(M) = 1.$$

Example. $\bar{w}(\mathbb{RP}^n) = 1/(1+a)^{n+1} \in \bar{H}^*(\mathbb{RP}^n;\mathbb{Z}_2)$.

So for example, when $n = 2^k$, $k > 0$, $w(\mathbb{RP}^{2^k}) = 1 + a + a^{2^k}$. This is true since by 6.24 $\binom{2^k+1}{r} \equiv 1 \bmod 2$ if and only if $r = 0, 1, 2^k$. Thus the total normal Stiefel - Whitney class is given by

$$\bar{w}(\mathbb{RP}^{2^k}) = 1/(1 + a + a^{2^k}) = 1 + a + a^2 + \cdots + a^{2^k-1}.$$

Note. The reason this series is truncated a a^{2^k-1} is because

$$(1 + a + a^{2^k})(1 + a + a^2 + \cdots + a^{2^k-1}) = 1 \in H^*(\mathbb{RP}^n;\mathbb{Z}_2)$$

since $H^q(\mathbb{RP}^n) = 0$ for $q > n$.

Corollary 6.34. *There is no immersion of $\mathbb{RP}^{2^k}$ in $\mathbb{R}^N$ for $N \le 2^{k+1} - 2$.*

Proof. The above calculation shows that $\bar{w}_{2^k-1}(\mathbb{RP}^{2^k}) \neq 0$. Thus it cannot have a normal bundle of dimension less than $2^k - 1$. The result follows. □

6.5 Pontrjagin Classes

In this section we define and study Pontrjagin classes. These are integral characteristic classes for real vector bundles and are defined in terms of the Chern classes of the complexification of the bundle. We will then show that polynomials in Pontrjagin classes and the Euler class define all possible characteristic classes for oriented, real vector bundles when the values of the characteristic classes is cohomology with coefficients in an integral domain R which contains $1/2$. By the classification theorem, to deduce this we must compute $H^*(BSO(n);R)$. For this calculation we follow the treatment given in Milnor and Stasheff [121].

6.5.1 Orientations and Complex Conjugates

We begin with a reexamination of certain basic properties of complex vector bundles.

Let V be an n - dimensional $\mathbb{C}$ - vector space with basis $\{v_1, \cdots, v_n\}$. By multiplication of these basis vectors by the complex number i, we get a collection of $2n$ - vectors $\{v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n\}$ which forms a basis for V as a real $2n$ - dimensional vector space. This basis then determines an orientation of the underlying real vector space V.

Exercise. Show that the orientation of V that the basis $\{v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n\}$ determines is independent of the choice of the original basis $\{v_1, \cdots, v_n\}$

Thus every complex vector space V has a canonical orientation. By choosing this orientation for every fiber of a complex vector bundle ζ, we see that every complex vector bundle has a canonical orientation. By the results of section 6.2 this means that every n - dimensional complex vector bundle ζ over a space X has a canonical choice of Thom class $u \in H^{2n}(T(\zeta))$ and hence Euler class

$$\chi(\zeta) = c_n(\zeta) \in H^{2n}(X).$$

Now given a complex bundle ζ there exists a *conjugate bundle* $\bar{\zeta}$ which is equal to ζ as a real, $2n$ - dimensional bundle, but whose complex structure is conjugate. More specifically, recall that a complex structure on a $2n$ - dimensional real bundle ζ determines and is determined by a linear transformation

$$J_\zeta : \zeta \to \zeta$$

with the property that $J_\zeta^2 = J_\zeta \circ J_\zeta = -id$. If ζ has a complex structure then J_ζ is just scalar multiplication by the complex number i on each fiber. If we replace J_ζ by $-J_\zeta$ we define a new complex structure on ζ referred to as the *conjugate* complex structure. We write $\bar{\zeta}$ to denote ζ with this structure. That is,

$$J_{\bar{\zeta}} = -J_\zeta.$$

Notice that the identity map

$$id : \zeta \to \bar{\zeta}$$

is anti-complex linear (or conjugate complex linear) in the sense that

$$id(J_\zeta \cdot v) = -J_{\bar{\zeta}} \cdot id(v).$$

We note that the conjugate bundle $\bar{\zeta}$ is often not isomorphic to ζ as complex vector bundles. For example, consider the two dimensional sphere as complex projective space

$$S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \infty.$$

The tangent bundle $\tau(\mathbb{CP}^1)$ has the induced structure as a complex line bundle.

Proposition 6.35. *The complex line bundles $\tau(S^2)$ and $\bar{\tau}(S^2)$ are not isomorphic.*

Proof. Suppose $\phi : \tau(S^2) \to \bar{\tau}(S^2)$ is a isomorphism as complex vector bundles. Then at every tangent space

$$\phi_x : T_x S^2 \to T_x S^2$$

is a an isomorphism that conjugates the complex structure. Any such isomorphism is given by reflection through a line ℓ_x in the tangent plane $T_x S^2$. Therefore for every x we have picked a line $\ell_x \subset T_x S^2$. This defines a (real) one dimensional subbundle ℓ of $\tau(S^2)$, which, by the classification theorem is given by an element of

$$[S^2, BO(1)] \cong H^1(S^2, \mathbb{Z}_2) = 0.$$

Thus ℓ is a trivial subbundle of $\tau(S^2)$. Hence we can find a nowhere vanishing vector field on S^2, which gives us a contradiction. □

Exercise. Let $\bar{\gamma}_n$ be the conjugate of the universal bundle γ_n over $BU(n)$. By the classification theorem, $\bar{\gamma}_n$ is classified by a map

$$q : BU(n) \to BU(n)$$

having the property that $q^*(\gamma_n) = \bar{\gamma}_n$. Using the Grassmannian model of $BU(n)$, find an explicit description of a map $q : BU(n) \to BU(n)$ with this property.

The following describes the effect of conjugating a vector bundle on its Chern classes.

Theorem 6.36. $c_k(\bar{\zeta}) = (-1)^k c_k(\zeta)$

Proof. Suppose ζ is an n - dimensional bundle. By the classification theorem and the functorial property of Chern classes it suffices to prove this theorem when ζ is the universal bundle γ_n over $BU(n)$. Now in our calculations of the cohomology of these classifying spaces, we proved that the inclusion $\iota : BU(k) \to BU(n)$ induces an isomorphism in cohomology in dimension k,

$$\iota^* : H^{2k}(BU(n)) \xrightarrow{\cong} H^{2k}(BU(k)).$$

Hence it suffices to prove this theorem for the universal k - dimensional bundle γ_k over $BU(k)$.

Now $c_k(\gamma_k) = \chi(\gamma_k)$ and similarly, $c_k(\bar{\gamma}_k) = \chi(\bar{\gamma}_k)$. So it suffices to prove that

$$\chi(\gamma_k) = (-1)^{-k}\chi(\bar{\gamma}_k).$$

But by the observations above, this is equivalent to showing that the canonical orientation of the underlying real $2k$ - dimensional bundle from the complex structures of γ_k and $\bar{\gamma}_k$ are the same if k is even, and opposite if k is odd. To do this we only need to compare the orientations at a single point. Let $x \in BU(k)$ be given by $\mathbb{C}^k \subset \mathbb{C}^\infty$ as the first k - coordinates. If $\{e_1, \cdots, e_k\}$ forms the standard basis for $\mathbb{C}^k$, then the orientations of $\gamma_k(x)$ determined by the complex structures of γ_k and $\bar{\gamma}_k$ are respectively represented by the real bases

$$\{e_1, ie_1, \cdots, e_k, ie_k\} \quad \text{and} \quad \{e_1, -ie_1, \cdots, e_k, -ie_k\}.$$

The change of basis matrix between these two basis has determinant $(-1)^k$. The theorem follows. □

Now suppose η is a real n - dimensional vector bundle over a space X, we then let $\eta_{\mathbb{C}}$ be its complexification

$$\eta_{\mathbb{C}} = \eta \otimes_{\mathbb{R}} \mathbb{C}.$$

$\eta_{\mathbb{C}}$ has the obvious structure as an n - dimensional complex vector bundle.

Proposition 6.37. *There is an isomorphism*

$$\phi : \eta_{\mathbb{C}} \xrightarrow{\cong} \bar{\eta}_{\mathbb{C}}.$$

Proof. Define

$$\begin{aligned} \phi : \eta_{\mathbb{C}} &\to \bar{\eta}_{\mathbb{C}} \\ \eta \otimes \mathbb{C} &\to \eta \otimes \bar{\mathbb{C}} \\ v \otimes z &\to v \otimes \bar{z} \end{aligned}$$

for $v \in \eta$ and $z \in \mathbb{C}$. Clearly ϕ is an isomorphism of complex vector bundles. □

Corollary 6.38. *For a real n - dimensional bundle η, then for k odd,*

$$2c_k(\eta_{\mathbb{C}}) = 0.$$

Proof. By 6.36 and 6.37

$$c_k(\eta_{\mathbb{C}}) = (-1)^k c_k(\eta_{\mathbb{C}}).$$

Hence for k odd $c_k(\eta_{\mathbb{C}})$ has order 2. □

6.5.2 Pontrjagin classes

We now use these results to define Pontrjagin classes for real vector bundles.

Definition 6.8. *Let η be an n - dimensional real vector bundle over a space X. Then define the i^{th} - Pontrjagin class*

$$p_i(\eta) \in H^{4i}(X;\mathbb{Z})$$

by the formula

$$p_i(\eta) = (-1)^i c_{2i}(\eta_{\mathbb{C}}).$$

Remark. The signs used in this definition are done to make calculations in the next section come out easily.

As we've done with Stiefel - Whitney and Chern classes, define the total Pontrjagin class

$$p(\eta) = 1 + p_1(\eta) + \cdots + p_i(\eta) + \cdots \in \bar{H}^*(X, \mathbb{Z}).$$

The following is the Whitney sum formula for Pontrjagin classes, and follows immediately for the Whitney sum formula for Chern classes and Corollary 6.38.

Theorem 6.39. *For real bundles η and ξ over X, we have*

$$2(p(\eta \oplus \xi) - p(\eta)p(\xi)) = 0 \in H^*(X;\mathbb{Z}).$$

In particular if R is a commutative integral domain containing $1/2$, then viewed as characteristic classes with values in $H^*(X;R)$, we have

$$p(\eta \oplus \xi) = p(\eta)p(\xi) \in \bar{H}^*(X : R).$$

Remark. Most often Pontryagin classes are viewed as having values in rational cohomology, and so the formula $p(\eta \oplus \xi) = p(\eta)p(\xi)$ applies.

We now study the Pontrjagin classes of a complex vector bundle. Let ζ be a complex n - dimensional bundle over a space X, and let $\zeta_{\mathbb{C}} = \zeta \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of its underlying real $2n$ - dimensional bundle. So $\zeta_{\mathbb{C}}$ is a complex $2n$ - dimensional bundle. We leave the proof of the following to the reader.

Proposition 6.40. *As complex $2n$ - dimensional bundles,*

$$\zeta_{\mathbb{C}} \cong \zeta \oplus \bar{\zeta}.$$

This result, together with 6.36 and the definition of Pontrjagin classes imply the following.

Corollary 6.41. *Let ζ be a complex n - dimensional bundle. Then its Pontryagin classes are determined by its Chern classes according to the formula*

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots + c_n) \\ &\in H^*(X, \mathbb{Z}). \end{aligned}$$

Example. We will compute the Pontrjagin classes of the tangent bundle of projective space, $\tau(\mathbb{CP}^n)$. Recall that the total Chern class is given by

$$c(\tau(\mathbb{CP}^n)) = (1+a)^{n+1}$$

where $a \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ is the generator. Notice that this implies that for the conjugate, $\bar{\tau}(\mathbb{CP}^n)$ we have

$$c(\bar{\tau}(\mathbb{CP}^n)) = (1-a)^{n+1}$$

Thus by the above formula we have

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1+a)^{n+1}(1-a)^{n+1} \\ &= (1-a^2)^{n+1}. \end{aligned}$$

We therefore have the formula

$$p_k(\mathbb{CP}^n) = \binom{n+1}{k} a^{2k} \in H^{4k}(\mathbb{CP}^n).$$

Notice that this example also implies the following:

Corollary 6.42. *The total Pontrjagin class of the complex projective space $\mathbb{CP}^n$ is given by*

$$p(\tau(\mathbb{CP}^n)) = (1+a^2)^{n+1}$$

Now let η be an oriented real n- dimensional vector bundle. Then the complexification $\eta_{\mathbb{C}} = \eta \otimes \mathbb{C} = \eta \oplus i\eta$ which is simply $\eta \oplus \eta$ as real vector bundles.

Lemma 6.43. *The above isomorphism*

$$\eta_{\mathbb{C}} \cong \eta \oplus \eta$$

of real vector bundles takes the canonical orientation of $\eta_{\mathbb{C}}$ to $(-1)^{\frac{n(n-1)}{2}}$ times the orientation of $\eta \oplus \eta$ induced from the given orientation of η.

Proof. Pick a particular fiber, η_x. Let $\{v_1, \cdots, v_n\}$ be a $\mathbb{C}$ - basis for V. Then the basis $\{v_1, iv_1, \cdots, v_n iv_n\}$ determines the orientation for $\eta_x \otimes \mathbb{C}$. However the basis $\{v_1, \cdots, v_n, iv_1, \cdots iv_n\}$ gives the natural basis for $(\eta \oplus i\eta)_x$. The change of basis matrix has determinant $(-1)^{\frac{n(n-1)}{2}}$. □

Corollary 6.44. *If η is an oriented $2k$ - dimensional real vector bundle, then*

$$p_k(\eta) = \chi(\eta)^2 \in H^{4k}(X).$$

Proof.

$$\begin{aligned}
p_k(\eta) &= (-1)^k c_{2k}(\eta \times \mathbb{C}) \\
&= (-1)^k \chi(\eta \otimes \mathbb{C}) \\
&= (-1)^k (-1)^{k(2k-1)} \chi(\eta \oplus \eta) \\
&= \chi(\eta \oplus \eta) \\
&= \chi(\eta)^2.
\end{aligned}$$

□

6.5.3 Oriented characteristic classes

We now use the results above to show that Pontrjagin classes and the Euler class yield all possible characteristic classes for oriented vector bundles, if the coefficient ring contains 1/2. More specifically we prove the following.

Theorem 6.45. *Let R be an integral domain containing $1/2$. Then*

$$\begin{aligned}
H^*(BSO(2n+1); R) &= R[p_1, \cdots, p_n] \\
H^*(BSO(2n); R) &= R[p_1, \cdots, p_{n-1}, \chi(\gamma_{2n})]
\end{aligned}$$

Remark. This theorem can be restated by saying that $H^*(BSO(n); R)$ is generated by $\{p_1, \cdots, p_{[n/2]}\}$ and χ, subject only to the relations

$$\begin{aligned}
\chi &= 0 \quad \text{if } n \text{ is odd} \\
\chi^2 &= p_{[n/2]} \quad \text{if } n \text{ is even.}
\end{aligned}$$

Proof. In this proof all cohomology will be taken with R coefficients. We first observe that since $SO(1)$ is the trivial group, $BSO(1)$ is contractible, and so $\tilde{H}^*(BSO(1)) = 0$. This will be the first step in an inductive proof. So we assume the theorem has been proved for $BSO(n-1)$, and we now compute $H^*(BSO(n))$ using the Gysin sequence:

$$\begin{array}{cccccc} \cdots \to H^{q-1}(BSO(n-1)) & \xrightarrow{\delta} & H^{q-n}(BSO(n)) & \xrightarrow{\cup\chi} & H^q(BSO(n)) & \xrightarrow{\iota^*} \\ H^q(BSO(n-1)) & \xrightarrow{\delta} & H^{q-n+1}(BSO(n)) & \xrightarrow{\cup\chi} & H^{q+1}(BSO(n)) \to \cdots & \end{array} \tag{6.12}$$

Case 1. n is even.

Since the first $n/2 - 1$ Pontrjagin classes are defined in $H^*(BSO(n))$ as well as in $H^*(BSO(n-1))$, the inductive assumption implies that $\iota^* : H^*(BSO(n)) \to H^*(BSO(n-1))$ is surjective. Thus the Gysin sequence reduces to short exact sequences

$$0 \to H^q(BSO(n)) \xrightarrow{\cup\chi} H^{q+n}(BSO(n)) \xrightarrow{\iota^*} H^{q+n}(BSO(n-1)) \to 0.$$

The inductive step then follows.

Case 2. n is odd, say $n = 2m+1$.

By Proposition 6.13 in this case the Euler class χ has order two in integral cohomology. Thus since R contains $1/2$, in cohomology with R coefficients, the Euler class is zero. Thus the Gysin sequence reduces to short exact sequences:

$$0 \to H^j(BSO(2m+1)) \xrightarrow{\iota^*} H^*(BSO(2m)) \to H^{j-2m}(BSO(2m+1)) \to 0.$$

Thus the map ι^* makes $H^*(BSO(2m+1))$ a subalgebra of $H^*(BSO(2m))$. This subalgeabra contains the Pontrjagin classes and hence it contains the graded algebra $A^* = R[p_1, \cdots p_m]$. By computing ranks we will now show that this is the entire image of ι^*. This will complete the inductive step in this case.

So inductively assume that the rank of A^{j-1} is equal to the rank of $H^j(BSO(2m+1))$. Now we know that every element of $H^j(BSO(2m))$ can be written uniquely as a sum $a + \chi b$ where $a \in A^j$ and $b \in A^{j-2m}$. Thus

$$H^j(BSO(2m)) \cong A^j \oplus A^{j-2m}$$

which implies that

$$rk(H^j(BSO(2m)) = rk(A^j) + rk(A^{j-2m}).$$

But by the exactness of the above sequence,

$$rk(H^j(BSO(2m)) = rk(H^j(BSO(2m+1)) + rk(H^{j-2m}(BSO(2m+1))).$$

Comparing these two equations, and using our inductive assumption, we conclude that

$$rk(H^j(BSO(2m+1)) = rk(A^j).$$

Thus $A^j = \iota^*(H^j(BSO(2m+1)))$, which completes the inductive argument.

□

6.6 Connections, Curvature, and Characteristic Classes

In this section we describe how Chern and Pontrjagin classes can be defined using connections (i.e covariant derivatives) on vector bundles. What we will describe is an introduction to the theory of Chern and Weil that describe the cohomology of a classifying space of a compact Lie group in terms of invariant polynomials on its Lie algebra. The treatment we will follow is from Milnor and Stasheff [121].

Definition 6.9. *Let $M_n(\mathbb{C})$ be the ring of $n \times n$ matrices over $\mathbb{C}$. Then an invariant polynomial* on $M_n(\mathbb{C})$ is a function

$$P : M_n(\mathbb{C}) \to \mathbb{C}$$

which can be expressed as a complex polynomial in the entries of the matrix, and satisfies,

$$P(ABA^{-1}) = P(B)$$

for every $B \in M_n(\mathbb{C})$ and $A \in GL(n, \mathbb{C})$.

Examples. The trace function $(a_{i,j}) \to \sum_{j=1}^n a_{j,j}$ and the determinant function are examples of invariant polynomials on $M_n(\mathbb{C})$.

Now let $D_A : \Omega^0(M; \zeta) \to \Omega^1(M; \zeta)$ be a connection (or covariant derivative) on a complex n - dimensional vector bundle ζ. Its curvature is a a two- form with values in the endomorphism bundle

$$F_A \in \Omega^2(M; End(\zeta))$$

The endomorphism bundle can be described alternatively as follows. Let E_ζ be the principal $GL(n, \mathbb{C})$ bundle associated to ζ. Then of course $\zeta = E_\zeta \otimes_{GL(n,\mathbb{C})} \mathbb{C}^n$. The endomorphism bundle can then be described as follows. The proof is an easy exercise that we leave to the reader.

Proposition 6.46.

$$End(\zeta) \cong ad(\zeta) = E_\zeta \times_{GL(n,\mathbb{C})} M_n(\mathbb{C})$$

where $GL(n,\mathbb{C})$ acts on $M_n(bc)$ by conjugation,

$$A \cdot B = ABA^{-1}.$$

Let ω be a differential p - form on M with values in $End(\zeta)$,

$$\omega \in \Omega^p(M; End(\zeta)) \cong \Omega^p(M; ad(\zeta)) = \Omega^p(M; E_\zeta \times_{GL(n,\mathbb{C})} M_n(\mathbb{C})).$$

Then on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ for ζ, and hence the induced coordinate chart and local trivialization for $ad(\zeta)$, ω can be viewed as an $n \times n$ matrix of p -forms on M. We write

$$\omega = (\omega_{i,j}).$$

Of course this description depends on the coordinate chart and local trivialization chosen, but at any $x \in U$, then by the above proposition, two trivializations yield conjugate matrices. That is, if $(\omega_{i,j}(x))$ and $(\omega'_{i,j}(x))$ are two matrix descriptions of $\omega(x)$ defined by two different local trivializations of $\zeta_{|_U}$, then there exists an $A \in GL(n,\mathbb{C})$ with

$$A(\omega_{i,j}(x))A^{-1} = (\omega'_{i,j}(x)).$$

Now let P be an invariant polynomial on $M_n(\mathbb{C})$ of degree d. Then using the wedge bracket we can apply P to a matrix of p forms, and produce a differential form of top dimension pd on $U \subset M$: $P(\omega_{i,j}) \in \Omega^{pd}(U)$. Now since the polynomial P is invariant under conjugation, the form $P(\omega_{i,j})$ is independent of the local trivialization of $\zeta_{|_U}$. These forms therefore fit together to give a well defined global form

$$P(\omega) \in \Omega^*(M). \tag{6.13}$$

If P is homogeneous of degree d, then

$$P(\omega) \in \Omega^{pd}(M). \tag{6.14}$$

An important example is when $\omega = F_A \in \Omega^2(M; End(\zeta))$ is the curvature form of a connection D_A on ζ. We have the following fundamental lemma, that will allow us to define characteristic classes in terms of these forms and invariant polynomials.

Lemma 6.47. *For any connection D_A and invariant polynomial (or invariant power series) P, the differential form $P(F_A)$ is closed. That is,*

$$dP(F_A) = 0.$$

Proof. (following Milnor and Stasheff [121]) Let P be an invariant polynomial or power series. We write $P(A) = P(a_{i,j})$ where the $a_{i,j}$'s are the entries of the matrix. We can then consider the matrix of partial derivatives $(\partial P/\partial(x_{i,j}))$ where the $x_{i,j}$'s are indeterminates. Let $F_A = (\omega_{i,j})$ be the curvature matrix of two - forms on an open set U with a given trivialization. Then the exterior derivative has the following local expression

$$dP(F_A) = \sum (\partial P/\partial \omega_{i,j}) d\omega_{i,j}. \tag{6.15}$$

In matrix notation this can be written as

$$dP(F_A) = trace(P'(F_A)dF_A)$$

Now as seen in Chapter 3, on a trivial bundle, and hence on this local coordinate patch, a connection D_A can be viewed as a matrix valued one form,

$$D_A = (\alpha_{i,j})$$

and with respect to which the curvature F_A has the formula

$$\omega_{i,j} = d\alpha_{i,j} - \sum_k \omega_{i,k} \wedge \omega_{k,j}.$$

In matrix notation we write

$$F_A = d\alpha - \alpha \wedge \alpha.$$

Differentiating yields the following form of the Bianchi identity

$$dF_A = \alpha \wedge F_A - F_A \wedge \alpha. \tag{6.16}$$

We need the following observation.

Claim. The transpose of the matrix of first derivatives of an invariant polynomial (or power series) $P'(A)$ commutes with A.

Proof. Let $E_{j,i}$ be the matrix with entry 1 in the (j,i)-th place and zeros in all other coordinates. Now differentiate the equation

$$P((I + tE_{j,i})A) = P(A(I + tE_{j,i}))$$

with respect to t and then setting $t = 0$ yields

$$\sum_k A_{i,k}(\partial P/\partial A_{j,k}) = \sum_k (\partial P/\partial A_{k,i})A_{k,i}.$$

Thus the matrix A commutes with the transpose of $(\partial P/\partial Ai, j)$ as claimed. □

We now complete the proof of the lemma. Substituting F_A for the matrix of indeterminates in the above claim means we have

$$F_A \wedge P'(F_A) = P'(F_A) \wedge F_A. \qquad (6.17)$$

Now for notational convenience let $X = P'(F_A) \wedge \alpha$. Then substituting the Bianchi identity 6.16 into 6.15 and using 6.17 we obtain

$$\begin{aligned} dP(F_A) &= trace\,(X \wedge F_A - F_A \wedge X) \\ &= \sum (X_{i,j} \wedge \omega_{j,i} - \omega_{j,i} \wedge X_{i,j}). \end{aligned}$$

Since each $X_{i,j}$ commutes with the 2 - form $\omega_{j,i}$, this sum is zero, which proves the lemma. □

Thus for any connection D_A on the complex vector bundle ζ over M, and invariant polynomial P, the form $P(F_A)$ represents a deRham cohomology class with complex coefficients. That is,

$$[P(F_A)] \in H^*(M : \mathbb{C}).$$

Theorem 6.48. *The cohomology class* $[P(F_A)] \in H^*(X, \mathbb{C})$ *is independent of the connection* D_A.

Proof. Let D_{A_0} and D_{A_1} be two connections on ζ. Pull back the bundle ζ over $M \times \mathbb{R}$ via the projection map $M \times \mathbb{R} \to M$. Call this pull - back bundle $\bar{\zeta}$ over $M \times \mathbb{R}$. We get the induced pull back connections $\bar{D}_{A_i}$, $i = 0, 1$ as well. We can then form the linear combination of connections

$$D_A = t\bar{D}_{A_1} + (1-t)D_{A_0}.$$

Then $P(F_A)$ is a deRham cocycle on $M \times \mathbb{R}$. Now let $i = 0$ or 1 and consider the inclusions $j_i : M = M \times \{i\} \hookrightarrow M \times \mathbb{R}$. The induced connection $j_i^*(D_A) = D_{A_i}$ on ζ. But since there is an obvious homotopy between j_0 and j_1 and hence the cohomology classes

$$[j_0^*(P(F_A)) = P(F_{A_0})] = [j_1^*(P(F_A)) = P(F_{A_1})].$$

This proves the theorem. □

Thus the invariant polynomial P determines a cohomology class given any bundle ζ over a smooth manifold. It is immediate that these classes are preserved under pull - back, and are hence characteristic classes for $U(n)$ bundles, and hence are given by elements of

$$H^*(BU(n); \mathbb{C}) \cong \mathbb{C}[c_1, \cdots, c_n].$$

In order to see how an invariant polynomial corresponds to a polynomial in the Chern classes we need the following bit of algebra.

Recall the elementary symmetric polynomials $\sigma_1, \cdots \sigma_n$ in n -variables, discussed in section 6.3. If we view the n - variables as the eigenvalues of an $n \times n$ matrix, we can write

$$det(I + tA) = 1 + t\sigma_1(A) + \cdots + t^n\sigma_n(A). \tag{6.18}$$

Lemma 6.49. *Any invariant polynomial on $M_n(\mathbb{C})$ can be expressed as a polynomial of $\sigma_1, \cdots, \sigma_n$.*

Proof. Given $A \in M_n(\mathbb{C})$, chose a B such that BAB^{-1} is in Jordan canonical form. Replacing B with $diag(\epsilon, \epsilon^2, \cdots, \epsilon^n)B$, we can make the off diagonal entries arbitrarily close to zero. By continuity it follows that $P(A)$ depends only on the diagonal entries of BAB^{-1}, i,e the eigenvalues of A. Since $P(A)$ is invariant, it must be a symmetric polynomial of these eigenvalues. Hence it is a polynomial in the elementary symmetric polynomials. □

So we now consider the elementary symmetric polynomials, viewed as invariant polynomials in $M_n(\mathbb{C})$. Hence by the above constructions they determine characteristic classes $[\sigma_r(F_A)] \in H^{2r}(M;\mathbb{C})$ where F_A is a connection on a vector bundle ζ over M.

Now we've seen the elementary symmetric functions before in the context of characteristic classes. Namely we've seen that $H^*(BU(n))$ can be viewed as the subalgebra of symmetric polynomials in $\mathbb{Z}[x_1, \cdots x_n] = H^*(BU(1) \times \cdots \times BU(1))$, with the Chern class C_r corresponding to the elementary symmetric polynomial σ_r. This was the phenomenon of the *splitting principle*.

We will now use a splitting principle argument to prove the following.

Theorem 6.50. *Let ζ be a complex n - dimensional vector bundle with connection D_A. Then the cohomology class $[\sigma_r(F_A)] \in H^{2r}(X;\mathbb{C})$ is equal to $(2\pi i)^r c_r(\zeta)$, for $r = 1, \cdots, n$.*

Proof. We first prove this theorem for complex line bundles. That is, $n = 1$. In this case $\sigma_1(F_A) = F_A$ which is a closed form in $\Omega^2(M; ad(\zeta)) = \Omega^2(M;\mathbb{C})$ because the adjoint action of $GL(1,\mathbb{C})$ is trivial since it is an abelian group. In particular F_A is closed in this case by Lemma 6.47. Thus F_A represents a cohomology class in $H^2(M;\mathbb{C})$. Moreover as seen above, this cohomology class $[F_A]$ is a characteristic class for line bundles and hence is an element of $H^2(BU(1);\mathbb{C}) \cong \mathbb{C}$ generated by the first Chern class $c_1 \in H^2(BU(1))$. So for this case we need to prove the following generalization of the Gauss - Bonnet theorem.

Lemma 6.51. *Let ζ be a complex line bundle over a manifold M with connection D_A. Then the curvature form F_A is a closed two - form representing the cohomology class*

$$[F_A] = 2\pi i c_1(\zeta) = 2\pi i \chi(\zeta).$$

Before we prove this lemma we show how this lemma can in fact be interpreted as a generalization of the classical Gauss - Bonnet theorem. So let D_A be a unitary connection on ζ. (That is, D_A is induced by a connection on an associated principal $U(1)$ - bundle.) If we view ζ as a two dimensional, oriented vector bundle which, to keep notation straight we refer to as $\zeta_{\mathbb{R}}$, then D_A induces (and is induced by) a connection $D_{A_{\mathbb{R}}}$ on the real bundle $\zeta_{\mathbb{R}}$. Notice that since $SO(2) \cong U(1)$ then orthogonal connections on oriented real two dimensional bundes are equivalent to unitary connections on complex line bundles.

Since $SO(2)$ is abelian, the real adjoint bundle

$$ad(\zeta_{\mathbb{R}}) = E_{\zeta_{\mathbb{R}}} \times_{SO(2)} M_2(\mathbb{R})$$

is trivial. Hence the curvature $F_{A_{\mathbb{R}}}$ is then a 2×2 matrix valued two - form.

$$F_{A_{\mathbb{R}}} \in \Omega^2(M; M_2(\mathbb{R})).$$

Moreover, since the Lie algebra of $SO(2)$ consists of skew symmetric 2×2 real matrices, then it is straightforward to check the following relation between the original complex valued connection $F_A \in \Omega^2(M; \mathbb{C})$ and the real curvature form $F_{A_{\mathbb{R}}} \in \Omega^1(M; M_2(\mathbb{R}))$.

Claim. If $F_{A_{\mathbb{R}}}$ is written as the skew symmetric matrix of $2 - forms$

$$F_{A_{\mathbb{R}}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \in \Omega^2(M; M_2(\mathbb{R}))$$

then

$$F_A = i\omega \in \Omega^2(M; \mathbb{C}).$$

When the original connection $D_{A_{\mathbb{R}}}$ is the *Levi - Civita* connection associated to a Riemannian metric on the tangent bundle of a Riemann surface, the curvature form

$$\omega \in \Omega^2(M, \mathbb{R})$$

is referred to as the "Gauss - Bonnet"" connection. If dA denotes the area form with respect to the metric, then we can write

$$\omega = \kappa \, dA$$

then κ is a scalar valued function called the "Gaussian curvature" of the

Riemann surface M. In this case, by the claim we have $[F_A] = 2\pi i \chi(T(M))$, and since

$$\langle \chi(T(M)), [M] \rangle = \chi_M,$$

Where χ_M the Euler characteristic of M, we have

$$\langle [F_A], [M] \rangle = \int_M F_A = i \int_M \omega = i \int_M \kappa \, dA.$$

Thus the above lemma applied to this case, which states that

$$\langle [F_A], [M] \rangle = 2\pi i \chi_M$$

is equivalent to the classical Gauss - Bonnet theorem which states that

$$\int_M \kappa \, dA = 2\pi \chi_M = 2\pi(2 - 2g) \tag{6.19}$$

where g is the genus of the Riemann surface M.

We now prove the above lemma.

Proof. As mentioned above, since $[F_A]$ is a characteristic class for line bundles, and so it is some multiple of the first Chern class, say $[F_A] = qc_1(\zeta)$. By the naturality, the coefficient q is independent of the bundle. So to evaluate q it is enough to compute it on a specific bundle. We choose the tangent bundle of the unit sphere $\tau(S^2)$, equipped with the Levi - Civita connection D_A corresponding to the usual round metric (or equivalently the metric coming from the complex strucure $S^2 = \mathbb{CP}^1$). In this case the Gaussian curvature is constant at one,

$$\kappa = 1.$$

Moreover since $\tau(S^2) \oplus \epsilon_1 \cong \gamma_1 \oplus \gamma_1$, the Whitney sum formula yields

$$\langle c_1(S^2), [S^2] \rangle = 2\langle c_1(\gamma_1), [S^2] \rangle = 2.$$

Thus we have

$$\begin{aligned} \langle [F_A], [S^2] \rangle &= q\langle c_1(S^2), [S^2] \rangle \\ &= 2q. \end{aligned}$$

Putting these facts together yields that

$$\begin{aligned} 2q &= \langle [F_A], [S^2] \rangle \\ &= \int_{S^2} F_A \\ &= i \int_{S^2} \kappa \, dA \\ &= i \int_{S^2} dA \quad = i \cdot \text{surface area of } S^2 \\ &= i \cdot 4\pi. \end{aligned}$$

Hence $q = 2\pi i$, as claimed. □

We now proceed with the proof of Theorem 6.50 in the case when the bundle is a sum of line bundles. By the splitting principal we will then be able to conclude the theorem is true for all bundles.

So let $\zeta = L_1 \oplus \cdots \oplus L_n$ where $L_1, \cdots, L_n$ are complex line bundles over M. Let $D_1, \cdots, D_n$ be connections on $L_!, \cdots, L_n$ respectively. Now let D_A be the connection on ζ given by the sum of these connections

$$D_A = D_1 \oplus \cdots \oplus D_n.$$

Notice that with respect to any local trivialization, the curvature matrix F_A is the diagonal $n \times n$ matrix with diagonal entries, the curvatures $F_1, \cdots, F_n$ of the connections $D_1, \cdots D_n$ respectively. Thus the invariant polynomial applied to the curvature form $\sigma_r(F_A)$ is given by the symmetric polynomial in the diagonal entries,

$$\sigma_r(F_A) = \sigma_r(F_1, \cdots, F_r).$$

Now since the curvatures F_i are closed 2 - forms on M, we have an equation of cohomology classes

$$[\sigma_r(F_A)] = \sigma_r([F_1], \cdots, [F_r]).$$

By the above lemma we therefore have

$$\begin{aligned}
[\sigma_r(F_A)] &= \sigma_r([F_1], \cdots, [F_n]) \\
&= \sigma_r((2\pi i)c_1(L_1), \cdots, (2\pi i)c_1(L_n)) \\
&= (2\pi i)^r \sigma_r(c_1(L), \cdots, c_1(L_n)) \quad \text{since } \sigma_r \text{ is symmetric} \\
&= (2\pi i)^r c_r(L_1 \oplus \cdots \oplus L_n) \quad \text{by the splitting principal 6.20} \\
&= (2\pi i)^r c_r(\zeta)
\end{aligned}$$

as claimed.

This proves the theorem when ζ is a sum of line bundles. As observed above, the splitting principal implies that the theorem then must be true for all bundles. □

We end this section by describing two corollaries of this important theorem.

Corollary 6.52. *For any real vector bundle η, the deRham cocycle $\sigma_{2k}(F_A)$ represent the cohomology class $(2\pi)^{2k} p_k(\eta) \in H^{4k}(M;\mathbb{R})$, while $[\sigma_{2k+1}(F_A)]$ is zero in $H^{4k+2}(M;\mathbb{R})$.*

Proof. This just follows from the definition of the Pontrjagin classes in terms of the even Chern classes of the complexification, and the fact that the odd Chern classes of the complexification have order two and therefore represent the zero class in $H^*(M;\mathbb{R})$. □

Recall that a flat connection is one whose curvature is zero. The following is immediate form the above theorem.

Corollary 6.53. *If a real (or complex) vector bundle has a flat connection, then all its Pontrjagin (or Chern) classes with rational coefficients are zero.*

We recall that a bundle has a flat connection if and only if its structure group can be reduced to a discrete group. Thus a complex vector bundle with a discrete structure group has zero Chern classes with rational coefficients. This can be interpreted as saying that if $\iota : G \subset GL(n, \mathbb{C})$ is the inclusion of a discrete subgroup, then the map in cohomology,

$$\mathbb{Q}[c_1, \cdots, c_n] = H^*(BU(n); \mathbb{Q}) = H^*(BGL_n(\mathbb{C}); \mathbb{Q}) \xrightarrow{\iota^*} H^*(BG; \mathbb{Q})$$

is zero.

7

Embeddings and Immersions in Euclidean Space

7.1 The existence of embeddings: The Whitney Embedding Theorem

The following result is often known as the "Easy Whitney Embedding Theorem". It tells us that we may view any manifold as a submanifold of Euclidean space.

Theorem 7.1. *Let M^n be a C^r manifold of dimension n. Then there is a C^r-embedding $e : M^n \hookrightarrow \mathbb{R}^L$ for L sufficiently large.*

Proof. We prove this theorem in the case when M^n is closed. We refer the reader to [72] for the general case. Since M^n is compact we can find a *finite* atlas $\{\phi_i, U_i\}_{i=1}^M$ with the following properties:

1. For all $i = 1, \cdots, m, B_2(0) \subset \phi_i(U_i) \subset \mathbb{R}^n$, and
2. $M^n = \bigcup_{i=1}^m Int\, \phi_i^{-1}(B_1(0))$.

Here $B_r(0) \subset \mathbb{R}^n$ is the open ball around the origin of radius r.

Let $\lambda : \mathbb{R}^n \to [0,1]$ be a C^∞ "bump function" such that

$$\lambda(x) = \begin{cases} 1 & \text{on} \quad B_1(0) \\ 0 & \text{on} \quad \mathbb{R}^n - B_2(0) \end{cases}$$

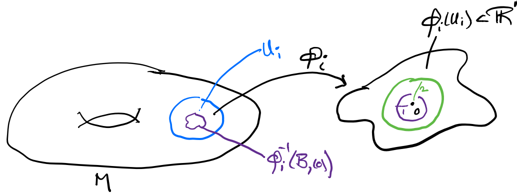
Define $\lambda_i : M^n \to [0,1]$ by

$$\lambda_i = \begin{cases} \lambda \circ \phi_i & \text{on} \quad U_i \\ 0 & \text{on} \quad M^n - U_i. \end{cases}$$

These are "local bump functions". Notice that the sets $S_i = \lambda_i^{-1}(1) \subset U_i$, $i = 1, \cdots, m$ cover M^n.

Now define $f_i : M^n \to \mathbb{R}^n$ by

$$f_i(x) = \begin{cases} \lambda_i(x)\phi_i(x) & \text{if} \quad x \in U_i \\ 0 \quad \text{if} \quad x \in M - U_i \end{cases}$$



Notice that f_i is C^r. Define $g_i(x) = (f_i(x), \lambda_i(x)) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$, and

$$g = (g_1, \cdots, g_m) : M^n \to \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} = \mathbb{R}^{m(n+1)}.$$

g is a C^r map. We claim it is an embedding.

If $x \in S_i$, g_i is immersive at x, so therefore g is immersive at x. Since the S_i's cover M^n, g_i is an immersion. We observe that g is one-to-one.

Suppose $x \neq y$ and $y \in S_i$. If x also lies in S_i, then since

$$f_{i_{|S_i}} = \phi_{i_{|S_i}}$$

then $f_i(x) \neq f_i(y)$ since ϕ_i is injective. If x *does not* lie in S_i, then

$$\lambda_i(y) = 1 \neq \lambda_i(x).$$

So $g(x) \neq g(y)$.

So $g : M^n \to \mathbb{R}^{n(m+1)}$ is an injective immersion. Since M^n is compact, g is an embedding. □

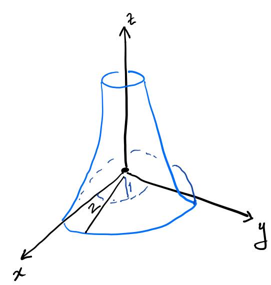


FIGURE 7.1
A graph of λ when $n = 2$.

Remark. Notice that this theorem implies that a compact n-manifold M^n can be embedding in *any* manifold N^m if the dimension of N^m is sufficiently large. This is because N^m looks locally like Euclidean space, and so by the above theorem N^m can be embedding in an open set inside M^n.

7.2 Obstructions to the existence of embeddings and immersions, the Hirsch-Smale theorem, and the immersion conjecture

A stronger version of Theorem 7.1 was proved by H. Whitney in a seminal paper published in 1944 [164].

Theorem 7.2. *[164] A. (Whitney Embedding Theorem) Let M^n be a compact C^r manifold of dimension n, with $r \geq 1$. Then there is a C^r-embedding $e : M^n \hookrightarrow \mathbb{R}^{2n}$. Furthermore there is a C^r-immersion $j : M^n \looparrowright \mathbb{R}^{2n-1}$.*

An extension of Whitney's theorem to the setting of manifolds with boundary is the following:

Theorem 7.3. *Let M^n be a C^r-n-dimensional compact manifold with boundary, with $r \geq 1$. Then there is a neat C^r embedding of M^n into $\mathbb{H}^{2n}$.*

It is natural to ask if Whitney's theorem is the best possible. More specfically, one can ask the following question. From now on all manifolds we consider are closed and C^∞, unless specifically stated otherwise.

Question 1. What is the smallest positive integer $\phi(n)$ so that every compact n-dimensional manifold can be embedded in $\mathbb{R}^{n+\phi(n)}$? Notice that Whitney's theorem says that $\phi(n) \leq n$.

Question 2. What is the smallest positive integer $\psi(n)$ so that every compact n-dimensional manifold can be immersed in $\mathbb{R}^{n+\psi(n)}$? Whitney's theorem says that $\psi(n) \leq n - 1$.

Question 1 poses a problem that as of this date is unsolved. There are many results of the best possible embedding dimension for particular n-manifolds, but in general the answer to Question 1 is unknown. However in the case when n is a power of 2 one can prove that Whitney's result is best possible. That is, if $n = 2^k$, then $\phi(2^k) = 2^k$. We give a sketch of a proof of this fact by proving the following.

Proposition 7.4. *The projective space $\mathbb{RP}^{2^k}$ embeds in $\mathbb{R}^{2^{k+1}}$ by Whitney's theorem, but it does not embedd in $\mathbb{R}^{2^{k+1}-1}$.*

Proof. *(Sketch)* We give a sketch of an argument that uses a theory of Haefliger developed in [68]. For X any space, consider the configuration space of k ordered, distinct points in X:

$$F(X, k) = \{(x_1, \cdots, x_k) \in X^k \, : \, x_i \neq x_j \quad \text{if} \quad i \neq j\}.$$

Notice that the symmetric group Σ_k acts freely on $F(X,k)$ by permuting the order of the elements.

Notice that if $e : M^n \hookrightarrow \mathbb{R}^L$ is an embedding of a manifold into Euclidean space, there is an induced map of configuration spaces

$$F(e) : F(M^n, 2)/\Sigma_2 \to F(\mathbb{R}^L, 2)/\Sigma_2.$$

We claim that $F(\mathbb{R}^L, 2)/\Sigma_2$ has the homotopy type of the projective space $\mathbb{RP}^{L-1}$. To see this, notice that $F(\mathbb{R}^L, 2)$ is diffeomorphic to $\mathbb{R}^L \times (\mathbb{R}^L - \{0\})$ via the map that sends (x_1, x_2) to $(x_1 + x_2,\, x_1 - x_2)$. This is a Σ_2-equivariant diffeomorphism, where the action on $\mathbb{R}^L \times (\mathbb{R}^L - \{0\})$ is given by $(u, v) \to (u, -v)$. But clearly with respect to this action $\mathbb{R}^L \times (\mathbb{R}^L - \{0\})$ is Σ_2-equivariantly homotopy equivalent to the sphere S^{L-1} with the antipodal Σ_2-action. The claim then follows.

Now since any compact n- manifold M^n embeds in $\mathbb{R}^L$ for L sufficiently large, and since any two embeddings into sufficiently large dimensional Euclidean space are isotopic (to be discussed below), then one always comes equipped with a map, well defined up to homotopy,

$$\omega : F(M^n, 2)/\Sigma_2 \to F(\mathbb{R}^\infty, 2)/\Sigma_2 \simeq \mathbb{RP}^\infty.$$

Furthermore, by the above claim, if M^n embeds in $\mathbb{R}^L$, this map factors, up to homotopy, through a map $\omega_L : F(M^n, 2)/\Sigma_2 \to \mathbb{RP}^{L-1}$. By Whitney's theorem, one can always find such a ω_L for $L = n$. However in the case of $M^n = \mathbb{RP}^{2^k}$, Haefliger showed using obstruction theory that there is no map $\omega_{2^k-1} : F(\mathbb{RP}^{2^k}, 2)/\Sigma_2) \to \mathbb{RP}^{2^k-2}$ that factors $\omega : F(\mathbb{RP}^{2^k}, 2)/\Sigma_2) \to \mathbb{RP}^\infty$. This means that $\mathbb{RP}^{2^k}$ cannot be embedded in $\mathbb{R}^{2^{k+1}-1}$. □

Notice that this proposition says that in the case $n = 2^k$ the answer to Question 1 above is $\phi(2^k) = 2^k$. But as was mentioned above, in general Question 1 is unresolved. However, as we have observed, Haefligger's theory supplies a homotopy theoretic obstruction to embedding manifolds in Euclidean space. We remark that in recent years Haefligger's theory has been generalized to a theory of "Embedding Calculus", as developed by T. Goodwillie, M. Weiss, and others [62], [63], [162] [163]. This is a beautiful and effective theory for studying spaces of embeddings of one manifold into an other, using sophisticated homotopy theoretic techniques. We encourage the reader to learn more about this theory.

The situation with immersions instead of embeddings is considerably easier, due to the following famous result of Hirsch and Smale [73]. This is an early example of the h-principle (where "h" stands for homotopy) as defined by Gromov [61] and developed further by Eliashberg and Mishachev [46]. We now describe the Hirsch-Smale result.

Suppose $f : M^n \looparrowright P^{n+k}$ is an immersion between smooth (C^∞) manifolds. Then one has the induced map of tangent bundles yielding the commutative diagram

$$\begin{array}{ccc} \tau M^n & \xrightarrow{Df} & \tau P^{n+k} \\ \downarrow & & \downarrow \\ M^n & \xrightarrow[f]{} & P^{n+k} \end{array}$$

This is an example of a *bundle monomorphism*, meaning a map of vector bundles

$$\begin{array}{ccc} \zeta & \xrightarrow{\bar{\gamma}} & \xi \\ \downarrow & & \downarrow \\ X & \xrightarrow[\gamma]{} & Y \end{array}$$

so that $\gamma_x : \zeta_x \to \xi_{\gamma(x)}$ is a linear monomorphism of vector spaces for each $x \in X$. We denote the space of such bundle monomorphisms by $Mono(\zeta, \xi)$. Let $Imm(M^n, P^{n+k})$ be the space of immersions, topologized in the space of all maps given the compact-open topology. Then differentiation induces a map

$$D : Imm(M^n, P^{n+k}) \to Mono(\tau M^n, \tau P^{n+k}).$$

Theorem 7.5. *(Hirsch and Smale [73]). Let M^n be a compact, smooth manifold of dimension n, and P^{n+k} be a smooth manifold of dimension $n+k$, with $k \geq 1$. Then the map*

$$D : Imm(M^n, P^{n+k}) \to Mono(\tau M^n, \tau P^{n+k}).$$

is a weak homotopy equivalence.

Notice that in particular, if $Mono(\tau M^n, \tau \mathbb{R}^{n+k})$ is nonempty, then there exists an immersion $M^n \looparrowright \mathbb{R}^{n+k}$, for $k \geq 1$.

Notice furthermore that a bundle monomorphism $\gamma : \tau M^n \to \tau \mathbb{R}^{n+k}$ determines a k-dimensional normal bundle,

$$\pi : \nu^k_\gamma \to M^n$$

where $\pi^{-1}(x) = \{v \in \mathbb{R}^{n+k} \text{ such that } v \perp \gamma(T_x M^n)\}$. That is ν^k_γ is the *orthogonal complement* to τM^n, inside $\tau \mathbb{R}^{n+k}$. In other words,

$$\tau M^n \oplus \nu^k_\gamma \cong M \times \mathbb{R}^{n+k}.$$

The following is is a direct consequence of the Hirsch-Smale theorem.

Corollary 7.6. *A compact n-manifold M^n immerses in $\mathbb{R}^{n+k}$ if and only if there is a k-dimensional bundle $\nu^k \to M^n$ such that*

$$\tau M^n \oplus \nu^k \cong M^n \times \mathbb{R}^{n+k}.$$

We now give an interpretation of these results in terms of classifying spaces. We use [30] as a reference. This allows one to recast the question of immersing manifolds into Euclidean space into a homotopy theoretic problem.

As above, let $BO(k)$ denote the classifying space of k-dimensional vector bundles, and let $BO = \lim_{k\to\infty} BO(k)$. Since every manifold immerses, and indeed embeds in sufficiently high dimensional Euclidean space, this means there is a map

$$\nu : M^n \to BO$$

representing this high dimensional (or "stable") normal bundle. This map is well-defined up to homotopy for the following reason. Given any compact space X with basepoint, the homotopy classes of basepoint preserving maps $[X, BO]$ represents the set of stable vector bundles $SVect(S)$, which is isomorphic to the reduced K-theory, $\tilde{KO}(X)$, and is therefore an abelian group. (We refer the reader to Theorem 3.17 for a discussion of this fact.) In particular the addition in this abelian group corresponds to the Whitney sum of vector bundles. In this abelian group structure, the stable normal bundle is the inverse of the stable tangent bundle represented by the composite

$$\tau M : M^n \to BO(n) \to BO.$$

Thus the stable normal bundle map is well-defined, up to homotopy. We may therefore restate Corollary 7.6 as follows.

Theorem 7.7. *Let M^n be a closed n-manifold and $\nu : M^n \to BO$ represent its stable normal bundle. Then M^n immerses in $\mathbb{R}^{n+k}$ if and only if there is a map $\nu^k : M^n \to BO(k)$ so that the composite*

$$M^n \xrightarrow{\nu^k} BO(k) \to BO$$

is homotopic to the stable normal bundle map $\nu : M^n \to BO$.

Using this theorem, the work of Brown and Peterson [22] [23] [25], and the author [28], combined to give a resolution of Question 2 above. We now outline how this was achieved.

In [101] Massey showed that for every closed n-manifold M^n, the homomorphism induced by the stable normal bundle map

$$\nu^* : H^*(BO; \mathbb{Z}/2) \to H^*(M^n; \mathbb{Z}/2)$$

factors through $H^*(BO(n - \alpha(n))$, where $\alpha(n)$ is the number of ones in the dyadic (base 2) expansion of n. That is to say, there is a homomorphism $\tilde{\nu}^* : H^*(BO(2n - \alpha(n); \mathbb{Z}/2) \to H^*(M^n; \mathbb{Z}/2)$ so that the composition

$$H^*(BO; \mathbb{Z}/2) \xrightarrow{\iota^*} H^*(BO(n - \alpha(n); \mathbb{Z}/2) \xrightarrow{\tilde{\nu}^*} H^*(M^n; \mathbb{Z}/2)$$

is equal to ν^*. Here $\iota : BO(n - \alpha(n) \to BO$ is the usual inclusion. Now recall from Theorem 6.15 that $H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega_1, \cdots, \omega_k, \cdots]$ and that $H^*(BO(m); \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega_1, \cdots, \omega_m]$ for every m. So Massey's result can be restated as the following.

Theorem 7.8. *(Massey [101]) Let M^n be a closed n-dimensional manifold, and let $\nu_M : M^n \to BO$ classify its stable normal bundle. Then*

$$\omega_i(\nu_M) = 0$$

for all $i > n - \alpha(n)$.

For a closed n-manifold M^n, let $I_{M^n} \subset H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \cdots, w_i, \cdots]$ be the kernel of the stable normal bundle homomorphism, ν^* : $H^*(BO; \mathbb{Z}/2) \to H^*(M^n; \mathbb{Z}/2)$. Let I_n be the intersection

$$I_n = \bigcap_{M^m} I_{M^n}.$$

Here the intersection is taken over all closed n-manifolds. I_n is an ideal in $\mathbb{Z}/2[w_1, \cdots, w_i, \cdots]$, and by Massey's result we know that $w_i \in I_n$ for all $i > n - \alpha(n)$. In [22] [23] Brown and Peterson computed I_n explicitly, thus refining Massey's theorem. In [25] they went further and constructed a "universal space" for normal bundles of n-manifolds, and proved the following theorem.

Theorem 7.9. *(Brown and Peterson [25]). For every n there is a space BO/I_n equipped with a map $\rho_n : BO/I_n \to BO$ satisfying the following properties.*

1. *In cohomology $\rho_n^* : H^*(BO; \mathbb{Z}/2) \to H^*BO/I_n; \mathbb{Z}/2)$ is surjective, with kernel I_n. That is, ρ_n^* induces an isomorphism*

$$H^*(BO/I_n; \mathbb{Z}/2) \cong H^*(BO; \mathbb{Z}/2)/I_n.$$

2. *Every closed n-manifold M^n admits a map $\tilde{\nu}_{M^n} : M^n \to BO/I_n$ such that the composition*

$$M^n \xrightarrow{\tilde{\nu}_{M^n}} BO/I_n \xrightarrow{\rho_n} BO$$

is homotopic to the stable normal bundle map $\nu_{M^n} : M^n \to BO$.

Notice that by combining the work of Massey and Brown-Peterson, we have the following commutative diagram for every closed n-manifold M^n:

$$\begin{array}{ccc} H^*(BO;\mathbb{Z}/2) & \xrightarrow{\iota^*_{n-\alpha(n)}} & H^*(BO(n-\alpha(n);\mathbb{Z}/2) \\ \nu^*_{M^n}\big\downarrow & & \big\downarrow \rho^*_n \\ H^*(M^n;\mathbb{Z}/2) & \xleftarrow[\tilde{\nu}^*_{M^n}]{} & H^*(BO;\mathbb{Z}/2))/I_n \end{array}$$

Brown and Peterson's work [25] can be viewed as realizing a part of this cohomology diagram as coming from a diagram of spaces:

$$\begin{array}{ccc} BO & \xleftarrow{\iota_{n-\alpha(n)}} & BO(n-\alpha(n)) \\ \nu_{M^n}\big\uparrow & & \\ M^n & \xrightarrow[\tilde{\nu}_{M^n}]{} & BO/I_n \end{array}$$

In [28] the topological realization of this cohomology diagram was made complete when the author proved the following.

Theorem 7.10. *([28]) For every n there is a map $\tilde{\rho}_n : BO/I_n \to BO(n-\alpha(n))$ such that the composition $BO/I_n \xrightarrow{\tilde{\rho}_n} BO(n-\alpha(n)) \xrightarrow{\iota_{n-\alpha(n)}} BO$ is homotopic to $\rho_n : BO/I_n \to BO$ as in Theorem 7.9.*

Now let M^n be an n-manifold, and let $\tilde{\nu}_{M^n} : M^n \to BO/I_n$ be as in Theorem 7.9. Combining Theorem 7.9 with Theorem 7.10 implies that the composition

$$\tilde{\nu}_{M^n} : M^n \xrightarrow{\tilde{\nu}_{M^n}} BO/I_n \xrightarrow{\tilde{\rho}_n} BO(n-\alpha(n))$$

factors (up to homotopy) the stable normal bundle map $\nu_{M^n} : M^n \to BO$. Then by Theorem 7.7 we can conclude the following theorem.

Theorem 7.11. *([28]) Every closed n-manifold M^n admits an immersion*

$$j_{M^n} : M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}.$$

We end this section by describing why this is the best possible result. That is, the answer to Question 2 above, which asks what is the smallest integer $\psi(n)$ such that every closed n-manifold immerses in $\mathbb{R}^{n+\psi(n)}$ is $\psi(n) = n - \alpha(n)$.

We will actually describe a closed manifold M^n whose normal Stiefel-Whitney class, $w_{n-\alpha(n)}(\nu_{M^n})$ is nonzero. This would then supply an obstruction to immersing M^n into $\mathbb{R}^{2n-\alpha(n)-1}$.

The manifold M^n can be described as follows. Write n as a sum of distinct powers of 2:

$$n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}.$$

Note that r, the number of distinct powers of 2 in this description, is equal to $\alpha(n)$. We then define

$$M^n = \mathbb{RP}^{2^{i_1}} \times \mathbb{RP}^{2^{i_2}} \times \cdots \times \mathbb{RP}^{2^{i_r}}.$$

We then need to prove the following.

Proposition 7.12. *The normal Stiefel-Whitney class*

$$w_{n-\alpha(n)}(\nu_{M^n}) \in H^{n-\alpha(n)}(M^n; \mathbb{Z}/2)$$

is nonzero.

Proof. . We first observe that the case when n is a power of 2 was proved in Corollary 6.34. This used the fact that

$$\bar{w}(\mathbb{RP}^m) = w(\nu_{\mathbb{RP}^m}) = \frac{1}{w(\mathbb{RP}^m)} = \frac{1}{(1+a)^{m+1}} \in \prod_k H^k(\mathbb{RP}^m; \mathbb{Z}/2),$$

where $a \in H^1(\mathbb{RP}^m; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is the generator. This was proved in the example following Proposition 6.33 above. In particular,

$$\bar{w}(\mathbb{RP}^{2^j}) = w(\nu_{\mathbb{RP}^{2^j}}) = 1 + a + a^2 \cdots + a^{2^j-1}.$$

We now turn to the general case.

Write $n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r}$ as above, and let $M^n = \mathbb{RP}^{2^{i_1}} \times \cdots \times \mathbb{RP}^{2^{i_r}}$. Then the total normal Stiefel-Whitney class is given by

$$\bar{w}(M^n) = w(\nu_{M^n}) = \otimes_{j=1}^r \bar{w}(\mathbb{RP}^{2^{i_j}}) = \otimes_{j=1}^r (1 + a_j + \cdots a_j^{2^{i_j}-1}) \in \otimes_{j=1}^r H^*(\mathbb{RP}^{2^{i_j}}; \mathbb{Z}/2)$$
$$\cong H^*(M^n; \mathbb{Z}/2).$$

Notice that the highest dimensional nonzero monomial in this expression is

$$a^{2^{i_1}-1} \otimes \cdots \otimes a^{2^{i_r}-1}$$

which lies in dimension $\sum_{j=1}^r (2^{i_j} - 1) = n - r = n - \alpha(n)$. Thus

$$\bar{w}_{n-\alpha(n)}(\nu_{M^n}) = a^{2^{i_1}-1} \otimes \cdots \otimes a^{2^{i_r}-1} \in H^{n-\alpha(n)}(\prod_{j=1}^r \mathbb{RP}^{2^{i_j}}; \mathbb{Z}/2) = H^{n-\alpha(n)}(M^n; \mathbb{Z}/2),$$

and this class is clearly nonzero. □

To summarize, this proposition says that for M^n defined as the product of projective spaces as above, then $w_{n-\alpha(n)}(\nu_{M^n}) \in H^{n-\alpha(n)}(M^n; \mathbb{Z}/2)$ is nonzero. Thus, even though M^n admits an immersion into $\mathbb{R}^{2n-\alpha(n)}$, there is no immersion of M^n into $\mathbb{R}^{2n-\alpha(n)-1}$. In particular this says that the answer to Question 2 above is $\psi(n) = n - \alpha(n)$.

7.3 "Turning a sphere inside-out".

In the last subsection we used the Hirsch - Smale theorem (Theorem 7.5) to discuss the existence or nonexistence of immersions of manifolds into Euclidean spaces of varying dimensions. In this subsection we discuss the first application of this theorem, which was to show that any two immersions of S^2 into $\mathbb{R}^3$ are *isotopic* (sometimes referred to as "regularly homotopic").

Specifically we will give Smale's proof of his famous theorem saying that the identity embedding $\iota : S^2 \hookrightarrow \mathbb{R}^3$ defined by $\iota(x, y, z) = (x, y, z)$, is isotopic as immersions to its opposite, $-\iota(x, y, z) = -(x, y, z)$. That is, there exists a one-parameter family of immersions connecting ι to $-\iota$. Such a one parameter family is called an "eversion" of the sphere. The fact that such an eversion exists is perhaps counter-intuitive. It is sometimes described as "turning the sphere inside out", and indeed there are now videos showing such eversions. However Smale's original proof was a nonconstructive one, which relied on (an early version of) Theorem 7.5.

Notice that the statement that two immersions $f, g : M \looparrowright N$ are isotopic (or "regularly homotopic") is equivalent to the statement that f and g lie in the same path component of $Imm(M, N)$. To prove that the immersions ι and j of S^2 into $\mathbb{R}^3$ are isotopic, Smale proved the following:

Theorem 7.13. *(Smale [138]) The space $Imm(S^2, \mathbb{R}^3)$ is path connected.*

Proof. By Theorem 7.5 one has a weak homotopy equivalence

$$D : Imm(S^2, \mathbb{R}^3) \xrightarrow{\simeq} Mono(\tau S^2, \tau\mathbb{R}^3).$$

We can think about the space $Mono(\tau S^2, \tau\mathbb{R}^3)$ in the following way. Consider the fiber bundle

$$Mono(\mathbb{R}^2, \mathbb{R}^3) \to I(\tau S^2, \mathbb{R}^3) \xrightarrow{p} S^2 \tag{7.1}$$

where $I(\tau S^2, \mathbb{R}^3)$ is defined to be the space

$$I(\tau S^2, \mathbb{R}^3) = \{(x, \psi) : x \in S^2, \quad \text{and } \psi : T_x S^2 \to \mathbb{R}^3 \text{ is a linear monomorphism.}\}$$

Then $p(x, \psi) = x \in S^2$. So each fiber of p is equivalent to the Stiefel manifold $V_{2,3} = Mono(\mathbb{R}^2, \mathbb{R}^3)$. Notice that $V_{2,3}$ has the homotopy type of $O(3)/O(1) \cong SO(3)$. This is true by the following reasoning. Using the Gram-Schmidt process, one sees that $Mono(\mathbb{R}^2, \mathbb{R}^3)$ is homotopy equivalent to the space of inner-product preserving monomorphisms, $Mono^{<,>}(\mathbb{R}^2, \mathbb{R}^3)$. Now this space has a transitive action of the orthogonal group $O(3)$, and the isotropy subgroup of the inclusion of $\mathbb{R}^2$ in $\mathbb{R}^3$ given by $(x, y) \to (0, x, y)$ is $O(1) < O(3)$.

Notice there is a natural homeomorphism

$$Mono(\tau S^2, \tau\mathbb{R}^3) \xrightarrow{\cong} \Gamma_{S^2}(I(\tau S^2, \mathbb{R}^3))$$

where $\Gamma_{S^2}(I(\tau S^2, \mathbb{R}^3))$ is the space of (differentiable) sections of the bundle (7.1). To prove the theorem it then suffices to prove the following.

Lemma 7.14. *The space of sections $\Gamma_{S^2}(I(\tau S^2, \mathbb{R}^3))$ is path connected.*

Proof. For ease of notation let Γ represent the space $\Gamma_{S^2}(I(\tau S^2, \mathbb{R}^3)) \cong Mono(\tau S^2, \tau\mathbb{R}^3) \simeq Imm(S^2, \mathbb{R}^3)$. Let α, and $\beta \in \Gamma$ be any two sectons. We will show that α and β live in the same path component of Γ. Write $S^2 = \mathbb{R}^2 \cup \infty$, and fix an identification of $T_\infty S^2$ with $\mathbb{R}^2$. Without loss of generality we may assume that

$$\alpha(\infty) = \beta(\infty) = (\infty, \iota) \in I(\tau S^2, \mathbb{R}^3)$$

where $\iota : T_\infty S^2 \cong \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ is the natural inclusion $(u, v) \to (0, u, v)$. This is because the group $SO(3)$ acts transitively on $V_{2,3}$, and so one may rotate α and β if necessary so that they satisfy this basepoint relation. Since $SO(3)$ is connected such rotations preserve the path components of α and β.

So we may assume that α and β lie in $\Gamma_b \subset \Gamma$ which we define to be the space of sections satisfying this basepoint condition. Notice that Γ_b can be viewed as a subspace of the space of all maps S^2 to $I(TS^2, \mathbb{R}^3)$ that take ∞ to (∞, ι). This is the two-fold based loop space $\Omega^2 I(TS^2, \mathbb{R}^3)$. Indeed Γ_b is exactly that subspace of $\Omega^2 I(TS^2, \mathbb{R}^3)$ which maps to the identity element in $\Omega^2 S^2$ under the map $\Omega^2 p : \Omega^2 I(TS^2, \mathbb{R}^3) \to \Omega^2 S^2$. This map, being the two-fold loop map of the fibration (7.1), defines a homotopy fibration

$$\Omega^2 V_{2,3} \to \Omega^2 I(\tau S^2, \mathbb{R}^3) \xrightarrow{\Omega^2 p} \Omega^2 S^2. \tag{7.2}$$

We make a couple of observations about this fibration. First recall that the homotopy group $\pi_2(V_{2,3}) = \pi_2(SO(3)) = 0$. This is because $SO(3) \cong \mathbb{RP}^3$ and the universal cover of $\mathbb{RP}^3$ is S^3, whose second homotopy group vanishes. This implies that $\Omega^2\mathbb{RP}^3 = \Omega^2 SO(3) \cong \Omega^2 V_{2,3}$ is path connected. By considering this fibration sequence one then deduces that there is a bijection between the path components of $\Omega^2 I(TS^2, \mathbb{R}^3)$ and of $\Omega^2 S^2$. In fact this bijection is an isomorphism between abelian groups. This is because the path components of two-fold loop spaces are abelian groups and $\Omega^2 p$ is a map that preserves this two-fold loop structure. Thus we may conclude that

$$\pi_0(\Omega^2 I(\tau S^2, \mathbb{R}^3)) \cong \pi_0(\Omega^2 S^2) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

Furthermore, observe that the path components of a two-fold loop space are all homotopy equivalent. This is seen as follows. Let $\Omega^2 Y$ be a two-fold loop space. Let g and h represent elements of this space and $\Omega^2_g Y$ and $\Omega^2_h Y$ be the path components of this space containing g and h respectively. "Multiplying by $g^{-1}h$ defines a map $\times g^{-1}h : \Omega^2_g Y \to \Omega^2_h Y$ which has homotopy inverse $\times h^{-1}g : \Omega^2_h Y \to \Omega^2_g Y$.

We conclude that we can restrict two-fold loop fibration (7.2) to any path component of $\Omega^2 S^2$ to obtain a homotopy fibration sequence

$$\Omega^2 V_{2,3} \to \Omega^2_{[n]} I(TS^2, \mathbb{R}^3) \xrightarrow{\Omega^2 p} \Omega^2_{[n]} S^2.$$

Here $\Omega^2_{[n]}S^2$ is the component of $\Omega^2 S^2$ containing maps of degree n. But notice that when $n = 1$, then by the definition of what a section means, Γ_b is the fiber of $\Omega^2 p$ over the identity map of S^2, $id \in \Omega^2_{[1]}S^2$. We may then conclude that $\Gamma_b \simeq \Omega^2 V_{2,3}$, which as just observed, is path connected. In particular our original sections α and β in γ live in the same path component. □

□

Exercises

Let M^n be a closed differentiable manifold, and let $e_0 : M^n \looparrowright \mathbb{R}^N$ and $e_1 \looparrowright \mathbb{R}^N$ be two immersions of M^n. We say that e_0 and e_1 are *isotopic* if there is a one-parameter family of immersions connecting e_0 and e_1. That is, e_0 and e_1 are isotopic if there is a continuous map $H : M^n \times [0,1] \to \mathbb{R}^N$ so that

- $H(x,0) = e_0(x)$ and $H(x,1) = e_1(x)$ for all $x \in M^n$
- The map $H_t : M^n \to \mathbb{R}^N$ defined by $H_t(x) = H(x,t)$ is a differentiable immersion for every $t \in [0,1]$.

Smale's theorem about "turning a sphere inside out" says that any two immersions $S^2 \looparrowright \mathbb{R}^3$ are isotopic.

1. Show, however, that there are infinitely many distinct isotopy classes of immersions $S^1 \looparrowright \mathbb{R}^2$. You may use Smale's theorem saying that the space of immersions $M \looparrowright \mathbb{R}^N$ is weakly homotopy equivalent to the space of bundle monomorphiisms $TM \to T\mathbb{R}^N$.
2. Describe a representative of each isotopy class you find.

Final Remark. In discussing eversions of spheres, we proved (ala Smale) that all immersion of S^2 in $\mathbb{R}^3$ are regularly homotopic (isotopic). Ultimately, using Hirsch-Smale theory, this was because $\pi_2(V_{2,3}) = 0$. However, somewhat surprisingly, there are infinitely many isotopy classes of immersions of S^2 into $\mathbb{R}^4$. This is because $\pi_2(V_{2,4}) \cong \mathbb{Z}$. We leave it to the reader to fill in the details of this striking result.

8

Tubular Neighborhoods, more on Transversality, and Intersection Theory

8.1 The tubular neighborhood theorem

We begin this chapter by proving another important, and basic result in differential topology: the "Tubular Neighborhood Theorem".

Theorem 8.1. *Suppose M^n is an n-dimensional smooth manifold, and suppose the $N^k \subset M^n$ is a k-dimensional submanifold. Then there exists an open neighborhood η of N^k in M^n that satisfies the following properties:*

1. *There is a neighborhood deformation retract*

$$p : \eta \to N^k.$$

   That is, p is a smooth map with the property that $p \circ \iota = id_{N^k}$ and $\iota \circ p : \eta \to \eta$ is homotopic to id_η. Here $\iota : N^k \hookrightarrow M^n$ is the inclusion.

2. *Let $\pi : \nu \to N^k$ be the normal bundle of N^k in M^n. Then there is a diffeomorphism $\Phi : \eta \to \nu$ making the following diagram commute:*

$$\begin{array}{ccc} \eta & \xrightarrow{\Phi} & \nu \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ N^k & \xrightarrow[=]{} & N^k \end{array}$$

Remark. The open set η in this theorem is referred to as a "tubular neighborhood" because, as the theorem states, it is diffeomorphic to the total space of a vector bundle, ν which locally looks like a "tube", $N^k \times \mathbb{R}^{n-k}$.

Observe that the statement of this theorem can be made in another way, which is often quite useful.

Theorem 8.2. *(Tubular neighborhood theorem, equivalent formulation.) Suppose $e : N^k \hookrightarrow M^n$ is an embedding of smooth manifolds with normal bundle ν. Assume that N^k is closed. Consider the inclusion of the zero section,*

$\zeta : N^k \hookrightarrow \nu$. Then the embedding e extends to an embedding $\tilde{e} : \nu \hookrightarrow M^n$ which is a diffeomorphism onto an open subset of M^k. By $\tilde{e}$ "extending" e we mean that the composition

$$N^k \xrightarrow{\zeta} \nu \xrightarrow{\tilde{e}} M^n$$

is equal to the embedding $e : N^k \hookrightarrow M^n$.

We leave it to the reader to check that this formulation is indeed equivalent to Theorem 8.1. We begin the proof of Theorem 8.1 by first proving it in the case where the ambient manifold is Euclidean space.

Theorem 8.3. *Let $e : N^k \hookrightarrow \mathbb{R}^n$ be an embedding of a closed manifold N^k. Then N^k has a "tubular neighborhood".*

Proof. Observe that it suffices to show that there is an open neighborhood V of the zero section $N^k \hookrightarrow \nu$ that supports an embedding into $\mathbb{R}^n$ that extends $e : N^k \to \mathbb{R}^n$. This is because, by the vector bundle structure of ν there is clearly an embedding of ν into any neighborhood of the zero section that fixes the zero section.

Let m be the codimension, $m = n - k$. Consider the map to the Grassmannian,

$$g : N^k \to Gr_m(\mathbb{R}^n)$$

defined by $g(x) = \nu_x \subset \mathbb{R}^n$. That is, $g(x)$ is the normal space to x in $\mathbb{R}^n$. More precisely,

$$\nu_x = \left(D_x e(T_x N^k)\right)^{\perp}.$$

Notice that the normal bundle $\nu \to N^k$ is the pullback, $\nu = g^*(\gamma_m)$, where $\gamma_m \to Gr_m(\mathbb{R}^n)$ is the canonical bundle. Specifically,

$$g^*(\gamma_m) = \{(x, v) \in N^k \times \mathbb{R}^n \ : \ v \in \nu_x\}.$$

Define a map $\phi : \nu \to \mathbb{R}^n$ by $\phi(x, v) = x + v \in \mathbb{R}^n$. As above, identify ν with $g^*(\gamma_m)$. Then notice that the tangent space to ν at $(x, 0)$ is given by

$$T_{(x,0)}\nu = T_x M \oplus \nu_x.$$

Furthermore, one immediately sees that the derivative of ϕ at $(x, 0)$,

$$D_{(x,0)}\phi : T_{(x,0)}\nu \to T_x\mathbb{R}^n$$

is the identity on on both $T_x M$ and on ν_x. Therefore $D\phi$ has rank n at all points on the zero section. It follows that ϕ is an immersion of a neighborhood U of the zero section in ν. Since the restriction of ϕ to the zero section itself is the given by the identity of $N^k \subset \mathbb{R}^n$, it implies that the restriction of ϕ to a perhaps smaller neighborhood V of the zero section in ν is an embedding.

□

We now proceed with the proof of Theorem 8.1.

Proof. By Whitney's embedding theorem 7.1 we can assume that $M^n \subset \mathbb{R}^N$ for some sufficiently large N. Let $W \subset \mathbb{R}^N$ be a tubular neighborhood of M^n, and $r : W \to M^n$ a retraction. Give M^n a metric induced by the Euclidean metric on $\mathbb{R}^n$. Notice we have an inclusion of vector bundles over N^k,

$$\nu \hookrightarrow TM^n_{|_{N^k}} \hookrightarrow T\mathbb{R}^N_{|_{N^k}} = N^k \times \mathbb{R}^N.$$

For $x \in N^k$, let $U_x = \{(x, v) \in \nu_x : x + v \in W\}$. Then the set $U = \bigcup_{x \in N^k} U_x$ can be viewed as a subset of $N^k \times \mathbb{R}^N$ and can then be given the subspace topology. Notice that by definition, $U \subset \nu$ and is an open subspace, because it is the inverse image of W under the map

$$\begin{aligned} \nu &\to \mathbb{R}^N \\ (x, v) &\to x + v. \end{aligned}$$

The map

$$\begin{aligned} \phi : U &\to M^n \\ \phi(x, v) &= r(x + v) \end{aligned}$$

is then easily checked to be a tubular neighborhood of $e : N^k \hookrightarrow M^n$. □

The tubular neighborhood theorem is extremely important in differential topology, and is used quite often. For example, it is crucial in knot theory, where one studies embeddings of S^1 in $\mathbb{R}^3 \subset S^3$. Let K be such a knot. That is, it is the image of such an embedding. Let $\eta(K)$ be a tubular neighborhood of K in S^3. Then the fundamental group of the complement, $S^3 - \eta(K)$ is an extremely important invariant of the isotopy class of the knot, and has been the main tool in studying knot theory for more than a century. This group is most often not abelian, but has abelianization $= \mathbb{Z}$. This is seen using the fact that the abelianiization of $\pi_1(S^3 - K)$ is equal to the first homology, $H_1(S^3 - K)$, and then using Alexander duality.

8.2 The genericity of transversality

In Chapter 3 we discussed the notions of regular values and transversality. In this section we will return to these notions and prove that they are *generic* in a sense that we will make precise. We will be following the discussion of these results given in Bredon's book [16] which is a very good reference for these concepts.

Recall that if $\phi : M^n \to N^n$ is smooth, then $p \in M$ is a *critical point* of ϕ

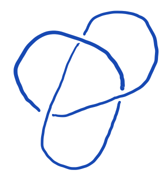


FIGURE 8.1
The trefoil knot

if the derivative $D_p\phi$ has rank strictly smaller than n. If p is critical, $\phi(p) \in N$ is a critical value. If $x \in N$ is not a critical value, it is called a *regular* value. So in particular, $x \in N$ is regular

- if $m \geq n$ and $D_p\phi$ is surjective for all $p \in M$ with $\phi(p) = x$, or
- $m < n$ and x is not in the image of ϕ.

The following theorem is well known in Analysis and Topology, and its proof is given in many texts, including the appendix of Bredon's book [16], as well as in Hirsch's book [72].

Theorem 8.4. *(Sard's theorem) If $\phi : M^m \to \mathbb{R}^n$ is C^∞, then the set of critical values has measure zero in $\mathbb{R}^n$.*

Before we state an important corollary to this theorem, which we will rely on heavily, we recall some terms from measure theory.

Definition 8.1. *A* **nowhere dense** subspace of a topological space is one whose set theoretic closure has empty interior. A subspace $E \subset X$ is **first category** if E is the countable union of subspaces that are nowhere dense. A **residual subspace** is the complement of a first category subspace. That is, it's complement is the countable union of nowhere dense subspaces. A residual subspace is sometimes called "everywhere dense".

Corollary 8.5. *(A. B Brown's theorem) If $\phi : M^m \to N^n$ is a C^∞ map, then the set of regular values of ϕ is residual in N^n.*

Proof. If C is the set of critical points of ϕ, and $K \subset M^m$ is compact, then $\phi(C \cap K)$ is a compact subspace of N^n, and its interior is empty by Sard's theorem. Therefore $\phi(C \cap K)$ is nowhere dense. Since M^m is covered by a countable union of such compace subspaces, $\phi(C)$ is first category and thus it's complement is residual. □

We note that if $m = 1$, then Sard's theorem says that there aren't any smooth, space-filling curves, unlike in the continuous setting.

We now apply Sard's theorem to the setting of transversality theory. We first show that zero sections of vector bundles can be perturbed to be transverse to any map.

Theorem 8.6. *Let $\xi \to Y$ be a smooth vector bundle over a smooth, compact manifold. Let X be a smooth manifold and $f : X \to \xi$ a smooth map. Then there is a smooth cross section $s : Y \to \xi$ as close to the zero section as desired, so that $f \pitchfork s(Y)$.*

Proof. Since Y is compact, we know that there exists a smooth vector bundle $\eta \to Y$ such that $\xi \oplus \eta$ is trivial. That is, there is an isomorphism of vector bundles over Y,

$$\Psi : \xi \oplus \eta \xrightarrow{\cong} Y \times \mathbb{R}^n,$$

which we can take to be smooth. Let $p : \xi \oplus \eta \to \mathbb{R}^n$ be the projection of Ψ onto the $\mathbb{R}^n$ factor. We then have a commutative diagram

$$\begin{array}{ccc} f^*(\xi \oplus \eta) & \xrightarrow{\bar{f}} & \xi \oplus \eta \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & \xi \end{array}$$

Here π is the projection, and $\bar{f}$ and π' are the obvious maps induced by f and π, respectively.

Let $z \in \mathbb{R}^n$ be a regular value of the composition

$$f^*(\xi \oplus \eta) \xrightarrow{\bar{f}} \xi \oplus \eta \xrightarrow{p} \mathbb{R}^n.$$

By Sard's theorem z can be chosen to be arbitrarily close to the origin. Since z is regular, the composition of the derivatives

$$Dp \circ D\bar{f} : T_v f^*(\xi \oplus \eta) \to \mathbb{R}^n$$

is surjective for any $v \in f^*(\xi \oplus \eta)$ such that $p \circ \bar{f}(v) = z$. Using the trivialization $\Psi : \xi \oplus \eta \xrightarrow{\cong} Y \times \mathbb{R}^n$, we may conclude that the image of $D\bar{f}$ must span the complement of that tangent space to $Y \times \{z\}$ at $(p_Y \bar{f}(v), z)$, where p_Y is the projection of the trivialization Ψ onto the Y factor. This means that $\bar{f}$ is transverse to the section $s' : Y \to \xi \oplus \eta$ given in terms of the trivialization Ψ, by $s'(y) = (y, z)$. Define the section $s : Y \to \xi$ by $s(y) = \pi(s'(y)$. Notice that the following diagram commutes:

$$\begin{array}{ccccc} f^*(\xi \oplus \eta) & \xrightarrow{\bar{f}} & \xi \oplus \eta & \xleftarrow{s'} & Y \\ \pi' \downarrow & & \downarrow \pi & & \downarrow = \\ X & \xrightarrow[f]{} & \xi & \xleftarrow{s} & Y \end{array}$$

We claim that f is transverse to $s(Y)$. To see this, let $x \in X$, $y \in Y$ be such that $f(x) = s(y)$. Then

$$\pi(s'(y)) = s(y) = f(x).$$

By the definition of the pullback bundle, $(x, s'(y)) \in f^*(\xi \oplus \eta)$ has $\bar{f}(x, s'(y)) = s'(y)$. Since $\bar{f} \pitchfork s'(Y)$, the images of $D_{(x,s'(y))}\bar{f}$ and $D_y s'$ span $T_{\bar{f}(x,s'(y))=s'(y)}(\xi \oplus \eta)$. Since π is a submersion, we may conclude that the images of $D_x f$ and $D_y s$ span $T_{f(x)=s(y)}\xi$. That is, $f \pitchfork s(Y)$. Notice that by Sard's theorem, the section s may be taken to be arbitrarily close to the zero section by choosing $z \in \mathbb{R}^n$ sufficiently close to the $0 \in \mathbb{R}^n$. □

Corollary 8.7. *Let $f : M \to W$ be a smooth map between smooth manifolds. Assume that M is compact. Let N be another compact, smooth manifold and suppose $g_0 : N \hookrightarrow W$ is a smooth embedding. Then there is an arbitrarily small isotopy of g_0 to a smooth embedding $g_1 : N \hookrightarrow W$ with the property that $f \pitchfork g_1(N)$.*

Proof. Let $\nu \to N$ be the normal bundle of the embedding $g_0 : N \hookrightarrow W$. By the tubular neighborhood theorem (8.2) g_0 extends to an embedding $g : \nu \hookrightarrow W$ which is a diffeomorphism onto an open subspace (the tubular neighborhood). Notice that if we define $M' = f^{-1}(g(\nu))$, then $M' \subset M$ is an open submanifold. We now apply the Theorem 8.6 to the restriction of f, $f_{|_{M'}} : M' \to \nu$. □

We will actually need another version of this corollary that says that transversal intersections are generic with respect to perturbations of the map f. But first we need the following:

Lemma 8.8. *Let N be a compact smooth submanifold of a smooth manifold W. Let T be a tubular neighborhood of N. It is equipped with a retraction $p : T \to N$¿ If $s : N \to T$ is any section (i.e $p \circ s = id$) then there is a diffeomorphism $h : T \to T$ that preserves fibers, extends continuously to the identity on the boundary ∂T, and takes s to the zero section. Moreover the diffeomorphism h can be taken to be homotopic to the identity of T.*

Proof. By the tubular neighborhood theorem, it suffices to work in the vector bundle setting. Let $p : \nu \to N$ be the normal bundle of N in W. Let $s : N \to \nu$ be any section, and let $z : N \to \nu$ be the zero section. Define

$$H : \nu \to \nu$$
$$H(v) = v - s(p(v)).$$

Notice that H is a map of fiber bundles in that it preserves fibers (i.e $p \circ H(v) = p(v)$). But notice also that H is not a map of vector bundles since it is not linear on each fiber. Rather, H is affine on each fiber. In any case, H is clearly a diffeomorphism.

Notice that $H \circ s(x) = z(x)$. Moreover H is homotopic to the identity through diffeomorphisms. To see this, define for $t \in [0,1]$ $H_t(v) = v - ts(p(v))$. Notice that H_0 is the identity, and $H_1 = H$. □

Corollary 8.9. *Let M be a closed, smooth manifold and $f_0 : M \to W$ a smooth map between smooth manifolds. Let $N \subset W$ be a smooth, closed submanifold and let T be any tubular neighborhood of N. Then there is a smooth map $f_1 : M \to W$ with the following properties.*

1. *$f_1 \pitchfork N$,*
2. *$f_1 = f_0$ outside of $f^{-1}(T)$,*
3. *f_1 is homotopic to f_0 on all of M via a homotopy that is constant outside of $f_0^{-1}(T)$.*

Proof. By Theorem 8.6 we know there exists a section s of a tubular neighborhood of N such that $f_0 \pitchfork s(N)$. Composing f_0 with the homotopy h described in the above lemma defines f_1. This f_1 may not be smooth at the boundary of the tubular neighborhood, but it can be smoothly approximated without changing it near the intersection with N, where f_1 is already smooth. □

Remarks. 1. This corollary says that one can perturb any map f_0 with as small of a perturbation as one would like, to make it transverse to N.

2. There exist strengthenings of this result saying that the set $\{f : M \to W$ such that $f \pitchfork N\}$ is "generic" (i.e a countable intersection of open, dense subsets) in the space of all smooth maps $C^\infty(M, W)$. Hirsch's book [72] gives a good exposition of this. For our purposes we only need that the space of transverse maps is *dense in the space of all smooth maps,* which is what the above results show.

8.3 Applications to intersection theory

One immediate application of transversality theory says that one can pull low dimensional submanifolds of a large dimensional manifolds apart, so that they do not interesect. More precisely, we have the following.

Proposition 8.10. *Let P^p and Q^q be closed submanifolds of M^n where $p+q < n$. Then one can perturb either P^p or Q^q by an arbitrarily small amount so that they do not intersect.*

More precisely, suppose $M^n \subset \mathbb{R}^N$. Let $e : P^p \hookrightarrow M^n$ be an embedding whose image is the submanifold in question. Then for any choice of $\epsilon > 0$, there exists another embedding $\tilde{e} : P^p \hookrightarrow M^n$, isotopic to e, so that for any $x \in P^p$, $\|e(x) - \tilde{e}(x)\| < \epsilon$ and $\tilde{e}(P^p) \cap Q^q = \emptyset$.

Proof. This follows from Corollary 8.7 and the fact that from Theorem 3.7 we see that the only transversal intersections of p-dimensional and q-dimensional submanifolds of an n-dimensional manifold when $p + q < n$, is the empty intersection. □

Here is another easy consequence of transversality theory. It is a statement about the homotopy groups of complements of submanifolds of Euclidean space.

Proposition 8.11. *Suppose M^m is a smooth, closed manifold, equipped with an embedding $e : M^m \hookrightarrow \mathbb{R}^n$. Then any smooth map of a sphere to the complement,*

$$f : S^k \to \mathbb{R}^n - M^m$$

can be extended to a map of the closed disk, $\tilde{f} : D^{k+1} \to \mathbb{R}^n - M^m$ if $k < n - m - 1$.

Proof. $f : S^k \to \mathbb{R}^n$ is null homotopic, since $\mathbb{R}^n$ is contractible. So there exists an extension $\tilde{f}_0 : D^{k+1} \to \mathbb{R}^n$. Perturb $\tilde{f}_0$ if necessary, to a map $\tilde{f}_1 : D^{k+1} \to \mathbb{R}^n$ that is homotopic to $\tilde{f}_0$ relative to its boundary, and such that $\tilde{f}_1 \pitchfork M^m$. But since $(k+1)+m < n$, this means that $\tilde{f}_1(D^{k+1}) \cap M^m = \emptyset$. In particular this means that the original map $f : S^k \to \mathbb{R}^n - M^m$ is null homotopic, and can therefore be extended to a map $\tilde{f} : D^{k+1} \to \mathbb{R}^n - M^m$. □

Another important application of transversality to intersection theory is when the sum of the dimensions of the submanifolds *equals* the dimension of the ambient manifold. So let P^p and Q^q be closed submanifolds of M^n, where $n = p+q$. Then basic transversality theory says that one can perturb either P^p or Q^q so that they intersect transversally. (At this point the reader should be able to make this statement precise.) In this setting the intersection $P^p \cap Q^q$ is a manifold of dimension $p + q - n = 0$. By compactness $P^p \cap Q^q$ is a finite number of points. When P^q, Q^q, and M^n are all oriented, $P^p \cap Q^q$ will inherit an orientation, and so each of the points making it up will have an orientation. This will just be a sign (± 1) and so one can count these points according to sign to obtain the "*intersection*" number. We now make this more precise.

Consider the following commutative diagram of embeddings:

$$\begin{array}{ccc} P^p & \xrightarrow{\subset} & M^n \\ \cup\uparrow & & \uparrow\cup \\ P^p \cap Q^q & \xrightarrow[\subset]{} & Q^q \end{array}$$

When P^p, Q^q, and M^n are all oriented, the normal bundle of $Q^q \hookrightarrow M^n$ has an induced orientation. Furthermore it restricts to give the (oriented) normal bundle of $P^p \cap Q^q \hookrightarrow P^p$. Since $P^p \cap Q^q$ is a finite set of points, $\{x_1, \cdots, x_k\}$, its normal bundle in P^p, being diffeomorphic to its tubular neighborhood, is just a finite collection of disjoint disks, $D_i \subset P^p$, $i = 1, \cdots, k$ each of which is oriented. In particular each tangent space $T_{x_i}D_i$ is oriented. But notice that $T_{x_i}D_i = T_{x_i}P^p$, which has an orientation coming from the original orientation of P^p. If these two orientations agree we say that $sgn(x_i) = +1$. If these orientations disagree we say that $sgn(x_i) = -1$. We can now make the following definition.

Definition 8.2. *Define the intersection number*

$$[P^p \cap Q^q] = \sum_{i=1}^{k} sgn(x_i) \in \mathbb{Z}$$

It is important to know that the intersection number is well defined. Of course we had to choose orientations and that can affect the ultimate sign of the intersection number. But it is important to also know that the intersection number does not depend on the particular perturbation (small isotopy) used in order to achieve transversal intersections. Once we know that we will be able to conclude the following:

Proposition 8.12. *Let P^p and Q^q be closed submanifolds of M^n, where $n = p+q$. Suppose these manifolds are all oriented. Then if the intersection number $[P^p \cap Q^q] \neq 0$, the neither P^p nor Q^q can be isotoped so that the resulting embeddings are disjoint. That is, P^p and Q^q cannot be "pulled off of each other" in M^n.*

To show that the intersection number is well defined, and to generalize it to study more complicated intersections, we will employ the use of Poincaré duality to develop the intersection theory homologically.

9

Poincaré Duality, Intersection theory, and Linking numbers

Our goal in this chapter is to use Poincaré duality to do intersection theory rigorously. A particular goal will be to prove that the intersection number of two submanifolds, the sum of whose dimensions equals the dimension of the ambient manifold, is well defined (see Definition 8.2). Along the way we relate intersection theory with such constructions as the "shriek" or "umkehr" map, the Pontrjagin-Thom "collapse map", and the Thom isomorphism.

9.1 Poincaré Duality, the "shriek map", and the Thom isomorphism

Let M^m and N^n be closed, oriented manifolds of dimensions m and n respectively. Their orientations determine (and are determined by) choices of fundamental classes $[M^m] \in H_m(M^m;\mathbb{Z})$ and $[N^n] \in H_n(N^n;\mathbb{Z})$ that in turn determine Poincaré duality isomorphisms

$$\cap[M^m] : H^q(M;\mathbb{Z}) \xrightarrow{\cong} H_{m-q}(M;\mathbb{Z}) \quad \text{and} \quad \cap[N^n] : H^q(N;\mathbb{Z}) \xrightarrow{\cong} H_{n-q}(N;\mathbb{Z}) \tag{9.1}$$

We refer to their inverse isomorphisms as

$$D_M : H_r(M;\mathbb{Z}) \xrightarrow{\cong} H^{m-r}(M;\mathbb{Z}) \quad \text{and} \quad D_N : H_r(N;\mathbb{Z}) \xrightarrow{\cong} H^{n-r}(N;\mathbb{Z}). \tag{9.2}$$

Given a map $f : M^m \to N^n$, we of course have the induced homomorphisms in both homology and cohomology, which would exist even if M and N were replaced by any topological spaces. However, given that they are closed, oriented manifolds, the existence of Poincaré duality allows one to define a "shriek" or "umkehr" map.

Definition 9.1. *Define the homomorphism $f^! : H^q(M^m;\mathbb{Z}) \to H^{n-m+q}(N^n;\mathbb{Z})$ to be the unique map making the following diagram commute:*

$$\begin{array}{ccc} H^q(M^m;\mathbb{Z}) & \xrightarrow{f^!} & H^{n-m+q}(N^n;\mathbb{Z}) \\ {\scriptstyle \cap[M^m]}\downarrow\cong & & \cong\downarrow{\scriptstyle \cap[N^n]} \\ H_{m-q}(M;\mathbb{Z}) & \xrightarrow[f_*]{} & H_{m-q}(N;\mathbb{Z}) \end{array}$$

Now suppose that M^m is a closed, oriented m-dimensional manifold and N^n is a compact, oriented, n-dimensional manifold with boundary. Then a map $f : M \to N$ defines a shriek map with values in *relative* cohomology,

$$f^! : H^q(M^m;\mathbb{Z}) \to H^{n-m+q}(N^n, \partial N;\mathbb{Z}). \tag{9.3}$$

This is defined by using the relative version of Poincaré duality ("Poincaré - Lefschetz duality"). We leave the details to the reader.

This relative version of the shriek map is important in many settings, but particularly so when one has an oriented vector bundle over a closed, oriented manifold

$$p : \xi \to M^m.$$

Assume the fiber dimension of this vector bundle is k. Give ξ a Euclidean structure, and as before, let $D(\xi)$ and $S(\xi)$ denote the associated unit disk bundle and sphere bundle respectively. Notice that the orientation on ξ as well as the orientation on the base manifold M^m gives $D(\xi)$ the structure of a compact $m+k$-dimensional oriented manifold with boundary. Its boundary is given by $\partial D(\xi) = S(\xi)$.

Now let $\zeta : M^m \to D(\xi)$ be the the zero section. Then as discussed above, this defines a shriek map

$$\begin{aligned} \zeta^! : H^q(M^n;\mathbb{Z}) \to H^{q+k}(D(\xi), \partial D(\xi);\mathbb{Z}) &\xrightarrow{=} H^{q+k}(D(\xi), S(\xi);\mathbb{Z}) \\ &= H^{q+k}(T(\xi);\mathbb{Z}) \end{aligned}$$

where $T(\xi)$ is the Thom space of the bundle ξ.

The following result relates this shriek map, which is defined via Poincaré duality, with the Thom isomorphism.

Proposition 9.1. *Given an oriented, k-dimensional vector bundle over a closed, oriented manifold, $p : \xi \to M^m$ the shriek map of the zero section*

$$\zeta^! : H^q(M^m;\mathbb{Z}) \to H^{q+k}(D(\xi), \partial D(\xi);\mathbb{Z}) = H^{q+k}(T(\xi);\mathbb{Z})$$

is equal to the Thom isomorphism

$$\cup u : H^q(M^m;\mathbb{Z}) \xrightarrow{\cong} H^{q+k}(T(\xi);\mathbb{Z}).$$

Here $u \in H^k(T(\xi);\mathbb{Z})$ is the Thom class.

Proof. The shriek map $\zeta^!$ is defined to be the composition,

$$\zeta^! : H^q(M^n;\mathbb{Z}) \xrightarrow{\cap[M]} H_{m-q}(M^m;\mathbb{Z}) \xrightarrow{\zeta_*} H_{m-q}(D(\xi);\mathbb{Z}) \xrightarrow{D_{D(\xi)}} H^{k+q}(D(\xi),\partial D(\xi)).$$

Here, as above, $D_{D(\xi)}$ is the inverse to the Poincaré duality isomorphism given by capping with the fundamental class. Since $\cap[M]$ and $D_{D(\xi)}$ are both isomorphisms, and because ζ_* is an isomorphism since the zero section ζ is a homotopy equivalence, we may conclude that the composition $\zeta^!$ is an isomorphism. Of course we know that cupping with the Thom class $\cup u : H^q(M^m;\mathbb{Z}) \xrightarrow{\cong} H^{q+k}(T(\xi);\mathbb{Z})$. is also an isomorphism. So we need only show that they are the same isomorphism.

Notice that when $q = 0$, $H^q(M^m;\mathbb{Z}) \cong \mathbb{Z}$ and so the two isomorphisms $\zeta^!$ and $\cup u$ must agree in this dimension, at least up to sign. We leave it to the reader to check that the signs in fact agree given the compatibility of the orientation of $D(\xi)$ with the orientation of the bundle $p : \xi \to M^n$ and the orientation of M.

In general dimensions, let $\beta \in H^q(M)$. Since the zero section ζ is a homotopy equivalence we may write $\beta = \zeta^*(\alpha)$ for a unique class $\alpha \in H^q(D(\xi);\mathbb{Z})$.

$$\begin{aligned}\zeta^!(\beta) &= D_{D(\xi)}(\zeta_*(\beta \cap [M]) \\ &= D_{D(\xi)}(\zeta_*(\zeta^*(\alpha) \cap [M]) \\ &= D_{D(\xi)}(\alpha \cap \zeta_*[M]) \quad \text{by the naturality of the cap product.}\end{aligned} \tag{9.4}$$

Now the Thom isomorphism in *homology* is given by capping with the Thom class $\cap u : H_r(D(\xi), S(\xi)) \xrightarrow{\cong} H_{r-k}(M)$. In particular $[M] \in H_m(M;\mathbb{Z}) \cong \mathbb{Z}$ is equal to $u \cap [D(\xi), \partial D(\xi)]$ where $[D(\xi), \partial D(\xi)] \in H_{m+k}(D(\xi), \partial D(\xi);\mathbb{Z})$ is the (relative) fundamental class. Thus

$$\begin{aligned}\zeta^!(\beta) = D_{D(\xi)}(\alpha \cap \zeta_*[M]) &= D_{D(\xi)}(\alpha \cap z_*(u \cap [D(\xi), \partial D(\xi)]) \\ &= D_{D(\xi)}((\zeta^*(\alpha) \cup u) \cap [D(\xi), \partial D(\xi)]) \\ &= \zeta^*(\alpha) \cup u \quad \text{since } D_{D(\xi)} \text{ is inverse to} \\ &\quad \text{capping with the fundamental class} \\ &= \beta \cup u.\end{aligned}$$

□

As a result of this proposition we will be able to prove a result relating the shriek map to so-called "Thom collapse map", which is crucial in intersection theory.

The Thom collapse map can be described as follows. Let $e : N^n \hookrightarrow M^m$ be a smooth embedding of closed, oriented, smooth manifolds. Let ν be tubular neighborhood of $e(N^n)$ in M^m. Notice that the quotient space, $M/(M-\nu)$ is the one point compactification $\nu \cup \infty$, which is in turn homeomorphic to the Thom space $T(\nu)$.

Definition 9.2. *The "Thom collapse map" $\tau : M^m \to T(\nu)$ is the projection*

$$\tau : M^m \to M/(M-\nu) \cong T(\nu).$$

Theorem 9.2. *As above let $e : N^n \hookrightarrow M^m$ be a smooth embedding of closed, oriented, smooth manifolds. Let ν be tubular neighborhood of $e(N^n)$ in M^m. Let $k = m-n$ be the codimension of the embedding. Then the composition in cohomology*

$$H^q(N) \xrightarrow[\cong]{\cup u} H^{q+k}(T(\nu)) \xrightarrow{\tau^*} H^{q+k}(M)$$

is equal to the shriek map $e^! : H^q(N) \to H^{q+k}(M)$. Here $u \in H^k(T(\nu))$ is the Thom class.

Proof. Notice that the disk bundle, $D(\nu)$, is an oriented $m = n+k$-dimensional manifold with boundary $\partial D(\nu) = S(\nu)$. Let $[D(\nu), S(\nu)] \in H_m(D(\nu), S(\nu)) = H_m(T(\nu))$ be the relative fundamental class.

Observe first that $\tau_*[M] = [D(\nu), S(\nu)] \in H_m(D(\nu), S(\nu))$. This is because the diagrams

$$\begin{array}{ccc} H_m(M) & \xrightarrow{\tau_*} & H_m(D(\nu), S(\nu)) \\ \cong\downarrow & & \downarrow\cong \\ H_m(M, M-x) & \xrightarrow[=]{} & H_m(D(\nu), D(\nu)-x) \end{array}$$

commute for every $x \in D(\nu) \subset M$. Now the fundamental class $[M] \in H_m(M)$ is the unique class that maps to the generator of $H_m(M, M-x) \cong \mathbb{Z}$ determined by the orientation. Therefore $\tau_*([M]) \in H_m(D(\nu), S(\nu))$ is a class that maps to the generator of $H_n(D(\nu), D(\nu)-x)) \cong \mathbb{Z}$ determined by the orientation. But this property characterizes $[D(\nu), S(\nu)] \in H_m(D(\nu), S(\nu))$.

Secondly, observe that the following diagram commutes:

$$\begin{array}{ccc} \tilde{H}^*(D(\nu)/S(\nu)) & \xleftarrow{=} & H^*(D(\nu), S(\nu)) \\ \cap[D(\nu)/S(\nu)]\downarrow & & \cong\downarrow \cap[D(\nu), S(nu)] \\ \tilde{H}_{m-*}(D(\nu)/S(\nu)) & \longleftarrow & H_{m-*}(D(\nu)) \\ \tau_*\uparrow & & \downarrow\tilde{e}_* \\ H_{m-*}(M) & \xleftarrow[=]{} & H_{m-*}(M). \end{array}$$

Here $[D(\nu)/S(\nu)]$ is the image of the (relative) fundamental class $[D(\nu), S(\nu)]$ under the isomorphism $H_m(D(\nu), S(nu)) \xrightarrow{\cong} \tilde{H}_m(D(\nu)/S(\nu))$. $\tilde{e} : D(\nu) \hookrightarrow M$ is the extension of the embedding e to its tubular neighborhood.

By the naturality of the cap product this diagram expands to the following commutative diagram.

$$
\begin{array}{ccccc}
H^*(D(\nu), S(\nu)) & \xrightarrow{=} & H^*(D(\nu), S(\nu)) & \xrightarrow[\cong]{\cap[D(\nu),S(\nu)]} & H_{m-*}(D(\nu)) \\
\downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle \cap[D(\nu)/S(\nu)]} & & \downarrow{\scriptstyle \tilde{e}_*} \\
H^*(M) & & H_{m-*}(D(\nu)/S(\nu)) & \xleftarrow[\tau_*]{} & H_{m-*}(M) \\
\downarrow{\scriptstyle =} & & & & \downarrow{\scriptstyle =} \\
H^*(M) & & \xrightarrow[\cap[M]]{} & & H_{m-*}(M)
\end{array}
$$

By the above proposition we can now add to the exterior of this diagram:

$$
\begin{array}{ccc}
H^{*-k}(N) & \xrightarrow[\cong]{\cap[N]} & > H_{m-*}(N) \\
\downarrow{\scriptstyle \cup u = \zeta^!} & & \downarrow{\scriptstyle \zeta_*} \\
H^r(D(\nu), S(\nu)) & \xrightarrow[\cong]{\cap[D(\nu),S(\nu)]} & H_{m-*}(D(\nu)) \\
\downarrow{\scriptstyle \tau^*} & & \downarrow{\scriptstyle \tilde{e}_*} \\
H^*(M) & \xrightarrow[\cap[M]]{\cong} & H_{m-*}M
\end{array}
$$

Thus $\tau^* \circ \cup u = D_M \circ \tilde{e}_* \circ \zeta_* \circ \cap[N]$. (Recall that the duality isomorphism $D_M = (\cap[M])^{-1}$.) But $\tilde{e} \circ \zeta = e$, so we have that

$$\tau^* \circ \cup u = D_M \circ e_* \circ \cap[N] = e^!, \quad \text{by definition.}$$

□

The following corollary gives a clear relation between the Thom collapse map and Poincaré duality. In particular it says that the Thom class of a normal bundle of an embedded submanifold is dual to the fundamental class of the submanifold.

Corollary 9.3. *Let M be a closed, oriented manifold, with oriented, closed submanifold $e : N \hookrightarrow M$ of codimension k. Let ν be a tubular neighborhood of N, which we identify with the normal bundle. Let $\tau : M \to M/(M - \nu) \cong T(\nu)$ be the Thom collapse map, and let $u \in H^k(T(\nu))$ be the Thom class. Then*

$$\tau^*(u) = D(N).$$

Said another way, $\tau^*(u) \cap [M] = [N]$.

Proof. By Theorem 9.2, $\tau^*(u) = e^!(1)$. But recall that $e^! : H^0(N) \to H^k(M)$ is defined to be the unique homomorphism that makes the following diagram commute:

$$\begin{array}{ccc} H^0(N) & \xrightarrow{e^!} & H^k(M) \\ {\scriptstyle \cap[N]}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \cap[M]} \\ H_n(M) & \xrightarrow[e_*]{} & H_n(M). \end{array}$$

Thus $\tau^*(u) \cap [M] = e^!(1) \cap [M] = e_*([N])$. □

9.2 The intersection product

One can define the "intersection product" in the homology of a closed, oriented manifold both geometrically, using transversality theory, and algebraically, using Poincaré duality and the cup product. Our goal in this section is to show that these constructions define the same homological product. The intersection number, defined earlier (Definition 8.2), will be shown to be a special example of this product, and the consequence of these results will show that this number does not depend on the various geometric choices one makes in defining it.

Definition 9.3. *Let M^m be a closed, oriented m-dimensional manifold. The intersection product* is the pairing

$$H_p(M) \times H_n(M) \to H_{p+n-m}(M)$$
$$\alpha \times \beta \to \alpha \cdot \beta$$

defined to be the unique homomorphism making the following diagram commute:

$$\begin{array}{ccc} H_p(M) \times H_n(M) & \xrightarrow{\cdot} & H_{p+n-m}(M) \\ {\scriptstyle \cap[M]\times\cap[M]}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \cap[M]} \\ H^{m-p}(M) \times H^{m-n}(M) & \xrightarrow[\cup]{} & H^{2m-p-n}(M). \end{array}$$

That is, the intersection product is Poincaré dual to the cup product.

The following is the main result of this section.

Again, let M^m be a closed, oriented m-dimensional manifold. Suppose it has two oriented, closed submanifolds P^p of dimension p and N^n of dimension n that intersect transversally. (Otherwise perturb one of them so that the interesection becomes transverse.) By abuse of notation we let $[P] \in H_p(M)$ and $[N^n] \in H_n(M)$ be the homology classes given by the images of the fundamental classes of these submanifolds under the homomorphisms induced by their embeddings. We say that these submanifolds *represent* these homology classes.

Theorem 9.4. *Under these assumptions the homology class represented by the intersection*

$$[P \cap N] \in H_{p+n-m}(M)$$

represents the intersection product of the classes represented by the submanifolds P^p and N^n:

$$[P^p] \cdot [N^n] = [P \cap N].$$

This theorem actually has a generalization, whose proof requires only small adjustments to the proof of Theorem 9.4. We leave the details to the reader.

Theorem 9.5. *Let M^n, P^p, and N^n be closed, oriented manifolds. Let $f : P^p \to M^n$ be a smooth map and $g : N^n \hookrightarrow M^n$ a smooth embedding. Assume that $f \pitchfork g(N^n)$. That is for every $x \in P$ and $y \in N$ with $f(x) = g(y) = z \in M$, then $Df_x(T_xP) \oplus Dg_y(T_yN) = T_zM$. Consider the submanifold $f^{-1}(g(N)) \subset P$. Then this is a closed, oriented submanifold of dimension $p+n-m$ and the image of its fundamental class in homology $f_*[f^{-1}(g(N))] \in H_{p+n-m}(M)$ is Poincaré dual to the cup product $D_M(f_*[P]) \cup D_M(g_*([N]) \in H^{2m-p-n}(M)$.*

Before we prove Theorem 9.4 we make a couple remarks:

Remarks.

- Let's generalize our notion of "representing" a homology class in a closed oriented manifold by a submanifold, to a homology class $\alpha \in H_q(M)$ being represented by a manifold if there exists a closed, oriented manifold Q^q and a map $\phi : Q \to M$ with $\phi_*([Q]) = \alpha$. Then we will see in Chapter 12 below, that not every integral homology class is represented by such a manifold. However, as we will see below, a consequence of Thom's calculation of the unoriented cobordism ring is that in homology with $\mathbb{Z}/2$-coefficients, indeed every homology class is represented by a manifold. In the presence of such representations, (in integral or $\mathbb{Z}/2$ homology), this theorem says that the Poincaré dual of the cup product is represented by (transversal) intersections of manifolds. This gives a rather remarkable geometric interpretation of the cup product.
- Historically, there is reason to believe that the development of cohomology and the cup product was motivated by the goal of representing intersections of submanifolds. S. Lefshetz, who did seminal work in the development of intersection theory in both algebraic geometry and algebraic topology, was instrumental in developing the cup product in singular cohomology.

Proof of Theorem 9.4.

Proof. Consider the following commutative diagram, where the maps are all embeddings:

$$\begin{array}{ccc} N & \xrightarrow[\subset]{e_N} & M \\ \cup\Big\uparrow e_{P\cap N,N} & & \cup\Big\uparrow e_P \\ P\cap N & \xrightarrow[e_{P\cap N,P}]{\subset} & P \end{array}$$

By examining this diagram one sees that when one restricts the normal bundle of N in M to $P \cap N$, one gets the normal bundle of $P \cap N$ in P:

$$(\nu_{e_N})_{|P\cap N} = \nu_{e_{P\cap N,P}}.$$

Equivalently, the intersection of a tubular neighborhood of e_N with P is a tubular neighborhood of $e_{P\cap N,P}$. We represent these tubular neighborhoods by η's. We therefore have a commutative diagram involving Thom collapse maps:

$$\begin{array}{ccc} M & \xrightarrow{\tau_N} & M/(M-\eta_N) \cong T(\nu_{e_N}) \\ e_P\Big\uparrow & & \Big\uparrow T(e_P) \\ P & \xrightarrow[\tau_{P\cap N,P}]{} & P/(P-\eta_{P\cap N,P}) = T(\nu_{e_{P\cap N,P}}). \end{array}$$

Here $T(\nu)$ denotes the Thom space of the corresponding normal bundle, and $T(e_P)$ denotes the map of Thom spaces induced by the embedding e_P.

In particular this means that on the level of Thom classes,

$$T(e_P)^*(u_N) = u_{P\cap N,P} \in H^{m-n}(T(\nu_{e_{P\cap N,P}})).$$

Now by Corollary 9.3

$$\tau^*_{P\cap N,P}(u_{P\cap N,P}) \cap [P] = [P\cap N] \in H_{p+n-m}(P).$$

So therefore

$$\tau^*_{P\cap N,P}(T(e_P)^*(u_N)) \cap [P] = [P\cap N] \in H_{p+n-m}(P),$$

and by the commutativity of the above diagram, this means

$$e_P^*(\tau_N^*(u_N)) \cap [P] = [P\cap N] \in H_{p+n-m}(P).$$

So we may conclude that

$$(e_P)_*(e_P^*(\tau_N^*(u_N)) \cap [P]) = (e_P)_*[P\cap N] \in H_{p+n-m}(M).$$

By the definition of the intersection product, this says that

$$[P]\cdot[N] = [P\cap N] \in H_{p+n-m}(M).$$

□

An immediate consequence of this theorem is that the (homological) intersection pairing gives an obstruction to separating two submanifolds. By "separating", we mean that there is an isotopy of one or both of the embeddings of the two submanifolds, so that the resulting submanifolds do not intersect. That is, we have the following immediate corollary.

Corollary 9.6. *Let M^m be a closed, oriented m-dimensional manifold. Suppose it has two oriented, closed submanifolds P^p of dimension p and N^n of dimension n, such that the intersection product, $[P] \cdot [N] \in H_{p+n-m}(M)$ is nonzero. Then P and N cannot be separated in M.*

Exercises.

(1). Let M^m be a C^∞ closed manifold, and let $N^n \subset M^m$ be a smooth embedded submanifold, where N^n is also assumed to be compact with no boundary. We say that N^n can be *"moved off of itself"* in M if a tubular neighborhood η of N with retraction map $\rho : \eta \to N$ admits a section $\sigma : N \to \eta$ that is disjoint from N. That is, $N \cap \sigma(N) = \emptyset \subset \eta \subset M$.

(a). Suppose the dimensions of the manifolds satisfy $2n < m$. Prove that N can be moved off of itself in M.

(b). To see that the dimension requirement above is necessary in general, show that

$$\mathbb{RP}^1 \subset \mathbb{RP}^2$$

cannot be moved off of itself. *Hint:* Compute the self intersection number (mod 2) of $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

(2). Write $\mathbb{CP}^n$ in its projective coordinates. $\mathbb{CP}^2 = \{[z_0, z_1, z_2] \in \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^\times\}$. That is $\mathbb{CP}^{n+1}$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action, via scalar multiplication, of the nonzero complex numbers $\mathbb{C}^\times$.

There are two natural copies of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$ given by $\{[z_0, z_1, 0]\}$ and by $\{[0, z_1, z_2]\}$. If we call one of these N and the other K,

Show that the intersection product $[N] \cdot [K] = 1 \in H_0(\mathbb{CP}^2)$. Conclude that each of these classes represent a generator of $H_2(\mathbb{CP}^2)$.

(3). Let M^n be a closed oriented n-dimensional manifold, and let $\Delta : M \to M \times M$ be the diagonal map. Let $\Delta_! : H_q(M \times M) \to H_{q-n}(M)$ be the shriek map in homology. Show that for any homology classes α and β of M, then $\alpha \cdot \beta = \pm \Delta_!(\beta \times \alpha)$.

9.2.1 Intersection theory via Differential Forms

We end this section by pointing out how to compute the intersection number of two submanifolds of complementary dimension using differential forms.

Let M^m be a closed oriented manifold, with submanifolds Q^q of dimension q and P^p of dimension p where $p + q = m$. Let η_Q and η_P be tubular neighborhoods of these submanifolds. These can be viewed as open manifolds of dimension n. The (DeRham) cohomology with compact supports, $H^*_{cpt}(\eta_Q)$ is equal to the cohomology of the one-point compactification, which is homeomorphic to the Thom space of the normal bundle. Therefore there is a Thom class $u_Q \in H^p_{cpt}(\eta_Q)$, and similarly $u_P \in H^q_{cpt}(\eta_P)$. The Thom collapse map gives classes $\nu_Q \in H^p(M)$ and $\nu_P \in H^q(M)$. By abuse of notation we let ν_Q and ν_P denote differential forms on M of dimension q and p respectively that represent these cohomology classes.

These "Thom forms" can be viewed a differential forms on M whose support lies in the relevant tubular neighborhood which yield the orientation forms of the corresponding normal bundles.

The following is a reinterpretation of Theorem 9.4 in this setting, using the DeRham theorem. We leave the job of filling in the details of its proof as an exercise to the reader,

Theorem 9.7. *In the setting described above,*

$$\begin{aligned}[Q]\cdot[P] &= \int_M \nu_Q \wedge \nu_P \\ &= \langle u_Q \cup u_P; [M]\rangle \\ &= \int_P \nu_Q = \pm \int_Q \nu_P.\end{aligned}$$

9.3 Degrees, Euler numbers, and Linking numbers

In this section we will discuss interesting applications of the results about intersection theory developed in the last section.

9.3.1 The Degree of a map

Let $f : N^n \to M^n$ be a smooth map between closed, oriented, connected smooth manifolds of the same dimension ($= n$). The *degree* of f is an oriented (signed) count of the number of elements in the preimage of a generic point. More specifically we make the following definition:

Definition 9.4. *The degree of f, written $Deg(f)$ is defined to be the intersection number of $f : N^n \to M^n$ and a regular value $x \in M^n$, viewed as a zero-dimensional submanifold. That is, $Deg(f) = f_*[N^n]\cdot[x] \in \mathbb{Z}$.*

.

Notice that the intersection number, as defined in Definition 8.2, in this setting is given by

$$f_*[N] \cdot [x] = \sum_{i=1}^{k} sgn(x_i) \in \mathbb{Z}, \quad \text{where the sum is taken over all points in } f^{-1}(x) \in N$$
$$= [f^{-1}(x)] \in H_0(N^n) = \mathbb{Z} \quad \text{by Theorem 9.5.}$$

Now by Theorem 9.5,

$$f^{-1}(x)] = f_*[N] \cdot [x] = f^*(D_M[x]) \cap [N]$$
$$= D_M[x] \cap f_*[N].$$

Since the fundamental class $[M] \in H_n(M) \cong \mathbb{Z}$ is a generator, we may interpret this as the following corollary to Theorem 9.5.

Corollary 9.8. *Write $f_*[N] = d[M] \in H_n(M)$. Then $d = Deg(f)$.*

This corollary allows for easier calcuations of degree, and also shows that the notion of degree does not depend on the choice of regular value $x \in M$. Moreover it allows the extension of the notion of degree to any continuous (not necessarily smooth) map.

9.3.2 The Euler class and self intersections

Recall from Definition 6.5 that if $\xi \to N$ is an oriented vector bundle of fiber dimension k, the *Euler class*

$$\chi(\xi) \in H^k(N)$$

is defined to be the image of the Thom class under the composition

$$H^k(T(\xi)) = H^k(D(\xi), S(\xi)) \to H^k(D(\xi)) \xrightarrow{\zeta^*} H^k(N)$$

where $\zeta : N \to D(\xi) \subset \xi$ is the zero section.

In the setting when N is a n-dimensional, closed, oriented manifold, we can relate the Euler class to the self intersection of the zero section.

First, we explain what we mean by "self intersection". If $e : N^n \hookrightarrow M^m$ is an embedding of N into a compact, connected, oriented manifold M^n (with or without boundary), then we can perturb (i.e find an isotopy) of the embedding e to an embedding $\tilde{e} : N^n \hookrightarrow M^m$ so that $e(N) \pitchfork \tilde{e}(N)$. By Theorem 9.4, the resulting intersection, $e(N) \cap \tilde{e}(N)$ represents the class $[N] \cdot [N] \in H_{2n-m}(M)$. This class is called the "self intersection class". In particular, if $m = 2n$, this is a zero dimensional homology class, and therefore an integer, which represents a (signed) count of the number of points in the intersection $e(N) \cap \tilde{e}(N)$.

In the setting of a k-dimensional, oriented, smooth vector bundle $p : \xi \to$

N^n we may view the disk bundle $D(\xi)$ as an $(n+k)$-dimensional, oriented, compact manifold with boundary, and the zero section $\zeta : N \hookrightarrow D(\xi)$ as an embedding. We then have the following result.

Theorem 9.9. *The self intersection class of the zero section*

$$[\zeta(N)] \cdot [\zeta(N)] \in H_{n-k}(D(\xi)) \cong H_{n-k}(N)$$

is Poincaré dual to the Euler class. That is,

$$\chi(\xi) \cap [N] = [\zeta(N)] \cdot [\zeta(N)].$$

In particular, when $k = n$, the evaluation of the Euler class on the fundamental class $\langle \chi(\xi); [N] \rangle$ is equal to the self intersection number of the zero section.

Before we prove this theorem we observe the following corollary.

Corollary 9.10. *If a smooth vector bundle $p : \xi \to N^n$ over a closed, oriented manifold has a nowhere zero section, then the Euler class $\chi(\xi)$ is zero.*

Proof. Notice that any section $\sigma : N \to \xi$ is a homotopy equivalence, and is homotopic, as a map of spaces, to the zero section ζ. Such a homotopy can be taken to be $(x,t) \to (1-t)\sigma(x)$. Therefore the homology classes represented by these sections, $[\sigma(N)]$ and $[\zeta(n)]$ are equal. If $\sigma(x)$ is never zero, then $\sigma(N) \cap \zeta(N) = \emptyset$. Therefore by Theorem 9.4

$$0 = [\zeta(n)] \cdot [\sigma(N)] = [\zeta(n)] \cdot [\zeta(n)].$$

By Theorem 9.9, the Euler class $\chi(\xi) = 0$. □

We now prove Theorem 9.9.

Proof. By Proposition 9.1 the following diagram commutes:

$$\begin{array}{ccc}
H^q(N) & \xrightarrow[\cong]{\cup u} & H^{q+k}(D(\xi), S(\xi)) \\
{\scriptstyle \cap[N]}\downarrow \cong & & \cong \downarrow {\scriptstyle \cap[D(\xi),S(\xi)]} \\
H_{n-q}(N) & \xrightarrow[\zeta_*]{\cong} & H_{n-q}(D(\xi)).
\end{array}$$

We now insert this into a larger diagram:

$$\begin{array}{ccccc}
H^k(D(\xi), S(\xi)) \times H^k(D(\xi), S(\xi)) & \xrightarrow{\cup} & H^{2k}(D(\xi), S(\xi)) & \xleftarrow[\cong]{\cup u} & H^k(N) \\
{\scriptstyle \cap[D(\xi),S(\xi)] \times \cap[D(\xi),S(\xi)]}\downarrow \cong & & \cong \downarrow {\scriptstyle \cap[D(\xi),S(\xi)]} & & \cong \downarrow {\scriptstyle \cap[N]} \\
H_n(D(\xi)) \times H_n(D(\xi)) & \xrightarrow[\cdot]{} & H_{n-k}(D(\xi)) & \xleftarrow[\zeta_*]{\cong} & H_{n-k}(N)
\end{array}$$

Notice that the left hand square defines the intersection product in $H_*(D(\xi))$. Now by Theorem 9.9, the product of the Thom classes $u \times u$ in the upper left corner of this diagram, maps to $\zeta_*([N]) \times \zeta_*([N])$ in the lower left corner. But this class in turn maps to the intersection product $\zeta_*([N]) \cdot \zeta_*([N])$ in the lower middle of the diagram $(H_{n-k}(D(\xi)))$.

Furthermore, by definition, the Euler class $\chi(\xi) \in H^k(N)$ in the upper right corner of the diagram, maps to $u \cup u \in H^{2k}(D(\xi), S(\xi))$, and so

$$(\chi(\xi) \cup u) \cap [D(\xi), S(\xi)] = \zeta_*([N]) \cdot \zeta_*([N]) \in H_{n-k}(D(\xi)).$$

By the commutativity of the right hand square we conclude that

$$\zeta_*(\chi(\xi) \cap [N]) = \zeta_*([N]) \cdot \zeta_*([N]) \in H_{n-k}(D(\xi)).$$

This is the statement of the theorem. □

We now turn our attention to the case when the bundle we are considering is the tangent bundle, $p : \tau N \to N$. A section of the tangent bundle is a *vector field* on N. Applying Corollary 9.10 to this situation gives us the following:

Proposition 9.11. *If a smooth, closed, orientable manifold N has a nowhere zero vector field, then the Euler class of its tangent bundle, $\chi(\tau N)$, which we denote by $\chi(N)$, is zero.*

We end this subsection with a well known result which relates the Euler class of a manifold (i.e of its tangent bundle), with its Euler characteristic.

Theorem 9.12. *Let N be a closed, oriented, n-dimensional smooth manifold. Then the evaluation of its Euler class on the fundamental class is the Euler characteristic of the manifold:*

$$\langle \chi(N), [N] \rangle = \sum_{i=0}^{n} (-1)^i rank\, H_i(N).$$

Proof. The proof of this theorem involves a few steps. First, consider the diagonal embedding,

$$\Delta : N \to N \times N.$$

We first observe that the normal bundle $\nu(\Delta)$ of this embedding is the tangent bundle τN. We leave the verification of this fact to the reader. In order not to confuse notation we now adopt the "exponential" notation for the Thom space of a bundle. That is if $\xi \to X$ is a vector bundle, we now use the notation X^ξ to denote its Thom space.

Let $\tau : N \times N \to N^\nu(\Delta) = N^{\tau N}$ be the Thom collapse map. We now compute this Thom collapse map in cohomology. To do this, notice that Poincaré duality defines a nonsingular pairing

$$\langle\ ,\ \rangle : H^*(N; k) \times H^*(N; k) \to k$$
$$\langle \alpha, \beta \rangle = (\alpha \cup \beta)([N])$$

Let $\{\alpha_i\}$ be a basis for $H^*(N;k)$. Since this pairing is nondegenerate, there is a corresponding *dual* basis $\{\alpha_i^*\}$. That is, $(\alpha_i^* \cup \alpha_j)([N]) = \delta_{i,j}$, the Kronecker delta. In particular notice that if $\alpha_i \in H^q(N;k)$, then $\alpha_i^* \in H^{n-q}(N;k)$.

Lemma 9.13. *Let $u \in H^n(N^{\tau N};k)$ be the Thom class of the tangent bundle. Then*

$$\tau^*(u) = \sum_i (-1)^{|\alpha_i|} \alpha_i^* \times \alpha_i \in H^n(N \times N; k),$$

where $|\alpha_i|$ denotes the degree of α_i.

Proof. We take the following computation from Bredon [16], proof of Theorem 12.4.

By the Kunneth theorem we can write

$$\tau^*(u) = \sum_{i,j} c_{i,j}\, \alpha_i^* \times \alpha_j$$

for some coefficients $c_{i,j}$. Notice that we need only add over those terms where $|\alpha_i^*| + |\alpha_j| = n$. Since $|\alpha_i^*| = n - |\alpha_i|$, we assume $|\alpha_j| = |\alpha_i|$. For the following calculation take basis elements α_i and α_j of degree p. We compute $((\alpha_i \times \alpha_j^*) \cup \tau^*(u))([N \times N])$ in two different ways.

$$\begin{aligned}
((\alpha_i \times \alpha_j^*) \cup \tau^*(u))([N \times N]) &= (\alpha_i \times \alpha_j^*)(\tau^*(u) \cap [N \times N]) \\
&= (\alpha_i \times \alpha_j^*)(\Delta_*([N]), \quad \text{by Corollary 9.3} \\
&= \Delta^*(\alpha_i \times \alpha_j^*)([N]) \\
&= (\alpha_i \cup \alpha_j^*)([N]) \\
&= (-1)^{p(n-p)}(\alpha_j^* \cup \alpha_i)([N]) \\
&= (-1)^{p(n-p)}\delta_{i,j}
\end{aligned}$$

On the other hand

$$\begin{aligned}
&((\alpha_i \times \alpha_j^*) \cup \tau^*(u))([N \times N]) \\
&\quad = ((\alpha_i \times \alpha_j^*) \cup (\sum_{r,s} c_{r,s}\, \alpha_r^* \times \alpha_s))([N \times N]) \\
&\quad = (-1)^{n-p} c_{i,j}((\alpha_i \cup \alpha_i^*) \times (\alpha_j^* \cup \alpha_j)([N] \times [N]) \\
&\quad \text{since one gets zero for } \alpha_i, \alpha_j \neq \alpha_r, \alpha_s, \text{ all of degree } p \\
&\quad = (-1)^{n-p+p(n-p)+n} c_{i,j}((\alpha_i \cup \alpha_i^*)([N]))((\alpha_j^* \cup \alpha_j)([N])) \\
&\quad = (-1)^{p(n-p)-p} c_{i,j}.
\end{aligned}$$

So we conclude that $c_{i,j} = (-1)^p \delta_{i,j}$. ∎

To complete the proof of Theorem 9.12, we make the following observation about the relation of the Thom collapse map and the Euler class. Let $e : N \hookrightarrow M$ be a codimension k embedding of oriented manifolds, with normal bundle ν_e, and let $\tau : M \to N^{\nu_e}$ is the Thom collapse map. The following comes from a quick check of definitions, which we leave for the reader.

Lemma 9.14. *If $u \in H^k(N^{\nu_e})$ be the Thom class. Then the Euler class of the normal bundle ν_e can be described by*

$$\chi(\nu_e) = e^*\tau^*(u) \in H^k(N).$$

Applying this lemma to the diagonal embedding $\Delta : N \to N \times N$, we have that $\Delta^*(\tau^*(u)) = \chi(N)$. Applying Lemma 9.3.2 with rational coefficients we have that

$$\begin{aligned}\chi(N)([N]) = \Delta^*(\tau^*(u))([N]) &= \sum_i (-1)^|\alpha_i|(\alpha_i^* \cup \alpha_i)([N]) \\ &= \sum_i (-1)^|\alpha_i|\langle \alpha_i^*, \alpha_i\rangle \\ &= \sum_i (-1)^|\alpha_i| \\ &= \text{Euler characteristic of } N\end{aligned}$$

□

Notice that as an application of this theorem and of Proposition 9.11 we get the following classical result:

Proposition 9.15. *If a closed, oriented manifold N has nonzero Euler characteristic, then every vector field on N must contain a zero.*

In particular every vector field on an even dimensional sphere must contain a zero. This famous result, when applied to S^2 is often referred to as the "Hairy Billiard Ball Theorem".

9.3.3 Linking Numbers

We now discuss one more application of intersection theory. This is the classical notion of *linking numbers*.

In the general setting, suppose we have embeddings of closed, oriented manifolds in Euclidean space,

$$\begin{array}{ccc} M^m & \xrightarrow[\subset]{K_1} & \mathbb{R}^{n+m+1} \\ & & \cup \uparrow K_2 \\ & & N^n. \end{array}$$

We will assume that these manifolds intersect transversally, which in these dimensions means that they have disjoint images. Consider the composition

$$\begin{aligned} \Psi_{M,N} : & M^m \times N^n \to \mathbb{R}^{n+m+1} - \{0\} \to S^{n+m} \\ & (x,y) \longrightarrow (K_1(x) - K_2(y)) \longrightarrow \frac{K_1(x) - K_2(y)}{|K_1(x) - K_2(y)|} \end{aligned}$$

Giving S^{n+m} the orientation coming from viewing it as the boundary of the ball D^{n+m+1} inside $\mathbb{R}^{n+m+1}$, we can make the following definition.

Definition 9.5. *Define the linking number, $Lk(K_1, K_2)$ to be the degree*

$$Lk(K_1, K_2) = Deg(\Psi_{M,N}).$$

This is an algebraic-topological definition based on the homological properties of the map $\Psi_{M,N}$. However this notion has important geometric significance as well, as we will see in considering the classical case when we have the link of two disjointly embedded circles in S^3. We have the following diagram of embeddings:

$$\begin{array}{ccc} S^1 & \xrightarrow[\subset]{K_1} & \mathbb{R}^3 \\ & & \cup \uparrow K_2 \\ & & S^1. \end{array}$$

For $p \in S^2$, let

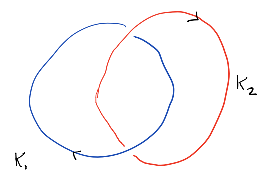
$$I(p) = \{(q_1, q_2) \in K_1 \times K_2 \,:\, q_2 - q_1 = \lambda p, \text{ where } \lambda > 0\}.$$

Notice that for $p \in S^2$, $I(p) = \Psi^{-1}_{K_1,K_2}(p)$.

Observation. Assume that $p = (0,0,1)$ is a regular value of Ψ_{K_1,K_2}. (If it is not, compose Ψ_{K_1,K_2} with a rotation of S^2 so that this condition is satisfied.) Project $K_1 \cup K_2$ onto $\mathbb{R}^2 = (x_1, x_2)-$ plane in $\mathbb{R}^3$, keeping track of the over and under-crossings:

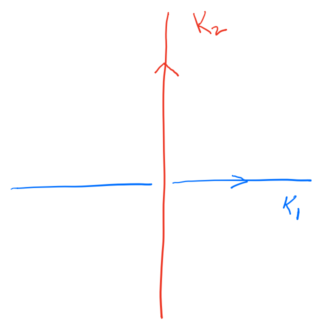
We claim that there is one element of $I(p)$ for every place that K_2 crosses *over* K_1. To see this, observe that if $(q_1, q_2) \in K_1 \times K_2$ is in $I(p)$, then the projections of q_1 and q_2 on $\mathbb{R}^2$ agree. This means that the first two coordinates of q_1 and of q_2 agree. Now since $q_2 - q_2 = \lambda p = (0,0,\lambda)$ with $\lambda > 0$, we must have that the third coordinate (the "z-coordinate") of q_2 is larger than the third coordinate of z_1. That is, K_2 crosses *over* K_1 at this point.

By Definition 9.5 of the linking number as the degree of Ψ_{K_1,K_2}, we can calculate this invariant either homologically, or, as seen after the discussion of the definition of degree (Definition 9.4) as the signed count of the points in the preimage of a regular value of Ψ_{K_1,K_2}. That is, it is a signed count of the

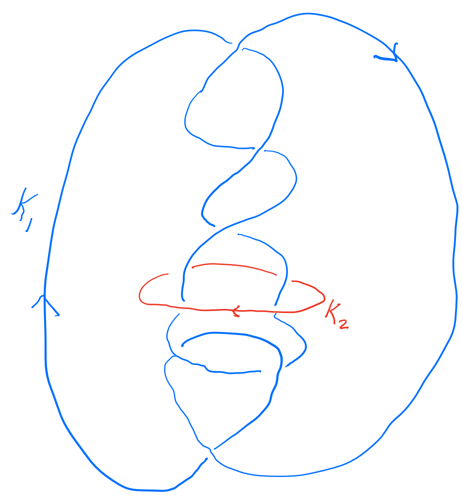


points of $I(0, 01)$. If $(q_1, q_2) \in I(0, 0, 1)$, then the sign $sgn(q_1, q_2)$ is determined by comparing the orientations of the curves, and the standard orientation of the plane. In the above example of the Hopf link, $I(0, 0, 1)$ consists of a single point, and the local orientations of the curves K_1 an K_2 at this point looks like the following. Therefore the linking number of the Hopf link is

$$Lk(K_1, K_2) = -1.$$



We now turn our attention to the following, more complicated link (figure 9.3.3).

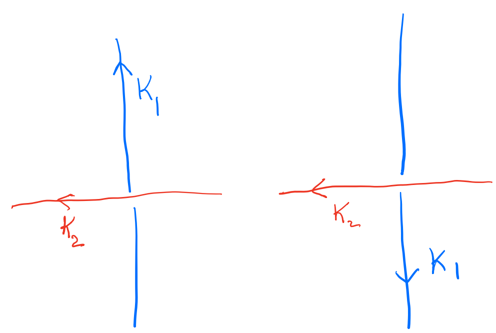


Notice that there are two places where K_2 crosses over K_1, and thus $I(0,0,1)$ has consists of two points.

The crossing on the left has $sgn = -1$ and the crossing on the right has $sgn = +1$. This means that the linking number,

$$Lk(K_1, K_2) = 0,$$

even though evidently the two embedded circles cannot be unlinked. This shows that while the linking number is a useful, computable invariant, it is not a *complete* invariant of a link of two embedded circles in $\mathbb{R}^3$.


K_1
K_2
K_2
K_1

10

Stable Homotopy

Throughout this chapter all spaces will equipped with basepoints, and will be assumed to be of the homotopy type of (based) CW-complexes.

Given a based space X, let ΣX denote its (reduced) suspension, and let

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

be the suspension homomorphism in homotopy groups. It is defined in the following way. If $\alpha : S^k \to X$ is a basepoint preserving map representing an element of $\pi_k(X)$, then define its suspension

$$\Sigma\alpha : S^{k+1} = \Sigma S^k = S^1 \wedge S^k \xrightarrow{1\wedge\alpha} S^1 \wedge X = \Sigma X. \tag{10.1}$$

The roots of stable homotopy theory go back to the following classical theorem of Freudenthal:

Theorem 10.1. *("Freudenthal Suspension Theorem" [52]) Let X be an n-connected based space, then the suspension homomorphism*

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism if $k \leq 2n$ and an epimorphism if $k = 2n+1$.

Many textbooks contain proofs of this theorem. A traditional reference is [159]. Below we will sketch a proof of this theorem using the Serre spectral sequence.

The Freudenthal Suspension Theorem naturally leads to the notion of the "stable range" for homotopy groups, i.e twice the connectivity of a space, in which homotopy data is preserved by suspension. An important example of this is the excision property. It is well known that homotopy groups, viewed as a functor from the category of pairs of based topological spaces to the category of groups, does not satisfy excision. For example, the fact that homology *does* satisfy excision implies that

$$\tilde{H}_*(X \vee Y) \xrightarrow{\cong} \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$$

and more generally the properties of excision and exactness provide the tremendously important calculational tool of the Mayer-Vietoris sequence.

But notice that the analogous homomorphism at the level of homotopy groups, is not, in general, an isomorphism. For example, $\pi_1(S^1 \vee S^1)$ is the free group on two generators, which is not commutative. On the other hand, $\pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, which certainly is commutative.

Nonetheless, as we will see later in this chapter, the Freudenthal theorem implies that the homotopy groups functor *does* satisfy excision in the stable range. This leads to the following definition.

Definition 10.1. *. Let X be a based space. Define its k^{th}-stable homotopy group $\pi_k^s(X)$ to be*

$$\pi_k^s(X) = \lim_{n\to\infty} \pi_{k+n}(\Sigma^n X).$$

where the colimit is taken over the suspension homomorphisms,

$$\Sigma : \pi_{k+n-1}(\Sigma^{n-1}(X)) \to \pi_{k+n}(\Sigma^n X).$$

Notice that by the Freudenthal Suspension Theorem, the limit in the definition of stable homotopy groups is achieved at a finite stage. More specifically, $\pi_k^s(X) \cong \pi_{k+q}(\Sigma^q X)$ for $q \geq k - 2c(X)$, where $c(X)$ is the connectivity of X (i.e the maximal nonnegative integer such that $\pi_r(X) = 0$ for $r \leq c(X)$).

Exercise Verify this claim. That is, show that $\pi_k^s(X) \cong \pi_{k+q}(\Sigma^q X)$ for $q \geq k - 2c(X)$, where $c(X)$ is the connectivity of X.

Stable homotopy groups are an invariant of the collection of spaces, $\{\Sigma^k X\}$, together with maps between the spaces in this collection $\Sigma(\Sigma^k X) \xrightarrow{\equiv} \Sigma^{k+1} X$. More generally, a collection of spaces $\{X_m\}$ together with maps $\epsilon_m : \Sigma X_m \to X_{m+1}$ is called a *spectrum*. Spectra are the objects of study of stable homotopy theory, and they were originally introduced and studied by Lima [92] and G. Whitehead [158]. We will introduce them and discuss some of their properties in this chapter. An important classical feature of spectra is that described by work of Brown [19] and Whitehead [158], where they classify *"generalized (co)homology theories"*. These are theories that satisfy all the Eilenberg-Steenrod except "dimension" (but including excision).We will discuss this classification in this chapter and discuss many important examples such as stable homotopy groups and K-theory. We will describe Bott periodicity, one of the great theorems of the twentieth century, and then discuss and apply the Atiyah-Hirzebruch spectra sequence for computing generalized (co)homology.

Another important, and more modern aspect of the study of spectra are their categorical aspects. We introduce symmetric spectra, describe ring spectra and module spectra, and then describe the Thom spectrum, viewed as a functor from the category of "spaces over BO" to the category of symmetric spectra. We discuss various products and generalized orientations of manifolds and then we have a discussion of Spanier-Whitehead duality and Atiyah duality for manifolds. We end this chapter with a discussion of the special properties of Eilenberg-MacLane spectra and the Steenrod algebras of cohomology operations.

10.0.1 Sketch of proof of the Freudenthal suspension theorem

We end this introductory section with a sketch of a proof of the Freudenthal suspension theorem. As you will see we rely on the Hurewicz theorem and the Serre spectral sequence.

Let X be an n-connected space of the homotopy type of a CW complex. If $n = 0$ then the theorem is trivial since both X and ΣX are path connected, and ΣX is simply connected. So we assume $n \geq 1$.

In our proof we will be considering based loop spaces, ΩY, where Y has the homotopy type of a based CW complex. We first recall that there is an adjunction isomorphism,

$$\pi_q(\Omega Y) \xrightarrow{\cong} \pi_{q+1}(Y).$$

This isomorphism is given as follows: Let $\alpha : S^q \to \Omega Y$ represent an element of $\pi_q(\Omega Y)$. So for every $x \in S^q$, $\alpha(x) : S^1 \to Y$ is a basepoint preserving loop. We may then consider the adjoint

$$\begin{aligned} \bar{\alpha} : S^{q+1} = S^q \wedge S^1 \to Y \\ \bar{\alpha}(x \wedge t) = \alpha(x)(t) \in Y \end{aligned}$$

Exercises.

1. Show that the correspondence

$$\begin{gathered} \pi_q(\Omega Y) \to \pi_{q+1}(Y) \\ \alpha \to \bar{\alpha} \end{gathered}$$

defines an isomorphism.

2. Consider the adjoint map

$$\begin{aligned} j : X \to \Omega\Sigma X \\ x \to j_x : S^1 \to S^1 \wedge X \end{aligned}$$

defined by $j_x(t) = t \wedge x$. Show that the induced composition

$$\pi_q(X) \xrightarrow{j} \pi_q(\Omega\Sigma X) \xrightarrow{\cong} \pi_{q+1}(\Sigma X)$$

is equal to the suspension homomorphism $\Sigma : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ defined above (10.1).

Lemma 10.2. *Let X be an n-connected space. The map $j : X \to \Omega\Sigma X$ induces an isomorphism in homology,*

$$j_* : H_q(X) \xrightarrow{\cong} H_q(\Omega\Sigma X)$$

for $q \leq 2n$.

Proof. For a based space Y let PY be the space of basepoint preserving paths. Such paths are maps $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_0$, where $y_0 \in Y$ is the basepoint. Consider the fibration

$$\Omega\Sigma X \to P\Sigma X \xrightarrow{\epsilon} \Sigma X$$

defined by $\epsilon(\gamma) = \gamma(1)$. We will consider the Serre spectral sequence for this fibration. Notice that the base space ΣX is simply connected and the total space $P\Sigma X$ is contractible. Therefore the E_∞ term of this spectral sequence must be identically zero, with the exception that $E_\infty^{0,0} = \mathbb{Z}$. This means that for every nonzero, positive dimensional class x in the E_2 term that is an infinite cycle, i.e $d_r(x) = 0$ for all r, then there must exist an element $y \in E_q$ for some q, with $d_q(y) = x$.

Consider the E_2-term. Recall that $E_2^{p,q} = H_p(\Sigma X; H_q(\Omega\Sigma X))$. Since X is n-connected, ΣX is $(n+1)$-connected, so $E_2^{p,q} = 0$ for $0 < p \leq n$. Also, since $\Omega\Sigma X$ is n-connected, we also have that $E_2^{p,q} = 0$ for $0 < q < n$.

Consider the differentials of the form

$$d_m : E_m^{n+q,0} \to E_m^{n+q-m,m-1}$$

for $q \leq n+1$. We claim that these differentials are all zero except when $m = n+q$, in which case

$$\begin{aligned} d_{n+q} : E_{n+q}^{n+q,0} &\to E_{n+q}^{0,n+q-1} \\ H_{n+q}(\Sigma X) &\to H_{n+q-1}(\Omega\Sigma X) \end{aligned}$$

is an isomorphism. To see this notice that $E_m^{n+q,0}$ is a subquotient of $H_{n+q}(\Sigma X)$ and $E_m^{n+q-m,m-1}$ is a subquotient of $H_{n+q-m}(\Sigma X; H_{m-1}(\Omega\Sigma X)$. When $m \leq n$, $H_{m-1}(\Omega\Sigma X) = 0$ since $\Omega\Sigma X$ is n-connected. If $n+q-1 \geq m > n$ then $H_{n+q-m}(\Sigma X) = 0$ since $n+q-m < q \leq n+1$ and ΣX is $(n+1)$-connected.

Since the spectral sequence converges to the zero E_∞-term (except $E_\infty^{0,0} = \mathbb{Z}$), we therefore must have that

$$\begin{aligned} d_{n+q} : E_{n+q}^{n+q,0} &\to E_{n+q}^{0,n+q-1} \\ H_{n+q}(\Sigma X) &\to H_{n+q-1}(\Omega\Sigma X) \end{aligned}$$

must be an isomorphism for $0 \leq q \leq n+1$. We leave it for the reader to check that the composition

$$H_{n+q-1}(X) \xrightarrow{\cong} H_{n+q}(\Sigma X) \xrightarrow{d_{n+q}} H_{n+q}(\Omega\Sigma X)$$

is the homology homomorphism induced by the map $j : X \to \Omega\Sigma X$. $\square$

We now complete the proof of the Freudenthal suspension theorem.

Proof. We continue to assume that X is n-connected and we again consider the map $j : X \to \Omega\Sigma X$. As defined in Chapter 7 (Definition 4.5), we may consider the homotopy fiber of this map

$$F_j = \{(x, \alpha) \in X \times (\Omega\Sigma X)^I \quad \text{such that} \quad \alpha(0) = j(x) \quad \text{and} \quad \alpha(1) = y_0\}$$

where $y_0 \in \Omega\Sigma X$ is the basepoint. Since the sequence

$$F_j \to X \xrightarrow{j} \Omega\Sigma X$$

is, up to homotopy, a fibration sequence we may consider its Serre spectral sequence. Recall that we are assuming that X (and therefore $\Omega\Sigma X$) is n-connected, with $n \geq 1$. Therefore the long exact sequence in homotopy groups ends with

$$\to \pi_2(X) \xrightarrow{j_*} \pi_2(\Omega\Sigma X) \xrightarrow{\partial_*} \pi_1(F_j) \to 0.$$

Since X and $\Omega\Sigma X$ are both simply connected, the Hurewicz theorem says that $\pi_2(X) \cong H_2(X)$ and $\pi_2(\Omega\Sigma X) \cong H_2(\Omega\Sigma X)$. But $j_* : H_2(X) \to H_2(\Omega\Sigma X)$ is an isomorphism by the above lemma. Thus $j_* : \pi_2(X) \to \pi_2(\Omega\Sigma X)$ is an isomorphism, and so we may conclude that $\pi_1(F_j) = 0$.

Now by examining the Serre spectral sequence for the homotopy fibration sequence $F_j \to X \xrightarrow{j} \Omega\Sigma X$, we see that since, by the above lemma, $j_* : H_p(X) \to H_p(\Omega\Sigma X)$ is an isomorphism for $p \leq 2n + 1$, then along the horizontal axis of this spectral sequence we have that

$$H_p(\Omega\Sigma X) = E_2^{p,0} \cong E_\infty^{p,0} = H_p(X)$$

for $p \leq 2n + 1$. Now along the vertical axis we have $H_q(F_j) = E_2^{0,q}$ which is equal to $E_\infty^{0,q}$ for $q \leq 2n$ because in this range there are no possible nonzero differentials. By the convergence of the spectral sequence to $H_*(X)$, and the fact that every element of $H_*(X)$ is represented in this spectral sequence by an element on the horizontal axis $E_\infty^{*,0}$ (in this range), we must conclude that $E_2^{0,q}$ must be zero for $q \leq 2n$. That is,

$$H_q(F_j) = 0$$

for $q \leq 2n$. Since, as observed above, $\pi_1(F_j) = 0$, then by the Hurewicz theorem we may conclude that $\pi_q(F_j) = 0$ for $q \leq 2n$. By the long exact sequence in homotopy groups for a fibration, this says that

$$j_* : \pi_q X \to \pi_q(\Omega\Sigma X)$$

is an isomorphism for $q \leq 2n$ and surjective for $q = 2n + 1$. Equivalently, the suspension homomorphism, $\pi_q X \to \pi_{q+1}(\Sigma X)$ is an isomorphism for $q \leq 2n$ and is surjective for $q = 2n + 1$. □

10.1 Spectra

The basic definition of a spectrum is the following.

Definition 10.2. *([92], [158]) A spectrum* is a sequence of (based) spaces $\{X_n,\, n \in \mathbb{Z}\}$ together with maps $\epsilon_n : \Sigma X_n \to X_{n+1}$. These maps are known as "structure maps". These structure maps can equivalently be given as maps $\bar{\epsilon}_n : X_n \to \Omega X_{n+1}$, where, as above, ΩY denotes the based loop space of a based space Y. The relation between these types of structure maps is given by the *adjunction* between a map $f : \Sigma X \to Y$ and the map $\bar{f} : X \to \Omega Y$, defined by $\bar{f}(x)(t) = f(t \wedge x) \in Y$.

In some settings one is only required to have spaces X_n for $n \geq 0$. This fits with the situation above, since we can simply define for $n < 0$, $X_n = point$.

Examples.

1. The *sphere spectrum* $\mathbb{S}$ is defined by
$$\mathbb{S}_n = S^n,$$
and $\epsilon_n : \Sigma\mathbb{S}_n = \Sigma S^n = S^{n+1} \to S^{n+1} = \mathbb{S}_{n+1}$ is the identity map. We will see that the sphere spectrum plays a crucial role in stable homotopy theory, analogous of the role the integers play in the theory of rings and modules.

2. The archetypical example of a spectrum is the *suspension spectrum* of a space, X. We denote the suspension spectrum by $\Sigma^\infty X$. Its definition is
$$(\Sigma^\infty X)_n = \Sigma^n X$$
and
$$\epsilon_n : \Sigma(\Sigma^n X) = \Sigma^{n+1} X \to \Sigma^{n+1} X$$
is the identity map for each n. Notice that the sphere spectrum $\mathbb{S}$ is a suspension spectrum, $\mathbb{S} = \Sigma^\infty S^0$.

3. An Eilenberg-MacLane spectrum $\mathbb{H}G$ for an abelian group G is a collection of other important examples. A spectrum $\mathbb{H}G$ is an Eilenberg-MacLane spectrum of type HG, if $\mathbb{H}G_n$ is an Eilenberg-MacLane space of type $K(G,n)$, and the structure maps $\epsilon_n : \Sigma\mathbb{H}G_n \to \mathbb{H}G_{n+1}$ are maps whose homotopy class in
$$\begin{aligned}[\Sigma K(G,n), K(G,n+1)] &\cong H^{n+1}(\Sigma K(G,n); G) \\ &\cong H^n(K(G,n); G) \\ &\cong Hom(H_n(K(G,n)); G) \\ &\cong Hom(G,G)\end{aligned}$$
corresponds to the identity homomorphism.

4. Let $U = \lim_{n\to\infty} U(n)$ be the colimit of the unitary groups. A famous theorem of R. Bott [14] known as *"Bott Periodicity"* says that there is a homotopy equivalence,

$$\beta : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U. \tag{10.2}$$

Furthermore, as we saw in Theorem 5.13, for any topological group G, there is a homotopy equivalence $\gamma : G \xrightarrow{\simeq} \Omega BG$. So in particular we have an equivalence $\gamma : U \xrightarrow{\simeq} \Omega BU = \Omega(\mathbb{Z} \times BU)$. (We take the basepoint of $\mathbb{Z} \times BU$ to be the basepoint of BU in $\{0\} \times BU \subset \mathbb{Z} \times BU$.)

One can then define the complex K-theory spectrum $\mathbb{KU}$ by

$$\mathbb{KU}_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even} \\ U & \text{if } n \text{ is odd} \end{cases}$$

The structure maps in the K-theory spectrum are given by the homotopy equivalences

$$\bar{\epsilon}_{2m} = \beta : \mathbb{Z} \times BU \xrightarrow{\simeq} \Omega U$$
$$\bar{\epsilon}_{2m+1} = \gamma : U \xrightarrow{\simeq} \Omega(\mathbb{Z} \times BU)$$

This spectrum is sometimes referred to as the "Bott spectrum" or the "Bott periodicity" spectrum. Notice in particular that the spectrum $\mathbb{KU}$ has the feature that the adjoints of the structure maps , $\bar{\epsilon}_n : \mathbb{KU}_n \xrightarrow{\simeq} \Omega(\mathbb{KU}_{n+1})$ are homotopy equivalences for every $n \in \mathbb{Z}$.

Definition 10.3. *Given a spectrum $\mathbb{X}$, we define its homotopy groups by $\pi_k(\mathbb{X}) = \lim_{q\to\infty} \pi_{k+q}(X_q)$. This colimit is defined via maps*

$$\pi_{k+q}(X_q) \xrightarrow{\Sigma} \pi_{k+q+1}(\Sigma X_q) \xrightarrow{(\epsilon_k)_*} \pi_{k+q+1}(X_{q+1}).$$

Its homology groups are defined similarlly:

$$H_k(\mathbb{X}) = \lim_{q\to\infty} H_{k+q}(X_q).$$

Notice that a spectrum may have nonzero negatively graded homotopy groups and homology groups. For example, $\pi_{-3}(\mathbb{X}) = \lim_{q\to\infty} \pi_{q-3} X_q$ which may not be zero.

Exercise. Show that for k any integer (positive, negative, or zero),

$$\pi_k(\mathbb{KU}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Hint. Use the fact that $\pi_k(\Omega Y) \cong \pi_{k+1}(Y)$ for any based space Y.

One important feature of spectra is that they can be suspended, or *desuspended* an arbitrary number of times. If $\mathbb{X}$ is a spectrum, then for $k \in \mathbb{Z}$ define the k-fold suspension. $\Sigma^k \mathbb{X}$ by letting

$$(\Sigma^k \mathbb{X})_m = \mathbb{X}_{m+k}, \tag{10.3}$$

and the structure maps for $\Sigma^k \mathbb{X}$ are defined in terms of the structure maps of $\mathbb{X}$.

Exercise. Show that there are suspension isomorphisms

$$\begin{aligned} \pi_q(\mathbb{X}) &\cong \pi_{q+k}(\Sigma^k \mathbb{X}) \quad \text{and} \\ H_q(\mathbb{X}) &\cong H_{q+k}(\Sigma^k \mathbb{X}) \end{aligned} \tag{10.4}$$

10.1.1 Morphisms

If we think categorically, we now have objects in a category of spectra. But what about morphisms? Naively, one might expect a morphism between two spectra $\mathbb{X}$ and $\mathbb{Y}$ should be a collection of maps $f_n : \mathbb{X}_n \to \mathbb{Y}_n$ that respect the structure maps. That is, the following diagrams should commute:

$$\begin{array}{ccc} \Sigma \mathbb{X}_n & \xrightarrow{\Sigma f_n} & \Sigma \mathbb{Y}_n \\ {\scriptstyle \epsilon_n}\downarrow & & \downarrow{\scriptstyle \epsilon_n} \\ \mathbb{X}_{n+1} & \xrightarrow[f_{n+1}]{} & \mathbb{Y}_n. \end{array}$$

Certainly such collections of maps $\{f_n\}$ will constitute a morphism (or map) of spectra, but here is an important example of what such a definition would exclude.

Consider the Hopf map $\eta : S^3 \to S^2$. We can suspend the Hopf map an arbitrary number of times to produce maps

$$\Sigma^{n-2}\eta : S^{n+1} \to S^n$$

for all $n \geq 2$. In terms of the spaces making up sphere spectra we have maps

$$\begin{aligned} \eta_n : (\Sigma \mathbb{S})_n &\to \mathbb{S}_n \\ \Sigma^{n-2}\eta : S^{n+1} &\to S^n \end{aligned}$$

for $n \geq 2$ that preserve the structure maps. But notice that no such maps exist for $n = 0$ or 1, because there is no map $S^2 \to S^1$ whose suspension is the Hopf map $\eta : S^3 \to S^2$. But surely we want the collection of maps defined by suspending η to define a map between spectra,

$$\eta : \Sigma \mathbb{S} \to \mathbb{S}.$$

More generally, we would like to have a definition of morphisms between spectra such that every element of the stable homotopy groups of any spectrum $\mathbb{X}$, $\alpha \in \pi_n^s(\mathbb{X})$ is represented by a map of spectra

$$\alpha : \Sigma^n \mathbb{S} \to \mathbb{X}$$

for any $n \in \mathbb{Z}$.

To produce such an appropriate definition of morphism of spectra, we use the notion of an "ω - spectrum".

Definition 10.4. *An "ω - spectrum" is a spectrum $\mathbb{Y}$ such that the adjoints of the structure maps*

$$\epsilon_n : \mathbb{Y}_n \to \Omega \mathbb{Y}_{n+1}$$

are homeomorphisms.

This might seem like a very restrictive definition, but we observe that each spectrum in the sense of Definition 10.2 naturally has an associated ω-spectrum.

Definition 10.5. *Let $\mathbb{X}$ be a spectrum. Define its associated ω-spectrum $\mathbb{X}^\omega$ by*

$$\mathbb{X}_n^\omega = \lim_{k \to \infty} \Omega^k \mathbb{X}_{n+k}.$$

The maps used in this limit are

$$\Omega^{k-1} \mathbb{X}_{n+k-1} \xrightarrow{\Omega^{k-1} \bar{\epsilon}_{n+k-1}} \Omega^{k-1} \Omega \mathbb{X}_{n+k} = \Omega^k \mathbb{X}_{n+k}.$$

The structure maps for the ω-spectrum $\mathbb{X}^\omega$ are given by

$$\begin{aligned} \epsilon_n^\omega : \Sigma \mathbb{X}_n^\omega = \Sigma \lim_{k \to \infty} \Omega^k \mathbb{X}_{n+k} \to \lim_{k \to \infty} \Sigma \Omega^k \mathbb{X}_{n+k} &= \lim_{q \to \infty} \Sigma \Omega (\Omega^q \mathbb{X}_{n+1+q}) \\ &\xrightarrow{ev} \lim_{q \to \infty} \Omega^q \mathbb{X}_{n+1+q} = \mathbb{X}_{n+1}^\omega. \end{aligned}$$

In this description q was substituted for $k-1$ and $ev : \Sigma \Omega Y \to Y$ is the evaluation map, $ev(t \wedge \theta) = \theta(t) \in Y$.

Exercise. Check that $\mathbb{X}^\omega$ is indeed an ω-spectrum, and that there are natural isomorphisms

$$\pi_k \mathbb{X} \xrightarrow{\cong} \pi_k(\mathbb{X}^\omega) \quad \text{and} \quad H_k \mathbb{X} \xrightarrow{\cong} H_k(\mathbb{X}^\omega)$$

for any spectrum $\mathbb{X}$.

We may now define what we mean by a map or morphism between spectra.

Definition 10.6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be spectra, as in Definition 10.2. We define a map of spectra $\phi : \mathbb{X} \to \mathbb{Y}$ to be a collection of maps between the spaces making up their associated ω-spectra,*

$$\phi_n : \mathbb{X}^\omega = \lim_{k\to\infty} \Omega^k X_{n+k} \to \lim_{k\to\infty} \Omega^k Y_{n+k} = \mathbb{Y}^\omega_n$$

that preserve the structure maps

$$\begin{array}{ccc} \Sigma\mathbb{X}^\omega_n & \xrightarrow{\Sigma\phi_n} & \Sigma\mathbb{Y}^\omega_n \\ {\scriptstyle \epsilon^\omega_n}\downarrow & & \downarrow{\scriptstyle \epsilon^\omega_n} \\ \mathbb{X}^\omega_{n+1} & \xrightarrow[\phi_{n+1}]{} & \mathbb{Y}^\omega_n. \end{array}$$

Remark. Since morphisms between spectra are made up out of maps between spaces in their corresponding ω-spectra, some texts refer to an object satisfying Definition 10.2 as a *"prespectrum"* and reserve the term "spectrum" to an object that we call an ω-spectrum.

Exercise. Let $\mathbb{X}$ be a spectrum. Show that every element of its homotopy groups $\alpha \in \pi_m\mathbb{X}$ represents, and is represented by a map of spectra

$$\alpha : \Sigma^m\mathbb{S} \to \mathbb{X}.$$

10.2 Generalized (co)homology and Brown's Representability Theorem

One of the most important applications of spectra over the many years since their original definition, has been to *generalized (co)homology theories*. Such a theory is a functor from the category of pairs of spaces to the category of graded abelian groups that satisfy all of the Eilenberg-Steenrod axioms with the possible exception of the dimension axiom. The *Brown Representability theorem* states that any such generalized cohomology theory is represented by a spectrum, and conversely, any spectrum represents a generalized cohomology theory. Considering the homological perspective, Whitehead [158] showed how spectra give rise to generalized homology theories as well, and he studied manifold orientations and Poincaré duality in the setting of these generalized theories. In this section we describe these major advances in homotopy theory.

10.2.1 Brown's Representability Theorem

We begin by describing Brown's representability theorem [19] [20]. We actually describe a variant of Brown's theorem proved by Adams in [6].

Let $\mathcal{CW}$ be the category whose objects are finite CW-complexes with basepoints, and whose morphisms are basepoint preserving maps. Let $\mathcal{G}$ be the category of abelian groups and homomorphisms. We consider contravariant functors

$$\mathcal{H} : \mathcal{CW} \to \mathcal{G}.$$

For ease of notation, if $f : X \to Y$ is a basepoint preserving map between finite CW complexes, we denote the homomorphism induced by applying $\mathcal{H}$ to this map by

$$f^* : \mathcal{H}(Y) \to \mathcal{H}(X).$$

We consider three interesting axioms on such contravariant functors. The first is the *homotopy axiom.*

Homotopy Axiom. If $f, g : X \to Y$ are basepoint preserving maps between finite CW-complexes that are homotopic via a basepoint preserving homotopy, then

$$f^* = g^* : \mathcal{H}(Y) \to \mathcal{H}(X).$$

For our next axiom let X and $Y \in \mathcal{CW}$, and let $X \vee Y$ be their wedge. Let $\iota_X : X \hookrightarrow X \vee Y$ and $\iota_Y : Y \hookrightarrow X \vee Y$ be the natural inclusions. A contravariant functor $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ defines a homomorphism

$$\iota_X^* \times \iota_Y^* : \mathcal{H}(X \vee Y) \to \mathcal{H}(X) \times \mathcal{H}(Y).$$

Wedge Axiom. $\iota_X^* \times \iota_Y^* : \mathcal{H}(X \vee Y) \to \mathcal{H}(X) \times \mathcal{H}(Y)$ is an isomorphism.

In order to state the third axiom, consider the following diagram, in which the homomorphisms are induced by the obvious inclusion maps.

$$\begin{array}{ccc} \mathcal{H}(X \cup Y) & \xrightarrow{c^*} & \mathcal{H}(X) \\ {\scriptstyle d^*}\downarrow & & \downarrow{\scriptstyle a^*} \\ \mathcal{H}(Y) & \xrightarrow[b^*]{} & .\mathcal{H}(X \cap Y) \end{array}$$

Mayer-Vietoris Axiom. . Suppose $x \in \mathcal{H}(X)$ and $y \in \mathcal{H}(Y)$ are such that $a^*(x) = b^*(y)$. Then there exists an element $z \in \mathcal{H}(X \cup Y)$ such that $c^*(z) = x$ and $d^*(z) = y$.

The following is the variant of Brown's representability theorem, proved in this form by Adams, that will be most useful to us.

Theorem 10.3. *[19][20][6] Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ be a contravariant functor satisfying the Homotopy Axiom, the Wedge Axiom, and the Mayer-Vietoris Axiom. Then $\mathcal{H}$ is "representable". That is, there is a based (not necessarily finite) CW-complex B and a natural bijection of sets*

$$T : [X, B] \xrightarrow{\cong} \mathcal{H}(X)$$

defined for all finite, based CW complexes X.

Comment. By "natural bijection" we mean that if $f : X \to Y$ is a morphism in $\mathcal{CW}$ (i.e a basepoint preserving map), then the following diagram commutes:

$$\begin{array}{ccc} [Y,B] & \xrightarrow[\cong]{T} & \mathcal{H}(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ [X,B] & \xrightarrow[\cong]{T} & \mathcal{H}(X). \end{array}$$

We will now sketch a proof of Brown's theorem. We will actually describe the proof of something stronger. Namely we will show that the "representing space" B is a "weak, group-like H-space", to be properly defined below, but it implies that B has a product map, $B \times B \to B$ which gives the set of homotopy classes of maps $[X, B]$ a group structure. We will then show that the set bijection

$$T : [X, B] \xrightarrow{\cong} \mathcal{H}(X)$$

is actually an isomorphism of groups.

The first step is to consider an extension of a contravariant functor $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ to *all* CW-complexes (i.e not necessarily finite), as described by Adams [6]. He defined a contravariant functor

$$\hat{\mathcal{H}}(X) = \varprojlim_{\alpha} \mathcal{H}(X_\alpha)$$

where the inverse limit runs over all finite subcomplexes $X_\alpha \subset X$. (All of our subcomplexes are assumed to contain the basepoint.). Of course if X is a finite CW-complex, $\hat{\mathcal{H}}(X) = \mathcal{H}(X)$. In order to understand the properties of the extended functor $\hat{\mathcal{H}}$, Adams introduced the notion of "weak homotopy".

Definition 10.7. *Two basepoint preserving maps between based CW-complexes $f, g : X \to Y$ are "weakly homotopic", written $f \sim_w g$ if fh is homotopic to gh for every map $h : K \to X$ where K is a based finite CW-complex and h is basepoint preserving.*

The following result is an easy exercise that we leave to the reader.

Lemma 10.4. . *Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ be a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms. Then if $f, g : X \to Y$ are basepoint preserving maps between (not necessarily finite) CW-complexes that are weakly homotopic, then*

$$f^* = g^* : \hat{\mathcal{H}}(Y) \to \hat{\mathcal{H}}(X).$$

Here is a rather straightforward result that the reader can verify or look up in Brown's papers [19][20].

Lemma 10.5. *Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ satisfy the Homotopy, Wedge, and Mayer-Vietoris axioms. Let*

$$K \xrightarrow{f} L \xrightarrow{i} L \cup_f c(K)$$

be a cofibration sequence of finite complexes. Then the sequence

$$\mathcal{H}(K) \xleftarrow{f^*} \mathcal{H}(L) \xleftarrow{i^*} \mathcal{H}(L \cup_f c(K))$$

is exact. That is the kernel of f^* is equal to the image of i^*.

For the next result we assume that K is a finite complex, containing subcomplexes L and M. We continue to assume that $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ is a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms.

Lemma 10.6. *Let $\iota_1 : L \to L \cup M$ and $\iota_2 : M \to L \cup M$ be the inclusion maps. Then there is an exact sequence*

$$\mathcal{H}(L) \times \mathcal{H}(M) \xleftarrow{\iota_1^* \times \iota_2^*} \mathcal{H}(L \cup M) \leftarrow \mathcal{H}(\Sigma(L \cap M)) \xleftarrow{g^*} \mathcal{H}(\Sigma(L \vee M))$$

which is natural with respect to maps $K, L, M \to K', L', M'$. Furthermore the homomorphism g^* is induced by a map of spaces, $g : \Sigma(L \cap M) \to \Sigma(L \vee M)$.

Proof. Consider the obvious map

$$L \vee M \to L \cup M$$

and the resulting cofibration sequence

$$L \vee M \to L \cup M \to (L \cup M) \cup c(L \vee M) \to \Sigma(L \vee M) \to \cdots$$

Notice that the third term is homotopy equivalent to $\Sigma(L \cap M)$. The lemma now follows from Lemma 10.5. □

Exercise. Prove the assertion made in this proof that $(L \cup M) \cup c(L \vee M)$ is homotopy equivalent to $\Sigma(L \cap M)$.

The next two results are immediate, and we leave their verifications to the reader. In both cases we continue to assume that $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ is a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms.

Lemma 10.7. *$\hat{\mathcal{H}}$ satisfies the Wedge axiom.*

Lemma 10.8. *Let X be a CW-complex and $\{X_\alpha\}$ a directed set of subcomplexes whose union is X. Then the natural map*

$$\hat{\mathcal{H}}(X) \to \varprojlim_{\alpha} \hat{\mathcal{H}}(X_\alpha)$$

is an isomorphism.

The following is quite a reasonable result, whose proof is in Adams's paper [6]. We refer the reader to that paper for the argument.

Proposition 10.9. *Let X be a based CW-complex with subcomplexes $U, V \subset X$, each containing the basepoint. Suppose furthermore that $U \cap V$ is a finite complex. Let $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ be a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms. Then the square*

$$\begin{array}{ccc} \mathcal{H}(U \cap V) = \hat{\mathcal{H}}(U \cap V) & \longleftarrow & \hat{\mathcal{H}}(U) \\ \uparrow & & \uparrow \\ \hat{\mathcal{H}}(V) & \longleftarrow & \hat{\mathcal{H}}(U \cup V) \end{array}$$

satisfies the Mayer-Vietoris axiom.

We now apply these results to prove Brown's representability theorem. For the remainder of this section we continue to assume that $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ is a contravariant functor satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms.

Let Y be a CW complex (not necessarily finite) and let $y \in \hat{\mathcal{H}}(Y)$. Given any CW-complex X, let $[X, Y]_w$ denote the set of weak homotopy clases of basepoint preserving maps, as defined in Definition 10.7. Consider the natural transformation

$$\hat{T} : [X, Y]_w \to \hat{\mathcal{H}}(Y)$$

given by

$$\hat{T}^*(f) = f^*(y).$$

Notice that $\hat{T}$ is well-defined by Lemma 10.4 and is natural for all CW-complexes X. By restricting to finite complexes one as a natural transformation

$$T : [K, Y] \to \mathcal{H}(Y).$$

Let $NatTrans(A, B)$ be the set of natural transformations between functors A and B. The following lemma is not difficult, and is Adams's interpretation ([6], Lemma 4.1) of a result of Brown ([19], p. 478).

Lemma 10.10. *The above construction gives bijective correspondences*

$$\begin{aligned} \hat{\mathcal{H}}(Y) &\cong NatTrans([X, Y]_w, \hat{\mathcal{H}}(X)) \\ &\cong NatTrans([K, Y], \mathcal{H}(K)). \end{aligned}$$

Furthermore these correspondences are natural with respect to maps of Y.

The following is the basic constructive idea in forming the representing space for a functor $\mathcal{H}$.

Lemma 10.11. *Let Y_n be a CW-complex provided with an element $y_n \in \hat{\mathcal{H}}(Y_n)$. Then there exists a complex Y_{n+1} with an element $y_{n+1} \in \hat{\mathcal{H}}(Y_{n+1})$ and an embedding $i : Y_n \hookrightarrow Y_{n+1}$ satisfying the following properties:*

1. $i^* y_{n+1} = y_n$
2. *If K is any finite CW-complex and $f, g : K \to Y_n$ are maps such that $f^*(y_n) = g^*(y_n)$, then $i \circ f$ is homotopic to $i \circ g$ as maps $K \to Y_{n+1}$.*

Proof. (Sketch) For each finite complex K and pair of homotopy classes of maps $f : K \to Y_n$ and $g : K \to Y_n$ such that $f^*(y_n) = g^*(y_n)$ choose representatives for f and g. Furthermore we let K range over representatives of all homotopy classes of finite complexes. Thus we have chosen a countable set of indices A and maps $f_\alpha, g_\alpha : K_\alpha \to Y_n$ for $\alpha \in A$. One then forms

$$Y_{n+1} = Y_n \cup \bigcup_{\alpha \in A} (I \times K_\alpha)/I \times point)$$

where. "point" refers to the basepoint in K_α and the reduced cylinder $I \times K_\alpha)/I \times point$ is attached to Y_n by the map f_α at one end and g_α at the other end. (This construction is called a "mapping cylinder".).

The embedding $i : Y_n \hookrightarrow Y_{n+1}$ is the obvious inclusion map. Clearly $i \circ f_\alpha$ is homotopic to $i \circ g_\alpha$ for each $\alpha \in A$. It remains to define the class $y_{n+1} \in \hat{\mathcal{H}}(Y_{n+1})$ such that $i^* y_{n+1} = y_n$. This is straightforward using the axioms established for $\hat{\mathcal{H}}$, and we refer the reader to Adams [6] or Brown [19] for details. □

We are now ready to prove the following slight generalization of Brown's representability theorem (Theorem 10.3 above).

Theorem 10.12. *Let Y_0 be a CW-complex equipped with a class $y_0 \in \hat{\mathcal{H}}(Y_0)$. Then there exists a CW-complex Y together with an embedding $i : Y_0 \hookrightarrow Y$ and an element $y \in \hat{\mathcal{H}}(Y)$ such that $i^*(y) = y_0$, and such that the corresponding natural transformation*

$$T : [K, Y] \to \mathcal{H}(K)$$

is a bijection of sets for all finite complexes K.

Remarks. 1. Theorem 10.3 is a special case of Theorem 10.12 reflecting the case $Y_0 = point$.

2. Y is called a *representing complex* for the functor $\mathcal{H}$.

Proof. Let K run over a countable set of representatives of finite complexes as in the proof of Lemma 10.11. For each K let h run over $\mathcal{H}(K)$. Form

$$Y_1 = Y_0 \vee \bigvee_{K,h} K$$

Using the fact that $\hat{\mathcal{H}}$ satisfies the Wedge Axiom, let $y_1 \in \hat{\mathcal{H}}(Y_1)$ be the element that restricts to y_0 in $\hat{\mathcal{H}}(Y_0)$, and to h on the $(K,h)^{th}$ summand of the wedge. This then implies that the natural transformation

$$T_1 : [K, Y_1] \to \mathcal{H}(K)$$

corresponding to $y_1 \in \hat{\mathcal{H}}(Y_1)$ is surjective for every K.

Now construct complexes

$$Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots Y_n \hookrightarrow \cdots$$

and elements $y_n \in \hat{\mathcal{H}}(Y_n)$, as in Lemma 10.11. Let $Y = \bigcup_n Y_n$, and let $y \in \hat{\mathcal{H}}(Y)$ be the element that restricts to $y_n \in \hat{\mathcal{H}}(Y_n)$ for every n. (This uses Lemma 10.8.) The corresponding natural transformation

$$T : [K, Y] \to \mathcal{H}(K)$$

is still surjective. But notice that it is also injective. To see this, let $f, g : K \to Y$ be any two maps such that $f^*(y) = g^*(y)$. Then since K is a finite complex f and g both must map into Y_n for some n. This means $f^*(y_n) = g^*(y_n)$ and f is homotopic to g in Y_{n+1} by Lemma 10.11. This completes the proof of Theorem 10.12. □

The following extension of Theorem 10.3 is straightforward, and we refer the reader to the paper by Adams [6] for details.

Theorem 10.13. *1. There is one and only one transformation*

$$\hat{T} : [X, Y]_w \to \hat{\mathcal{H}}(X)$$

defined and natural for all CW-complexes X, that reduces to T when X is finite.

2. The natural transformation $\hat{T}$ is a bijection of sets for any CW-complex X.

Lemma 10.10 together with the Brown Representability Theorem 10.12 allows us to prove the following quite easily. (See [6], Addendum 1.5.)

Proposition 10.14. *Let X be any CW complex (not necessarily finite), and let Y be a representing complex for the contravariant functor $\mathcal{H} : \mathcal{CW} \to \mathcal{G}$ satisfying the Homotopy, Wedge, and Mayer-Vietoris axioms. Let*

$$U : [K, X] \to [K, Y]$$

be a natural transformation of sets defined for finite CW complexes K. Then there is a map $f : X \to Y$ inducing U, and is unique up to weak homotopy.

Proof. We first show that such a map $f : X \to Y$ exists. Consider the composition

$$[K, X] \xrightarrow{U} [K, Y] \xrightarrow{T} \mathcal{H}(K),$$

where T is the natural "representing transformation" defined in Theorem 10.12. By Lemma 10.10 this composition corresponds to an element $\alpha \in \hat{\mathcal{H}}(X)$, while T itself corresponds to an element $\beta \in \hat{\mathcal{H}}(Y)$. By Theorem 10.13 there is a map $f : X \to Y$, which is well-defined up to weak homotopy, such that $f^*(\beta) = \alpha$. By the naturality statement in Lemma 10.10 this means that $Tf_* : [K, X] \to \mathcal{H}(K)$ is equal to $T \circ U$. Since T is a bijection, $f_* = U$.

To check the uniqueness statement, suppose $f_* = g_* : [K, X] \to [K, Y]$ for every finite complex K. By definition this means that f is weakly homotopic to g. □

We next show that the natural transformation $\hat{T} : [X, Y] \to \hat{\mathcal{H}}(Y)$ defined in Theorem 10.13 is an isomorphism of groups. In order to show this we need to show that the representing space Y has a multiplicative structure that endows $[X, Y]$ with a group structure with respect to which T is a homomorphism of groups.

Consider the product structure

$$\hat{\mathcal{H}}(X) \times \hat{\mathcal{H}}(X) \xrightarrow{\mu} \hat{\mathcal{H}}(X)$$

given by the group structure of $\mathcal{H}(X)$. Here X can be any CW complex. By Theorem 10.13 this defines a product

$$[X, Y \times Y]_w \xrightarrow{\cong} [X, Y]_w \times [X, Y]_w \xrightarrow{\mu} [X, Y]_w$$

for any CW-complex. By letting $X = Y \times Y$, we can let $\nu : Y \times Y \to Y$ represent the image of the identity in $[Y \times Y, Y \times Y]_w$ under this composition. $\nu : Y \times Y \to Y$ is well-defined up to weak homotopy. By construction, this product induces the product structure on $[X, Y]_w$ for any CW-complex and therefore on $[K, Y] \cong \mathcal{H}(K)$ for any finite CW-complex K. Also by construction, the natural transformation

$$T : [X, Y]_w \to \hat{\mathcal{H}}(X)$$

respects this product structure, and therefore by Theorem 10.13 is an isomorphism of groups.

10.2.2 Generalized (co)homology theories

We now apply the Brown representability theorem to classify *generalized cohomology theories*. We refer the reader to [19] for details.

Definition 10.8. *Let $\mathcal{CW}_2$ be the category of pairs (X, A) of finite CW-complexes. A (generalized) cohomology theory E is a collection of contravariant functors $E^q : \mathcal{CW}_2 \to \mathcal{G}$ and a collection of natural homomorphisms $\delta^q : E^q(A) \to E^{q+1}(X, A)$ defined for each pair $(X, A) \in \mathcal{CW}_2$, satisfying the following Eilenberg-Steenrod axioms:*

- **Homotopy:** If $f, g : (X, A) \to (Y, B)$ are homotopic, $f^* = g^* : E^q(Y, B) \to E^q(X, A)$. (Here, as above, we are using the superscript $*$ to denote the homomorphism of groups induced by the contravariant functor E applied to the map (morphism) in $\mathcal{CW}_2$.)
- **Exactness:** Let $\iota : (X, \emptyset) \to (X, A)$ and $j : A \hookrightarrow X$ be the natural inclusion maps. Then the following sequence is exact.

$$\cdots \to E^{q-1}(A) \xrightarrow{\delta^{q-1}} E^q(X, A) \xrightarrow{\iota^*} E^q(X) \xrightarrow{j^*} E^q(A) \to \cdots$$

- **Excision:** If $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2)$ are CW-pairs in $\mathcal{CW}_2$, the map

$$E^q(X_1 \cup X_2, X_2) \to E^q(X_1, X_1 \cap X_2)$$

  induced by the inclusion map is an isomorphism for all q.

We now describe Brown's theorem stating that all generalized cohomology theories determine, and are determined by a spectrum.

Let $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q} : \Sigma\mathrm{E_q} \to \mathrm{E_{q+1}})\}$ be an ω-spectrum. (Recall that this means that the adjoint mappings $\bar{\epsilon}_q : E_q \to \Omega E_{q+1}$ are homeomorphisms.). For a finite CW-complex X, with subcomplex $A \subset X$, let $E^q(X, A) = [X/A, E_q]$. Here, as above, we mean based homotopy classes of basepoint preserving maps.

Note. If A is not a subcomplex of X, but one rather simply has a map $\iota : A \to X$, one can replace the quotient X/A by the mapping cone $X \cup_\iota c(A)$. When A is the empty set, we use the notation X/A to mean the space $X \sqcup$ *point*, where the basepoint is the disjoint point. In this case the set

$$[X/\emptyset, E_q] = [X \sqcup point, E_q]$$

where the last set can simply be viewed as the set of homotopy classes of *unbased* maps from X to E_q. Let $S : X/A \simeq X \cup_\iota c(A) \to \Sigma A$ be the map that collapses $X \subset X \cup_\iota c(A)$ to a point. Let $\sigma : [A \sqcup point, \Omega E_{q+1}] \xrightarrow{\simeq} [\Sigma(A \sqcup point), E_{q+1}]$ be the usual adjunction isomorphism. We can then define

$$\delta^q : [A \sqcup point, E_q] \to [X/A, E_{q+1}]$$

by letting $\delta(f) : X/A \to E_{q+1}$ be the composition

$$X/A \xrightarrow{S} \Sigma(A \sqcup point) \xrightarrow{\Sigma f_+} \Sigma E_q \xrightarrow{\epsilon_q} E_{q+1}.$$

In this composition $f_+ : A \sqcup point \to E_q$ is equal to f on A and sends the disjoint basepoint to the basepoint of E_q.

Notice that the set $[X, E_q]$ has a natural group structure since this set is equal to $[X, \Omega E_{q+1}]$. Indeed it is naturally an abelian group. (Consider the following exercise.)

Exercise. 1. Show that if X and Y are any based spaces, $[X, \Omega Y]$ has a natural group structure.

Hint. Recall how one defines the group structure on the fundamental group $\pi_1(Y)$, to show that ΩY has a product that defines a group structure "up to homotopy".

2. Show that the set $[X, \Omega^2 Z]$ is an abelian group. Here $\Omega^2(Z) = \Omega(\Omega(Z))$.

Hint. Recall how one shows that the second homotopy group $\pi_2(Z)$ is abelian.

The following is now a straightforward exercise.

Theorem 10.15. *For* $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q} : \Sigma \mathrm{E_q} \to \mathrm{E_{q+1}})\}$ *an* ω*-spectrum, the contravariant functor* $(E^q(X, A), \delta^q)$ *as defined above, forms a generalized cohomology theory. That is, it satisfies Definition 10.8 above.*

Exercise. Prove Theorem 10.15.

We now consider the converse.

Theorem 10.16. *(Brown [19]) Let* $E = \{(E^q, \delta^q)\}$ *be a generalized cohomology theory as defined above. Then there is an* ω*-spectrum* $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q})\}$ *and natural equivalences* $\tau_q : [(X/A, E_q] \to E^q(X, A)$ *defined for all pairs of finite CW-complexes* (X, A)*. Furthermore we have the relation* $\delta^q \tau_q = \tau_{q+1} \delta^q$ *for each* $q \in \mathbb{Z}$*.*

Proof. Consider the contravariant functors

$$\begin{aligned} \tilde{E}^q : \mathcal{CW} &\to \mathcal{G} \\ X &\to E^q(X, x_0) \end{aligned}$$

where $x_0 \in X$ is the basepoint. Since E satisfies the Eilenberg-Steenrod axioms as given in Definition 10.8, clearly $\tilde{E}^q$ satisfies the Homotopy Axiom. Furthermore, standard arguments show that since E satisfies exactness and excision, each $\tilde{E}^q$ satisfies the Wedge and Mayer-Vietoris axioms. Therefore by the Brown Representability Theorem 10.3, there is a representing space E_q for the functor $\tilde{E}^q$. That is, there is a natural equivalence

$$\tau_q : [X, E_q] \xrightarrow{\cong} \tilde{E}^q(X) = E^q(X, x_0).$$

We now show that the collection $\{E_q\}$ fit together to define a spectrum. We first prove a suspension isomorphism in the following lemma.

Lemma 10.17. *There is a natural isomorphism*

$$\delta^q : \tilde{E}^q(X) \xrightarrow{\cong} \tilde{E}^{q+1}(\Sigma X).$$

Proof. . Consider the triple $(c(X), X, x_0)$. Here $c(X)$ is the cone, $c(X) = X \times I / X \times \{1\} \cup x_0 \times I$ where $I = [0, 1]$ is the unit interval. X is viewed as a subspace of $c(X)$ as $X \times \{0\}$. By excision and exactness one has that

$$E^q(c(X), X) \cong E^q(\Sigma X, x_0) = \tilde{E}^q(\Sigma X)$$

By substituting this in to the exact sequence of a triple we have an exact sequence

$$\cdots \to E^q(c(X), x_0) \to E^q(X, x_0) \xrightarrow{\delta^q} E^{q+1}(c(X), X) \to \cdots .$$

Since the cone $c(X)$ is contractible and the cohomology theory E satisfies the Homotopy Axiom, $E^r(c(X), x_0) = 0$ for every r. Therefore this sequence becomes

$$\cdots \to 0 \to \tilde{E}^q(X) \xrightarrow{\delta^q} \tilde{E}^{q+1}(\Sigma X) \to 0 \to \cdots$$

Thus the connecting homomorphisms $\delta^q : \tilde{E}^q(X) \to \tilde{E}^{q+1}(\Sigma X)$ is an isomorphism. □

We now continue our proof of Theorem 10.16.

The natural suspension isomorphism given by the above lemma can be interpreted as a natural isomorphism

$$\delta^q : [X, E_q] \xrightarrow{\cong} [\Sigma X, E_{q+1}] \xrightarrow{\cong} [X, \Omega E_{q+1}].$$

By Proposition 10.14 above, the natural transformation δ^q is realized by a map we call $\epsilon_q : E_q \to \Omega E_{q+1}$ which is uniquely defined up to weak homotopy. Notice furthermore that these maps are weak homotopy equivalences, since for any finite complex K the map

$$[K, E_q] \xrightarrow{\delta^q = (\epsilon_q)_*} [K, \Omega E_{q+1}]$$

is an isomorphism. So the collection $\{(E_q, \epsilon_q)\}$ defines a spectrum which represents the cohomology theory E. Notice that we might think of this spectrum as a "weak homotopy ω-spectrum", since the adjoints of the structure maps $\bar{\epsilon}_q : E_q \to \Omega E_{q+1}$ are weak homotopy equivalences, where in the definition of an actual ω-spectrum, these maps are required to be homeomorphisms. In any case we can now replace this spectrum by its associated ω-spectrum as in Definition 10.5 which we call $\mathbb{E}$. This completes the proof. □

An analogous statement for *"generalized homology theories"*, which is to say that *covariant functors* $\{E_q, \delta_q\} : \mathcal{CW}_2 \to \mathcal{G}$ that satisfy the covariant analogues of the Eilenberg-Steenrod axioms 10.8, was proven by G. Whitehead [158]. In order to state his theorem we first introduce the following definition.

Definition 10.9. *Let X be a space with basepoint, and $\mathbb{E} = \{E_k, \epsilon_k\}$ be a spectrum. The smash product spectrum $X \wedge \mathbb{E}$ has as its k^{th} space,*

$$(X \wedge \mathbb{E})_k = X \wedge E_k$$

with structure maps $1 \wedge \epsilon_k : \Sigma(X \wedge E_k) = X \wedge \Sigma E_k \xrightarrow{1\wedge\epsilon_k} X \wedge E_{k+1}$.

Exercise. Show that for any based space X, the spectrum $\mathbb{S} \wedge X$ is weakly homotopy equivalent to the suspension spectrum, $\Sigma^\infty X$. That is to say, there is a morphism of spectra, $\phi : \mathbb{S} \wedge X \to \Sigma^\infty X$ that induces an isomorphism on homotopy groups.

G. Whitehead [158] proved the following analogue of Theorem 10.16 about about the representability of *generalized homology* theories.

Theorem 10.18. *(Whitehead [158]) Let $E_* = \{(E_q, \delta_q)\}$ be a generalized homology theory. That is, it is a collection of covariant functors $E_q : \mathcal{CW}_2 \to \mathcal{G}$ and a collection of natural homomorphisms $\delta_q : E_q(X, A) \to E_{q-1}(A)$ defined for each pair $(X, A) \in \mathcal{CW}_2$, satisfying the covariant analogues of the Eilenberg-Steenrod axioms: Homotopy, Exactness, and Excision (see Definition 10.8). Then there is an ω-spectrum $\mathbb{E} = \{(\mathrm{E_q}, \epsilon_\mathrm{q})\}$ and natural equivalences*

$$\tau_q : \pi_q((X/A) \wedge \mathbb{E}) \to \mathrm{E_q(X, A)}$$

defined for all pairs of finite CW-complexes (X, A). Furthermore we have the relation $\delta_q \tau_q = \tau_{q+1} \delta_q$ for each $q \in \mathbb{Z}$.

Examples.

1. Let $\mathbb{S}$ be the sphere spectrum. Since, by the above exercise, for any space X, $X \wedge \mathbb{S} \simeq \Sigma^\infty X$, we have that the generalized homology theory represented by $\mathbb{S}$ is *stable homotopy groups*. The generalized cohomology theory represented by $\mathbb{S}$ is known as *stable cohomotopy*. Notice that

$$\mathbb{S}^k(X) = \varinjlim_n [\Sigma^n X, S^{n+k}],$$

and may in particular be nonzero for $k < 0$.

2. Let G be an abelian group and $\mathbb{H}G$ the corresponding Eilenberg-MacLane spectrum. The cohomology theory this spectrum represents is ordinary cohomology with coefficients in G. This is due to the well-known result of Hopf stating that

$$[X, K(G, m)] \cong H^m(X; G).$$

In homology, Whitehead's theorem gives that $\mathbb{H}G_*(X) \cong H_*(X; G)$ which is the very non-intuitive result that

$$H_k(X; G) \cong \pi_k(X_+ \wedge \mathbb{H}G) = \varinjlim_n \pi_{k+n}(X \wedge K(G, k+n)),$$

where $X_+ = X/\emptyset$ is X with a disjoint basepoint.

3. Let $\mathbb{KU}$ be the complex K-theory spectrum as defined earlier. Recall that $\mathbb{KU}_q = \mathbb{Z} \times BU$ if q is even, and $\mathbb{KU}_q = U$ if q is odd. As mentioned above, Bott periodicity implies that $\mathbb{KU}$ is an ω-spectrum. The associated cohomology theory it represents is referred to as *complex* K-theory, denoted by $K^*(X)$. Notice that

$$K^0(X) \cong [X, \mathbb{Z} \times BU]$$

and as studied earlier, for X compact this is the Grothendieck group completion of the abelian monoid $Vect^{\mathbb{C}}(X)$ of complex vector bundles over X. The periodic nature of this spectrum tells us that

$$K^q(X) \cong \begin{cases} K^0(X) \cong [X_+, \mathbb{Z} \times BU] & \text{if } q \text{ is even, and} \\ K^1(X) \cong [X_+, U] & \text{if } q \text{ is odd.} \end{cases}$$

Generalized (co)homology theories are required to satisfy all the Eilenberg-Steenrod axioms, except the Dimension Axiom. Recall that the Dimension Axiom says that the (co)homology of a point is zero except in dimension zero. For a generalized theory, this need not be the case. As we see above,

$$E_*(point) = \pi_*(point_+ \wedge \mathbb{E}) = \pi_*(S^0 \wedge \mathbb{E}) = \pi_*(\mathbb{E}),$$

and this group is often nonzero in many dimensions.

Exercise. Show that $\pi_q(\mathbb{E}) = 0$ for all $q \neq 0$ if and only if $\mathbb{E}$ is an Eilenberg-MacLane spectrum.

With the classification of generalized (co)homology theories by spectra, we can now understand the notions of (co)homology of a spectrum, as well as of a space.

Definition 10.10. *Let $\mathbb{E}$ be an ω-spectrum representing a generalized cohomology theory E^* and generalized homology theory E_*. Let $\mathbb{X}$ be any connective spectrum (i.e a spectrum with $\pi_q(\mathbb{X}) = 0$ for $q < 0$). We defined the generalized homology groups*

$$E_q(\mathbb{X}) = \pi_q(\mathbb{E} \wedge \mathbb{X}).$$

We similarly define the generalized cohomology groups by

$$E^q(\mathbb{X}) = [\mathbb{X}, \mathbb{E}]_q$$

where by this notation we mean weak homotopy classes of maps from $\mathbb{X}$ to $\Sigma^q\mathbb{E}$.

Exercise. Show that if X is a finite CW-complex and $\mathbb{E}$ is a spectrum representing (co)homology theories E_* and E^*, then

$$E_*(X) \cong E_*(\Sigma^\infty(X_+)) \quad \text{and} \quad E^*(X) \cong E^*(\Sigma^\infty(X_+)).$$

10.2.3 Application: The finiteness of the positive dimensional stable homotopy groups of spheres.

A famous theorem of Serre [136] is that the stable homotopy groups of spheres are finite in positive dimensions:

Theorem 10.19. *(Serre) [136]*

$$\varinjlim_{k} \pi_{q+k}(S^k) = \pi_q(\mathbb{S})$$

are finite abelian groups for $q > 0$.

As an application of the theory of spectra and generalized homology theories, we sketch a proof of Serre's theorem, modulo one result that Serre proved along the way.

Lemma 10.20. *(Serre)[136] $\pi_k(\mathbb{S})$ is a finitely generated abelian group for every k.*

Let $\mathbb{S}_{\mathbb{Q}}$ be the spectrum that represents the generalized homology theory, given by "rational stable homotopy", $(X, A) \to \pi^s_*(X, A) \otimes \mathbb{Q}$.

Exercise. Show that the homotopy groups of this representing spectrum are the rational stable homotopy groups of spheres.

$$\pi_s(\mathbb{S}_{\mathbb{Q}}) \cong \pi_*(\mathbb{S}) \otimes \mathbb{Q}.$$

We now observe that the homology of $\mathbb{S}_{\mathbb{Q}}$ is quite simple.

Lemma 10.21.

$$H_*(\mathbb{S}_{\mathbb{Q}}; \mathbb{Z}) = \begin{cases} \mathbb{Q}, & \text{for } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} H_*(\mathbb{S}_{\mathbb{Q}}; \mathbb{Z}) &= \pi_*(\mathbb{S}_{\mathbb{Q}} \wedge H\mathbb{Z}) \\ &\cong (\mathbb{S}_{\mathbb{Q}})_*(H\mathbb{Z}) \\ &= \pi_*(H\mathbb{Z}) \otimes \mathbb{Q} \end{aligned}$$

The result follows. □

Now consider the rational Hurewicz map, viewed as a map of generalized homology theories:

$$h : \pi_*(-) \otimes \mathbb{Q} \to H_*(-; \mathbb{Q}).$$

This is induced by a map of representing spectra,

$$h : \mathbb{S}_{\mathbb{Q}} \to H\mathbb{Q}.$$

The following is immediate from the lemma.

Proposition 10.22. *The map of spectra $h : \mathbb{S}_{\mathbb{Q}} \to H\mathbb{Q}$ is an equivalence. Therefore there is an isomorphism*

$$h_*; \pi_*(\mathbb{E}) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(\mathbb{E}; \mathbb{Q})$$

for any spectrum $\mathbb{E}$.

Applying this proposition to the sphere spectrum $\mathbb{S}$, we have that

$$\pi_*(\mathbb{S}) \otimes \mathbb{Q} \cong H_*(\mathbb{S}; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

In particular this means that $\pi_q(\mathbb{S}) \otimes \mathbb{Q} = 0$ for $q > 0$. By Lemma 10.20, this implies that $\pi_q(\mathbb{S})$ is a finite abelian group for $q > 0$.

10.3 The Atiyah-Hirzebruch spectral sequence

The Atiyah - Hirzebruch spectral sequence provides one with a computational technique for computing the generalized (co)homology of a CW complex X in terms of its ordinary (co)homology and the homotopy groups of the spectrum representing the generalized theory. It is based on filtering X by its skeletal filtration.

10.3.1 The spectral sequence

Let X be a finite, n-dimensional based CW complex, with basepoint $x_0 \in X$. Let h^* be a generalized cohomology theory represented by a spectrum $\mathbb{E}$. Let $\tilde{h}^*$ be the reduced theory,

$$\tilde{h}^*(X) = h^*(X, x_0) = ker\,(h^*(X) \to h^*(x_0)).$$

Consider the skeletal filtration of X:

$$x_0 = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n = X. \tag{10.5}$$

Let $F_m\tilde{h}^p(X) = ker\,(\tilde{h}^p(X) \to \tilde{h}^p(X^m))$. We then have a filtration on $\tilde{h}^p(X)$:

$$0 = F_n\tilde{h}^p(X) \subset F_{n-1}\tilde{h}^p(X) \subset \cdots \subset F_0\tilde{h}^p(X) \subset F_{-1}\tilde{h}^p(X) = \tilde{h}^p(X). \tag{10.6}$$

The Atiyah Hirzebruch spectral sequence (AHSS) will have as its E_1-term the homology of the subquotients of the skeletal filtration

$$E_1^{p,n-p} = h^n(X^p, X^{p-1}) \cong \tilde{h}^n(X^p, X^{p-1}).$$

It will converge to $\tilde{h}^*(X)$ in the sense that the E_∞-term is given by the subquotients of the above filtration,

$$E_\infty^{p,n-p} = F_p\tilde{h}^n(X)/F_{p-1}\tilde{h}^n(X)).$$

To understand the basic idea, first notice that the subquotient of the skeletal filtration X^p/X^{p-1} is a wedge of $(p-1)$-dimensional spheres, and these spheres form a basis of the cellular chain group $C_p(X)$. We call that basis β_p. That is,

$$X^p/X^{p-1} \simeq \bigvee_{\beta_p} S^p.$$

We therefore have

$$\begin{aligned}\tilde{h}^n(X^p/X^{p-1}) = \bigoplus_{\beta_p}\tilde{h}^n(S^p) &\cong \bigoplus_{\beta_p}\tilde{h}^{n-p}(S^0) \\ &\cong \bigoplus_{\beta_p} h^{n-p}(pt) = Hom(C_p(X), h^{n-p}(pt)) \\ &= C^p(X; h^{n-p}(pt)) \cong C^p(X; \pi_{n-p}(\mathbb{E})).\end{aligned} \tag{10.7}$$

These cochain groups form the E_1-term of the *Atiyah - Hirzebruch spectral sequence*. Here is the statement of the theorem asserting the existence of this spectral sequence.

Theorem 10.23. *(Atiyah and Hirzebruch [10]). Let X be a finite CW-complex and h^* a generalized cohomology theory represented by a spectrum $\mathbb{E}$. Then there is a spectral sequence converging to $\tilde{h}^*(X)$, satisfying the following properties:*

1. $E_1^{p,q} = C^p(X; h^q(pt)) = C^p(X; \pi_q(\mathbb{E}))$. *These are the cellular cochains of X.*
2. $E_2^{p,q} = \tilde{H}^p(X; h^q(pt))$.
3. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$
4. $E_\infty^{p,q} = F_{p-1}\tilde{h}^{p+q}(X)/F_p\tilde{h}^{p+q}(X))$.

We remark that there is a similar spectral sequence converging to the generalized homology.

10.3.2 The spectral sequence of an exact couple and the construction of the AHSS

The construction of the Atiyah-Hirzebruch spectral sequence (AHSS) is explained clearly in Adam's well-known book [7]. Here we indicate its construction as an example of a spectral sequence arising from an *exact couple*. We begin by describing this general construction, and then show how it can be used to construct the AHSS with the properties described in Theorem 10.23.

Definition 10.11. *An exact couple is a triangle of abelian groups or chain complexes and homomorphisms between them that are exact.*

$$\begin{array}{ccc} D_1 & \xrightarrow{\alpha} & D_1 \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta_1} \\ & E_1 & \end{array}$$

In other words, $ker\,\alpha = Image\,\gamma$, $ker\,\gamma = Image\,\beta_1$, and $ker\,\beta_1 = Image\,\alpha$.

One might think of E_1 as the first term of a spectral sequence. If one lets

$$d_1 = \beta_1 \circ \gamma : E_1 \to E_1$$

then one can define $E_2 = H_*(E_1, d_1)$, $D_2 = Image\,\alpha$, and $\beta_2 = \beta_1 \circ \alpha^{-1} : D_2 \to E_2$, and then one can check that

$$\begin{array}{ccc} D_2 & \xrightarrow{\alpha} & D_2 \\ {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta_2} \\ & E_2 & \end{array}$$

is an exact couple as well (called the *derived* exact couple). We then think of E_2 as the second term of the spectral sequence. Continuing in this fashion produces all the terms in a spectral sequence.

Applying the reduced generalized cohomology to the skeletal filtration of a finite CW complex $\tilde{h}^*(X)$ leads to an exact couple in the following way.

We let $D_1 = \bigoplus_{p,q} \tilde{h}^{p+q}(X^p)$ and $E_1 = \bigoplus_{p,q} \tilde{h}^{p+q}(X^p/X^{p-1}) \cong C^p(X; h^q(pt))$. For each p one has a long exact sequence in generalized cohomology,

$$\cdots \xrightarrow{\beta_1} h^{p+q}(X^p, X^{p-1}) \xrightarrow{\gamma} h^{p+q}(X^p) \xrightarrow{\alpha} h^{p+q}(X^{p-1}) \xrightarrow{\beta_1} h^{p+q+1}(X^p, X^{p-1}) \xrightarrow{\gamma} \cdots$$

One can easily check that this defines an exact couple, with the spaces and maps being the direct sum of all long exact sequences associated to the various skeletal pairs (X^p, X^{p-1}). The resulting spectral sequence is the *Atiyah-Hirzebruch spectral sequence.*

To compute the E_2-term of this spectral sequence we need to compute the homology $H_*(E_1, d_1)$ where

$$d_1 = \beta_1 \circ \gamma : \bigoplus_{p,q} \tilde{h}^{p+q}(X^p/X^{p-1}) \to \bigoplus_{p,q} h^{p+q+1}(X^{p+1}, X^p) \qquad (10.8)$$

$$\bigoplus_{p,q} C^p(X; h^q(pt)) \to \bigoplus_{p,q} C^{p+1}(X; h^q(pt)).$$

Exercise. Prove that $d_1 : \bigoplus_{p,q} C^p(X; h^q(pt)) \to \bigoplus_{p,q} C^{p+1}(X; h^q(pt))$ as defined above is the coboundary map in the cellular cochain complex and therefore the E_2-term in the Atiyah-Hirzebruch spectral sequence is

$$E_2^{p,q} = \tilde{H}^p(X; h^q(pt)).$$

The remaining properties of the Atiyah-Hirzebruch spectral sequence as described in Theorem 10.23 are proved in a rather straightforward way. See [7] for a clear treatment. We now apply the AHSS to compute the K-theory of some important, familiar spaces.

10.3.3 Some K-theory calculations with the AHSS

In this subsection we will use the Atiyah-Hirzebruch spectral sequence to calculate the (complex) K-theory of certain important manifolds. We begin with the calculation of the K-theory of closed orientable surfaces.

Proposition 10.24. *Let Σ_g be a closed, orientable surface of genus g. Then*

$$K^0(\Sigma_g) \cong \mathbb{Z}^2 \quad \textit{and} \quad K^1(\Sigma_g) \cong \mathbb{Z}^{2g}.$$

Note. By Bott periodicity this result determines the K-cohomology of Σ_g in all dimensions. Namely,

$$K^q(\Sigma_g) = \begin{cases} \mathbb{Z}^2 & \text{if } q \text{ is even} \\ \mathbb{Z}^{2g} & \text{if } q \text{ is odd.} \end{cases}$$

Proof. The cohomology of Σ_g is nonzero in only three dimensions: $H^0(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$, $H^1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$, and $H^2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$. Therefore the E_2-term of the Atiyah-Hirzebruch spectral sequence has only the following nonzero groups:

$$E_2^{0,2m} \cong \mathbb{Z}, \quad E_2^{1,2m} \cong \mathbb{Z}^{2g}, \quad E_2^{2,2m} \cong \mathbb{Z}$$

for each m. Again, all other groups in the E_2-term are zero. Since $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, one immediately sees that all differentials must be zero. Therefore the spectral sequence "collapses", i.e $E_2^{p,q} = E_\infty^{p,q}$ and the result follows. □

Proposition 10.25. *The K-theory of $\mathbb{CP}^n$ is given by*

$$K^0(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1} \quad K^1(\mathbb{CP}^n) = 0,$$

Proof. The E_2- term of the AHSS is given by

$$E_2^{p,q} = H^p(\mathbb{CP}^n; K^q(pt)) = \begin{cases} \mathbb{Z} & \text{if } p \text{ and } q \text{ are both even and } 0 \le p \le n \\ 0 & \text{otherwise.} \end{cases}$$

If we call the total degree of an element of $E_r^{p,q}$ $p+q$, then we see that the only nonzero terms in this spectral sequence have even total degree. Yet the differentials change the total degree of an element by $+1$. Therefore the differentials are all zero and the spectral sequence collapses at the E_2-term. Now

$$\bigoplus_{p+q=0} E_2^{p,q} = \mathbb{Z}^{n+1}$$

so the result follows. □

Exercises. .

1. What is $K^q(\mathbb{CP}^n)$ for all q? *Hint.* Use Bott periodicity.

2. Show that if X is any space with $H^q(X;\mathbb{Z}) = 0$ for q odd, then the AHSS collapses at the E_2-term, which is to say

$$\begin{aligned} E_\infty^{p,q} &= H^p(X; K^q(pt)) \\ &= \begin{cases} H^p(X;\mathbb{Z}) & \text{if } p \text{ and } q \text{ are both even} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{10.9}$$

10.4 Symmetric spectra, ring spectra and module spectra

The graded abelian group $E_*(point) \cong \pi_*(\mathbb{E})$ is called the *coefficients* of the generalized homology theory E. Now motivated by structures in ordinary (co)homology, one might expect to find structures such as a evaluation map of a generalized cohomology theory on its corresponding generalized homology theory, taking values in the coefficients, or perhaps a cup product in the generalized cohomology. Notice that even in ordinary (Eilenberg-MacLane) (co)homology, $H^*(X;G)$ has these structures only if G is a ring. In a generalized theory we will need the representing spectrum $\mathbb{E}$ to be a "ring spectrum", which means there is a monoid structure

$$\mathbb{E} \wedge \mathbb{E} \to \mathbb{E}.$$

Of course we don't have a definition of the smash product of spectra yet. So far we only know how to take the smash product of a space with a spectrum. Defining an associative smash product has the effect of giving the category of spectra a "monoidal structure". For the purposes of defining structures at the level of generalized (co)homology such as cup product, having a ring structure "up to homotopy" suffices, and that was all that existed from the time of Whitehead's seminal paper [158] until the 1990's. Defining such a structure that is actually associative, instead of just associative up to homotopy, is quite a technical challenge. It was first accomplished by Hovey, Shipley, and Smith in [76]. Such a structure allows one to talk about "ring spectra", "module spectra", and roughly speaking, to do homological algebra in the category of spectra. In this section we describe a monoidal structure on a category of spectra, define the notion of a "ring spectrum", and discuss several applications. In the next chapter this structure will prove quite useful in studying cobordisms of manifolds in the setting of generalized (co)homology theories.

Our goal in this subsection is to show that a category of spectra exists which is in some sense equivalent to the one described above, and that has a monoidal structure defined by smash product of spectra. We begin with a definition of the type of categorical monoidal structure we are looking for.

Definition 10.12. *A monoidal category is a category $\mathcal{C}$ equipped with a monoidal* structure. A monoidal structure consists of the following:

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product or monoidal product,
- an object I called the unit object or identity object,
- three natural isomorphisms subject to certain coherence conditions expressing the fact that the tensor operation
  - is associative: there is a natural (in each of three arguments A, B, C) isomorphism α called the *associator*, with components
  $$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
  - has I as left and right identity: there are two natural isomorphisms λ and ρ called the left and right unitor respectively, with components $\lambda_A : I \otimes A \cong A$ and $\rho_A : A \otimes I \cong A$.
- The coherence conditions for these natural transformations are:
  - for all A, B, C, and D in $\mathcal{C}$, the following diagram commutes
  $$\begin{array}{ccccc} (A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A \otimes B) \otimes C) \otimes D \\ {\scriptstyle I_A \otimes \alpha_{B,C,D}}\downarrow & & & & \uparrow{\scriptstyle \alpha_{A,B,C} \otimes I_D} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$
  - for all A and B in $\mathcal{C}$, the following diagram commutes
  $$\begin{array}{ccc} A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B \\ {\scriptstyle I_A \otimes \lambda_B}\downarrow & & \downarrow{\scriptstyle \rho_A \otimes I_B} \\ A \otimes B & \xrightarrow[=]{} & A \otimes B \end{array}$$

A **strict monoidal category** is one for which the natural isomorphisms α, λ, and ρ are identities. It turns out that every monoidal category is monoidally equivalent to a strict monoidal category.

Examples:

1. $Vect_k$, the category of finite dimensional vector spaces over a field k, with morphisms being k-linear transformations. The monoidal structure is tensor product (over k) of vector spaces. The unit is the one-dimensional vector space k.

2. $\mathcal{G}_{ab}$, category of Abelian groups and group homomorphisms. The monoidal structure is tensor product of abelian groups, and the unit is the group of integers $\mathbb{Z}$. More generally, the category $R - mod$ of modules over a commutative ring R is a monoidal category, with tensor product (over R) the monoidal structure, and with the unit being R itself.

3. Set, the category of finite sets and set maps. The monoidal structure is cartesian product, and the one-element set is the unit.

4. Cat, the category of small categories (i.e categories where the objects and morphisms both form sets) is a monoidal category, where the monoidal structure is the cartesian product of categories. The category with one object and whose only morphism is the identity morphism is the unit.

A *symmetric monoidal category* is a monoidal category where the monoidal structure $\otimes$ is commutative up to coherent isomorphism. Here is a strict definition.

Definition 10.13. *A symmetric monoidal category is a monoidal category $(\mathcal{C}, \otimes, I)$ such that, for every pair A, B of objects in $\mathcal{C}$, there is an isomorphism $s_{A,B} : A \otimes B \to B \otimes A$ that is natural in both A and B and such that the following coherence diagrams commute:*

- *The unit coherence:*

$$\begin{array}{ccc} A \otimes I & \xrightarrow{s_{A,I}} & I \otimes A \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \lambda_A} \\ A & \xrightarrow[=]{} & A \end{array}$$

- *The associativity coherence:*

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{s_{A,B} \otimes 1_C} & (B \otimes A) \otimes C \\ {\scriptstyle \alpha_{A,B,C}}\downarrow & & \downarrow{\scriptstyle \alpha_{B,A,C}} \\ A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\ {\scriptstyle s_{A,B \otimes C}}\downarrow & & \downarrow{\scriptstyle 1_B \otimes s_{A,C}} \\ (B \otimes C) \otimes A & \xrightarrow[\alpha_{B,C,A}]{} & B \otimes (C \otimes A) \end{array}$$

- *The inverse law:*

$$\begin{array}{ccc} B \otimes A & \xrightarrow{=} & B \otimes A \\ {\scriptstyle s_{A,B}} \uparrow & & \downarrow {\scriptstyle s_{B,A}} \\ A \otimes B & \xrightarrow[1_{A\otimes B}]{} & A \otimes B \end{array}$$

Exercises.

1. Show that all of the categories in the above examples of monoidal categories in fact are symmetric monoidal.
2. Find an example of a monoidal category that is not symmetric monoidal.

Our goal in this subsection is to show that there is a category of spectra that has a symmetric monoidal structure, where the monoidal structure is a representation of smash product of spectra. Such symmetric monoidal categories of spectra were found in the late 1990's and early 2000's (see, for example [76], [47], [100]), and thankfully, they were all eventually shown to be equivalent in an appropriate sense. Here we describe the notion of *symmetric spectra* of [76]. Actually in [76] the authors work in the setting of simplicial sets, but here we work in the setting of topological spaces.

Definition 10.14. *. A symmetric spectrum $\mathbb{X}$ is a sequence of spaces $\{X_n, n \geq 0\}$ together with structure maps $\epsilon_n : \Sigma X_n \to X_{n+1}$, and actions of the symmetric groups $\Sigma_n \times X_n \to X_n$ so that if we think of S^p as the p-fold smash product*

$$S^p = S^1 \wedge \cdots \wedge S^1$$

with the action of Σ_p given by permutation of coordinates, then the composition

$$S^p \wedge X_q \xrightarrow{1\wedge\epsilon_q} S^{p-1} \wedge X_{1+q} \xrightarrow{1\wedge\epsilon_{1+q}} \cdots \xrightarrow{1\wedge\epsilon_{p-1+q}} S^1 \wedge X_{p-1+q} \xrightarrow{\epsilon_{p+q}} X_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant. Here $(\Sigma_p \times \Sigma_q)$ acts on X_{p+q} as it is naturally a subgroup of Σ_{p+q} consisting of those permutations of $p+q$ letters that fix the first p letters and the last q letters as sets.

A map (morphism) of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$ is a sequence of maps $f_n : X_n \to Y_n$ that is Σ_n equivariant, and which respect the structure maps. That is the following diagrams commute:

$$\begin{array}{ccc} \Sigma\mathbb{X}_n & \xrightarrow{\epsilon_n} & \mathbb{X}_{n+1} \\ {\scriptstyle \Sigma f_n} \downarrow & & \downarrow {\scriptstyle f_{n+1}} \\ \Sigma\mathbb{Y}_n & \xrightarrow[\epsilon_n]{} & \mathbb{Y}_{n+1} \end{array}$$

In order to complete the definition of the category of symmetric spectra, which we call $\mathbf{Sp}^{\boldsymbol{\Sigma}}$, we observe that the collection of morphisms between two symmetric spectra, $Map^{\mathbf{Sp^\Sigma}}(\mathbb{X},\mathbb{Y})$ is itself a symmetric spectrum. For this we define

$$Map^{\mathbf{Sp^\Sigma}}(\mathbb{X},\mathbb{Y})_n \subset \prod_i Map(\mathbb{X}_i, \mathbb{Y}_{i+n})$$

to be the subspace of all collections of Σ_i-equivariant maps $\{\phi_i : \mathbb{X}_i \to \mathbb{Y}_{i+n}\}$, as i varies, that respect the structure maps. That is, the following diagrams commute:

$$\begin{array}{ccc} \Sigma\mathbb{X}_i & \xrightarrow{\Sigma f_i} & \Sigma\mathbb{Y}_{i+n} \\ {\scriptstyle \epsilon_i}\downarrow & & \downarrow{\scriptstyle \epsilon_i} \\ \mathbb{X}_{i+1} & \xrightarrow[f_{i+1}]{} & \mathbb{Y}_{i+1+n} \end{array}$$

Notice that $Map^{\mathbf{Sp^\Sigma}}(\mathbb{X},\mathbb{Y})_0$ is just the space of all maps of symmetric spectra.

We leave it to the reader to check that the collection of spaces $\{Map^{\mathbf{Sp^\Sigma}}(\mathbb{X},\mathbb{Y})_n\}$ support natural symmetric group actions and structure maps to define a symmetric spectrum structure.

This definition defines a category of symmetric spectra that we call $\mathbf{Sp^\Sigma}$.

Note: By replacing the symmetric groups Σ_n by the orthogonal groups $O(n)$ in the above definition one gets the notion of an *orthogonal spectrum*, and the category of such, $\mathbf{Sp^O}$. Like symmetric spectra, orthogonal spectra have been very useful in homotopy theory. However for the purposes of this book we emphasize symmetric spectra.

As mentioned above, one of the main reasons to consider symmetric or orthogonal spectra, is that the categories of such are symmetric monoidal, where the monoidal structure is an operation that defines smash product of such spectra. We now define the smash product of two symmetric spectra, $\mathbb{X}$ and $\mathbb{Y}$.

Definition 10.15. *The smash product of two symmetric spectra $\mathbb{X}$ and $\mathbb{Y}$ is the symmetric spectrum $\mathbb{X}\wedge\mathbb{Y}$ defined by*

$$(\mathbb{X}\wedge\mathbb{Y})_n = \bigvee_{p+q=n} \Sigma_{n_+} \wedge_{\Sigma_p\times\Sigma_q} (\mathbb{X}_p\wedge\mathbb{Y}_q)/\sim$$

where Σ_{n_+} denotes the symmetric group Σ_n with an additional disjoint base-point, and the quotient relation identifies the images, for every r, of the two maps

$$\alpha : \Sigma_{(p+q+r)_+} \wedge (S^p \wedge \mathbb{X}_q \wedge \mathbb{Y}_r) \longrightarrow \Sigma_{(p+q+r)_+} \wedge_{\Sigma_q\times\Sigma_{p+r}} (\mathbb{X}_q \wedge \mathbb{Y}_{p+r})$$

and

$$\beta : \Sigma_{(p+q+r)_+} \wedge (S^p \wedge \mathbb{X}_q \wedge \mathbb{Y}_r) \longrightarrow \Sigma_{(p+q+r)_+} \wedge_{\Sigma_{p+q}\times\Sigma_r} (\mathbb{X}_{p+q} \wedge \mathbb{Y}_r)$$

where $\alpha(\sigma, t, x, y) = (\sigma \circ \tau_{q,p}, x, ty)$ *and* $\beta(\sigma, t, x, y) = (\sigma, tx, y)$. *Here* $(t, x) \to tx$ *is shorthand for the structure map*

$$S^p \wedge \mathbb{X}_q \to \mathbb{X}_{p+q}$$

and $\tau_{q,p} \in \Sigma_{p+q+r}$ *is the permutation that moves the first block of* q *letters past the second block of* p *letters and leaves the last block of* r *letters alone.*

The action of the symmetric group Σ_n *on* $(\mathbb{X} \wedge \mathbb{Y})_n$ *is induced by the action on the left hand coordinate of* $\Sigma_{n_+} \wedge_{\Sigma_p \times \Sigma_q} (\mathbb{X}_p \wedge \mathbb{Y}_q)$.

Exercise.
Define the structure maps $\epsilon_n : \Sigma(\mathbb{X} \wedge \mathbb{Y})_n \to (\mathbb{X} \wedge \mathbb{Y})_{n+1}$ and verify that $\mathbb{X} \wedge \mathbb{Y}$ is indeed a symmetric spectrum.

Note. The construction in this definition of identifying the images of the two maps α and β is known in category theory as the *"coequalizer"* of α and β.

The following, proved in [76] is not too difficult to prove, but is extremely important.

Theorem 10.26. *(Hovey, Shipley, and Smith [76], corollary 2.2.4) The smash product* $\mathbb{X} \wedge \mathbb{Y}$ *is a symmetric monoidal structure on the category of symmetric spectra,* $\mathbf{Sp}^{\mathbf{\Sigma}}$.

Exercise. Verify that the unit in this symmetric monoidal structure is the sphere spectrum $\mathbb{S}$, where $\mathbb{S}_k$ is the k-fold smash product

$$\mathbb{S}_k = S^k = S^1 \wedge \cdots \wedge S^1$$

and the action of the symmetric group Σ_k is given by permuting the coordinates.

It is important to understand when a morphism of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$, is in an appropriate sense, an (homotopy) equivalence. The appropriate notion of equivalence is important because one would like to consider the associated "homotopy category", where one takes the same objects (symmetric spectra) and one "inverts" the equivalences. That is, in the homotopy category one formally adds inverse morphisms to every equivalence. This is a construction due to Quillen [128] and can be done whenever one has what is called a "model" structure on a category. A model category is one that has three distinguished types of morphisms, called "fibrations", "cofibrations", and "weak equivalences", satisfying several axioms. The associated homotopy category is defined by "localizing" with respect to the weak equivalences. This is a fascinating and important area of study, and there are several good texts on the subject. We refer the reader to [128], [75], and [108].

A model structure for the category of symmetric spectra, $\mathbf{Sp}^{\mathbf{\Sigma}}$, was described and studied in detail in [76]. It turns out that there is a subtlety when defining the notion of (stable) equivalence in the category $\mathbf{Sp}^{\mathbf{\Sigma}}$. Following the notion of weak equivalence of spaces or of (ordinary) spectra, one might be inclined to declare that a morphism of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$ is a stable equivalence if it is a weak homotopy equivalence as a map of ordinary spectra; that is, if it induces an isomorphism of homotopy groups. However as is pointed out clearly in [76], this will not work. Namely our goal is to find a good notion of stable equivalence of symmetric spectra that has the property that when one takes the associated homotopy category, one obtains a category equivalent to taking the homotopy category of the category of ordinary spectra. So in particular, when one wants to do calculations depending only on the homotopy type of spectra and maps between them, it would not matter if one was using symmetric spectra or ordinary spectra. The above naive notion of weak equivalence simply won't satisfy this property, as pointed out in [76]. Essentially, the way the authors of [76] found to deal with this issue was to declare that a map of symmetric spectra $f : \mathbb{X} \to \mathbb{Y}$ is a stable equivalence if the induced map E^*f of cohomology groups is an isomorphism for every generalized cohomology theory E^*. We refer the reader to [76] for details of these issues. The main upshot for our purposes is that there is now a symmetric monoidal category of spectra, with unit the sphere spectrum, whose associated homotopy theory is equivalent to what one would expect from the naive notions of spectra that go all the way back to Lima and Whitehead in the 1950's and early 1960's.

With the existence of a symmetric monoidal structure, one can begin doing "algebra" in our category of spectra. For example, a ring spectrum (with unit) $\mathbb{X}$ is one that is equipped with a pairing

$$\mu : \mathbb{X} \wedge \mathbb{X} \to \mathbb{X}$$

together with a unit maps $\eta : \mathbb{S} \to \mathbb{X}$ that satisfy the usual associativity conditions. In other words, a "ring spectrum" is a monoid in the category of spectra. For example, the sphere spectrum $\mathbb{S}$ has the pairing given by the equality

$$\mathbb{S} \wedge \mathbb{S} = \mathbb{S}$$

which makes $\mathbb{S}$ a commutative ring spectrum. Another important class of examples comes from the suspension spectrum of a group (with a disjoint basepoint), $\Sigma^\infty(G_+)$. This is because

$$\Sigma^\infty(G_+) \wedge \Sigma^\infty(G_+) = \Sigma^\infty((G \times G)_+).$$

The group multiplication defines the ring structure.

If Y is a space with a group action $G \times Y \to Y$, then its suspension spectrum $\Sigma^\infty(Y_+)$ becomes a *module spectrum* over the ring spectrum $\Sigma^\infty(G_+)$. In general a (left) module spectrum $\mathcal{M}$ over a ring spectrum $\mathbb{X}$ is a symmetric spectrum that is equipped with a pairing map

$$\mathbb{X} \wedge \mathcal{M} \to \mathcal{M}$$

that satisfies the usual associativity conditions. As one can see, once one has the structure of a symmetric monoidal category, one can start doing algebra in the category!

Now observe that one does not really need a group structure on G for $\Sigma^\infty(G_+)$ to have a ring structure. Indeed G just needs to be a monoid. An important class of such examples comes from the based loop space ΩX where X is any based space. In order that ΩX be a (strict) monoid, we take ΩX to refer to the space of "Moore loops". This is the space of pairs (r, α), where $r \geq 0$ and $\alpha : [0, r] \to X$ is a map that sends the endpoints 0 and r to the basepoint $x_0 \in X$. The multiplication in ΩX is given by juxtaposition:

$$(r, \alpha) \cdot (s, \beta) = (r + s, \alpha \cdot \beta) : [0, r + s] \to X$$

where

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(t) & \text{if} \quad 0 \leq t \leq r \\ \beta(t - r) & \text{if} \quad r \leq t \leq r + s \end{cases}$$

The study of the spectrum $\Sigma^\infty(\Omega X_+)$ as a ring spectrum was initiated by Waldhausen [152]. It was shown how the study of the category of modules over $\Sigma^\infty(\Omega X_+)$ leads to an understanding of various automorphism groups of X if X is a manifold (eg diffeomorphism groups, homeomorphism groups, PL homeomorphism groups, etc.). It has lead to the study of what is now known as "Waldhausen K-theory" which has been a major area of research in algebraic and differential topology since the 1970's. The reader is referred to [152], [153] to learn more.

The Eilenberg MacLane spectrum $\mathbb{H}R$ where R is any ring, is also a ring spectrum. The ring structure is induced up to homotopy by the pairings

$$K(R, q) \times K(R, s) \to K(R, q + s)$$

which represents the cohomology class given by the cross product $\iota_q \times \iota_s \in H^{q+s}(K(R, q) \times K(R, s); R)$ where $\iota_q \in H^q(K(R, q); R)$ and $\iota_s \in H^s(K(R, s); R)$ are the fundamental classes.

In a similar fashion, if P is a right module over a ring R, $\mathbb{H}P$ has the structure of a right module spectrum over the ring spectrum $\mathbb{H}R$.

In these notes we will not further pursue the homological algebra that is possible in the category of spectra. But understanding this structure is a very active area of research and it has had many applications.

From here on out, when we refer to "spectra", we will mean symmetric spectra, and we will most often leave out the reference to the category $\mathbf{Sp}^{\boldsymbol{\Sigma}}$. So for example when we write $Map(\mathbb{X}, \mathbb{Y})$ we will mean the mapping spectrum $Map^{\mathbf{Sp}^{\boldsymbol{\Sigma}}}(\mathbb{X}, \mathbb{Y})$.

10.5 Generalized cup and cap products

In this section we will study product structures on generalized cohomology theories represented by ring spectra. Our goal will be to apply them to the study of generalized orientations and duality structures on manifolds.

Let R be a ring, then a basic construction in algebraic topology is the cup product in the cohomology $H^*(X;R)$. If R is a commutative ring, then this product inherits a graded-commutative structure. Recall that the ingredients involved in this construction are the diagonal map

$$\Delta : X \to X \times X$$

and the ring multiplication $\mu : R \times R \to R$. More specifically the cup product is defined by

$$\cup : H^q(X;R) \times H^s(X;R) \xrightarrow{\times} H^{q+s}(X \times X;R) \xrightarrow{\Delta^*} H^{q+s}(X;R)$$

where the first map in this composition is the "cross product". This cross product is induced on the level of the representing Elienberg-MacLane spaces via the map $K(R,q) \times K(R,s) \to K(R,q+s)$ which represents the cross product class $\iota_q \times \iota_s \in H^{q+s}(K(R,q) \times K(R,s);R)$. Furthermore these classes define (up to homotopy) the ring spectrum structure on the representing Eilenberg-MacLane spectrum $\mathbb{H}R$.

This suggests that whenever we have a generalized cohomology theory E^* represented by a ring spectrum $\mathbb{E}$, then one can use that ring structure to define a "cross product" map, and a "cup product" structure in the generalized cohomology theory. This is indeed the case.

Definition 10.16. *Let E^* be a generalized cohomology theory represented by a ring spectrum $\mathbb{E}$. Let X and Y be spaces, and consider generalized cohomology classes $\alpha \in E^q(X)$, and $\beta \in E^s(Y)$. Let*

$$\phi_\alpha : \Sigma^\infty(X_+) \to \Sigma^q\mathbb{E} \quad \textit{and} \quad \phi_\beta : \Sigma^\infty(Y_+) \to \Sigma^s\mathbb{E}$$

be maps of spectra representing α and β respectively. The "cross product" $\alpha \times \beta \in E^{q+s}(X \times Y)$ is defined to be the cohomology class represented by the composition

$$\begin{aligned}\phi_{\alpha\times\beta} : \Sigma^\infty((X \times Y)_+) = \Sigma^\infty(X_+) \wedge \Sigma^\infty(Y_+) &\xrightarrow{\phi_\alpha \wedge \phi_\beta} \Sigma^q\mathbb{E} \wedge \Sigma^s\mathbb{E} \\ = \Sigma^{q+s}(\mathbb{E} \wedge \mathbb{E}) &\xrightarrow{\mu} \Sigma^{q+s}\mathbb{E}\end{aligned}$$

Here $\mu : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$ is the ring multiplication.

Definition 10.17. *Let E^* be a generalized cohomology theory represented by a ring spectrum $\mathbb{E}$. Let X be a space and $\alpha \in E^q(X)$ and $\beta \in E^s(X)$ generalized cohomology classes. The cup product $\alpha \cup \beta \in E^{q+s}(X)$ is defined to be the class represented by the composition*

$$\phi_{\alpha\cup\beta} : \Sigma^\infty(X_+) \xrightarrow{\Delta} \Sigma^\infty((X \times X)_+) \xrightarrow{\phi_{\alpha\times\beta}} \Sigma^{q+s}\mathbb{E}$$

where $\phi_{\alpha\times\beta}$ is the map representing the cross product as above.

Exercise. Verify that with the above definition, the generalized cohomology $E^*(X)$ has the structure of a graded ring. If $\mathbb{E}$ is a commutative ring spectrum, then verify that this ring structure on $E^*(X)$ is graded commutative, like it is for ordinary cohomology with coefficients in a commutative ring.

Notice that a key ingredient in the construction of these generalized cup products is the diagonal map $\Delta : X \to X \times X$ which induces a map on the level of spectra $\Delta : \Sigma^\infty(X_+) \to \Sigma^\infty(X_+) \wedge \Sigma^\infty(X_+)$. This map is called a coproduct, and this structure is often referred to as a *"coalgebra"* structure on the suspension spectrum $\Sigma^\infty(X_+)$. In general not all connective spectra $\mathbb{X}$ have this structure, and so their generalized cohomologies, $E^*(\mathbb{X})$ do not have cup products.

One might form a more general, "twisted" form of this construction in the following way. Suppose $p : \zeta \to X$ is a vector bundle over a finite CW complex. The diagonal map on $\Delta : X \to X \times X$ defines maps on the level of vector bundles

$$\begin{array}{ccc} \zeta & \xrightarrow{\Delta_R^\zeta} & \zeta \times X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\times 1} \\ X & \xrightarrow[\Delta]{} & X \times X, \end{array}$$

and similarly $\Delta_L^\zeta : \zeta \to X \times \zeta$.

Notice that the Thom space of $\zeta \times X$ is given by

$$T(\zeta \times X) = T\zeta \wedge X_+$$

and similarly $T(X \times \zeta) = X_+ \wedge T\zeta$.

The diagonal maps then induce maps on the Thom spaces for which, by abuse of notation, we use the same notation,

$$\Delta_R^\zeta : T\zeta \to T\zeta \wedge X_+ \quad \text{and} \quad \Delta_L^\zeta : T\zeta \to X_+ \wedge T\zeta.$$

Now let $\mathbb{E}$ be a commutative ring spectrum representing the generalized cohomology theory E^*.

Exercises.

1. Define a cross product map

$$E^q(\Sigma^\infty(T\zeta)) \times E^s(X) \to E^{q+s}(\Sigma^\infty(T\zeta \wedge X_+)).$$

2. Using the map of Thom spaces Δ_R^ζ, show that there is an induced pairing

$$E^q(\Sigma^\infty(T\zeta)) \times E^s(X) \xrightarrow{\mu} E^{q+s}(\Sigma^\infty(T\zeta))$$

that gives $E^*(\Sigma^\infty(T\zeta))$ the structure of right module over the graded ring $E^*(X)$. Similarly $E^*(\Sigma^\infty(T\zeta))$ is a left module over $E^*(X)$ using the map Δ_L^ζ.

3. Show that if ζ is the trivial zero dimensional bundle over X, i.e $\zeta = X \xrightarrow{p=id} X$, then $T\zeta = X_+$ and the above module structures are the cup product structure in $E^*(X)$.

There are other constructions from ordinary cohomology theory that also have analogues in generalized cohomology. The evaluation map of cohomology on homology, and more generally, the cap product maps are important examples.

As above let $\mathbb{E}$ be a ring spectrum representing the generalized cohomology theory E^* and the generalized homology theory E_*. Let X be a space (of the homotopy type of a CW-complex) and consider classes $\theta \in E_q(X) = \pi_q(X_+ \wedge \mathbb{E})$ and $\alpha \in E^q(X) = [X, \Sigma^q\mathbb{E}]$, which we take to mean weak homotopy classes of maps.

Definition 10.18. *Let $\psi_\theta : S^q \to X_+ \wedge \mathbb{E}$ and $\phi_\alpha : \Sigma^\infty(X_+) \to \Sigma^q\mathbb{E}$ be maps representing the classes $\theta \in E_q(X)$ and $\alpha \in E^q(X)$, respectively, The evaluation class $\langle \alpha, \theta \rangle \in \pi_0\mathbb{E}$ is defined to be the class represented by the composition*

$$S^q \xrightarrow{\psi_\theta} X_+ \wedge \mathbb{E} \xrightarrow{\phi_\alpha \wedge 1} \Sigma^q\mathbb{E} \wedge \mathbb{E}. \xrightarrow{\mu} \Sigma^q\mathbb{E}$$

where $\mu : \mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$ is the ring multiplication.

Exercise. Show that the evaluation pairing defines a ring homomorphism

$$E^*(X) \to Hom(E_*(X), \pi_0(\mathbb{E})).$$

Like in ordinary (co)homology theory, this evaluation pairing extends to define a "cap product" in generalized cohomology theories represented by ring spectra.

Definition 10.19. *Let $\mathbb{E}$ and X be as above. Suppose $\theta \in E_q(X)$ is represented by $\psi_\theta : S^q \to X_+ \wedge \mathbb{E}$, and $\beta \in E^r(X)$ is represented by $\phi_\beta : \Sigma^\infty(X_+) \to \Sigma^r\mathbb{E}$. We define the cap product $\theta \cap \beta \in E_{q-r}(X) = \pi_{q-r}(X_+ \wedge \mathbb{E})$ to be the class represented by the composition*

$$\psi_{\theta\cap\beta} : S^q \xrightarrow{\psi_\theta} X_+ \wedge \mathbb{E} \xrightarrow{\Delta\wedge 1} X_+ \wedge X_+ \wedge \mathbb{E} \xrightarrow{1\wedge\phi_\beta\wedge 1} X_+ \wedge \Sigma^r\mathbb{E} \wedge \mathbb{E} \xrightarrow{1\wedge\mu} X_+ \wedge \Sigma^r\mathbb{E}.$$

Note. This cap product pairing is a map $\cap : E_q(X) \times E^r(X) \to E_{q-r}(X)$. We leave it to the reader to verify that this definition also applies to give slightly more general pairings (compare (1.6))

$$E_q(X, A) \times E^r(X) \xrightarrow{\cap} E_{q-r}(X, A) \quad \text{and} \quad E_q(X, A) \times E^r(X, A) \xrightarrow{\cap} E_{q-r}(X).$$

10.6 Thom spectra

Our next goal is to apply generalized cohomology theory to the study of manifolds, and in particular to prove a generalized form of Poincaré duality with respect to a generalized cohomology theory E^*. As we recall, the notion of orientations played a crucial role in Poincaré duality for usual (co)homology. So we need to study the notion of orientations with respect to generalized cohomology theories. For this we will begin by generalizing the notion of Thom spaces, to *"Thom spectra"*.

Let $\zeta \to X$ be a k-dimensional vector bundle over a finite CW-complex. As before, let $T\zeta$ be it's Thom space. Let $\epsilon^m \to X$ be an m-dimensional trivial bundle, $\epsilon^m = X \times \mathbb{R}^m$. Observe, as we have earlier, that

$$T\epsilon^m = (X \times D^k)/(X \times S^{k-1}) \cong \Sigma^m(X_+).$$

Consider the Whitney sum bundle $\zeta \oplus \epsilon^m \to X$.

Exercise. Prove that there is a natural homeomorphism,

$$T(\zeta \oplus \epsilon^m) \cong \Sigma^m T\zeta.$$

Given the result of this exercise we can think of the suspension spectrum $\Sigma^\infty(T\zeta)$ as having its m^{th}-space equal to $T(\zeta \oplus \epsilon^m) = T(\zeta \times \mathbb{R}^m)$.

Exercise. Show that the natural symmetric spectrum structure on the suspension spectrum can be described in terms of the symmetric groups Σ_m acting on $\mathbb{R}^m$ by permuting the coordinates.

The above observation tells us that we have a natural equivalence of spectra,

$$\Sigma^m \Sigma^\infty(T\zeta) \simeq \Sigma^\infty(T(\zeta \oplus \epsilon^m)).$$

Suppose the k-dimensional bundle ζ is classified by a map

$$f_\zeta : X \to BO(k).$$

Then the above observations say that $\Sigma^m \Sigma^\infty(T\zeta)$ is the suspension spectrum of the Thom space of the $(k+m)$-dimensional vector bundle $\zeta \oplus \epsilon^m$ which is represented by the composition

$$f_{\zeta\oplus\epsilon^m} : X \xrightarrow{f_\zeta} BO(k) \to BO(k+m).$$

By allowing m to be negative in the above discussion, we are motivated to define the following.

Definition 10.20. *As above, let $\zeta \to X$ be a k dimensional vector bundle over a finite CW-complex X, classified by a map*

$$f_\zeta : X \to BO(k).$$

The Thom spectrum, which we denote using the exponential notation X^ζ, is defined to be the k-fold desuspension of the suspension spectrum of the Thom space,

$$X^\zeta = \Sigma^{-k}\Sigma^\infty(T\zeta).$$

We say that X^ζ is the Thom spectrum of the virtual zero-dimensional bundle $\zeta - \epsilon^k$ classified by the map

$$\phi_\zeta : X \to BO$$

defined to be the composition $\phi_\zeta : X \xrightarrow{f_\zeta} BO(k) \to BO$.

Let us consider Thom spectra from a different perspective. Suppose X is a finite CW-complex, and we are given a map

$$f : X \to BO.$$

The question we would like to now address is the following:

Question. Can we define a Thom spectrum X^f associated to the map $f : X \to BO$?

Notice that if we were given a factorization of f through a finite $BO(k)$, i.e a map $f_k : X \to BO(k)$ such that the composition $X \xrightarrow{f_k} BO(k) \to BO$ is homotopic to f, then, of course, f_k classifies a k-dimensional vector bundle ζ_{f_k}, and we can define the Thom spectrum X^f to be the Thom spectrum of this factorization,

$$X^f = X^{\zeta_{f_k}} = \Sigma^{-k}\Sigma^\infty(T(\zeta_{f_k})). \tag{10.10}$$

But is this spectrum, or at least its homotopy type, independent of the choice of factorization?

To address this, suppose $\tilde{f}_q : X \to BO(q)$ is another factorization of f. (The integer q may or not be the same as k.)

Proposition 10.27. *The spectra* $X^{\zeta_{f_k}} = \Sigma^{-k}\Sigma^{\infty}(T(\zeta_{f_k}))$ *and* $X^{\zeta_{\tilde{f}_q}} = \Sigma^{-q}\Sigma^{\infty}(T(\zeta_{\tilde{f}_q}))$ *are (weakly) homotopy equivalent.*

Proof. Since the two compositions

$$X \xrightarrow{f_k} BO(k) \to BO \quad \text{and} \quad X \xrightarrow{\tilde{f}_q} BO(q) \to BO$$

are both homotopic to $f : X \to BO$, they are therefore homotopic to each other. Let $X^{(m)}$ be any finite subcomplex of X. Then there must be a finite N larger than both k and q such that the compositions

$$f_k^N : X^{(m)} \xrightarrow{f_k} BO(k) \to BO(N) \quad \text{and} \quad \tilde{f}_q^N : X^{(m)} \xrightarrow{\tilde{f}_q} BO(q) \to BO(N)$$

are homotopic. Therefore they classify isomorphic N-dimensional vector bundles over $X^{(m)}$, and so have homotopy equivalent Thom spaces. Notice that the bundle classified by f_k^N is equal to $\zeta_{f_k} \oplus \epsilon^{N-k}$, whereas the bundle classified by $\tilde{f}_q^N$ is equal to $\zeta_{\tilde{f}_q} \oplus \epsilon^{N-q}$. We therefore have a bundle isomorphism over $X^{(m)}$

$$\zeta_{f_k} \oplus \epsilon^{N-k} \cong \zeta_{\tilde{f}_q} \oplus \epsilon^{N-q}.$$

On the level of Thom spaces we have a homotopy equivalence

$$\begin{aligned} T(\zeta_{f_k} \oplus \epsilon^{N-k}) &\simeq T(\zeta_{\tilde{f}_q} \oplus \epsilon^{N-q}) \\ \Sigma^{N-k} T(\zeta_{f_k}) &\simeq \Sigma^{N-q} T(\zeta_{\tilde{f}_q}). \end{aligned}$$

We therefore have an equivalence of spectra,

$$\begin{aligned} (X^{(m)})^{\zeta_{f_k}} &= \Sigma^{-k}\Sigma^{\infty}(T(\zeta_{f_k})) = \Sigma^{-N}\Sigma^{N-k}\Sigma^{\infty}(T(\zeta_{f_k})) = \Sigma^{-N}\Sigma^{\infty}\Sigma^{N-k}T(\zeta_{f_k})) \\ &\simeq \Sigma^{-N}\Sigma^{\infty}\Sigma^{N-q}T(\zeta_{\tilde{f}_q}) \simeq \Sigma^{-N}\Sigma^{N-q}\Sigma^{\infty}T(\zeta_{\tilde{f}_q}) \\ &= \Sigma^{-q}\Sigma^{\infty}T(\zeta_{\tilde{f}_q}) = (X^{(m)})^{\zeta_{\tilde{f}_q}} \end{aligned}$$

□

Since this equivalence is true for any subcomplex $X^{(m)}$ of X, we can conclude that $X^{\zeta_{f_k}}$ and $X^{\zeta_{\tilde{f}_q}}$ have the same weak homotopy type.

The following exercise is proved in a similar manner.

Exercise. Let $f : X \to BO$ and $g : Y \to BO$ be maps where X and Y are finite CW complexes. Suppose there is a map $\phi : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ BO & \xrightarrow[=]{} & BO \end{array}$$

Show that there is an induced map on the level of Thom spectra

$$T\phi : X^f \to Y^g$$

that is well defined up to weak homotopy.

At this point we have associated to every map $f : X \to BO$, where X is a finite CW-complex, a Thom spectrum X^f, which is well-defined up to homotopy. And to every "map over BO", that is a map $\phi : X \to Y$ respecting maps $f : X \to BO$ and $g : Y \to BO$ as in the exercise, we have associated a map of Thom spectra, $T\phi : X^f \to X^g$, which again, is well-defined up to homotopy. This suggests that there might be a functoriality result where we can remove the "up-to-homotopy" restriction. Indeed there is such a result, and it is fairly recent. In order to describe it, we first consider Thom spectra for maps $f : X \to BO$ where X is a CW-complex that is not necessarily finite. Consider the skeletal filtration of X:

$$X^0 \hookrightarrow X^{(1)} \hookrightarrow \cdots \hookrightarrow X^{(k-1)} \hookrightarrow X^{(k)} \hookrightarrow \cdots \hookrightarrow X.$$

We first observe the following:

Theorem 10.28. *Any map $f : X \to BO$ is homotopic to one which takes the k^{th}-skeleton $X^{(k)}$ to $BO(k)$. That is, there is a commutative diagram*

$$\begin{array}{ccccccccccc}
X^{(0)} & \xrightarrow{\hookrightarrow} & \cdots & \xrightarrow{\hookrightarrow} & X^{(k)} & \xrightarrow{\hookrightarrow} & X^{(k+1)} & \xrightarrow{\hookrightarrow} & \cdots & \xrightarrow{\hookrightarrow} & X \\
{\scriptstyle f^{(0)}}\downarrow & & & & {\scriptstyle f^{(k)}}\downarrow & & \downarrow{\scriptstyle f^{(k+1)}} & & & & \downarrow{\scriptstyle f} \\
BO(0) & \xrightarrow[\hookrightarrow]{} & \cdots & \xrightarrow[\hookrightarrow]{} & BO(k) & \xrightarrow[\hookrightarrow]{} & BO(k+1) & \xrightarrow[\hookrightarrow]{} & \cdots & \xrightarrow[\hookrightarrow]{} & BO
\end{array}$$

Proof. This follows from obstruction theory and in particular Theorem 4.11 and Proposition 5.14.

□

Consider a skeletal filtration preserving map $f : X \to BO$ as in the statement of this theorem. Let $\zeta^{(k)} \to X^{(k)}$ be the k-dimensional vector bundle classified by $f^{(k)} : X^{(k)} \to BO(k)$. Consider the composition

$$X^{(k)} \xrightarrow{f^{(k)}} BO(k) \hookrightarrow BO(k+1).$$

This classifies the $(k+1)$-dimensional bundle $\zeta^{(k)} \oplus \epsilon^1$ over $X^{(k)}$. The Thom space of this bundle is the suspension

$$T(\zeta^{(k)} \oplus \epsilon^1) = \Sigma T(\zeta^{(k)}).$$

Now by the commutativity of the diagram in this theorem we have maps of vector bundles,

$$\begin{array}{ccc} \zeta^{(k)} \oplus \epsilon^1 & \longrightarrow & \zeta^{(k+1)} \\ \downarrow & & \downarrow \\ X^{(k)} & \xrightarrow[\hookrightarrow]{} & X^{(k+1)} \end{array}$$

and hence we have a map of Thom spaces that we call

$$\epsilon_k : \Sigma T(\zeta^{(k)}) \to T(\zeta^{(k+1)}).$$

We can then make the following definition:

Definition 10.21. *Given the above situation we define the spectrum*

$$X^f = \{T(\zeta^{(k)}); \quad \epsilon_k : \Sigma T(\zeta^{(k)}) \to T(\zeta^{(k+1)})\}.$$

Exercises.

1. Show that the weak homotopy type of X^f is well defined. That is, it does not depend on the choices of homotoping $f : X \to BO$ into a skeletal filtration preserving map, as in the statement of Theorem 10.28.

2. Give the Thom spectrum X^f the structure of a symmetric spectrum.

3. Suppose X is a finite CW-complex and $f : X \to BO$ is given by a factorization $X \xrightarrow{f_k} BO(k) \to BO$, and that the map f_k classifies the k-dimensional vector bundle

$$\zeta^k \to X.$$

Show that the definition of the Thom spectrum given above (10.10)

$$X^f = \Sigma^{-k}\Sigma^{\infty}(T(\zeta^k))$$

agrees, up to homotopy, with Definition 10.21.

4. Suppose $f : X \to BO$ and $g : Y \to BO$ are maps from (not-necessarily-finite) CW complexes. Suppose furthermore that

$$\phi : X \to Y$$

is a map making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ f\downarrow & & \downarrow g \\ BO & \xrightarrow[=]{} & BO. \end{array}$$

Define an induced map of Thom spectra $T\phi : X^f \to Y^g$ that extends the definition given in the previous exercise set when X and Y are assumed to be finite.

Examples.

1. Consider the identity map $id : BO \to BO$. Then the construction above defines a Thom spectrum BO^{id} for which we use the standard notation $\mathbb{MO}$. If $MO(n)$ denotes the Thom space of the universal bundle over $BO(n)$, then as a spectrum $\mathbb{MO}$ is made up of the spaces $MO(n)$ together with the structure maps $\epsilon_n : \Sigma MO(n) \to MO(n+1)$, defined as in Definition 10.21.

2. The inclusion of the unitary group into the orthogonal group $U(n) \hookrightarrow O(2n)$ defines a map on classifying spaces $\gamma_n : BU(n) \to BO(2n)$. On the level of bundles it takes an n-dimensional complex vector bundle, forgets its complex structure and views it as a $2n$-dimensional real vector bundle. The maps γ_n fit together to define a map

$$\gamma : BU \to BO.$$

   It has a corresponding Thom spectrum which we denote by $\mathbb{MU}$.

The Thom spectra $\mathbb{MO}$ and $\mathbb{MU}$ were originally introduced by R. Thom in [150] and play an essential role in cobordism theory which we will see in the next chapter.

We now consider certain multiplicative properties of Thom spectra. First suppose that $\zeta^k \to X$ and $\xi^q \to Y$ are k and q dimensional vector bundles, respectively, where the base spaces are CW complexes. Let $f_\zeta : X \to BO(k)$ and $g_\xi : Y \to BO(q)$ be classifying maps for these bundles. One can consider the external product bundle

$$\zeta^k \times \xi^q \to X \times Y.$$

As we've observed before, this $(k+q)$-dimensional vector bundle is classified by the composition map

$$X \times Y \xrightarrow{f_\zeta \times g_\xi} BO(k) \times BO(q) \xrightarrow{\mu_{k,q}} BO(k+q)$$

where $\mu_{k,q} : BO(k) \times BO(q) \to BO(k+q)$ is the "Whitney sum" pairing induced by the "block addition" homomorphism

$$O(k) \times O(q) \to O(k+q)$$

given by sending a $k \times k$ matrix A and a $q \times q$ matrix B to the $(k+q) \times (k+q)$ matrix that has A in the upper left $k \times k$ block, B in the lower right $q \times q$ block, and zero's elsewhere.

The following is simply a parameterized form of the fact that

$$(D^k \times D^q)/\partial(D^k \times D^q) \cong D^k/\partial D^k \wedge D^q/\partial D^q.$$

We leave its proof to the reader.

Proposition 10.29. *The Thom space of $\zeta^k \times \xi^q$ is given by*

$$T(\zeta^k \times \xi^q) \cong T(\zeta^k) \wedge T(\xi^q).$$

The pairing maps $\mu_{k,q} : BO(k) \times BO(q) \to BO(k+q)$ fit together to give a pairing

$$\mu : BO \times BO \to BO \tag{10.11}$$

This can be verified directly, which we encourage the reader to do. This pairing also follows as a consequence of (real) Bott periodicity. Recall that (complex) Bott periodicity says that

$$\mathbb{Z} \times BU \simeq \Omega U \quad \text{and, of course} \quad U \simeq \Omega BU = \Omega(\mathbb{Z} \times BU)$$

which implies the two-fold periodicity

$$\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU).$$

In the case of BO, in [14] Bott proved that there is an eight-fold periodicity

$$\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO). \tag{10.12}$$

Indeed Bott showed that

$$\mathbb{Z} \times BO \simeq \Omega(U/O) \tag{10.13}$$

where $U/O = \varinjlim_n U(n)/O(n)$. Here $O(n) \subset U(n)$ is the subspace of all unitary matrices with the property that all of their entries are real (i.e have zero imaginary parts).

Using Moore loops, the loop space $\Omega(U/O)$ has the structure of an associative monoid. This gives a homotopy theoretic model of $\mathbb{Z} \times BO$ with the structure of an associative monoid. We call this product

$$\mu : (\mathbb{Z} \times BO) \times (\mathbb{Z} \times BO) \to \mathbb{Z} \times BO.$$

It restricts on components to give pairings

$$\mu_{m,n} : (\{m\} \times BO) \times (\{n\} \times BO) \to \{m+n\} \times BO.$$

In particular it defines a monoid structure on $BO = \{0\} \times BO \hookrightarrow \mathbb{Z} \times BO$. This monoid structure corresponds, up to homotopy, with the Whitney sum pairings $\mu_{k,q} : BO(k) \times BO(q) \to BO(k+q)$ described above, and hence the (abuse of) notation. BU has a similar monoid structure. We refer the reader to [105] for a much more complete discussion of these structures.

Corollary 10.30. *. If $f : X \to BO$ and $g : Y \to BO$ are maps from CW-complexes to BO, consider the composition,*

$$f \cdot g : X \times Y \xrightarrow{f \times g} BO \times BO \xrightarrow{\mu} BO.$$

Then there is an equivalence of Thom spectra

$$(X \times Y)^{f \cdot g} \simeq X^f \wedge X^g.$$

We now define a category $\mathcal{C}_{BO}$ of "spaces over BO". The objects of $\mathcal{C}_{BO}$ are maps from CW-complexes, $f : X \to BO$, and a morphisms between objects $f : X \to BO$ and $g : Y \to BO$ are maps $\phi : X \to Y$ making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ BO & \xrightarrow[=]{} & BO. \end{array}$$

Since BO is a monoid, the category $\mathcal{C}_{BO}$ inherits a monoidal structure. In particular the product of two objects $f : X \to BO$ and $g : Y \to BO$ is the composition

$$f \times g : X \times Y \xrightarrow{f \times g} BO \times BO \xrightarrow{\mu} BO.$$

We leave it to the reader to check that $\mathcal{C}_{BO}$ satisfies the properties of being a monoidal category. (This is true of the category $\mathcal{C}_M$ of spaces over any monoid M.)

Notice that a monoid in the category $\mathcal{C}_{BO}$ is an object $f : X \to BO$ together with a monoid structure on X, $\nu : X \times X \to X$ that lives above BO. That is, the following diagram commutes:

$$\begin{array}{ccc} X \times X & \xrightarrow{\nu} & X \\ {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle f} \\ BO \times BO & \xrightarrow[\mu]{} & BO \end{array}$$

If $f : X \to BO$ is a monoid in $\mathcal{C}_{BO}$, which we refer to as a "monoid over BO", then the commutativity of this diagram and Corollary 10.30 implies the following.

Proposition 10.31. *If $f : X \to BO$ is a monoid over BO with monoid product $\nu : X \times X \to X$, then there is a map of spectra $T\nu$ which is well-defined up to homotopy,*

$$T\nu : X^f \wedge X^f \to X^f.$$

Furthermore this map is associative up to homotopy, and there exists a "unit map" $u : \mathbb{S} \to X^f$ so that the compositions

$$X^f = X^f \wedge \mathbb{S} \xrightarrow{1 \wedge u} X^f \wedge X^f \xrightarrow{T\nu}, \text{ and}$$
$$X^f = \mathbb{S} \wedge X^f \xrightarrow{u \wedge 1} X^f \wedge X^f \xrightarrow{T\nu} \text{ and}$$

are homotopic to the identity map. In other words, X^f is a "ring spectrum up to homotopy."

Note. The unit map $u : \mathbb{S} \to X^f$ is the map of Thom spectra induced by the inclusion of the basepoint (i.e the unit of the monoid), $x_0 \hookrightarrow X$.

This proposition suggests that there might be a functor $Th : \mathcal{C}_{BO} \to \mathbf{Sp}^{\mathbf{\Sigma}}$ that assigns to a space over BO, $f : X \to BO$, its associated Thom spectrum, X^f. Furthermore this functor should preserves products. That is, it should send a monoid over BO to a ring spectrum. This proposition says that this can be done "up to homotopy". But in recent years it has been proven that one indeed can define a "Thom functor" that preserves this monoidal structure. Equivalent forms of the following result were proved in [91], [1], and [13].

Theorem 10.32. *[91], [1], [13] There is a monoidal functor*

$$Th : \mathcal{C}_{BO} \to \mathbf{Sp}^{\mathbf{\Sigma}}$$

that takes an object $f : X \to BO$ to its Thom spectrum X^f.

The following is a more descriptive way of stating this result.

Corollary 10.33. *If $f : X \to BO$ is a monoid over BO, its Thom spectrum X^f is a ring spectrum. If $Y \to BO$ is another monoid over BO and $\phi : X \to Y$ is a morphism in $\mathcal{C}_{BO}$ that preserves the monoid structures, then the induced map on the level of Thom spectra,*

$$T\phi : X^f \to Y^g$$

is a map of ring spectra.

Examples

$\mathbb{MO}$, the Thom spectrum of the identity map $id : BO \to BO$, is a ring spectrum, as is the Thom spectrum $\mathbb{MU}$ of the canonical map $BU \to BO$.

Note. These spectra are essential to the study of cobordisms of manifolds, as we will see in the next chapter. We will also see that their ring spectrum structures are crucial for being able to do cobordism calculations.

10.7 The ring structure of $H_*(BO; \mathbb{Z}/2)$, $H_*(BU; \mathbb{Z})$, $H_*(\mathbb{MO}; \mathbb{Z}/2)$, and $H_*(\mathbb{MU}; \mathbb{Z})$

In the previous section we observed that as a result of Bott periodicity, BO and BU are infinite loop spaces. This in particular means they have homotopy commutative product maps

$$BO \times BO \to BO \quad \text{and} \quad BU \times BU \to BU.$$

This implies that the homologies $H_*(BO;\mathbb{Z}/2)$ and $H_*(BU;\mathbb{Z})$ are graded commutative rings.

We also saw that the Thom spectra $\mathbb{MO}$ and $\mathbb{MU}$ have the induced structure of homotopy commutative ring spectra. This implies that their homologies $H_*(\mathbb{MO};\mathbb{Z}/2)$ and $H_*(\mathbb{MU};\mathbb{Z})$ also are graded commutative rings. The goal of this section is to compute these rings.

We begin with a calculation of $H_*(BO;\mathbb{Z}/2)$.

Theorem 10.34. *There is an isomorphism of graded algebras,*

$$H_*(BO;\mathbb{Z}/2) \cong \mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty;\mathbb{Z}/2)] = \mathbb{Z}/2[a_1, a_2, \cdots, \text{ where} |a_i| = i]$$

Proof. For ease of notation we leave off the coefficients in (co)homology. All coefficients will be $\mathbb{Z}/2$.

Recall that the "Splitting Principle" (Theorem 6.20) says that the product map

$$\mu : BO(1)^{\times m} \to BO(m)$$

induces a monomorphism in cohomology, $\mu^* : H^*(BO(m)) \to H^*(BO(1))^{\otimes m}$, or equivalently, the map in homology, $\mu_* : H_*(BO(1))^{\otimes m} \to H_*(BO(m))$ is surjective.

Using the facts that $BO(1) = \mathbb{RP}^\infty$ and that $\iota_* : H_q(BO(m)) \to H_q(BO)$ is an isomorphism through dimension m (see Theorem 6.15), we can conclude that

$$\mu_* : H_*(\mathbb{RP}^\infty)^{\otimes m} \to H_*(BO)$$

is surjective through dimension m. Since μ_* induces the product structure in $H_*(BO)$, this says that in the algebra structure, every element in $H_q(BO)$ can be written as a linear combination of monomials in the image of $\tilde{H}_*(\mathbb{RP}^\infty)$ of length $\leq m$ for $q \leq m$. Since $H_*(BO)$ is a commutative algebra, this says that μ_* induces a surjective map of algebras,

$$\mu_* : \mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty)] \to H_*(BO).$$

Now since $\mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty)] = \mathbb{Z}/2[a_i,\, i \geq 1 \,:\, |a_i| = i]$ and from Theorem 6.15 we know that the cohomology, $H^*(BO) \cong \mathbb{Z}/2[w_i,\, i \geq 1, : |w_i| = i]$, we can conclude that as $\mathbb{Z}/2$ vector spaces, $\mathbb{Z}/2[\tilde{H}_*(\mathbb{RP}^\infty)]$ and $H_*(BO)$ have the same rank in each dimension. Therefore μ_*, being a surjective map of algebras over $\mathbb{Z}/2$ that have the same rank in every dimension, must be an isomorphism. □

Exercise. Show, using an argument like above, that

$$H_*(BU;\mathbb{Z}) \cong \mathbb{Z}[\tilde{H}_*(\mathbb{CP}^\infty;\mathbb{Z})] = \mathbb{Z}[u_i,\, i \geq 1 \,:\, |u_i| = 2i].$$

We now discuss the homology of the corresponding Thom spectra, $H_*(\mathbb{MO};\mathbb{Z}/2)$ and $H_*(\mathbb{MU};\mathbb{Z})$.

We continue with the notational convention that if we do not put coefficients in (co)homology, we mean the coefficients are $\mathbb{Z}/2$. Recall the Thom isomorphism,

$$\cup u_k : H^*(BO(k)) \xrightarrow{\cong} H^{*+k}(MO(k)).$$

Now consider the image of the Thom class $u_k \in H^k(MO(k))$ in $H^{k-1}(MO(k-1))$ under the composition

$$H^k(MO(k)) \xrightarrow{\epsilon_{k-1}^*} H^k(\Sigma MO(k-1)) \xrightarrow{\cong} H^{k-1}(MO(k-1))$$

where $\epsilon_{k-1} : \Sigma MO(k-1) \to MO(k)$ is the structure map of the spectrum $\mathbb{MO}$, and the second map in this composition is the suspension isomorphism. Since the structure map ϵ_{k_1} is the map induced on Thom spaces by the map $BO(k-1) \to BO(k)$, then the Thom classes are preserved. That is, the image of u_k under this composition is u_{k-1}. Therefore the Thom classes fit together to define a zero dimensional cohomology class in the spectrum,

$$u \in H^0(\mathbb{MO}),$$

and so the Thom isomorphism can be viewed as an isomorphism between the cohomology of the base space BO and that of the spectrum $\mathbb{MO}$:

$$\cup u : H^*(BO) \xrightarrow{\cong} H^*(\mathbb{MO}). \tag{10.14}$$

Notice that when viewed in this way the Thom isomorphism does not shift degrees.

Now recall that the dual of the cup product with the Thom class in cohomology, is taking the cap product with the Thom class,

$$\cap u_k : H_*(MO(k)) \xrightarrow{\cong} H_{*-k}(BO(k)).$$

On the spectrum level the dual of the Thom isomorphism 10.14 can therefore be written

$$\cap u : H_*(\mathbb{MO}) \xrightarrow{\cong} H_*(BO). \tag{10.15}$$

Again, from this perspective there is no dimension shift.

Lemma 10.35. *Taking the cap product with the Thom class*

$$\cap u : H_*(\mathbb{MO}) \xrightarrow{\cong} H_*(BO)$$

is an isomorphism of graded rings.

Proof. Since we know that $\cap u$ is an isomorphism, we need only show that it preserves the product structure.

As discussed earlier, the product structure on $H_*(BO)$ is induced by the product maps $m_{k,r} : BO(k) \times BO(r) \to BO(k+r)$ and the product structure on $H_*(\mathbb{MO})$ is induced by the ring spectrum structure on $\mathbb{MO}$, which in turn is induced by the maps of Thom spaces $\mu_{k,r} : MO(k) \wedge MO(r) \to MO(k+r)$ induced by the maps $m : BO(k) \times BO(r) \to BO(k+r)$. In particular this means that the homomorphisms $\mu^*_{k,r}$ preserve Thom classes. That is,

$$\mu^*_{k,r}(u_{k+r}) = u_k \otimes u_r \in \tilde{H}^*(MO(k) \wedge MO(r)) = \tilde{H}^*(MO(k)) \otimes \tilde{H}^*(MO(r)).$$

Because of the relationship between cup and cap product, this means that for every $\alpha \in H_q(MO(k))$ and $\beta \in H_s(MO(r))$, we have

$$u_{k+r} \cap (\mu_{k,r})_*(\alpha \otimes \beta) = m_*((u_k \cap \alpha) \otimes (u_r \otimes \beta).$$

Translated to the spectrum level, this is exactly the statement that $\cap u : H_*(\mathbb{MO}) \xrightarrow{\cong} H_*(BO)$ is a ring homomorphism. □

Let $\mathbb{MO}(k)$ be the spectrum $\Sigma^{-k}\Sigma^*(MO(k))$. Note that there are maps of spectra. $\mathbb{MO}(1) \to \mathbb{MO}(2) \to \cdots \mathbb{MO}(k) \to \cdots \mathbb{MO}$. Moreover, using these spectra, the Thom isomorphisms do not shift degrees:

$$\cup u_k : H^*(BO(k)) \xrightarrow{\cong} H^*(\mathbb{MO}(k)).$$

From this lemma and Theorem 10.34 we can conclude the following.

Theorem 10.36. *There is an isomorphism of graded algebras,*

$$H_*(\mathbb{MO}) \cong \mathbb{Z}/2[H_*(\mathbb{MO}(1))] = \mathbb{Z}/2[e_1, e_2, \cdots, \textit{ where } |e_i| = i].$$

Exercise. Adapt the above arguments to prove the following:

Theorem 10.37. *There is an isomorphism of graded algebras,*

$$H_*(\mathbb{MU}; \mathbb{Z}) \cong \mathbb{Z}[H_*(\mathbb{MU}(1); \mathbb{Z})] = \mathbb{Z}[t_1, t_2, \cdots, \textit{ where } |t_i| = 2i].$$

10.8 Generalized orientations, the generalized Thom isomorphism, and the generalized Poincaré and Alexander duality theorems

As we saw in Chapter 1, orientations are a crucial property for studying Poincaré duality for manifolds. For a commutative ring R we described the notion of R-orientability of a manifold (Definition 1.4), and in particular we proved that if a manifold is $\mathbb{Z}$-orientable, then it is R-orientable for any commutative ring R. We also proved that if a closed topological manifold is R-orientable, it satisfies Poincaré duality with respect to (co)homology with R-coefficients.

In this subsection we verify the analogues of these results with respect to *generalized* (co)homology theories. In particular we define the notion of orientability of a manifold with respect to a generalized (co)homology theory, when the representing spectrum for that theory is a ring spectrum. We prove that if a manifold is orientable with respect to stable (co)homotopy, the generalized theories represented by the sphere spectrum $\mathbb{S}$, then the manifold is orientable with respect to any generalized (co)homology theory $\mathbb{E}$ represented by a ring spectrum. We then prove the appropriate version of Poincaré duality for $\mathbb{E}$-oriented manifolds. All of these results were originally proved by G. Whitehead in [158].

10.8.1 Orientations

Recall from Chapter 1, Definition 1.2, that if M^n is a (topological) manifold, a local orientation of M^n at $x \in M^n$ is a choice of generator of

$$\begin{aligned} H_n(M^n, M^n - \{x\}) &\cong H_n(U_x, U_x - x) \\ &\cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \\ &\cong H_n(S^n, point) \\ &\cong \mathbb{Z} \end{aligned}$$

Here U_x is an open neighborhood (chart) of x, homeomorphic to $\mathbb{R}^n$.

Now let E_* be a generalized homology theory represented by a ring spectrum $\mathbb{E}$. To make an analogous definition of local E_*-orientation, we need to consider

$$\begin{aligned} E_n(S^n, point) &= \pi_n(S^n \wedge \mathbb{E}) \\ &\cong \pi_0(S^0 \wedge \mathbb{E}) = \pi_0(\mathbb{E}) \quad \text{by the suspension isomorphism.} \end{aligned}$$

Now since $\mathbb{E}$ is a ring spectrum, $\pi_*(\mathbb{E})$ is a graded ring, and $\pi_0(\mathbb{E})$ is a (nongraded) subring. This leads us to the following more general definition.

Definition 10.22. *Let E_* be a generalized homology theory represented by a ring spectrum $\mathbb{E}$, and let M^n be a topological manifold. Then an E_*-local orientation (equivalently referred to as an $\mathbb{E}$-local orientation) of M^n at $x \in M^n$ is a choice of unit (generator) in the ring $E_n(M^n, M^n - \{x\}) \cong E_n(S^n, point) \cong \pi_0(\mathbb{E})$.*

Now recall from the observation after the statement of Theorem 1.3 in Chapter 1 that if M^n is a closed, connected manifold, it has a global orientation if an only if there is a "fundamental class" $[M^n] \in H_n(M^n; \mathbb{Z})$ whose image in $H_n(M^n, M^n - \{x\})$ defines a local orientation for every $x \in M^n$ (i.e is a generator of $H_n(M^n, M^n - \{x\})$ for every $x \in M^n$). This leads to the following definition.

Definition 10.23. *Let E_* be a generalized homology theory represented by a connective ring spectrum $\mathbb{E}$ (remember this means that $\pi_q(\mathbb{E}) = 0$ for $q < 0$), and let M^n be a closed topological manifold. Then M^n is E_*-orientable (or equivalently, $\mathbb{E}$-orientable) if there is a class $[M^n]_E \in E_n(M^n)$ so that the restriction to $E_n(M^n, M^n - \{x\}) \cong \pi_0(\mathbb{E})$ defines a local E_*-orientation for every $x \in M^n$ (i.e is a unit of $E_n(M^n, M^n - \{x\}) \cong \pi_0(\mathbb{E})$).*

We note that this condition can be described more homotopy theoretically in the following way. A class $[M^n]_E \in E_n(M^n)$ is represented by a map

$$\zeta_M : S^n \to M^n_+ \wedge \mathbb{E}.$$

For $x \in M^n$, let U_x be an open neighborhood of x homeomorphic to $\mathbb{R}^n$ as above, and consider the projection map

$$p_x : M^n \to M^n/M^n - U_x \cong D^n/\partial D^n = S^n$$

where D^n is the closed n-dimensional disk. Now consider the composition

$$\rho_x^\zeta : S^n \xrightarrow{\zeta_M} M^n_+ \wedge \mathbb{E} \xrightarrow{p_x \wedge 1} (M^n/M^n - U_x) \wedge \mathbb{E} \cong S^n \wedge \mathbb{E}.$$

This composition represents a class in $\pi_n(S^n \wedge \mathbb{E}) \cong \pi_0(\mathbb{E})$. So the condition of $\mathbb{E}$-orientability is that there is a class $[M^n]_E \in E_n(M^n)$ represented by a map $\zeta_M : S^n \to M^n_+ \wedge \mathbb{E}$ so that the induced map $\rho_x^\zeta : S^n \to S^n \wedge \mathbb{E}$ represents a unit in the ring $\pi_0(\mathbb{E})$ for every $x \in M^n$.

Using the language we used for ordinary homology, a class $[M^n]_E \in E_n(M^n)$ satisfying the above property is called an E_*-fundamental class (or $\mathbb{E}$-fundamental class) of M^n. It is also often referred to as an E_*-orientation class of M^n.

Exercises.

1. Show that if a closed manifold M^n is $\mathbb{S}$-orientable, then it is $\mathbb{E}$-orientable for any ring spectrum $\mathbb{E}$. In particular it is orientable with respect to integral homology.
2. Show that the sphere S^n is $\mathbb{S}$-orientable for every n.
3. Let $\mathbb{H}\mathbb{Z}$ be the integral Eilenberg-MacLane spectrum. Let $u : \mathbb{S} \to \mathbb{H}\mathbb{Z}$ be the unit. Notice that u induces a map in generalized homology theories, called the "stable Hurewicz homomorphism",

$$u_* : \pi_*^s(X) \to H_*(X; \mathbb{Z}).$$

Now let M^n be a closed, connected, $\mathbb{S}$-oriented n-dimensional manifold. Show that the stable Hurewicz homomorphism in dimension n,

$$u_* : \pi_n^s(M^n) \to H_n(M^n; \mathbb{Z})$$

is surjective.

4. Continuing to assume that M^n is a closed, connected, $\mathbb{S}$-oriented n-dimensional manifold, let $[M^n]_{\mathbb{S}} \in \pi_n^s(M^n)$ be a $\mathbb{S}$-fundamental class, represented by a map of spectra

$$[M^n]_{\mathbb{S}} : \Sigma^\infty S^n \to \Sigma^\infty(M^n_+).$$

Let $x_0 \in M^n$ be a basepoint, with an open neighborhood U_{x_0} homeomorphic to $\mathbb{R}^n$. Let $\tilde{M}^n$ be M^n punctured at x_0. That is, $\tilde{M}^n$ is the complement

$$\tilde{M}^n = M^n - U_{x_0}.$$

Show that there is a weak homotopy equivalence of suspension spectra

$$\phi : \Sigma^\infty((S^n \vee \tilde{M}^n) \xrightarrow{\simeq} \Sigma^\infty(M^n).$$

Hint. Use the result of the previous exercise.

We observe that the notion of E_*-orientability can also be described via an orientation covering space as was done for orientation with respect to an ordinary homology theory in chapter one. Namely, one can construct E_*-orientation covering space over any connected (not necessarily compact) n-manifold M^n

$$p : Or_{E_*}(M^n) \to M^n$$

where the fiber over a point $x \in M^n$ is the set of units in the ring $E_n(M^n, M^n - \{x\}) \cong \pi_0(\mathbb{E})$. Details of the construction are left to the reader.

A global E_*-orientation is then a section of the covering space $Or_{E_*}(M^n)$. Because its proof only relied on the Eilenberg-Steenrod axioms, we immediately have the following generalization of Theorem 1.3.

Theorem 10.38. *Let M^n be an n-manifold and $A \subset M^n$ a compact subspace. Let E_* be a generalized homology theory represented by a connective ring spectrum $\mathbb{E}$. Then if $\alpha : A \to Or_{E_*}(M^n)$ is a section of the orientation covering space over A, then there exists a unique homology class $\alpha_A \in E_n(M, M - A)$ whose image in $E_n(M, M - x)$ is $\alpha(x)$ for every $x \in A$.*

In particular if M^n is closed, then by taking $A = \emptyset$, this gives the equivalence of having a section of $Or_{E_*}(M^n)$ and the existence of a fundamental class $\alpha_\emptyset = [M^n]_{\mathbb{E}} \in E_n(M^n)$.

10.8.2 Poincaré and Alexander duality, and the Thom isomorphism for generalized (co)homology

Our goal is to use the notion of E_*-orientation and derive, like we did in chapter 1 for ordinary (co)homology, Poincaré duality for generalized (co)homology theories. Throughout this subsection we continue to assume that E_* is a generalized homology theory represented by a ring spectrum, $\mathbb{E}$.

Our first step is to understand the notion of generalized cohomology with compact supports. When we defined ordinary cohomology with compact supports in Chapter 1, we used cochains. For generalized cohomology we make use of mapping spectra.

Recall from Definition 10.14 that for symmetric spectra $\mathbb{X}$ and $\mathbb{Y}$ we have an associated morphism spectrum $Map(\mathbb{X}, \mathbb{Y})$. In the setting when $\mathbb{X}$ is the suspension spectrum of a space X, this has a particularly easy definition. Namely $Map(X, \mathbb{Y})_n = Map(X, \mathbb{Y}_n)$ with the obvious structure maps. (These mapping spaces consist of *basepoint preserving* maps.) Now given our ring spectrum $\mathbb{E}$, notice that the generalized cohomology group is given by

$$\begin{aligned} E^n(X) &= [X_+, \Sigma^n \mathbb{E}] \\ &= \pi_0(Map(X_+, \Sigma^n \mathbb{E})) \\ &= \pi_{-n}(Map(X_+, \mathbb{E})). \end{aligned} \tag{10.16}$$

For this reason we will choose to define generalized cohomology with compact supports using mapping spectra. In particular let M^n be an n-dimensional manifold, not necessarily compact. Let $K \subset M^n$ be a compact subspace. Notice that if Y is some other space, the mapping space $Map(M^n/M^n - K, Y)$ can be interpreted as the space of basepoint preserving maps from M^n to Y that map the complement of K to the basepoint of Y. Notice furthermore that if K_1 and K_2 are compact subspaces of M^n with $K_1 \subset K_2$, then $(M^n - K_2) \subset (M^n - K_1)$ we have an induced map, which is an inclusion,

$$Map(M^n/(M^n - K_1), Y) \to Map(M^n/(M^n - K_2), Y).$$

Definition 10.24. *We define the space of compactly supported maps, $Map^c(M^n, Y)$ to be the colimit,*

$$Map^c(M^n; Y) = \mathrm{colim}_{K \subset M^n} Map(M^n/(M^n - K), Y)$$

where the colimit is taken over all compact subsets K of M^n.

Notice that $Map^c(M^n; Y) \subset Map(M^n, Y)$ consists of all maps that send the complement of some compact subspace $K \subset M^n$ to the basepoint of Y. This allows us to define the compactly supported mapping spectrum $Map^c(M^n_+, \mathbb{E})$ as follows:

Definition 10.25. *We define the spectrum*

$$Map^c(M^n_+, \mathbb{E})$$

by $Map^c(M^n_+, \mathbb{E})_k = Map^c(M^n_+; \mathbb{E}_k) \subset Map(M^n_+; \mathbb{E}_k)$ with the induced structure maps.

Motivated by observation (10.16), we make the following definition of *generalized cohomology with compact supports.*

Definition 10.26. *Let M^n be an n-dimensional manifold and E^* a generalized cohomology theory represented by a connective ring spectrum $\mathbb{E}$. We define the E^*-cohomology with compact supports to be,*

$$E_c^q(M^n) = \pi_{-q} Map^c(M^n_+, \mathbb{E})$$

Notice there is a natural map $E_c^*(M^n) \to E^*(M^n)$ which is an isomorphism if M^n is compact.

Exercise. (Compare with the exercise after the statement of the Poincaré Duality Theorem 1.5 in Chapter 1.) Show that

$$E_c^*(\mathbb{R}^n) \cong \tilde{E}^*(S^n)$$

and more generally that if X is a space whose one-point compactification $X \cup \infty$ has the property that the point at infinity in the one-point compactification has a contractible open neighborhood, as is the case if X is a manifold, then

$$E_c^*(X) \cong \tilde{E}^*(X \cup \infty).$$

We can now state the Poincaré Duality Theorem for generalized cohomology. First observe that if E^* is a generalized cohomology theory represented by a ring spectrum $\mathbb{E}$, and if M^n is a $\mathbb{E}$-oriented n-dimensional manifold, then if $K \subset M^n$ is any compact space, we have an orientation class $\alpha_K \in E_n(M^n, M^n - K)$, which induces a cap product operation (see Definition 10.19)

$$\cap \alpha_K : E^q(M^n, M^n - K) \to E_{n-q}(M^n).$$

As seen in ordinary (co)homology, these operations respect the inclusions of one compact subspace into another, and define a map

$$D_{M^n} : E_c^q(M^n) \to \mathrm{colim}_{K \subset M^n} E^q(M^n, M^n - K) \xrightarrow{\mathrm{colim}_K \{\cap \alpha_K\}} E_{n-q}(M^n).$$

Theorem 10.39. *Let E^* be a generalized cohomology theory represented by a connective ring spectrum $\mathbb{E}$. Let M^n be a $\mathbb{E}$-oriented manifold. Then the duality map*

$$D_{M^n} : E_c^k(M^n) \to E_{n-k}(M^n).$$

is an isomorphism for all k.

Proof. This theorem is a generalization of Theorem 1.6. As you will recall in the proof of that theorem, the argument just needed that the theorem is true for $M^n = \mathbb{R}^n$, which we know in the generalized setting by the exercises above, as well as the fact that (co)homology satisfies the homotopy, exactness, and excision Eilenberg-Steenrod axioms, so that, for example, we get Mayer-Vietoris sequences. Of course, generalized (co)homology theories also satisfy these axioms, and so the proof of Poincaré duality goes through for such generalized theories. We leave the exercise of going through that proof and showing that all the steps are satisfied by generalized (co)homology theories to the reader. We remark that this was first proved by Whitehead in [158]. □

Now recall that in Chapter 2 the notion of orientability was generalized from manifolds to vector bundles. In particular a manifold is orientable if and only if its tangent bundle is orientable. (See Definition 2.10.) The idea was to assign to a vector bundle $\zeta \to X$ an "orientation double cover", Or_ζ. An orientation of ζ is a section of this covering space. If no such section exists, the bundle ζ is not orientable.

There is a similar notion of $\mathbb{E}$-orientability of a k-dimensional vector bundle $\zeta \to X$, where $\mathbb{E}$ is a ring spectrum representing a generalized homology theory E_*. To define this, we consider the covering space $Or^\zeta_{E_*} \to X$, where the fiber over $x \in X$ is the set of units of $E_k(\zeta_x, \zeta_x - \{0\}) \cong E_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \cong \pi_0(\mathbb{E})$. We leave it to the reader to adapt the methods used in Chapters 1 and 2 to define the topology of the space $Or^\zeta_{E_*}$. With this orientation cover we can make the following definition:

Definition 10.27. *Let $\mathbb{E}$ be a connective ring spectrum representing the generalized cohomology theory E^*. Let $\zeta \to X$ be a k-dimensional vector bundle. A $\mathbb{E}$-orientation of ζ is a section of the orientation cover $Or^\zeta_{E_*}$.*

Exercises.

1. Show that a manifold M^n is $\mathbb{E}$-orientable if and only if its tangent bundle $TM^n \to M^n$ is $\mathbb{E}$-orientable.

2. Show that a closed manifold M^n equipped with an embedding or immersion into $\mathbb{R}^L$ for some L, is $\mathbb{E}$-orientable if and only if the normal bundle to this immersion is $\mathbb{E}$-orientable. Indeed show that an $\mathbb{E}$-orientation of its tangent bundle induces an $\mathbb{E}$-orientation of its normal bundle, and vice versa.

An important property of oriented vector bundles is the Thom isomorphism theorem (6.10). There is an analogous Thom isomorphism theorem for $\mathbb{E}$-oriented vector bundles $\zeta \to X$, which we now state. The proof follows the proof of Theorem 6.10 at every step.

Theorem 10.40. *Let ζ be an $\mathbb{E}$- oriented n - dimensional real vector bundle over a connected space X, where $\mathbb{E}$ is a connective ring spectrum representing the generalized cohomology theory E^*. The orientation gives generators (units)*

$u_x \in E^n(\zeta_x, \zeta_x - \{0\}) \cong \pi_0(\mathbb{E})$. *Then there is a unique class, called the* $\mathbb{E}$-*Thom class, in the cohomology of the Thom space*

$$u \in E^n(T(\zeta))$$

so that for every $x \in X$, *if*

$$j_x : \zeta_x/(\zeta_x - \{0\}) \hookrightarrow \zeta/(\zeta - zero(X)) \cong T(\zeta)$$

is the natural inclusion, where $zero(X)$ *is the image of the zero section, then under the induced homomorphism in* E^*- *cohomology,*

$$j_x^* : E^n(T(\zeta)) \to E^n(\zeta_x, \zeta_x - \{0\}) \cong \pi_0(\mathbb{E}),$$

$j_x^*(u) = u_x$.

Furthermore the induced cup product map

$$\gamma : E^q(X) \xrightarrow{\cup u} \tilde{E}^{q+n}(T(\zeta))$$

is an isomorphism for every $q \in \mathbb{Z}$.

This generalized Thom isomorphism theorem has many applications, but a particularly interesting one that we will discuss is an analogue of Alexander duality for $\mathbb{E}$-oriented manifolds.

Theorem 10.41. *(Alexander Duality) Let* $e : M^n \subset \mathbb{R}^N$ *be a regular embedding of a closed,* $\mathbb{E}$-*oriented manifold into Euclidean space. Here* $\mathbb{E}$ *is a connective ring spectrum representing the generalized cohomology theory* E^* *and homology theory* E_*. *Then there is an isomorphism*

$$E^r(M^n) \cong \tilde{E}_{N-r-1}(\mathbb{R}^N - M^n).$$

Before we prove this theorem, we note that one of the most striking applications of this duality theorem is to knot theory. Recall that a "knot" is the image of a regular embedding $e : S^1 \hookrightarrow \mathbb{R}^3$. We call the image of this embedding $K \subset \mathbb{R}^3$. As above, assume $\mathbb{E}$ is a connective ring spectrum. Then the Alexander Duality theorem, combined with Poincaré duality calculates the E_*-homology of the complement of the knot in terms of the E_* homology of S^1:

Corollary 10.42.

$$\tilde{E}_q(\mathbb{R}^3 - K) \cong E_{q-1}(S^1).$$

Notice that in the case of ordinary integral homology this corollary says that $H_1(\mathbb{R}^3 - K) \cong \mathbb{Z}$. This in particular says that the fundamental group of the complement of the knot, which can be quite complicated, always has the integers $\mathbb{Z}$ as its abelianization.

We now proceed with the proof of the Alexander duality theorem.

Proof. . Let η_e be a tubular neighborhood of the embedding $e : M^n \hookrightarrow \mathbb{R}^N$, and let $\nu_e \to M^n$ be its normal bundle. Since e is a regular embedding, the complement of the tubular neighborhood is a deformation retract of the complement of the manifold,

$$\mathbb{R}^N - \eta_e \xrightarrow{\simeq} \mathbb{R}^n - M^n.$$

Now that the quotient space $\mathbb{R}^N/(\mathbb{R}^N - \eta_e)$ is homeomorphic to the one-point compactification $\eta_e \cup \infty$. But by the Tubular Neighborhood Theorem, this is homeomorphic to the one-point compactification of the normal bundle, $\nu_e \cup \infty$. Now notice that since M^n is compact, the one-point compactification of the normal bundle is homeomorphic to its Thom space, $T(\nu_e)$. So we have

$$\mathbb{R}^N/(\mathbb{R}^N - \eta_e) \cong \eta_e \cup \infty \cong \nu_e \cup \infty \cong T(\nu_e).$$

We use this observation in the following way. Since M^n is assumed to be $\mathbb{E}$-orientable, then as a vector bundles, its tangent bundle is $\mathbb{E}$-orientable. But by an exercise above this is equivalent to its normal bundle ν_e being orientable. Now the $\mathbb{E}$-Thom isomorphism theorem 10.40, interpreted for homology (rather than cohomology) says that taking the cap product with the Thom class $u_e \in E^{N-n}((T(\nu_e))$ gives an isomorphism

$$\cap u_e : \tilde{E}_{q+N-n}(T(\nu_e)) \xrightarrow{\cong} E_q(M^n). \tag{10.17}$$

Combining this with the above homeomorphisms, together with the fact that E_* satisfies the Excision Axiom, we get an isomorphism

$$E_{q+N-n}(\mathbb{R}^N, \mathbb{R}^N - \eta_e) \cong E_q(M^n). \tag{10.18}$$

Now using the long exact sequence in reduced homology for the pair $(\mathbb{R}^N, \mathbb{R}^N - \eta_e)$ together with the fact that by the Homotopy Axiom which implies that $\tilde{E}_*(\mathbb{R}^N) = 0$, we see that the connecting homomorphism is an isomorphism,

$$\partial : E_r(\mathbb{R}^N, \mathbb{R}^N - \eta_e) \xrightarrow{\cong} \tilde{E}_{r-1}(\mathbb{R}^N - \eta_e)$$

for all $r \in \mathbb{Z}$. Combining this with isomorphism (10.18) we get an isomorphism

$$E_q(M^n) \cong \tilde{E}_{q+N-n-1}(\mathbb{R}^N - \eta_e) \cong \tilde{E}_{q+N-n-1}(\mathbb{R}^N - M^n).$$

But Poincaré duality gives us an isomorphism

$$\cap[M^n]_{\mathbb{E}} : E^{n-q}(M^n) \xrightarrow{\cong} E_q(M^n).$$

The theorem now follows by combining these isomorphisms. □

10.8.3 Spanier-Whitehead duality and Atiyah duality

An important result in the study of closed differentiable manifolds says that if a manifold M^n is embedded in $\mathbb{R}^N$, then the Thom spectrum of the normal bundle and the manifold itself, are in a sense that can be made precise, dual to each other. This is a stable homotopy theoretic generalization of the Alexander Duality theorem, and was proved by Atiyah in [8]. The type of duality that is appropriate in this setting is known as "Spanier-Whitehead" duality (see [141]). In this subsection we introduce and explore these concepts.

The notion of Spanier-Whitehead duality is a direct analogue of the notion of duality in linear algebra. Recall that if V and W are finite dimensional vector spaces over a field k, then they are said to be *dual* to each other if there is a bilinear pairing

$$V \times W \to k$$

whose adjoints define isomorphisms

$$V \xrightarrow{\cong} Hom(W, k) \quad \text{and}$$
$$W \xrightarrow{\cong} Hom(V, k).$$

In the setting of spectra, the notion of a finite dimensional vector space is replaced by the notion of a *"finite spectrum"*. Such a spectrum $\mathbb{X}$ is one whose homology is finite in the sense that

1. $H_q(\mathbb{X})$ is nonzero for only finitely many $q \in \mathbb{Z}$, and

2. $H_q(\mathbb{X})$ is a finitely generated abelian group for every $q \in \mathbb{Z}$.

The archetypical example of a finite spectrum is the suspension spectrum of a finite, based CW-complex, $\mathbb{X} = \Sigma^\infty(X)$. This example is quite general because of the result of the following exercise:

Exercise. Show that every finite spectrum $\mathbb{X}$ is weakly homotopy equivalent to an iterated suspension or desuspension of the suspension spectrum of a finite CW-complex.

Definition 10.28. *Two finite spectra $\mathbb{X}$ and $\mathbb{Y}$ are said to be "Spanier-Whitehead dual" to each other, (or simply $\mathbb{S}-$ dual) if there is a pairing of spectra*

$$\mathbb{X} \wedge \mathbb{Y} \to \mathbb{S}$$

whose adjoints define weak homotopy equivalences,

$$\mathbb{Y} \xrightarrow{\simeq} Map(\mathbb{X}, \mathbb{S}) \quad \textit{and}$$
$$\mathbb{X} \xrightarrow{\simeq} Map(\mathbb{Y}, \mathbb{S}).$$

An equivalent definition is that $\mathbb{X}$ and $\mathbb{Y}$ are said to be Spanier-Whitehead dual if there are maps of spectra

$$\mu : \mathbb{X} \wedge \mathbb{Y} \to \mathbb{S} \quad \textit{and} \quad \eta : \mathbb{S} \to \mathbb{Y} \wedge \mathbb{X}$$

so that compositions

$$\begin{aligned} &\mathbb{X} = \mathbb{X} \wedge \mathbb{S} \xrightarrow{1\wedge\eta} \mathbb{X} \wedge \mathbb{Y} \wedge \mathbb{X} \xrightarrow{\mu\wedge 1} \mathbb{S} \wedge \mathbb{X} = \mathbb{X} \quad \textit{and} \\ &\mathbb{Y} = \mathbb{S} \wedge \mathbb{Y} \xrightarrow{\eta\wedge 1} \mathbb{Y} \wedge \mathbb{X} \wedge \mathbb{Y} \xrightarrow{1\wedge\mu} \mathbb{Y} \wedge \mathbb{S} = \mathbb{Y} \end{aligned}$$

are homotopic to the identity.,

Exercise. Show that these two definitions are equivalent.

If $\mathbb{X}$ is a finite spectrum, we denote its Spanier-Whitehead dual by $D\mathbb{X}$.

Observations.

1. The sphere spectrum $\mathbb{S}$ is Spanier-Whitehead dual to itself, via the identity map

$$\mathbb{S} \wedge \mathbb{S} \xrightarrow{=} \mathbb{S}.$$

2. If $\mathbb{X}$ is Spanier-Whitehead dual to $\mathbb{Y}$, then the iterated suspensions $\Sigma^k\mathbb{X}$ and $\Sigma^{-k}\mathbb{Y}$ are also Spanier-Whitehead dual.

Exercises.

1. Let $\mathbb{X}$ be a finite spectrum, and let $\mathbb{E}$ be a connective spectrum. (Recall that a connective spectrum is one which has zero homotopy groups in negative dimensions.) Prove that there is a weak homotopy equivalence of spectra

$$Map(\mathbb{X}, \mathbb{S}) \wedge \mathbb{E} \xrightarrow{\simeq} Map(\mathbb{X}, \mathbb{E}).$$

2. Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are finite spectra that are Spanier-Whitehead dual to each other. Suppose that $\mathbb{E}$ is a connective spectrum representing cohomology and homology theories E^* and E_*, then

$$\begin{aligned} &E^q(\mathbb{X}) \cong E_{-q}(\mathbb{Y}) \quad \text{and} \\ &E^q(\mathbb{Y}) \cong E_{-q}(\mathbb{X}) \end{aligned}$$

for all $q \in \mathbb{Z}$.

3. Show that if $\mathbb{X}$ and $\mathbb{Y}$ are finite spectra,

$$D(\mathbb{X} \wedge \mathbb{Y}) \simeq D\mathbb{X} \wedge D\mathbb{Y}.$$

4. Show that the dual of the dual is the original spectrum. That is, if $\mathbb{X}$ is a finite spectrum then $DD\mathbb{X} \simeq \mathbb{X}$.

Note. In exercises 3 and 4 the equivalences mean the same weak homotopy type).

5. If $f : \mathbb{X} \to \mathbb{Y}$ is a map of finite spectra, then there is a natural map $D(f) : D(\mathbb{Y}) \to D(\mathbb{X})$ with $DD(f) = f : \mathbb{X} \to \mathbb{Y}$.

6. If $\mathbb{X} \xrightarrow{f} \mathbb{Y} \xrightarrow{g} \mathbb{Z}$ is a homotopy cofibration sequence of finite spectra, then $D(\mathbb{X}) \xleftarrow{D(f)} D(\mathbb{Y}) \xleftarrow{D(g)} D(\mathbb{Z})$ is also a homotopy cofibration sequence of spectra.

Let X be a finite CW-complex, and assume that X is embedded, in a nonsurjective way, in the sphere S^n, such that the complement $S^n - X$ has the homotopy type of a finite CW-complex. We actually assume that this embedding has a regular neighborhood η, meaning an open subset of S^n which contains the image of X as a deformation retract. For example if X is a smooth manifold smoothly and regularly embedded, then η can be taken to be a tubular neighborhood. Every finite CW complex does have such a "regular embedding" in a sphere of sufficiently high dimension. See, for example, [67]. Then the following gives a more general form of Alexander duality:

Theorem 10.43. *The Spanier-Whitehead dual of the suspension spectrum of X, which we denote by DX, is given by the $(n-1)$-fold desuspension of the suspension spectrum of the complement:*

$$DX \simeq \Sigma^{-(n-1)}\Sigma^{\infty}(S^n - X).$$

Notice that in this setting the complement $S^n - X$ has the homotopy type of the complement of a regular neighborhood, $S^n - \eta$.

Proof. We think of the sphere S^n as the one-point compactification, $S^n = \mathbb{R}^n \cup \infty$. By rotating S^n if necessary, we may assume without loss of generality that $X \subset \mathbb{R}^n \subset S^n$. Consider the map

$$\alpha : (\mathbb{R}^n - X) \times X \to S^{n-1}$$
$$(v, x) \to \frac{v - x}{\|v - x\|}$$

Now suspend that map:

$$\Sigma\alpha : \Sigma((\mathbb{R}^n - X) \times X) \to \Sigma S^{n-1} = S^n.$$

We now need the following basic homotopy theoretic lemma.

Lemma 10.44. *Let A and B be have the homotopy type of CW-complexes. Then there is a natural "splitting" of the suspension of the product,*

$$\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B).$$

Proof. We leave the proof of this lemma as an exercise for the reader. Use the following hint:

Hint. Consider the natural projection maps $p_A : \Sigma(A \times B) \to \Sigma A$, $p_B : \Sigma(A \times B) \to \Sigma B$, and $p_{A\wedge B} : \Sigma(A \times B) \to \Sigma(A \wedge B)$. Use the (iterated) "pinch map" $S^1 \to S^1 \vee S^1 \vee S^1$ in the suspension coordinate to define a map

$$\Sigma(A\times B) \to \Sigma(A\times B)\vee\Sigma(A\times B)\vee\Sigma(A\times B) \xrightarrow{p_A\vee p_B\vee p_{A\wedge B}} \Sigma A\vee\Sigma B\vee\Sigma(A\wedge B)$$

Show that this map is a homotopy equivalence. It might be easiest to first show that it induces an isomorphism in homology. □

We now return to the proof of Theorem 10.43. Using Lemma 10.44 we have a natural map which gives an inclusion of a wedge summand,

$$\iota : \Sigma((\mathbb{R}^n - X) \wedge X) \to \Sigma((\mathbb{R}^n - X) \times X).$$

We therefore may consider the composition

$$\Sigma((\mathbb{R}^n - X) \wedge X) \xrightarrow{\iota} \Sigma((\mathbb{R}^n - X) \times X) \xrightarrow{\Sigma\alpha} S^n.$$

Taking the n-fold iterated desuspension of the corresponding map of suspension spectra we produce a map of spectra,

$$\bar{\mu} : \Sigma^{-(n-1)}\Sigma^{\infty}(\mathbb{R}^n - X) \wedge \Sigma^{\infty}X \to \mathbb{S} = \Sigma^{\infty}S^0.$$

Taking adjoints we get a map of spectra

$$\mu : \Sigma^{-(n-1)}\Sigma^{\infty}(\mathbb{R}^n - X) \to D(X). \qquad (10.19)$$

Our goal is to show the map μ is a weak equivalence of spectra for every finite CW complex X. We will use an induction argument on the skeleta of X. We begin by showing that μ is an equivalence when X is a sphere S^k embedded in S^n. Now since we are assuming that $S^k \hookrightarrow S^n$ has a regular neighborhood η, then $S^n - S^k$ is homotopy equivalent to $S^k - \eta$. Since we may take an arbitrarily small perturbation of the embedding and make it smooth, we may assume that η is a tubular neighborhood of a smooth embedding of S^k in S^n. Indeed, since the embedding is not surjective, we may assume that its image does not include $\infty \in \mathbb{R}^n \cup \infty = S^n$. In this case we have that

$$\Sigma^{\infty}\left(\mathbb{R}^n/\mathbb{R}^n - \eta\right) = \Sigma^{\infty}T(\eta),$$

where by abuse of notation $T(\eta)$ refers to the Thom space of the normal bundle. (Our admittedly bad notation comes from identifying the tubular neighborhood with the normal bundle.) But the spectrum $\Sigma^{\infty}T(\eta)$ is the $(n-k)$-fold suspension of the Thom spectrum $\Sigma^{n-k}(S^k)^{\eta}$. But the stable normal bundle of S^k is trivial, so we have that

$$\Sigma^{\infty}\left(\mathbb{R}^n/\mathbb{R}^n - \eta\right) = \Sigma^{n-k}(S^k)^{\eta} \simeq \Sigma^{\infty}(S^n \vee S^{n-k}). \qquad (10.20)$$

Now notice that

$$\mathbb{R}^n/\mathbb{R}^n - \eta = S^n/S^n - \eta, \tag{10.21}$$

so that

$$\Sigma^\infty\left(S^n/S^n - \eta\right) \simeq \Sigma^\infty(S^{n-k} \vee S^n).$$

To finish the argument for the case of $S^k \subset S^n$, we study the following diagram of homotopy cofibration sequences of spectra

$$\begin{array}{ccccccc}
\Sigma^\infty S^{n-1} & \longrightarrow & \Sigma^\infty \mathbb{R}^n & \longrightarrow & \Sigma^\infty S^n & \xrightarrow{=} & \Sigma^\infty S^n \\
{\scriptstyle =}\uparrow & & \uparrow & & {\scriptstyle .}\uparrow & & \uparrow{\scriptstyle =} \\
\Sigma^\infty S^{n-1} & \longrightarrow & \Sigma^\infty(\mathbb{R}^n - S^k) & \longrightarrow & \Sigma^\infty(S^n - S^k) & \longrightarrow & \Sigma^\infty S^n \\
 & & {\scriptstyle \simeq}\uparrow & & \uparrow & & \\
 & & \Sigma^{-1}\Sigma^\infty\left(\mathbb{R}^n/\mathbb{R}^n - S^k\right) & \xrightarrow[=]{} & \Sigma^{-1}\Sigma^\infty\left(S^n/S^n - S^k\right) & &
\end{array}$$

The vertical map $\Sigma^{-1}\Sigma^\infty\left(\mathbb{R}^n/\mathbb{R}^n - S^k\right) \to \Sigma^\infty(\mathbb{R}^n - S^k)$ is an equivalence, because its cofiber, $\Sigma^\infty\mathbb{R}^n$ is contractible. Combining this with (10.35) implies that

$$\Sigma^\infty(\mathbb{R}^n - S^k) \simeq \Sigma^{-1}(\Sigma^\infty(S^{n-k} \vee S^n)) = \Sigma^\infty(S^{n-k-1} \vee S^{n-1}). \tag{10.22}$$

Now the horizontal map $\Sigma^\infty(S^n - S^k) \to \Sigma^\infty(S^n)$ is null homotopic since the inclusion $S^n - S^k \hookrightarrow S^n$ is not surjective, and hence its image lies in $S^n - point$ which is contractible. This implies there is a splitting $\sigma : \Sigma^\infty(S^n - S^k) \to \Sigma^\infty(\mathbb{R}^n - S^k)$. (By a *"splitting"* we mean that the composition $\Sigma^\infty(S^n - S^k) \xrightarrow{\sigma} \Sigma^\infty(\mathbb{R}^n - S^k) \to \Sigma^\infty(S^n - S^k)$ is homotopic to the identity,). This means that there is an equivalence

$$\begin{aligned}
\Sigma^\infty(\mathbb{R}^n - S^k) &\simeq \Sigma^\infty S^{n-1} \vee \Sigma^\infty(S^n - S^k), \quad \text{and by (10.22)} \\
&\simeq \Sigma^\infty S^{n-1} \vee \Sigma^\infty S^{n-k-1}
\end{aligned}$$

From this it is easy to conclude that

$$\Sigma^\infty(S^n - S^k) \simeq \Sigma^\infty S^{n-k-1} \simeq \Sigma^{n-1} D(S^k) \tag{10.23}$$

and that this equivalence is induced by the duality map described above (10.19).

We now continue our proof of Theorem 10.43 using an induction argument on the skeleta of a finite CW-complex X. It is an easy exercise to see that the theorem holds if X is a zero-dimensional finite complex, meaning it is a finite collection of points. So assume the theorem is true for complexes of dimension less than q, and let X be a q-dimensional finite complex. Let $X^{(q-1)}$ be its $(q-1)$-dimensional skeleton, and assume we have an embedding $X \hookrightarrow S^n$. Then by our inductive assumption we have that

$$DX^{(q-1)} \simeq \Sigma^{-(n-1)}\Sigma^\infty(S^n - X^{(q-1)})$$

and that the equivalence is induced by the pairing 10.19.

Now write

$$X = X^{(q-1)} \cup_{\alpha_1} D^q \cup_{\alpha_2} \cdots \cup_{\alpha_r} D^q$$

where $\alpha_1, \cdots, \alpha_r : \partial D^q = S^{q-1} \to X^{(q-1)}$ are the attaching maps.

For ease of notation we will assume that $r = 1$, which is to say $X = X^{(q-1)} \cup_\alpha D^q$. The general case, i.e when X has an arbitrary finite number q-cells can be handled in the same way.

Let $\tilde{X}$ be the space obtained from $X^{(q-1)}$ by attaching a thin cylinder $S^{q-1} \times [1-\epsilon, 1]$ via the map $\alpha : S^{q-1} \times \{1\} \to X^{(q-1)}$. $\tilde{X}$ is homotopy equivalent to $X^{(q-1)}$, so we know that

$$D\tilde{X} \simeq \Sigma^{-(n-1)}\Sigma^\infty(S^n - \tilde{X}). \tag{10.24}$$

If we let $\tilde{\alpha} : S^{q-1} \to \tilde{X}$ be the inclusion

$$\tilde{\alpha} : S^{q-1} \times \{1-\epsilon\} \subset S^{q-1} \times [1-\epsilon, 1] \subset \tilde{X}$$

we then have a description

$$X = \tilde{X} \cup_{\tilde{\alpha}} D^q.$$

In particular the composition $S^{q-1} \xrightarrow{\tilde{\alpha}} \tilde{X} \subset X \subset S^n$ is an embedding.

Now consider the cofibration sequence

$$S^{q-1} \xrightarrow{\tilde{\alpha}} \tilde{X} \to X$$

It's Spanier - Whitehead dual gives a cofibration sequence of spectra

$$D(S^{q-1}) \xleftarrow{D(\tilde{\alpha})} D(\tilde{X}) \leftarrow D(X)$$

We also have the commutative diagram of spectra

$$\begin{array}{ccccc}
\Sigma^{n-1}\Sigma^\infty(S^n - \tilde{\alpha}(S^{q-1})) & \longleftarrow & \Sigma^{n-1}\Sigma^\infty(S^n - \tilde{X}) & \longleftarrow & \Sigma^{n-1}\Sigma^\infty(S^n - X) \\
\simeq \downarrow \mu & & \simeq \downarrow \mu & & \downarrow \mu \\
D(S^{q-1}) & \xleftarrow{D(\tilde{\alpha})} & D(\tilde{X}) & \longleftarrow & D(X)
\end{array}$$

In order to prove that the right vertical map $\mu : \Sigma^{n-1}\Sigma^\infty(S^n - X) \to D(X)$ is a weak homotopy equivalence, it suffices to show that the top row

$$\Sigma^{n-1}\Sigma^\infty(S^n - \tilde{\alpha}(S^{q-1})) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - \tilde{X}) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - X)$$

is a cofibration sequence of spectra. This is because if that were the case, then the above diagram would be a map between cofibration sequences, where two of the terms are equivalences. This would imply that the third term is an equivalence.

We can see that this sequence is a cofibration sequence at the level of spaces. Namely, since $\tilde{\alpha} : S^{q-1} \to X \to X \to S^n$ is an embedding, its complement can be described by

$$S^n - \tilde{\alpha}(S^{q-1}) = (S^n - \tilde{X}) \sqcup_{S^n - X} (S^n - D^q).$$

Thus the cofiber of $(S^n - \tilde{X}) \to (S^n - \tilde{\alpha}(S^{q-1}))$ is the quotient $(S^n - D^q)/(S^n - X)$. But since $S^n - D^q$ is contractible, we have $\Sigma^\infty(S^n - D^q)/(S^n - X) \simeq \Sigma^\infty\Sigma(S^n - X)$. In other words, $\Sigma^{n-1}\Sigma^\infty(S^n - \tilde{\alpha}(S^{q-1})) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - \tilde{X}) \leftarrow \Sigma^{n-1}\Sigma^\infty(S^n - X)$ is a cofibration sequence of spectra. As mentioned before this is what was needed to complete the proof. □

We often have a situation where the embedding of a finite complex X is given inside a Euclidean space, $X \subset \mathbb{R}^n \subset \mathbb{R}^n \cup \infty = S^n$. So it is natural to ask how the the homotopy type of the complement $\mathbb{R}^n - X$ is related to the Spanier-Whitehead dual. For this notice that

$$\mathbb{R}^n - X = S^n - (X_+))$$

where the disjoint basepoint in X_+ is embedded in S^n as the point at ∞. So Theorem 10.43 has the following corollary.

Corollary 10.45. *Let X be a finite CW-complex regularly embedding in $\mathbb{R}^n$. then there are weak homotopy equivalences of spectra*

$$\begin{aligned}\Sigma^{-(n-1)}\Sigma^\infty(\mathbb{R}^n - X) = \Sigma^{-(n-1)}\Sigma^\infty(S^n - X_+) &\simeq D(X_+)\\ &\simeq D(X_+)\\ &= D(X) \vee \mathbb{S}\end{aligned}$$

An important result regarding the topology of manifolds, proved by Atiyah in [8], relates the Thom spectrum of the normal bundle of an embedding into Euclidean space, $e : M^n \hookrightarrow \mathbb{R}^N$, to the Spanier-Whitehead dual of M^n. This duality property is sometimes known as "Atiyah duality", and it now follows quickly from the generalized version of Alexander duality that we've proved (Theorem 10.43) and its Corollary 10.45.

Theorem 10.46. *(Atiyah [8]) Let M^n be a closed n-dimensional manifold and $e : M^n \hookrightarrow \mathbb{R}^{n+k}$ an embedding with normal bundle $\nu_e^k \to M^n$ and tubular neighborhood η_e. Then there is a weak homotopy equivalence of spectra*

$$\Sigma^\infty T(\nu_e^k) \simeq \Sigma^{n+k} D(M^n_+)$$

Proof. Recall that we have a homeomorphism of the Thom space of the normal bundle,

$$(M^n)^{\nu_e^k} = \nu_e^k \cup \infty \cong \eta_e \cup \infty \cong \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_e).$$

But since $\mathbb{R}^{n+k}$ is contractible there is a homotopy equivalence

$$\mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_e) \simeq \Sigma(\mathbb{R}^{n+k} - \eta_e).$$

So we have

$$(M^n)^{\nu_e^k} \simeq \Sigma(\mathbb{R}^{n+k} - \eta_e) \simeq \Sigma(\mathbb{R}^{n+k} - M^n).$$

The theorem now follows from Corollary 10.45. □

As in the constructions of the maps yielding Alexander duality (Theorem 10.43) one can give a conceptual, explicit map yielding Atiyah duality, Suppose the tubular neighborhood η_e of M^n in $\mathbb{R}^{n+k}$ is small enough so that every point $y \in \eta_e$ has Euclidean distance less that some number $\epsilon > 0$. Now consider the subtraction map

$$\begin{aligned} M^n \times \mathbb{R}^{n+k} &\xrightarrow{\alpha} \mathbb{R}^{n+k} \\ (x, v) &\to e(x) - v \end{aligned} \tag{10.25}$$

This map restricts to give a map

$$M^n \times \left(\mathbb{R}^{n+k} - \eta_e\right) \xrightarrow{\alpha} \mathbb{R}^{n+k} - B_\epsilon^{n+k}$$

where B_ϵ^{n+k} is the open ball around the origin $\mathbb{R}^{n+k}$ of radius ϵ. We therefore have a map of the quotient space

$$M^n \times \mathbb{R}^{n+k}/\left(M^n \times \left(\mathbb{R}^{n+k} - \eta_e\right)\right) \xrightarrow{\alpha} \mathbb{R}^{n+k}/\left(\mathbb{R}^{n+k} - B_\epsilon^{n+k}\right) \cong S^{n+k}. \tag{10.26}$$

The left hand quotient space is equal to the smash product with a disjoint basepoint, $M^n_+ \wedge \left(\mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_e)\right)$ which in turn, via the tubular neighborhood theorem, is homeomorphic to $M^n_+ \wedge M^{\nu^k}$. Thus this subtraction map defines a map

$$\alpha : M^n_+ \wedge M^{\nu^k} \to S^{n+k}$$

and therefore map of spectra, which by abuse of notation we still call α,

$$\alpha : M^n_+ \wedge \Sigma^{-(n+k)}\Sigma^\infty(M^{\nu^k}) \to \Sigma^\infty(S^0) = \mathbb{S}. \tag{10.27}$$

We leave it to the reader to verify, by running through the above proof, that this subtraction map α yields the Spanier-Whitehead duality between M^n_+ and the Thom space of the normal bundle $\Sigma^{-(n+k)}\Sigma^\infty(T(\nu^k))$.

Inspired by this we make the following definition.

Definition 10.29. *Let M^n be a closed n-dimensional manifold embedded in Euclidean space $M^n \subset \mathbb{R}^{n+k}$ with normal bundle $\nu^k \to M^n$. Define the spectrum M^{-TM} to be the desuspension of the Thom space*

$$M^{-TM} = \Sigma^{-(n+k)}\Sigma^\infty(T(\nu^k)).$$

Exercise. Show that the homotopy type of the spectrum M^{-TM} does not depend on the choice of embedding.

Atiyah duality can then be restated as follows.

Corollary 10.47. *There is an equivalence of spectra*

$$\begin{aligned} M^{-TM} &\simeq D(M^n_+) \\ &= Map(M^n_+, \mathbb{S}) \end{aligned}$$

We now observe that the Spanier-Whitehead dual of any space X with a disjoint basepoint, $D(X_+) = Map(X_+, \mathbb{S})$ has a natural ring spectrum structure. This is because there is a natural diagonal map

$$\Delta : X_+ \to (X \times X)_+ = X_+ \wedge X_+ \tag{10.28}$$

and of course a (commutative) ring structure on the sphere spectrum

$$\mathbb{S} \wedge \mathbb{S} \xrightarrow{=} \mathbb{S}.$$

This allows us to make the following ring structure on $D(X_+) = Map(X_+, \mathbb{S})$:

$$\begin{aligned} \mu : Map(X_+, \mathbb{S}) \wedge Map(X_+, \mathbb{S}) &\xrightarrow{\gamma} Map(X_+ \wedge X_+, \mathbb{S} \wedge \mathbb{S}) \\ &\xrightarrow{\Delta^*} Map(X_+, \mathbb{S} \wedge \mathbb{S}) \xrightarrow{=} Map(X_+, \mathbb{S}) \end{aligned} \tag{10.29}$$

where $\gamma(\phi_1 \wedge \phi_2)(x_1 \wedge x_2) = \phi_1(x_1) \wedge \phi_2(x_2)$ and $\Delta^*(\psi)(x) = \psi(\Delta(x)) = \psi(x \wedge x)$.

Actually this ring structure is commutative in the sense of [76] essentially because the diagonal map is cocommutative and because the ring structure on $\mathbb{S}$ is commutative. We refer the reader to [76] for a discussion of commutative ring (symmetric) spectra.

Notice that by Corollary 10.47 this ring structure translates to give the Thom spectrum M^{-TM} the structure of a commutative ring structure. In [29] the author described an explicit ring structure on M^{-TM} defined in terms of an embedding $M^n \hookrightarrow \mathbb{R}^{n+k}$.

10.9 Eilenberg-MacLane spectra and the Steenrod algebra

When we first introduced the notion of a spectrum toward the beginning of this chapter, one of the first examples described was that of the Eilenberg-MacLane spectrum $\mathbb{H}G$, where G is an abelian group. The n^{th} space in this spectrum $(\mathbb{H}G)_n$ is an Eilenberg-MacLane space

$$(\mathbb{H}G)_n = K(G, n).$$

The main reason that these spaces, and the resulting spectrum are so important is that they classify ordinary (co)homology, In particular, if X is any based space of the homotopy type of a CW-complex

$$H^n(X;G) \cong [X, K(G,n)]. \tag{10.30}$$

Because the Eilenberg-Steenrod axioms are satisfied, this means that, via Brown's Representability Theorem 10.16 and Whitehead's Theorem 10.18, that the Eilenberg-MacLane spectrum $\mathbb{H}G$ represents ordinary (co)homology with G-coefficients. This means that given any pair of spaces $A \subset X$ of the homotopy type of CW complexes

$$H^q(X,A;G) \cong [X/A, \Sigma^q \mathbb{H}G] \quad \text{and} \quad H_q(X,A;G) \cong \pi_q(X/A \wedge \mathbb{H}G).$$

In fact if $\mathbb{E}$ is a spectrum its homology and cohomology are given by

$$H^q(\mathbb{E};G) \cong [\mathbb{E}, \Sigma^q \mathbb{H}G] \quad \text{and} \quad H_q(\mathbb{E};G) \cong \pi_q(\mathbb{E} \wedge \mathbb{H}G).$$

In particular notice that

$$H^*(\mathbb{H}G;G) \cong [\mathbb{H}G, \mathbb{H}G]^*$$

where the superscript $*$ represents the degree of the maps to be taken. That is, $[\mathbb{H}G, \mathbb{H}G]^q = [\mathbb{H}G, \Sigma^q \mathbb{H}G]$.

This observation describes a special case of the generalized cohomology of a representing spectrum. Namely, suppose $\mathbb{E}$ is a spectrum representing a generalized cohomology theory E^*. Then

$$E^*(\mathbb{E}) = [\mathbb{E}, \mathbb{E}]^*.$$

When $\mathbb{E}$ is a ring spectrum these cohomology groups form a ring via composition. Indeed they form an algebra over the ground ring $E_* = E_*(point) = \pi_0(\mathbb{E})$. As we will see, the importance of this algebra is due to the fact that it forms the algebra of E^*-cohomology operations, in a sense that we will now make precise.

10.9.1 Cohomology operations

Recall that according to Definition 10.8, a generalized cohomology theory E^* consists of a collection of functors from the category of CW-pairs CW_2 to the category of abelian groups $\mathcal{G}$, as well as a collection of natural "coboundary" homomorphisms $\delta^q : E^q(A) \to E^{q+1}(X,A)$ for any CW-pair (X,A), that satisfy the Homotopy, Excision, and Exactness Eilenberg-Steenrod axioms.

Definition 10.30. *Let E^* be a generalized cohomology theory. An E^*-cohomology operation of degree k is a collection of natural transformations*

$\alpha^q : E^q \to E^{q+k}$ *that respect the coboundary homomorphisms. That is, for any CW pair* (X, A),

$$\delta^{q+k}\alpha^q(x) = \alpha^{q+1}\delta^q(x)$$

for any $x \in E^q(A)$.

Given a cohomology operation $\alpha = \{\alpha^q\}$ of degree k, we simply write

$$\alpha : E^* \to E^{*+k}.$$

Remark. The type of cohomology operations we are considering are sometimes referred to as "stable cohomology operations" since our definition implies that such a cohomology operation commutes with the suspension isomorphism $E^*(X) \cong E^*(\Sigma X, point)$.

Exercise. Verify this statement. That is, verify that according to our definition of a cohomology operation, such an operation commutes with the suspension isomorphism.

In this book we are mostly concerned with cohomology operations for ordinary cohomology with coefficients in $\mathbb{Z}/p$ where p is a prime number. Notice that the set of these operations form an algebra over the field $\mathbb{Z}/p$. The multiplication is given by composition of cohomology operations. This algebra is called the "**mod p Steenrod algebra"** which we denote by $\mathcal{A}_p$.

Notice that if (X, A) is any pair in CW_2, its cohomology $H^*(X, A; \mathbb{Z}/p)$ forms a module over the Steenrod algebra $\mathcal{A}_p$, under application of cohomology operations. This structure is extremely important in homotopy theory, and so we explore it further here.

The following is the basic connection between the Steenrod algebra $\mathcal{A}_p$ of mod p cohomology operations, and the mod p Eilenberg-MacLane spectrum.

Theorem 10.48. *There is an isomorphism of algebras over* $\mathbb{Z}/p$

$$\phi : \mathcal{A}_p \xrightarrow{\cong} H^*(\mathbb{HZ}/p; \mathbb{Z}/p) \cong [\mathbb{HZ}/p, \mathbb{HZ}/p]^*.$$

Proof. (Sketch). In some ways the proof of this theorem is formal. We suggest the book by Mosher and Tangora [124] for details.

Let $a \in \mathcal{A}_p$ be an element of degree k. Since a is a cohomology operation, it acts on the mod p cohomology of every space, and in particular of Eilenberg-MacLane spaces. So $a \in \mathcal{A}_p$ defines homomorphisms

$$a_n : H^n(K(\mathbb{Z}/p, n); \mathbb{Z}/p) \to H^{n+k}(K(\mathbb{Z}/p; n); \mathbb{Z}/p)$$

for every $n \geq 0$.

Now $H^n(K(\mathbb{Z}/p,n);\mathbb{Z}/p) = \mathbb{Z}/p$ so consider the image of the generator $a_n(\iota) \in H^{n+k}(K(\mathbb{Z}/p;\mathbb{Z}/p)$. Since cohomology is classified by Eilenberg-MacLane spaces, we can represent these cohomology classes by maps which are well-defined up to homotopy, which by abuse of notation we call

$$a_n : K(\mathbb{Z}/p,n) \to K(\mathbb{Z}/p,n+k)$$

for each n.

Furthermore, and the reader should check this, because the cohomology operation $a \in \mathcal{A}_p$ respects the suspension homomorphism, the following diagrams homotopy commute:

$$\begin{array}{ccc} K(\mathbb{Z}/p,n) & \xrightarrow{a_n} & K(\mathbb{Z}/p,n+k) \\ \simeq\downarrow & & \downarrow\simeq \\ \Omega K(\mathbb{Z}/p,n+1) & \xrightarrow[\Omega a_{n+1}]{} & \Omega K(\mathbb{Z}/p,n+k+1). \end{array}$$

The a_n's then fit together to give a map of ω-spectra, which by abuse of notation we again call

$$a : \mathbb{HZ}/p \to \Sigma^k \mathbb{HZ}/p,$$

We leave it to the reader to check that this map of spectra is well-defined up to homotopy, and this correspondence defines a map of graded algebras,

$$\phi : \mathcal{A}_p \to [\mathbb{HZ}/p;\mathbb{HZ}/p]^*.$$

To see that this map is an isomorphism, we note that the above procedure is completely reversible. Namely, given $\alpha \in [\mathbb{HZ}/p;\mathbb{HZ}/p]^k$, we represent α by a map of ω-spectra, which defines maps

$$\alpha_n : K(\mathbb{Z}/p,n) \to K(\mathbb{Z}/p,n+k)$$

such that the following diagrams commute:

$$\begin{array}{ccc} K(\mathbb{Z}/p,n) & \xrightarrow{\alpha_n} & K(\mathbb{Z}/p,n+k) \\ \simeq\downarrow & & \downarrow\simeq \\ \Omega K(\mathbb{Z}/p,n+1) & \xrightarrow[\Omega \alpha_{n+1}]{} & \Omega K(\mathbb{Z}/p,n+k+1). \end{array} \tag{10.31}$$

If $(X,A)$is any CW-pair, then by composition the maps α_n then define maps

$$[X/A, K(\mathbb{Z}/p,n)] \xrightarrow{\alpha_n} [X/A, K(\mathbb{Z}/p,n+k)]$$
$$H^n(X/A;\mathbb{Z}/p) \xrightarrow{\alpha_n} H^{n+k}(X/A;\mathbb{Z}/p)$$

We leave it to the reader to check that the commutativity of the squares (10.31) says that these operations are homomorphisms that commute with the suspension homomorphism, and therefore define a cohomology operation $\psi(\alpha) \in \mathcal{A}_p$, We leave it to the reader to fill in the details of this argument and to check that $\phi : \mathcal{A}_p \to [\mathbb{HZ}/p, \mathbb{HZ}/p]^*$ and $\psi : [\mathbb{HZ}/p, \mathbb{HZ}/p]^* \to \mathcal{A}_p$ are both algebra homomorphisms that are inverse to each other.

□

Historically, the Steenrod algebra $\mathcal{A}_p$ was discovered in two main steps. First, in approximately 1950, N. Steenrod described cohomology operations Sq^i with coeffiencts in $\mathbb{Z}/2$ that became known as "Steenrod squares" and he studied many of their properties. His student J. Adem found the multiplicative relations the Steenrod operations satisfied. Steenrod produced similarly defined cohomology operations $\mathcal{P}^i$ with coefficients in $\mathbb{Z}/p$ for p an odd prime, which he called "reduced powers". The reduced powers were shown to satisfy similar "Adem relations".

The second step, which we see is necessary by Theorem 10.48, is a calculation of $[\mathbb{HZ}/p, \mathbb{HZ}/p]^*$. To do this one needs to compute the cohomology of the Eilenberg-MacLane spaces, $H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. This was carried out by Cartan and Serre. A very nice account of that calculation is given in [124]. It is a beautiful example of a calculation using Serre's spectral sequence. In any case, the result of these calculations was that the Steenrod algebra $\mathcal{A}_p (= [\mathbb{HZ}/p, \mathbb{HZ}/p]^*)$ is precisely the algebra generated by the Steenrod squares at $p = 2$, and by the reduced powers together with the "Bockstein operator" $\beta : H^q(X, A; \mathbb{Z}/p) \to H^{q+1}(X, A; \mathbb{Z}/p)$ when p is odd.

To understand the Bockstein operator, recall that given any short exact sequence of abelian groups

$$0 \to H \xrightarrow{\iota} G \xrightarrow{p} K \to 0 \tag{10.32}$$

there is an associate long exact sequence of cohomology groups,

$$\xrightarrow{\delta} H^q(X, A; H) \xrightarrow{\iota_*} H^q(X, A; G) \xrightarrow{p_*} H^q(X, A; K) \xrightarrow{\delta} H^{q+1}(X, A; H) \xrightarrow{\iota_*} \dots$$

The connecting homomorphism $\delta : H^q(X, A; K) \to H^{q+1}(X, A; H)$ is known as the *"Bockstein operator"* associated to the short exact sequence (10.32). Of particular importance are the Bockstein operators associated to the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

for p a prime.

Exercise. Show that the Bockstein operator $\beta : H^q(X, A; \mathbb{Z}/p) \to H^{q+1}(X, A; \mathbb{Z}/p)$ associated to this short exact sequence is a cohomology operation in the sense of Definition 10.30.

10.9.2 The axioms and some consequences

In the famous book by Steenrod and Epstein on cohomology operations [144], they showed that the Steenrod squaring operations satisfy the following axioms, and moreover, they are completely characterized by these axioms.

Axioms. (10.33)

1. There are cohomology operations in the sense of Definition 10.30 known as "Steenrod squares"

$$Sq^i : H^n(-;\mathbb{Z}/2) \to H^{n+i}(-;\mathbb{Z}/2)$$

for all integers $i \geq 0$.

2. $Sq^0 = 1$ the identity transformation
3. $Sq^i(x) = 0$ if the dimension of x is less than i
4. $Sq^i(x) = x^2$ if the dimension of x equals i
5. The Steenrod squares satisfy the product formula known as the "Cartan formula":

$$Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y).$$

6. Sq^1 is the Bockstein homomorphism associated to the coefficient sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

7. The Steenrod squares satisfy the "Adem relations":
For $a < 2b$,

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$

where the binomial coefficients are taken mod 2.

Axioms (6) and (7) can be shown to be consequences of axioms (1)-(5). Since they commute with the suspension isomorphism, the Steenrod operations act on the cohomology of spectra as well as spaces.

A consequence of Cartan and Serre's calculation of the cohomology of the Eilenberg-MacLane spaces $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ and the resulting calculation of the cohomology of the Eilenberg-MacLane spectra, $H^*(\mathbb{HZ}/2;\mathbb{Z}/2) = [\mathbb{HZ}/2, \mathbb{HZ}/2]^*$ one has the following theorem.

Theorem 10.49. *The algebra of $\mathbb{Z}/2$-cohomology operations $\mathcal{A}_2 = [\mathbb{HZ}/2, \mathbb{HZ}/2]^*$ is the algebra over $\mathbb{Z}/2$ generated by the Steenrod squaring operations Sq^i subject to the Adem relations.*

In this book we will mostly be concerned with mod 2 cohomology operations, but in Steenrod and Epstein's book [144] they also describe the following mod p cohomology operations for p an odd prime.

Let p be an odd prime and let

$$\beta : H^q(X, A; \mathbb{Z}/p) \to H^{q+1}(X, A; \mathbb{Z}/p)$$

be the Bockstein operator associated to the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

We have the following axioms:

Axioms. (10.34)

1. There are cohomology operation in the sense of Definition 10.30

$$P^i : H^q(X, A; \mathbb{Z}/p) \to H^{q+2i(p-1)}(X, A; \mathbb{Z}/p)$$

   known as "Steenrod reduced power operations" for all integers $i \geq 0$.

2. $P^0 = 1$ the identity transformation
3. If $dim(x) = 2k$, then $P^k(x) = x^p$.
4. If $2k > dim(x)$, then $P^k(x) = 0$.
5. The reduced power operations satisfy a product formula known as the "Cartan formula":

$$P^k(xy) = \sum_i P^i(x)P^{k-i}(y).$$

6. The reduced powers satisfy the "Adem relations"": If $a < pb$ then

$$P^a P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t}P^t.$$

   If $a \leq b$ then

$$P^a \beta P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t}P^t + \sum_{t=0}^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t}\beta P^t.$$

Again, by the calcuation of Cartan and Serre of the cohomology of the Eilenberg-MacLane spaces $H^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$ and the resulting calculation of the cohomology of the Eilenberg-MacLane spectrum, $H^*(\mathbb{HZ}/p;\mathbb{Z}/p) \cong [\mathbb{HZ}/p, \mathbb{HZ}/p]^*$ one has the following theorem:

Theorem 10.50. *For p an odd prime, the algebra of $\mathbb{Z}/p$-cohomology operations $\mathcal{A}_p = [\mathbb{HZ}/p, \mathbb{HZ}/p]^*$ is the algebra over $\mathbb{Z}/p$ generated by the Steenrod reduced power operations P^i and the Bockstein operator β, subject to the Adem relations.*

As it turns out, the axioms for the Steenrod squares and Steenrod's reduced power operations completely characterize these cohomology operations (see [144] for a verification). We now observe that calculations can be directly made using these axioms.

Proposition 10.51. *. Let X be a space and $u \in H^1(X;\mathbb{Z}/2)$. Then*

$$Sq^i(u^k) = \binom{k}{i} u^{k+i}.$$

Proof. If $k = 0$, then the proposition follows immediately from Axioms 2 and 3 given in (10.33). Now we use induction on k, and observe that

$$\begin{aligned} Sq^i(u^k) &= Sq^i(u \cdot u^{k-1}) = Sq^0(u) \cdot Sq^i(u^{k-1}) + Sq^1 u \cdot Sq^{i-1}(u^{k-1}) \\ &= \left[\binom{k-1}{i} + \binom{k-1}{i-1}\right] u^{k+i} = \binom{k}{i} u^{k+i}. \end{aligned}$$

□

For ease of notation let $\mathbb{P}$ denote the infinite dimensional real projective space $\mathbb{P} = \mathbb{RP}^\infty$. Recall that its cohomology is the polynomial algebra,

$$H^*(\mathbb{P};\mathbb{Z}/2) \cong \mathbb{Z}/2[a]$$

where $a \in H^1(\mathbb{P};\mathbb{Z}/2)$. Proposition 10.51 then gives a complete calculation of $H^*(\mathbb{P};\mathbb{Z}/2)$ as a module over $\mathcal{A}_2$.

10.9.3 Basic algebraic properties

We now discuss some basic properties of the Steenrod algebra $\mathcal{A}_p$. For a more detailed discussion we refer the reader to the book by Steenrod and Epstein [144].

We begin with a purely combinatorial identity which is extremely useful in making calculations with the Steenrod algebra.

Proposition 10.52. *Let p be a prime and let $a = \sum_{i=0}^{k} a_i p^i$ and $b = \sum_{i=0}^{k} b_i, p^i \quad (0 \leq a_i, b_i < p)$. Then*

$$\binom{b}{a} = \prod_{i=0}^{m} \binom{b_i}{a_i} \quad \textit{mod } p.$$

We leave the proof of this proposition as an exercise for the reader, or the reader can refer to [144] for a proof. The main observation needed for the proof is that because for $0 < i < p$, the binomial coefficient $\binom{p}{i}$ is congruent to zero mod p, and so

$$(1+x)^p = 1 + x^p \mod p,$$

and by induction,

$$(1+x)^{p^i} = 1 + x^{p^i} \mod p$$

for all i.

We now focus our attention on the mod 2 Steenrod algebra, $\mathcal{A}_2$.

Given a finite sequence of nonnegative integers, $I = (i_1, \cdots, i_k)$, k is called the *length* of I, $k = \ell(I)$. We write

$$Sq^I = Sq^{i_1} \cdots Sq^{i_k}.$$

We say that a sequence I is *admissible* if $i_q \geq 2i_{q+1}$ for $q = 1, \cdots, k-1$ and if $i_k \geq 1$.

Theorem 10.53. *The collection $\{Sq^I : I \text{ is admissible}\}$ forms a $\mathbb{Z}/2$-vector space basis for $\mathcal{A}_2$.*

Proof. Given a sequence $I = (i_1, \cdots, i_k)$ we define its *moment* to be

$$m(I) = \sum_{q=1}^{k} k i_k.$$

We first show that any Sq^I, for any inadmissible sequence I is a sum of Sq^J's where the sequences J have smaller moment than I. This will show that the admissible monomials span the Steenrod algebra.

Let $I = (i_1, \cdots, i_k)$ be an inadmissible sequence with no zeros. Then for some q, $i_q < 2i_{q+1}$. Now by the Adem relations,

$$Sq^I = Sq^L Sq^{i_q} Sq^{i_{q+1}} Sq^M = \sum_j a_j Sq^L Sq^{i_q + i_{q+1} - j} Sq^j Sq^M$$

where $a_j \in \mathbb{Z}/2$. It is easy to check that each of the monomials in the sum have smaller moment than $m(I)$. Thus the admissible monomials span $\mathcal{A}_2$

In order to prove that the admissible monomials in $\mathcal{A}_2$ are linearly independent we need the following lemma, which involves an independently interesting calculation.

Let $\mathbb{P}^n$ denote the n-fold cartesian product of the infinite dimensional projective space $\mathbb{P}$ with itself. Let $w = a \times \cdots \times a \in H^n(\mathbb{P}^n; \mathbb{Z}/2)$. Notice that the following lemma will prove that the admissible monomials are linearly independent, and will complete the proof of this theorem.

Lemma 10.54. *The map $\mathcal{A}_2 \to H^*(\mathbb{P}^n; \mathbb{Z}/2)$ defined by*

$$Sq^I \to Sq^I(w)$$

sends admissible monomials of dimension $\leq n$ to linearly independent elements.

Proof. We prove this lemma by induction on n. For $n = 1$ it follows from the fact that $Sq^1(a) = a^2 \neq 0$. So we now assume the lemma is true for $n-1$. Our goal is to prove it for n. So suppose that

$$\sum_I a_I Sq^I(w) = 0$$

where the sum is taken over monomials of a fixed dimension q, where $q \leq n$. Our job is to prove that this implies that the coefficients a_I are all zero. We do this by decreasing induction on the length $\ell(I)$. Suppose that $a_I = 0$ for $\ell(I) > k$. We can rewrite the above equality as

$$\sum_{\ell(I)=k} a_I Sq^I(w) + \sum_{\ell(I)<k} a_I Sq^I(w) = 0. \tag{10.35}$$

Now the Kunneth formula says that

$$H^{q+n}(\mathbb{P}^n; \mathbb{Z}/2) \cong \sum_s H^s(\mathbb{P}; \mathbb{Z}/2) \otimes H^{q+n-s}(\mathbb{P}^{n-1}; \mathbb{Z}/2).$$

Let π be the projection onto the summand with $s = 2^k$. Let $w = u \times w'$, where $w' \in H^{n-1}(\mathbb{P}^{n-1}; \mathbb{Z}/2)$ is the generator. Then by the Cartan formula

$$Sq^I(w) = Sq^I(a \times w') = \sum_{I \leq J} Sq^J(u) \times Sq^{I-J}(w') \tag{10.36}$$

where $J \leq I$ means that $0 \leq j_r \leq i_r$ for all r. Let J_k be the sequence $(2^{k-1}, \cdots, 2^1, 2^0)$.

We claim that

$$\pi Sq^I(w) = \begin{cases} 0 \quad \text{if} \quad \ell(I) < k \\ a^{2^k} \times Sq^{I-J_k}(w') \quad \text{if} \quad \ell(I) = k. \end{cases} \tag{10.37}$$

To see this, notice that since $dim\, a = 1$, $Sq^J(a) = 0$ unless J has the form $(2^{q-1}, 2^{q-2}, \cdots, 2, 1)$ for some q. We call this sequence J_q. Notice furthermore $Sq^{J_q}(a) = a^{2^q}$.

To prove (10.37), notice that if $\ell(I) < k$ then $J \leq I$ implies that $\ell(J) < k$ and so $\pi Sq^i(w) = 0$. If $\ell(I) = k$, then $\pi(Sq^J(a) \times Sq^{I-J}(w')) = 0$ unless $J = J_k \leq I$. This verifies (10.37).

If we apply π to equation (10.35) and use (10.37), we find that

$$a^{2^k} \times \sum_{\ell(I)=k} a_I Sq^{I-J_k}(w') = 0. \tag{10.38}$$

Now one can easily check that as I ranges over all admissible sequences of length k and dimension q, $I - J_k$ will range over all admissible sequences of length $\leq k$, and dimension $q - 2^k + 1$, and the correspondence is one-to-one. Since $k \geq 1$, we have that $q - 2^k + 1 \leq n - 1$. So the inductive assumption on n implies that each coefficient in equation (10.38) is zero. Thus $a_I = 0$ for $\ell(I) = k$. This completes the proof of the lemma and therefore of Theorem 10.53. □

□

We now have an additive basis for the Steenrod algebra $\mathcal{A}_2$. Our next goal is to find a convenient set of multiplicative generators.

Suppose that A is an associative nonnegatively graded algebra over a field k, with $A^0 = k$. (Here A^0 is the subalgebra of elements of grading zero.) Then recall that the set of decomposable elements of A is the image of the multiplication map,

$$\mu : A_{>0} \otimes A_{>0} \to A,$$

where $A_{>0}$ consists of those elements of positive grading. This image is a two-sided ideal, and the quotient,

$$Q(A) = A/\mu(A_{>0} \otimes A_{>0})$$

is called the set of *indecomposable elements* of A. Our next goal is to compute the set of indecomposable elements in the Steenrod algebra, $\mathcal{A}_2$.

Lemma 10.55. *The Steenrod square Sq^i is decomposable if and only if i is not a power of 2.*

Proof. We write the Adem relations in the form

$$\binom{b-1}{a} Sq^{a+b} = Sq^a Sq^b + \sum_{j>0} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

where $0 < a < 2b$. One then immediately sees that if $\binom{b-1}{a} = 1 \in \mathbb{Z}/2$, then Sq^{a+b} is decomposable. Now suppose that i is not a power of 2. Then there is a unique k such that $i = a + 2^k$, $0 < a < 2^k$. Let $b = 2^k$. Then $b - 1 = 1 + 2 + \cdots + 2^{k-1}$. But as is immediate from Proposition 10.52, $\binom{b-1}{a} = 1 \in \mathbb{Z}/2$. Thus Sq^i is decomposable.

To prove the converse, let $i = 2^k$. Suppose by way of contradiction that Sq^{2^k} is decomposable, so we can write

$$Sq^{2^k} = \sum_{j=1}^{2^k-1} m_j Sq^j.$$

Then if $u \in H^1(\mathbb{P}; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator, we would have that

$$u^{2^{k+1}} = Sq^{2^k} u^{2^k} = \sum_{j=1}^{2^k-1} m_j Sq^j (u^{2^k}) = 0$$

since for $1 \leq j \leq 2^k - 1$, $dim\, Sq^j < dim\, u^{2^k}$. This contradiction completes the proof of the lemma. □

From this we now have a set of multiplicative generators of $\mathcal{A}_2$:

Theorem 10.56. *The elements Sq^{2^k} multiplicatively generate $\mathcal{A}_2$.*

We record the analogous results for $\mathcal{A}_p$, for p an odd prime. Again we refer the reader to [144] for details.

A monomial in $\mathcal{A}_p$ can be written in the form

$$\beta^{\epsilon_0} P^{s_1} \beta^{\epsilon_2} \cdots P^{s_k} \beta^{\epsilon_k}$$

where $\epsilon_i = 0, 1$ and the s_i's are positive integers. We denote this monomial by P^I, where $I = (\epsilon_0, s_1, \epsilon_2, s_2, \cdots, s_k, \epsilon_k)$.

A sequence I is called *admissible* if $s_i \geq p s_{i+1} + \epsilon_i$ for all $i \geq 1$. We call the corresponding element P^I and refer to it as an *admissible monomial.*

Theorem 10.57. *For p an odd prime the admissible monomials $P^I \in \mathcal{A}_p$ form an additive basis for $\mathcal{A}_p$.*

Theorem 10.58. *The elements P^{p^j} and the Bockstein β multiplicatively generate $\mathcal{A}_p$, for p an odd prime.*

10.9.4 Hopf algebras, the canonical antiautomorphism, and the dual of the Steenrod algebra

The Steenrod algebra $\mathcal{A}_p$ has an additional structure stemming from the Cartan formula. It is a *Hopf algebra.*

Definition 10.31. *A graded Hopf algebra $\mathcal{B}$ over a field k is both a unital, associative graded algebra and a counital coassociative coalgebra coalgbra over k such that the coproduct map,*

$$\psi : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

is a map of algebras.

All Hopf algebras considered in this book will be graded. We often will delete the term "graded" in our discusions of these Hopf algebras.

We observe that the Steenrod algebra $\mathcal{A}_p$ is a Hopf algebra over $\mathbb{Z}/p$, where the coproduct is induced by the map on generators,

$$\psi(Sq^k) = \sum_{i=0}^{k} Sq^i \otimes Sq^{k-i}, \text{ for } p = 2. \text{ For } p \text{ odd,} \tag{10.39}$$

$$\psi(P^j) = \sum_{i=0}^{j} P^i \otimes P^{j-i}, \text{ and } \psi(\beta) = \beta \otimes 1 + 1 \otimes \beta. \tag{10.40}$$

Exercise. Show that with the coproduct as defined above, $\mathcal{A}_p$ is a Hopf algebra. That is, show that the coproduct ψ is a map of algebras over $\mathbb{Z}/p$.

Hint. For $p = 2$ let $\bar{A}_2$ be the free algebra over $\mathbb{Z}/2$ generated by Sq^i : $i > 0$. There is a natural surjective map $\pi : \bar{A}_2 \to \mathcal{A}_2$ sending Sq^i to Sq^i that has kernel generated by the Adem relations. The map ψ defined above defines an algebra homomorphism $\bar{\psi} : \bar{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_2$. You have to show that $\bar{\psi}$ vanishes on $ker\, \pi$. If you have trouble carrying this out, see the argument at the beginning of Chapter II of [144].

We make three more observations about the Hopf algebra structure of the Steenrod algebra, $\mathcal{A}_p$.

- $\mathcal{A}_p$ is a *connected* Hopf algebra. Recall that a Hopf algebra $\mathcal{A}$ over a field k is *connected* if it is connected as a coalgebra. This means that $\mathcal{A}$ has no nonzero terms of negative grading, and the counit map $\epsilon : \mathcal{A} \to k$ is an isomorphism in degree zero.

  Exercise. Show that in a connected Hopf algebra over k, the coproduct map satisfies
  $$\psi(a) = a \otimes 1 + 1 \otimes a + \sum_i a_i' \otimes a_i''$$
  where the degrees of all the terms a_i' and a_i'' in this summation are all positive. Notice that in the case of $\mathcal{A}_p$, this follows immediately from the definition of ψ on the generators P^k and β.

- As a connected Hopf algebra, the Steenrod algebra $\mathcal{A}_p$ has a *canonical antiautomorphism*
$$\chi : \mathcal{A}_p \to \mathcal{A}_p \tag{10.41}$$
defined via the following formulas. If $\psi(a) = a \otimes 1 + 1 \otimes a + \sum_i a_i' \otimes a_i''$ Then
$$\chi(1) = 1 \quad \text{and} \quad 0 = \sum_{i=0}^{k} \chi(a_i')a_i''.$$

Exercises.

1. Show that $\chi : \mathcal{A}_p \to \mathcal{A}_p$ is an *antiautomorphism.* That is, $\chi(1) = 1$ and $\chi(a \cdot b) = \chi(b)\chi(a)$.

2. For $p = 2$ consider the total Steenrod square $Sq = 1 + Sq^1 + Sq^2 + \cdots + Sq^j + \cdots$ and its image under the antiautomorphism, $\chi(Sq) = 1 + \chi(Sq^1) + \chi(Sq^2) + \cdots + \chi(Sq^j) + \cdots$. Show that they are "inverse" to each other in the sense that
$$\chi(Sq)Sq = 1.$$

- As a coalgebra, $\mathcal{A}_2$ is cocommutative. That is, the following diagram commutes:
$$\begin{array}{ccc} \mathcal{A}_2 & \xrightarrow{\psi} & \mathcal{A}_2 \otimes \mathcal{A}_2 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{A}_2 & \xrightarrow{\psi} & \mathcal{A}_2 \otimes \mathcal{A}_2 \end{array}$$
where $\tau(a \times b) = b \otimes a$. This follows from the symmetry of the Cartan product formula upon which the coproduct ψ is based (10.39).

Notice that since $\mathcal{A}_p$ is a connective Hopf algebra for each prime p, we can consider the algebra structure on its dual, $\mathcal{A}_p^*$. By the dual of the Steenrod algebra $\mathcal{A}_p^*$ we mean the *graded* dual. that is,
$$\mathcal{A}_p^* = \bigoplus_{d=0}^{\infty} (\mathcal{A}_p)_d^*$$
where $(\mathcal{A}_p)_d^* = Hom((\mathcal{A}_p)_d, \mathbb{Z}/p)$ with $(\mathcal{A}_p)_d$ being the degree d component of $\mathcal{A}_p$.

It turns out, by a beautiful calculation of Milnor in [117], the dual algebra structure on $\mathcal{A}_p^*$ is very tractable, even though the algebra structure on $\mathcal{A}_p$ is quite complicated. We refer the reader to Milnor's orignial paper [117] or the account in Steenrod and Epstein's book [144] for a proof of the following theorem.

Theorem 10.59. *(Milnor [117]).*
Let $\{(Sq^I)^* : \quad I$ is admissible$\}$ be the additive basis of $\mathcal{A}_2^*$ dual to the basis of admissible sequences of $\mathcal{A}_2$. Let I_k be the admissible sequence $I_k = (2^{k-1}, 2^{k-2}, \cdots, 2, 1)$, and let $\xi_k = (Sq^{I_k})^* \in \mathcal{A}_2^*$. Then as an algebra, $\mathcal{A}_2^*$ is the polynomial algebra on the ξ_j's,

$$\mathcal{A}_2^* \cong \mathbb{Z}/2[\xi_1, \xi_2, \cdots, \xi_j, \cdots]$$

To state the analogous result for p an odd prime, consider the admissible sequences

$$J_k = (0, p^{k-1}, 0, p^{k-2}, \ldots, 0, p^1, 0, p^0),$$

and

$$J'_k = (0, p^{k-1}, 0, p^{k-2}, \ldots, 0, p^1, 0, p^0, 1).$$

Let P^{J_k} and $P^{J'_k}$ be the corresponding admissible monomials in $\mathcal{A}_p$. Notice that P^{J_k} involves no Bocksteins, and $P^{J'_k} = P^{J_k}\beta$.

Let $\xi_k \in \mathcal{A}_p^*$ be the dual of P^{J_k} and τ_k be the dual of $P^{J'_k}$ with respect to the basis of admissible monomials in $\mathcal{A}_p$. Notice that ξ_k has degree $2(p^k - 1)$ and τ_k has degree $2p^k - 1$ in $\mathcal{A}_p^*$.

Theorem 10.60. *(Milnor [117]) There is an isomorphism of graded algebras,*

$$E(\tau_0, \tau_1, \ldots) \otimes \mathbb{Z}/p[\xi_1, \xi_2, \ldots] \cong \mathcal{A}_p^*$$

where $E(\tau_0, \tau_1, \ldots)$ is the exterior algebra over $\mathbb{Z}/p$ generated by the τ_i's, and $\mathbb{Z}/p[\xi_1, \xi_2, \ldots]$ is the polynomial algebra over $\mathbb{Z}/p$ generated by the ξ_j's.

10.9.5 The Hopf Invariant

We now describe a classical application of the the Steenrod algebra to the homotopy groups of spheres. In particular we study the question of the existence of elements of the homotopy groups of spheres having *Hopf invariant* one.

Given a map $\phi : S^{2n-1} \to S^n$, it's *Hopf invariant,* $h(\phi)$, is defined as follows. Consider the mapping cone,

$$C(\phi) = S^n \cup_\phi D^{2n}$$

where here D^{2n} represents the closed disk of dimension $2n$ which is attached to S^n along its boundary $\partial D^{2n} = S^{2n-1}$ via the map ϕ. That is, $C(\phi)$ is the CW complex built out of the union of S^n with D^{2n}, subject to the identification of $x \in S^{2n-1} = \partial D^{2n}$ with $\phi(x) \in S^n$.

Now compute in mod 2 cohomology

$$\tilde{H}^q(C(\phi); \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } q = n \text{ or } 2n \\ 0 & \text{otherwise} \end{cases}$$

Let $\sigma_n \in H^n(C(\phi);\mathbb{Z}/2)$ and $\sigma_{2n} \in H^{2n}(C(\phi);\mathbb{Z}/2)$ be the generators. Now take the cup square,

$$\sigma_n^2 = \epsilon \cdot \sigma_{2n} \in H^{2n}(C(\phi)\mathbb{Z}/2)$$

where $\epsilon \in \mathbb{Z}/2$. Then the Hopf invariant of ϕ is defined to be the coefficient

$$h(\phi) = \epsilon \in \mathbb{Z}/2.$$

Notice that we could have equivalently defined the Hopf invariant $h(\phi)$ by

$$Sq^n(\sigma_n) = h(\phi)\sigma_{2n}.$$

Exercises.

1. Show that the Hopf invariant is a homotopy invariant. That is, if ϕ_1 and $\phi_2 : S^{2n-1} \to S^n$ are homotopic, then $h(\phi_1) = h(\phi_2)$, and moreover,

$$h : \pi_{2n-1}(S^n) \to \mathbb{Z}/2$$

is a homomorphism.

2. Extend the definition of the Hopf invariant(s) to the stable homotopy groups of spheres

$$h_k : \pi_{k-1}(\mathbb{S}) \to \mathbb{Z}/2,$$

where if $\psi : \Sigma^{k-1}\mathbb{S} \to \mathbb{S}$ represents a class in $\pi_{k-1}(\mathbb{S})$, and it has mapping cone $C(\psi)$, then define $h_k(\psi)$ by the equation in cohomology

$$Sq^k(\sigma_0) = h_k(\psi) \cdot \sigma_k$$

where σ_0 and σ_k are the generators of $H^q(C(\psi)\mathbb{Z}/2)$ in dimensions *zero* and k respectively.

Show that $h_k : \pi_{k-1}(\mathbb{S}) \to \mathbb{Z}/2$ is well-defined.

3. Consider the self map of the sphere spectrum $t : \mathbb{S} \to \mathbb{S}$ of degree 2. That is, $t \in \pi_0(\mathbb{S}) = \mathbb{Z}$ represents $2 \in \mathbb{Z}$. Show that t has Hopf invariant one,

$$h_1(t) = 1 \in \mathbb{Z}/2.$$

We now describe an immediate application of the Steenrod algebra to the problem of the existence of elements of the stable homotopy groups of spheres having Hopf invariant one.

Theorem 10.61. *If there exists an element $\phi \in \pi_{k-1}(\mathbb{S})$ with Hopf invariant $h_k(\phi) = 1 \in \mathbb{Z}/2$, then k is a power of 2.*

Proof. Suppose $\phi \in \pi_{k-1}(\mathbb{S})$ has Hopf invariant one. Then in the mapping cone $C(\phi)$, $Sq^k(\sigma_0) = \sigma_k \in H^i(C(\phi);\mathbb{Z}/2)$. Suppose k is not a power of 2. The by Lemma 10.55, Sq^k is decomposable. So we may write

$$Sq^k = \sum_{j=1}^{k-1} a_j b_j$$

where for each j, the dimension of b_j is equal to j and the dimension of a_j is $k-j$. Now $Sq^k(\sigma_0) \neq 0$ implies that $b_j(\sigma_0) \neq 0$ for some j. But $b_j(\sigma_0) \in H^j(C(\phi);\mathbb{Z}/2) = 0$ since $1 \leq j \leq k-1$. This contradiction implies the theorem. □

This result has an important application to the question of the existence of certain multiplicative structures on spheres and on Euclidean spaces.

Let S_i, $i = 1,2,3$ be spheres of dimension $n-1$, and suppose one has a pairing

$$\mu : S_1 \times S_2 \to S_3.$$

We say that μ has bidegree (α, β) if the restriction of μ to $S_1 \times x_2$ has degree α and the restriction of μ to $x_1 \times S_2$ has degree β. Here $x_i \in S_i$ are basepoints. Notice that the degree is independent of the choices of $x_i \in S_i$. We observe that if we think of $S^1 \in \mathbb{C}$ as the unit complex numbers, then complex multiplication defines a map

$$\mu_1 : S^1 \times S^1 \to S^1$$

of bidegree $(1,1)$. Similiarly multiplication of quaternions defines a map $\mu_3 : S^3 \times S^3 \to S^3$ and multiplication of the octonians defines a map $\mu_7 : S^7 \times S^7 \to S^7$, both having bidegree $(1,1)$.

Now go back to the general case of a map $\mu : S_1 \times S_2 \to S_3$ of bidegree (α, β). Let D_i, $i = 1,2,3$ be closed n-dimensional disks so that

$$\partial D_i = S_i.$$

Notice that $\partial(D_1 \times D_2) = (S_1 \times D_2) \cup (D_1 \times S_2)$ which is a $(2n-1)$ dimensional sphere, and that

$$(D_1 \times S_2) \cap (S_1 \times D_2) = S_1 \times S_2.$$

Consider the suspension ΣS_3 which is an n-dimensional sphere. This suspension consists of an upper and lower cone which we denote by C_+ and C_-. These are n-dimensional cells with $C_+ \cap C_- = S_3$. We extend the map $\mu : S_1 \times S_2 \to S_3$ to a map

$$C(\mu) : (D_1 \times S_2) \cup (S_1 \times D_2) \to C_+ \cup C_- = \Sigma S_3 \cong S^n$$

in such a way that $C(\mu)(D_1 \times S_2) \subset C_+$ and $C(\mu)(S_1 \times D_2) \subset C_-$. (We leave it to the reader to verify that such and extension can be produced.) Then $C(\mu)$ is a map

$$C(\mu) : S^{2n-1} \to S^n.$$

We now prove the following theorem about this construction.

Theorem 10.62. *The Hopf invariant of the map $C(\mu) : S^{2n-1} \to S^n$ is the product of the components of the bidegree mod 2:*

$$h(C(f)) = \alpha\beta \in \mathbb{Z}/2.$$

Proof. (See [144]) The product of the disks $D_1 \times D_2$ has boundary equal to $(D_1 \times S_2) \cup (S_1 \times D_2)$. So may consider the space

$$X = (D_1 \times D_2) \cup_{C(\mu)} S^n$$

where the attaching is along the boundary $\partial(D_1 \times D_2)$ via the map $C(\mu)$. Notice that the attaching map gives rise to a map of triples

$$g : (D_1 \times D_2, D_1 \times S_2, S_1 \times D_2) \to (X, C_+, C_-).$$

Let $u \in H^n(X; \mathbb{Z}/2) = \mathbb{Z}/2$ be the generator. Define u_+ and u_- to be the inverse images of u under the isomorphisms $H^n(X, C_+; \mathbb{Z}/2) \xrightarrow{\cong} H^n(X; \mathbb{Z}/2)$ and $H^n(X, C_-; \mathbb{Z}/2) \xrightarrow{\cong} H^n(X; \mathbb{Z}/2)$ respectively. Consider the commutative diagram (all coefficients are taken to be $\mathbb{Z}/2$)

$$\begin{array}{ccc} H^n(X) \otimes H^n(X) & \xrightarrow{\times} & H^{2n}(X) \\ \cong\uparrow & & \uparrow\cong \\ H^n(X, C_+) \otimes H^n(X, C_-) & \xrightarrow{\times} & H^{2n}(X, \Sigma S_3) \end{array}$$

Thus the cup product $u_+ \cup u_-$ has image u^2 under the map $H^{2n}(X; \Sigma S_3) \to H^{2n}(X)$.

Now easy diagram chases that we leave to the reader (or refer to [144]) show that the map of triples g restricts to maps in cohomology

$$g^* : H^n(X, C_-) \to H^n(D_1 \times D_2, S_1 \times D_2) \quad \text{and}$$
$$g^* : H^n(X, C_+) \to H^n(D_1 \times D_2, D_1 \times S_2)$$

such that $g^*(u_+) = \alpha v_+$ and $g^*(u_-) = \beta v_-$ where $v_+ \in H^n(D_1 \times D_2, S_1 \times D_2)$ and $v_- \in H^n(D_1 \times D_2, D_1 \times S_2)$ are the generators. Notice that v_+ determines a class $v_1 \in H^n(D_1, S_1)$ and v_- determines a class $v_2 \in H^n(D_2, S_2)$ under the obvious projection maps. Now

$$v_+ \cup v_- = (v_1 \times 1) \cup (1 \times v_2) = v_1 \times v_2.$$

Therefore

$$g^*(u_+) \cup g^*(u_-) = \alpha\beta(v_1 \times v_2)$$

and $(v_1 \times v_2)$ generates $H^{2n}(D_1 \times D_2; D_1 \times S_2 \cup S_1 \times D_2)$.

Now $g : (D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2) \to (X, \Sigma S_3)$ is a relative homeomorphism and so induces an isomorphism in cohomology. Therefore we have isomorphisms

$$H^{2n}(X) \xleftarrow{\cong} H^{2n}(X, \Sigma S_3) \xrightarrow{\cong} H^{2n}(D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2)$$

Under these isomorphisms, $u^2 \in H^{2n}(X)$ corresponds to $u_- \cup u_+ \in H^{2n}(X; \Sigma S_3)$ and to $\alpha\beta(u_1 \times u_2) \in H^{2n}(D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2)$. Since $(u_1 \times u_2) \in H^{2n}(D_1 \times D_2, D_1 \times S_2 \cup S_1 \times D_2)$ corresponds to the generator of $H^{2n}(X)$ under these isomorphisms, this completes the proof of the theorem. □

Notice that Theorem 10.62 says that if there is a pairing

$$\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$$

that has bidegree $(1, 1)$ then the resulting construction $C(\mu) : S^{2n-1} \to S^n$ has Hopf invariant one. But by Theorem 10.61 we know that this cannot happen unless n is a power of 2. That is to say we have the following application of these results.

Corollary 10.63. *If there is a pairing $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree $(1, 1)$, then $n = 2^k$ for some $k \geq 0$.*

Finally we remark that if $\mathbb{R}^n$ has the structure of a division algebra (even a non-associative one) then its unit sphere S^{n-1} would admit a pairing of bidigree $(1, 1)$. This is given by the restriction of the multiplication map

$$\mu : S^{n-1} \times S^{n-1} \subset (\mathbb{R}^n - \{0\}) \times (\mathbb{R}^n - \{0\}) \xrightarrow{multiply} \mathbb{R}^n - \{0\} \xrightarrow{\simeq} S^{n-1}$$

where the last map is the homotopy equivalence given by radial retraction of $\mathbb{R}^n - \{0\}$ onto the unit sphere. Notice that the image of the multiplication of two nonzero elements of $\mathbb{R}^n$ is nonzero is because a division algebra contains no zero divisors.

Exercise. Show that the pairing of the unit sphere S^{n-1} described above when $\mathbb{R}^n$ is a not-necessarily commutative or associative division algebra, has bidegree $(1, 1)$.

From these arguments we know that the only dimensions in $\pi_{n-1}(\mathbb{S})$ that can possibly contain elements of Hopf invariant one are when $n = 2^k$ for some $k \geq 0$. In one of the most striking algebraic topology results of the 20th century, J. F. Adams showed that there are no elements of $\pi_{n-1}(\mathbb{S})$ of Hopf invariant one unless $n = 1, 2, 4, 8$ [3]. In particular this means that the only dimensions n for which $\mathbb{R}^n$ can have the structure of a division algebra are $n = 1, 2, 4$, or 8. Of course such structures in these dimensions are well known: the real numbers when $n = 1$, the complex numbers when $n = 2$, the Hamiltonians when $n = 4$, and the octonions when $n = 8$. What was startling was that sophisticated techniques from algebraic topology could be used to show that no such structures exist in other dimensions.

Adams's technique for the solution of this problem is what became known as the Adams spectral sequence. We will say more about this spectral sequence later in this chapter.

10.9.6 Definitions

So far our discussion of the Steenrod algebra and its applications were based on the assumption that the Steenrod squares and reduced powers exist, and satisfy the axioms 10.33 and 10.34. We now give a quick definition of the Steenrod squaring operations.

Given a space X, consider the diagonal mapping $\Delta : X \to X \times X$. This map is clearly equivariant with respect to the trivial $\mathbb{Z}/2$- action on the source X and the action on the target $X \times X$ given by permuting the coordinates. Indeed Δ embeds X as the subspace of fixed points of this action. By this equivariance we can extend this map, which by abuse of notation we also call Δ,

$$\Delta : E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X \to E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X) \tag{10.42}$$
$$B\mathbb{Z}/2 \times X \xrightarrow{\Delta} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X).$$

In this notation $E\mathbb{Z}/2$ refers to the total space of the universal principal bundle

$$\mathbb{Z}/2 \to E\mathbb{Z}/2 \to B\mathbb{Z}/2$$

a model of which can be taken to be the $\mathbb{Z}/2$-covering space

$$\mathbb{Z}/2 \to S^\infty \to \mathbb{RP}^\infty.$$

The subscript $\mathbb{Z}/2$ under the product sign means taking the orbit space of the induced diagonal action on the product space. Notice that since in the source space the action on X is trivial,

$$E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X = B\mathbb{Z}/2 \times X \simeq \mathbb{RP}^\infty \times X.$$

One way to define the mod 2 Steenrod squares is by computing this map in cohomology. To do this, we recall that S^∞ has a $\mathbb{Z}/2$-equivariant cell decomposition with two cells e_i and e'_i in each dimension i. The $\mathbb{Z}/2$-action interchanges these two cells. Let $C_*(S^\infty)$ be the resulting cellular chain complex with coefficients in $\mathbb{Z}/2$. Since S^∞ is contractible and its $\mathbb{Z}/2$-action is free, $C_*(S^\infty)$ is a free acyclic resolution of the ground field $\mathbb{Z}/2$ as a module over the group ring $\mathcal{R}_2 = \mathbb{Z}/2[\mathbb{Z}/2]$. Explicitly it is the complex

$$\to \cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_0} C_0 \xrightarrow{\epsilon} \mathbb{Z}/2$$

where C_i is the 2-dimensional vector space generated by e_i and e'_i where the $\mathbb{Z}/2$ action interchanges these generators. That is, if $t \in \mathbb{Z}/2$ is the nonzero element, the the module structure of C_i is given by $t \cdot e_i = e'_i$ and $t \cdot e'_i = e_i$. The boundary homomorphism is given by

$$\partial_i(x) = (1 + t)x$$

for every i.

Exercise. Show that this complex is a free acyclic resolution of the ground field $\mathbb{Z}/2$ over $\mathcal{R}_2$.

If we let $\mathcal{S}_*(X)$ and $\mathcal{S}^*(X)$ respectively denote the singular chains and cochains with coefficients in $\mathbb{Z}/2$ of a space X, then using the Alexander-Whitney correspondence, which gives a chain equivalence $\mathcal{S}_*(X \times Y) \xrightarrow{\simeq} \mathcal{S}_*(X) \otimes \mathcal{S}_*(Y)$, one sees that the cohomology $H^*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2)$ can be computed using the cochain complex $C^*(S^\infty) \otimes_{\mathcal{R}_2} \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$. If $\alpha \otimes \beta \in \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$ then the $\mathcal{R}_2$ action is given by $t(\alpha \otimes \beta) = \beta \otimes \alpha$.

Exercises.

1. Verify this claim. That is, show that the cochain complex $C^*(S^\infty) \otimes_{\mathcal{R}_2} \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$ computes $H^*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2)$.
2. Show that for any cohomology class $\alpha \in H^q(X; \mathbb{Z}/2)$ represented by a cocycle $\tilde{\alpha} \in \mathcal{S}^q(X)$, the class $1 \otimes \tilde{\alpha} \otimes \tilde{\alpha} \in C^*(S^\infty) \otimes_{\mathcal{R}_2} \mathcal{S}^*(X) \otimes \mathcal{S}^*(X)$ is a cocycle and so represents an element

$$1 \otimes \alpha \otimes \alpha \in H^{2q}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2).$$

Verify that this correspondence gives a well-defined homomorphism

$$\omega : H^q(X; \mathbb{Z}/2) \to H^{2q}(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2).$$

Now consider the equivariant diagonal map in cohomology:

$$\Delta^* : H^*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} (X \times X); \mathbb{Z}/2) \to H^*(B\mathbb{Z}/2 \times X; \mathbb{Z}/2).$$

For $\alpha \in H^q(X; \mathbb{Z}/2)$, the Kunneth theorem says that we can write

$$\Delta^*(\omega(\alpha)) = \sum_{i=0}^{2q} a^i \otimes \beta_i$$

where $a \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2) = H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator, $a^i \in H^i(\mathbb{RP}^\infty; \mathbb{Z}/2)$ is the i-fold cup product, and $\beta_i \in H^{2q-i}(X; \mathbb{Z}/2)$ is some cohomology class. We define the Steenrod square Sq^{q-i} by letting

$$Sq^{q-i}(\alpha) = \beta_i.$$

A shorthand description of the above definition is

$$\Delta^*(1 \otimes \alpha \otimes \alpha) = \sum_{i=0}^{2q} a^i \otimes Sq^{q-i}(\alpha). \tag{10.43}$$

Exercise Show that this definition satisfies the following axiom from (10.33):

Axiom: If $\alpha \in H^q(X; \mathbb{Z}/2)$, $Sq^q(\alpha) = \alpha^2 \in H^{2q}(X l \mathbb{Z}/2)$.

We now have a map $Sq^k : H^q(X;\mathbb{Z}/2) \to H^{q+k}(X;\mathbb{Z}/2)$ for each space X and for each q and k. Of course now one needs to check that this defines a cohomology operation satisfying all the Axioms 10.33. This is essentially done in [144]. By "essentially" we mean that the above description is a topological version of an algebraic definition of the Steenrod squares given by Steenrod and Epstein in [144]. A closely related approach using the notion of "cup - i" products is given in the book by Mosher and Tangora [124]. An elegant, more general approach to Steenrod operations is given by J. P. May in [104]. We encourage the reader to consult these sources for more thorough developments of the Steenrod algebras.

10.9.7 Free modules over $\mathcal{A}_p$

We end this section on the Steenrod algebra with an observation about what it means for the cohomology of a spectrum to be a *free module* over $\mathcal{A}_p$. From Theorem 10.48 we know that $H^*(\mathbb{HZ}/p;\mathbb{Z}/p)$ is isomorphic to the Steenrod algebra $\mathcal{A}_p$. More generally we can conclude the following:

Corollary 10.64. *Let $\mathbb{E}$ be a spectrum that is weakly homotopy equivalent to a wedge of suspensions of the Eilenberg-MacLane spectrum $\mathbb{HZ}/p$. More precisely, suppose there is a graded $\mathbb{Z}/p$ - vector space with basis $\mathcal{B} = \{\eta\}$ such that*

$$\mathbb{E} \simeq \bigvee_{\eta \in \mathcal{B}} \Sigma^{|\eta|}\mathbb{HZ}/p$$

where $|\eta|$ denotes the grading (dimension) of a basis element $\eta \in \mathcal{B}$. Then $H^*(\mathbb{E};\mathbb{Z}/p)$ is a free module over the Steenrod algebra $\mathcal{A}_p$ with basis $\mathcal{B}$. That is,

$$H^*(\mathbb{E};\mathbb{Z}/p) \cong \bigoplus_{\eta \in \mathcal{B}} \Sigma^{|\eta|}\mathcal{A}_p.$$

We now observe that the converse to this corollary is also true.

Theorem 10.65. *Let p be a prime and suppose $\mathbb{E}$ is a spectrum such that $H^*(\mathbb{E};\mathbb{Z}/q) = 0$ for all primes $q \neq p$ and that $H^*(\mathbb{E};\mathbb{Q}) = 0$. (Such a spectrum is called "p-local".) Suppose furthermore that $H^*(\mathbb{E};\mathbb{Z}/p)$ is a free module over the Steenrod algebra $\mathcal{A}_p$, with a countable basis. Then $\mathbb{E}$ is weakly homotopy equivalent to a wedge of suspensions of the Eilenberg-MacLane spectrum $\mathbb{HZ}/p$.*

Proof. Let $\mathcal{B} = \{\eta\}$ be a basis for $H^*(\mathbb{E};\mathbb{Z}/p)$ as an $\mathcal{A}_p$-module. By assumption it is countable. As cohomology classes these classes can be represented by maps of spectra, which we call

$$b_\eta : \mathbb{E} \to \Sigma^{|\eta|}\mathbb{HZ}/p.$$

Now recall that any spectrum $\mathbb{X}$ has a "pinch map" $\mathbb{X} \to \mathbb{X} \vee \mathbb{X}$. Since one can suspend and desuspend a spectrum, this pinch map can be viewed as being induced by applying the pinch map $S^1 \to S^1 \vee S^1$ in the suspension coordinate. (Check that the homotopy type of the pinch map on a spectrum $\mathbb{X}$ is well-defined.) By iterating the pinch map a countable number of times one then has a map

$$\bigvee_{\eta \in \mathcal{B}} b_\eta : \mathbb{E} \to \bigvee_{\eta \in \mathcal{B}} \Sigma^{|\eta|} \mathbb{H}\mathbb{Z}/p.$$

It is immediate that this map induces an isomorphism in (co)homology with $\mathbb{Z}/p$-coefficients. Since both the source and target of this map have zero (co)homology with $\mathbb{Z}/q$-coefficients for q any prime other than p, and also have zero rational (co)homology, this means that this map induces an isomorphism in integral homology, and therefore by the Hurewicz theorem, in homotopy groups. □

10.10 The Adams Spectral Sequence

In this section we give a brief introduction to the (classical) Adams spectral sequence. There are many good references for further study, including [7] part III section 15, [132] chapter 3, and [124] chapter 18.

Let $\mathbb{X}$ be a spectrum of finite type. Fix a prime p. Until otherwise stated, all cohomology will be taken with $\mathbb{Z}/p$-coefficients. The mod- p cohomology $H^*(\mathbb{X}; \mathbb{Z}/p)$ is a module over the Steenrod algebra $\mathcal{A}_p$ of finite type. So let $\mathcal{B}_0 = \{b_1, \dots, b_i, \dots\}$ be a set of cohomology classes, with only finitely many of these classes in any given dimension, that generate $H^*(\mathbb{X}; \mathbb{Z}/p)$ over $\mathcal{A}_p$. These can be represented by maps to Eilenberg-MacLane spectra. Taking a wedge of all of them produces a map

$$\beta_0 = \bigvee_{b_i \in \mathcal{B}_0} b_i : \mathbb{X} \to \bigvee_{b_i \in \mathcal{B}_0} \Sigma^{|b_i|} \mathbb{H}\mathbb{Z}/p.$$

In cohomology this defines (and, up to homotopy, is defined by) a map which we give the same name to,

$$\beta_0 : \bigoplus_{b_i \in \mathcal{B}_0} \Sigma^{|b_i|} \mathcal{A}_p \to H^*(\mathbb{X}; \mathbb{Z}/p)$$

which is a surjective map of $\mathcal{A}_p$-modules.

Let $\mathbb{K}_0 = \bigvee_{b_i \in \mathcal{B}_0} \Sigma^{|b_i|} \mathbb{H}\mathbb{Z}/p$, and consider the homotopy cofiber (mapping cone) $\mathbb{K}_0 \cup_{\beta_0} c(\mathbb{X})$. Let $j_0 : \mathbb{K}_0 \to \mathbb{K}_0 \cup_{\beta_0} c(\mathbb{X})$ be the map to the homotopy cofiber. The projection map to the suspension $\pi_0 : \mathbb{K}_0 \cup_{\beta_0} c(\mathbb{X}) \to \Sigma\mathbb{X}$ is the next map Barratt-Puppe extension of this cofibration sequence. Now since we are in the category of spectra, where objects and morphisms can be desuspended as well as suspended, we can define $\mathbb{X}_1$ to be the desuspension

$$\mathbb{X}_1 = \Sigma^{-1}\left(\mathbb{K}_0 \cup_{\beta_0} c(\mathbb{X})\right)$$

and by abuse of notation we let π_0 also denote the desuspension of the map just defined,

$$\pi_0 : \mathbb{X}_1 \to \mathbb{X}.$$

Similarly we let j_0 also denote its desuspension,

$$j_0 : \Sigma^{-1}\mathbb{K}_0 \to \mathbb{X}_1.$$

We therefore have a diagram

$$\begin{array}{ccccc} \Sigma^{-1}\mathbb{K}_0 & \xrightarrow{j_0} & \mathbb{X}_1 & & \\ & & \downarrow{\scriptstyle \pi_0} & & \\ & & \mathbb{X} & \xrightarrow[\beta_0]{} & \mathbb{K}_0 \end{array}$$

where each two successive maps form a homotopy cofibration sequence of spectra.

To continue with the construction of an *"Adams resolution"* of $\mathbb{X}$, we make the same constructions with $\mathbb{X}_1$ replacing $\mathbb{X}$. That is we define a spectrum $\mathbb{K}_1$ which is a wedge of Eilenberg-MacLane spectra of type $\mathbb{H}\mathbb{Z}/p$, and a map $\beta_1 : \mathbb{X}_1 \to \mathbb{K}_1$ which is surjective in cohomology. Doing the same procedure as above we produce a diagram

$$\begin{array}{ccccc} \Sigma^{-1}\mathbb{K}_1 & \xrightarrow[j_1]{} & \mathbb{X}_2 & & \\ & & \downarrow{\scriptstyle \pi_1} & & \\ \Sigma^{-1}\mathbb{K}_0 & \xrightarrow[j_0]{} & \mathbb{X}_1 & \xrightarrow[\beta_1]{} & \mathbb{K}_1 \\ & & \downarrow{\scriptstyle \pi_0} & & \\ & & \mathbb{X} & \xrightarrow[\beta_0]{} & \mathbb{K}_0 \end{array}$$

where in the sequences $\Sigma^{-1}\mathbb{K}_{i-1} \xrightarrow{j_{i-1}} \mathbb{X}_i \xrightarrow{\pi_{i-1}} \mathbb{X}_{i-1} \xrightarrow{\beta_{i-1}} \mathbb{K}_{i-1}$ for $i = 1, 2$ each two successive maps form homotopy cofibration sequences. (Here we are letting $\mathbb{X} = \mathbb{X}_0$.)

Let

$$\delta_0 = \Sigma\beta_1 \circ \Sigma j_0 : \mathbb{K}_0 \to \Sigma\mathbb{K}_1.$$

Exercise. Show that, in cohomology, the homomorphism induced by δ_0 which we call $\partial_0 : H^*(\Sigma\mathbb{K}_1) \to H^*(\mathbb{K}_0)$ is surjective onto the kernel of $\beta_0 : H^*(\mathbb{K}_0) \to H^*(\mathbb{X})$.

By continuing this process we build a tower of the form

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow{\scriptstyle \pi_{i+1}} & & \\
\Sigma^{-1}\mathbb{K}_i & \xrightarrow[j_i]{} & \mathbb{X}_{i+1} & \xrightarrow[\beta_{i+1}]{} & \mathbb{K}_{i+1} \\
 & & \downarrow{\scriptstyle \pi_i} & & \\
\Sigma^{i-1}\mathbb{K}_{i-1} & \xrightarrow[j_{i-1}]{} & \mathbb{X}_i & \xrightarrow[\beta_i]{} & \mathbb{K}_i \\
 & & \downarrow{\scriptstyle \pi_{i-1}} & & \\
 & & \vdots & & \\
 & & \downarrow{\scriptstyle \pi_2} & & \\
\Sigma^{-1}\mathbb{K}_1 & \xrightarrow[j_1]{} & \mathbb{X}_2 & \xrightarrow[\beta_2]{} & \mathbb{K}_2 \\
 & & \downarrow{\scriptstyle \pi_1} & & \\
\Sigma^{-1}\mathbb{K}_0 & \xrightarrow[j_0]{} & \mathbb{X}_1 & \xrightarrow[\beta_1]{} & \mathbb{K}_1 \\
 & & \downarrow{\scriptstyle \pi_0} & & \\
 & & \mathbb{X} & \xrightarrow[\beta_0]{} & \mathbb{K}_0
\end{array} \tag{10.44}$$

with the following properties:

1. Each $\mathbb{K}_i$ is a wedge of Eilenberg-MacLane spectra of type $\mathbb{HZ}/p$, and each map β_i, in cohomology, induces a surjection
$$\beta_i : H^*(\mathbb{K}_i) \to H^*(\mathbb{X}_i).$$
2. In the sequences $\Sigma^{-1}\mathbb{K}_{i-1} \xrightarrow{j_{i-1}} \mathbb{X}_i \xrightarrow{\pi_{i-1}} \mathbb{X}_{i-1} \xrightarrow{\beta_{i-1}} \mathbb{K}_{i-1}$ each two successive maps form homotopy cofibration sequences, for all i.

This type of tower is called a *"mod p Adams resolution"* of the spectrum $\mathbb{X}$. It is the basic homotopy theoretic construction that yields the Adams spectral sequence.

Let $\delta_i : \Sigma^{-1}\mathbb{K}_i \to \mathbb{K}_{i+1}$ be the composition

$$\delta_i = \beta_{i+1} \circ j_i : \Sigma^{-1}\mathbb{K}_i \to \mathbb{X}_{i+1} \to \mathbb{K}_{i+1}.$$

Let $C_i = H^*(\mathbb{K}_i)$. Since $\mathbb{K}_i$ is a wedge of Eilenberg-MacLane spectra of type $\mathbb{HZ}/p$, C_i is a free module over the mod p Steenrod algebra, $\mathcal{A}_p$. Let $\partial_i = \delta_i^* : C_{i+1} \to C_i$,

Exercise. Show that the sequence

$$\cdots \xrightarrow{\partial_{i+1}} C_{i+1} \xrightarrow{\partial_i} C_i \xrightarrow{\partial_{i-1}} C_{i-1} \to \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\beta_0} H^*(\mathbb{X})$$

is a *free resolution* of $H^*(\mathbb{X})$ as modules over the Steenrod algebra $\mathcal{A}_p$. That is,

1. All the homomorphisms in this sequence are maps of $\mathcal{A}_p$-modules,
2. the C_i's are all free modules,
3. The sequence is exact, and β_0 is surjective.

Consider the $\mathbb{Z}/p$ vector space of $\mathcal{A}_p$-module homomorphisms, $Hom^*_{\mathcal{A}_p}(C_i;\mathbb{Z}/p)$, where $\mathbb{Z}/p$ is given the trivial $\mathcal{A}_p$-module structure. This is actually a graded vector space, where the grading, denoted by the superscript, is the degree of the homomorphism. More precisely, if we assume that $\mathbb{Z}/p$ is in degree 0, then elements of $Hom^q_{\mathcal{A}_p}(C_i;\mathbb{Z}/p)$ are $\mathcal{A}_p$ module homomorphisms $\phi : C^q_p = H^q(\mathbb{K}_p) \to \mathbb{Z}/p$. Notice that the above free resolution induces a sequence

$$Hom^*_{\mathcal{A}_p}(C_0;\mathbb{Z}/p) \xrightarrow{\partial_0^*} Hom^*_{\mathcal{A}_p}(C_1;\mathbb{Z}/p) \xrightarrow{\partial_1^*} Hom^*_{\mathcal{A}_p}(C_2;\mathbb{Z}/p) \xrightarrow{\partial_2^*} \cdots$$
$$\xrightarrow{\partial_{i-1}^*} Hom^*_{\mathcal{A}_p}(C_i;\mathbb{Z}/p) \xrightarrow{\partial_i^*} Hom^*_{\mathcal{A}_p}(C_{i+1};\mathbb{Z}/p) \xrightarrow{\partial_{i+1}^*} \cdots$$

The cohomology of this complex are the so-called "Ext" groups. More precisely, if we let $Z^{s.t} = \operatorname{Ker} \partial_s^* : Hom^t(C_s, \mathbb{Z}/p) \to Hom^t(C^j_{s+1}, \mathbb{Z}/p)$ and $B^{s,t} = \operatorname{Image} \partial_{s-1}^* : Hom^t(C_{s-1}, \mathbb{Z}/p) \to Hom^t(C_s, \mathbb{Z}/p)$, we then define

$$Ext^{s,t}_{\mathcal{A}_p}(H^*(\mathbb{X});\mathbb{Z}/p) = Z^{s.t}/B^{s,t}. \tag{10.45}$$

For those who are not familiar with this type of homological algebra, we quickly review some properties of this construction.

10.10.0.1 An aside on homological algebra

Let R be a graded ring with a unit, and let P and Q be graded left R-modules. Recall that this means that we have sequences $\{P_i\}$ and $\{Q_j\}$ of left R_0-modules, together with associative actions $R \otimes P \to P$ and $R \otimes Q \to Q$. For purposes of this discussion we will assume that the modules P and Q are *connective,* meaning that $P_i = 0$ for $i < 0$ and $Q_j = 0$ for $j < 0$. The group of homomorphisms $Hom^t_R(P,Q)$ is the abelian group of R-module homomorphisms $P \to Q$ of degree $-t$. That is it is a sequence of R_0-module homomorphisms $\phi_j : P_j \to Q_{j-t}$ that respect the R-actions on P and Q.

Definition 10.32. *A sequence of graded $R-$ module homomorphisms*

$$\cdots C_{i+1} \xrightarrow{\partial_i} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-2}} \cdots \to C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\epsilon} P$$

is called a **projective resolution** *of the module P if the following conditions are satisfied.*

1. *Each $C_r = \{C_{r,s}\}$ is a projective left graded R-module*
2. *All the homomorphisms $\{\partial_i\}$ are R-module homomorphisms of degree zero, and*
3. *The sequence is exact.*

It is a standard result that such projective resolutions exist and that any two projective resolutions of the same module are chain homotopy equivalent. Since a free module is projective, a free resolution, which has the same definition except all the C_r's are graded *free* modules, is a special type of projective resolution.

Given a projective resolution $\{C_i, \partial_i\}$ of P one can consider the sequence

$$0 \to Hom_R^t(C_0, Q) \xrightarrow{\partial_0^*} Hom^t(C_1, Q) \to \cdots \xrightarrow{\partial_{i_1}^*} Hom_R^t(C_i, Q) \xrightarrow{\partial_i^*} \cdots .$$

Since $\partial_s^* \circ \partial_{s-1}^* = 0$, we can take the cohomology group at $Hom_R^t(C_s, Q)$, that is take $Ker\,\partial_s^*/Image\,\partial_{s-1}^*$ and the resulting group is denoted $Ext_R^{s,t}(P,Q)$. Since any two projective resolutions of P are chain homotopy equivalent, these Ext-groups are independent of the choice of resolution, Calculating these Ext-groups can be quite difficult. However we can quickly conclude the results in the following exercises:

Exercises. 1. Prove that $Ext_R^{0,t}(P,Q) \cong Hom_R^t(P,Q)$.

Hint. Show that when one applies the Hom functor to the exact sequence

$$C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\epsilon} P \to 0$$

one obtains an exact sequence

$$0 \to Hom_R^t(P,Q) \xrightarrow{\epsilon^*} Hom_R^t(C_0,Q) \xrightarrow{\partial_0^*} Hom_R^t(C_1,Q)$$

for each t.

2. Show that if P is a projective R-module, and Q is any R-module, then $Ext_R^{s,t}(P,Q) = 0$ for $s \geq 1$ and all t.

Of course the cases of primary interest to us are the Ext-groups $Ext_{\mathcal{A}_p}^{s,t}(H^*(\mathbb{X}); \mathbb{Z}/p)$, where $\mathbb{X}$ is a spectrum of finite type. In the case when $\mathbb{X}$ is the sphere spectrum, $\mathbb{S}$, then $H^*(\mathbb{S}) = \mathbb{Z}/p$. Furthermore $Ext_{\mathcal{A}_p}^{*,*}(\mathbb{Z}/p; \mathbb{Z}/p)$

becomes a bigraded algebra over $\mathbb{Z}/p$. The multiplication is induced by the composition pairing

$$Hom_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes Hom_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p) \to Hom_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p).$$

Indeed such an algebra structure exists on $Ext_R^{*,*}(k, k)$ whenever R is an augmented algebra over a field k. We will not go into the construction of this multiplication pairing here. We refer the reader to [94] for a detailed description.

We now return to our discussion about the Adams spectral sequence. This spectral sequence is obtained by considering maps of a spectrum $\mathbb{Y}$ into the various spectra occurring in an Adams resolution (10.44). As we will see, when $\mathbb{Y}$ is a finite spectrum this will lead to a spectral sequence useful for computing the graded groups $[\mathbb{Y}, \mathbb{X}]_*$. (The subscript denotes the degree of the map. That is, $[\mathbb{Y}, \mathbb{X}]_k = [\Sigma^k\mathbb{Y}, \mathbb{X}]$.) Of particular interest is the case when $\mathbb{Y}$ is the sphere spectrum $\mathbb{S}$, in which case our goal is to use this spectral sequence to compute the homotopy groups of the spectrum $\mathbb{X}$.

The first step is an understanding of the groups $[\mathbb{Y}, \mathbb{K}_i]_*$ This is particularly easy.

Lemma 10.66. *If $\mathbb{K}$ is a wedge of Eilenberg-MacLane spectra of type $\mathbb{HZ}/p$, then*

$$[\mathbb{Y}, \mathbb{K}]_* \cong Hom^*_{\mathcal{A}_p}(H^*(\mathbb{K}); H^*(\mathbb{Y}))$$

where, as above, all cohomology is taken with $\mathbb{Z}/p$ - coefficients.

Proof. We first consider the spectrum $\mathbb{K} = \mathbb{HZ}/p$. Every map from $f : \Sigma^k\mathbb{Y} \to \mathbb{HZ}/p$ induces a homomorphism $f^* : H^*(\mathbb{K}) \to H^{*-k}(\mathbb{Y})$ which preserves the $\mathcal{A}_p$-module structures. This defines a homomorphism

$$\begin{aligned}\phi_* : [\mathbb{Y}, \mathbb{HZ}/p]_k \to Hom^k_{\mathcal{A}_p}(H^*(\mathbb{HZ}/p), H^*(\mathbb{Y})) &\cong Hom^k_{\mathcal{A}_p}(\mathcal{A}_p, H^*(\mathbb{Y})) \\ &\cong H^{*-k}(\mathbb{Y})\end{aligned}$$

which we established was an isomorphism earlier. The fact that this generalizes to give an isomorphism

$$\phi_* : [\mathbb{Y}, \mathbb{K}]_k \xrightarrow{\cong} Hom^k_{\mathcal{A}_p}(H^*(\mathbb{K}), H^*(\mathbb{Y}))$$

for any spectrum $\mathbb{K}$ that is a wedge of suspensions of copies of $\mathbb{HZ}/p$ is immediate. □

We can therefore conclude that

$$\begin{aligned}[\mathbb{Y}, \mathbb{K}_i]_* &\cong Hom^*_{\mathcal{A}_p}(H^*(\mathbb{K}_i); H^*(\mathbb{Y})) \\ &\cong Hom^*_{\mathcal{A}_p}(C_i; H^*(\mathbb{Y}))\end{aligned} \tag{10.46}$$

Now as above, if we let $\delta_i = \beta_{i+1} \circ j_i : \Sigma^{-1}\mathbb{K}_i \to \mathbb{X}_{i+1} \to \mathbb{K}_{i+1}$ in the diagram (10.44), and $\pi_i = (\delta_i)^* : C_{i+1} \to C_i$, we can consider the following chain complex:

$$\begin{aligned}&Hom^*_{\mathcal{A}_p}(C_0; H^*(\mathbb{Y})) \xrightarrow{\partial_0^*} Hom^*_{\mathcal{A}_p}(C_1; H^*(\mathbb{Y})) \xrightarrow{\partial_1^*} Hom^*_{\mathcal{A}_p}(C_2; H^*(\mathbb{Y})) \xrightarrow{\partial_2^*} \cdots \\ &\xrightarrow{\partial_{i-1}^*} Hom^*_{\mathcal{A}_p}(C_i; H^*(\mathbb{Y})) \xrightarrow{\partial_i^*} Hom^*_{\mathcal{A}_p}(C_{i+1}; H^*(\mathbb{Y})) \xrightarrow{\partial_{i+1}^*} \cdots\end{aligned} \tag{10.47}$$

As a result of the above exercise we can conclude the following:

Corollary 10.67. *The cohomology of chain complex (10.47) is* $Ext^{*,*}_{\mathcal{A}_p}(H^*(\mathbb{X}), H^*(\mathbb{Y}))$. *More specifically, if we consider the piece of the chain complex*

$$Hom^t_{\mathcal{A}_p}(C_{s-1}; H^*(\mathbb{Y})) \xrightarrow{\partial_{s-1}^*} Hom^t_{\mathcal{A}_p}(C_s; H^*(\mathbb{Y})) \xrightarrow{\partial_s^*} Hom^t_{\mathcal{A}_p}(C_{s+1}; H^*(\mathbb{Y}))$$

then

$$Ker\, \partial_s^* / Image\, \partial_{s-1}^* = Ext^{s,t}_{\mathcal{A}_p}(H^*(\mathbb{X}); H^*(\mathbb{Y})).$$

We now consider more fully the diagram obtained by mapping a finite spectrum $\mathbb{Y}$ into the Adams resolution 10.44:

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow{\scriptstyle \pi_{i+1}} & & \\
[\mathbb{Y}, \Sigma^{-1}\mathbb{K}_i] & \xrightarrow[j_i]{} & [\mathbb{Y}, \mathbb{X}_{i+1}] & \xrightarrow[\beta_{i+1}]{} & [\mathbb{Y}, \mathbb{K}_{i+1}] \\
 & & \downarrow{\scriptstyle \pi_i} & & \\
[\mathbb{Y}, \Sigma^{i-1}\mathbb{K}_{i-1}] & \xrightarrow[j_{i-1}]{} & [\mathbb{Y}, \mathbb{X}_i] & \xrightarrow[\beta_i]{} & [\mathbb{Y}, \mathbb{K}_i] \\
 & & \downarrow{\scriptstyle \pi_{i-1}} & & \\
 & & \vdots & & \\
 & & \downarrow{\scriptstyle \pi_2} & & \\
[\mathbb{Y}, \Sigma^{-1}\mathbb{K}_1] & \xrightarrow[j_1]{} & [\mathbb{Y}, \mathbb{X}_2] & \xrightarrow[\beta_2]{} & [\mathbb{Y}, \mathbb{K}_2] \\
 & & \downarrow{\scriptstyle \pi_1} & & \\
[\mathbb{Y}, \Sigma^{-1}\mathbb{K}_0] & \xrightarrow[j_0]{} & [\mathbb{Y}, \mathbb{X}_1] & \xrightarrow[\beta_1]{} & [\mathbb{Y}, \mathbb{K}_1] \\
 & & \downarrow{\scriptstyle \pi_0} & & \\
 & & [\mathbb{Y}, \mathbb{X}] & \xrightarrow[\beta_0]{} & [\mathbb{Y}, \mathbb{K}_0]
\end{array} \tag{10.48}$$

This diagram allows one to define a decreasing filtration of $[\mathbb{Y}, \mathbb{X}]$ in the following natural way.

Definition 10.33. *We define $F^s[\mathbb{Y}, \mathbb{X}]$ to be $Image\,([\mathbb{Y}, \mathbb{X}_s] \to [\mathbb{Y}, \mathbb{X}])$ under the composite projection $\mathbb{X}_s \to \mathbb{X}$. Notice that $F^{s+1} \subset F^s$ for each s.*

We define $F^\infty[\mathbb{Y}, \mathbb{X}]$ to be the intersection $\bigcap_{s=0}^\infty F^s[\mathbb{X}, \mathbb{Y}]$.

Diagram 10.48 also allows us to form an exact couple with

$$\begin{aligned} D_1^{s,t} &= [\mathbb{Y}, \Sigma^{s-t}\mathbb{X}_s] \\ E_1^{s,t} &= [\mathbb{Y}, \Sigma^{s-t}\mathbb{K}_s] \end{aligned} \tag{10.49}$$

The maps in the exact couple are given by the maps in the above diagram. As described in section 10.3.2 an exact couple leads to a spectral sequence. This is known as the (classical) Adams Spectral Sequence, and Adams proved following theorem:

Theorem 10.68. *(Adams [2]) If $\mathbb{X}$ is a spectrum of finite type and $\mathbb{Y}$ is a finite spectrum, then the spectral sequence induced from the exact couple 10.49 has the following properties:*

1. *Each E_r term is a bigraded abelian group, and the r^{th} differential d_r is a homomorphism of bigraded groups,*

   $$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

   such that $d_r \circ d_r = 0$.

2. $E_2^{s,t} \cong Ext_{\mathcal{A}_p}^{s,t}(H^*(\mathbb{X}), H^*(\mathbb{Y}))$

3. *There is a natural monomorphism $E_{r+1}^{s,t} \to E_r^{s,t}$ whenever $r > s$, and*

   $$E_\infty^{s,t} \cong F^s[\Sigma^{t-s}\mathbb{Y}, \mathbb{X}]/F^{s+1}[\Sigma^{t-s}\mathbb{Y}, \mathbb{X}]$$

4. *$F^\infty[\mathbb{Y}, \mathbb{X}]$ consists of the subgroup of elements of finite order prime to p.*

Note. The language used to express items 3 and 4 is that the Adams spectral sequence converges to the "p-primary completion of $[\mathbb{Y}, \mathbb{X}]$".

In the important special case when $\mathbb{Y} = \mathbb{X} = \mathbb{S}$, then the Adams spectral sequence converges to the p-primary completion of the stable homotopy groups of spheres, $\pi_*(\mathbb{S})$. Notice that since $H^*(\mathbb{S}) \cong \mathbb{Z}/p$ as $\mathcal{A}_p$-modules, the E_2 term of this spectral sequence is given by

$$E_2^{s,t} \cong Ext_{\mathcal{A}_p}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p).$$

We record the following further properties of the Adams spectral sequence in this important case:

Theorem 10.69. *1. Every E_r is equipped with a natural multiplicative structure with respect to which d_r is a derivation (i.e $d_r(\alpha \cdot \beta) = \pm\alpha \cdot d_r(\beta) \pm d_r(\alpha) \cdot \beta$).*

2. *The product in E_2 is the usual product in Ext, which is associative and commutative.*

3. *The product structure of E_{r+1} is induced by the product structure on E_r, and hence is also associative and commutative.*

4. *The induced product on E_∞ is equivalent to the product structure obtained by passing to quotients in the product structure of $\pi_s(\mathbb{S})$ coming from the commutative ring spectrum structure of $\mathbb{S}$.*

We will not verify all of the properties in these theorem here. We refer the reader to Adams original paper [2], the excellent texts [7],[132], and in the form described here, the book by Mosher and Tangora, [124].

The Adams spectral sequence has been one of the main tools of stable homotopy theory since its discovery. However it has been notoriously difficult to compute explicitly. In the remainder of this section we describe a few basic results and give references for further study.

We first discuss the E_2-term, $Ext^{*,*}_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p)$. This is purely an algebraic entity, depending entirely on the structure of the Steenrod algebra, $\mathcal{A}_p$. To get a feel for the structure of this *Ext*-group, we look for a *minimal resolution* of $\mathbb{Z}/p$ as an $\mathcal{A}_p$- module. By this we mean the following:

Definition 10.34. *Let $\epsilon : \mathcal{A}_p \to \mathbb{Z}/p$ be the augmentation homomorphism. This is an isomorphism in degree zero and is the zero homomorphism in positive degrees. Let $I(\mathcal{A}_p) = ker(\epsilon)$. $I(\mathcal{A}_p)$ is called the augmentation ideal of $\mathcal{A}_p$. A homomorphism of $\mathcal{A}_p$-modules $\phi : P \to Q$ is said the minimal if $ker(f) \subset I(\mathcal{A}_p) \cdot P$. A projective resolution is called minimal if all the homomorphisms in the resolution are minimal.*

Exercise. Show that if

$$\cdots \xrightarrow{\partial_j} C_j \xrightarrow{\partial_{j-1}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\epsilon} P$$

is a minimal projective resolution of P as an $\mathcal{A}_p$-module, then

1. $\partial_s^* : Hom^t_{\mathcal{A}_p}(C_s, \mathbb{Z}/p) \to Hom^t_{\mathcal{A}_{p+1}}(C_{s+1}, \mathbb{Z}/p)$ is the zero homomorphism for every $s \geq 0$, and as a result

2. $Ext^{s,t}_{\mathcal{A}_p}(P, \mathbb{Z}/p) = Hom^t_{\mathcal{A}_p}(C_s, \mathbb{Z}/p)$.

We now consider a minimal resolution of $\mathbb{Z}/p$ as an $\mathcal{A}_p$-module. We focus on the case $p = 2$.

Notice that C_0 in a minimal resolution of $\mathbb{Z}/2$ is simply $\mathcal{A}_2$ and the map $\epsilon : C_0 \to \mathbb{Z}/2$ is the augmentation. Thus

$$Ext^{0,t}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2) = Hom^t_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } t = 0 \\ 0 & \text{if } t > 0 \end{cases} .$$

Since the kernel of $\epsilon : C_0 \to \mathbb{Z}/2$ is the augmentation ideal $I(\mathcal{A}_2)$, then C_1 in a minimal resolution is a free $\mathcal{A}_2$- module with one generator for every element of a minimal set of $\mathcal{A}_2$-module generators of $I(\mathcal{A}_2)$. As we saw earlier, such a minimal generating set is $\{Sq^{2^i}, i \geq 0\}$ since these elements are both indecomposable, and as a set generate $\mathcal{A}_2$ as an algebra. Therefore C_1 is a free $\mathcal{A}_2$-module with one generator in every dimension of the form $t = 2^i$ for $i \geq 0$. Each generator defines a homomorphism of $\mathcal{A}_2$-modules of degree $-t$ from C_1 to $\mathbb{Z}/2$. We therefore have the following:

Proposition 10.70.

$$Ext^{1,t}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \textit{if } t = 2^i, \textit{ whose generator is usually called } h_i \\ 0 & \textit{otherwise} \end{cases}$$

It turns out that the C_j for $j \geq 2$ in a minimal resolution of $\mathbb{Z}/2$ over $\mathcal{A}_2$ are much more difficult to compute. Other, more global algebraic methods have shown to be more effective at computing these *Ext*-groups, but they still remain very difficult. Recall, however that $Ext^{*,*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$ is a graded commutative algebra, and the following result can be proved.

Proposition 10.71. *. $Ext^{2,*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$ has a basis (over $\mathbb{Z}/2$) given by the products $h_i h_j \in Ext^{2,2^i+2^j}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$ subject only to the relations $h_i h_{i+1} = 0$ for $i \geq 0$.*

We also point out that the threefold products $h_i h_j h_k \in Ext^{3,*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$ are subject only to the additional relations $h_j h_{j+2}^2 = 0$ and $h_i^3 = h_{i-1}^2 h_{i+1}$. Besides these classes there are also indecomposable elements in $Ext^{3,*}$.

For these results and more general discussions of traditional (i.e non-computer based) techniques for computing these *Ext* groups we refer the reader to [3], [102], [147].

The elements of $Ext^{*,*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$ discussed here represent classes in the E_2-term of the Adams spectral sequence. It is natural to ask about their behaviors within the spectral sequence. We list a few results with their references here.

Proposition 10.72. *The element $h_i \in Ext^{1,2^i}_{\mathcal{A}_2}(\mathbb{Z}/2, /bz/2) = E_2^{1,2^i}$ is an infinite cycle in the Adams spectral sequence if and only if there is an element* *in the stable homotopy group of spheres $\beta_i \in \pi_{2^i-1}(\mathbb{S})$ with Hopf invariant one. By an "infinite cycle" we mean that all differentials are zero, $d_r(h_i) = 0$, $r \geq 2$, so that h_i represents an element of $E_\infty^{1,2^i}$.*

Proof. (idea). The idea of the proof is to observe that h_i is an infinite cycle if and only if there is a map of sphere spectra $\beta_i : \Sigma^\infty S^{2^i-1} \to \mathbb{S}$ so that in the cohomology of the mapping cone, $\mathbb{S} \cup_{\beta_i} \Sigma^\infty D^{2^i}$, the Steenrod operation Sq^{2^i} is nonzero. Given the definition of h_i in terms of a minimal resolution of $\mathbb{Z}/2$ over $\mathcal{A}_2$ (see the discussion before the statement of Proposition 10.70), this observation is not too difficult to prove, and we encourage the reader to try to do so as an exercise. The proof can be found in [2], [3]. We note that since Sq^{2^i} would be nonzero in the cohomology of $\mathbb{S} \cup_{\beta_i} \Sigma^\infty D^{2^i}$, this implies that β_i would have Hopf invariant one. □

As was discussed in §10.9.4, there exist four elements of Hopf invariant one, and the names usually given to them

- $\times 2 \in \pi_0(\mathbb{S})$, represented by $h_0 \in E_2^{1,1}$,
- $\eta \in \pi_1(\mathbb{S})$, represented by $h_1 \in E_2^{1,2}$,
- $\nu \in \pi_3(\mathbb{S})$, represented by $h_2 \in E_2^{1,4}$, and
- $\sigma \in \pi_7(\mathbb{S})$, repreented by $h_3 \in E_2^{1,8}$.

As mentioned earlier, Adams solved the Hopf invariant one problem in a landmark paper [3]. Interpreted in terms of the Adams spectral sequence, using the notion of secondary cohomology operations he proved the following:

Theorem 10.73. *(Adams [3]). For $i \geq 4$*

$$d_2(h_i) = h_0 h_{i-1}^2 \neq 0 \in E_2^{3,2^i+1}.$$

One may continue along these lines and ask which elements of $Ext_{\mathcal{A}_2}^{s,*}(\mathbb{Z}/2, \mathbb{Z}/2) = E_2^{s,*}$ can be infinite cycles in the Adams spectral sequence. Such elements are said to have Adams - *algebraic* filtration s. For $s = 2$ there is now a nearly complete answer based on the following results.

In 1967 Mahowald and Tangora proved the following [98]:

Theorem 10.74. *(Mahowald and Tangora [98]) Except for a finite number of special cases, the only elements of $E_2^{2,*}$ that can possibly be infinite cycles are $\{h_1 h_j,\ j \geq 0\}$ and $\{h_j^2,\ j \geq 0\}$.*

In a very important paper published in 1977, Mahowald proved the following, which gave the first infinite family of elements in $\pi_*(\mathbb{S})$ having the same Adams - algebraic filtration.

Theorem 10.75. *(Mahowald [97]). For each $j \neq 2$, $h_1h_j \in Ext^{2,2^j+2}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2)$ is an infinite cycle in the Adams spectral sequence and represents a nonzero class $\eta_j \in \pi_{2^j}(\mathbb{S})$.*

And finally, in truly groundbreaking work, Hill, Hopkins, and Ravenel [70] proved the following result published in 2016.

Theorem 10.76. *(Hill, Hopkins, and Ravenel [70]) For $j \geq 7$, $h_j^2 \in Ext^{2,2^{j+1}}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2)$ supports a nonzero differential in the Adams spectral sequence, and therefore does not represent an element in $\pi_*(\mathbb{S})$.*

Comments.

- This theorem disproved the famous *"Kervaire invariant one conjecture"* that states that in every dimension of the form $2^{j+1}-2$ there exists a framed manifold of Kervaire invariant one. The Kervaire invariant is a framed surgery obstruction that will be explained in our discussion of framed cobordism in chapter 11. The fact that this conjecture is equivalent to the elements $h_j^2 \in E_2^{2,2^{j+1}}$ all being infinite cycles in the Adams spectral sequence was a famous theorem of Browder [17] in 1969. The Kervaire invariant one conjecture was a centerpiece of study in both differential topology and homotopy theory for fifty years. The fact that h_0^2, h_1^2, h_2^2, and h_3^2 are infinite cycles follows immediately from the fact that h_0, h_1, h_2, and h_3 are infinite cycles. Mahowald and Tangora proved that h_4^2 is an infinite cycle in [99]. The fact that h_5^2 is an infinite cycle was a difficult and technical theorem proved by Barratt, Jones, and Mahowald in [12]. Whether h_6^2 is an infinite cycle or not is still unknown. If it is, it would represent an element in $\pi_{126}(\mathbb{S})$.

  The Hill-Hopkins-Ravenel theorem was a true tour-de-force and used a wide variety of techniques from homotopy theory including equivariant homotopy theory, motivic homotopy, and chromatic homotopy. We strongly recommend the reader consult [70] for a fascinating journey into this deep and beautiful mathematics.

- There are odd primary analogues of many of the 2-primary results stated here. For example, the odd primary analogue of Adams's proof of the Hopf invariant one conjecture was proved by Liulevicius in [93]. There is an analogue of Mahowald's η_j family at odd primes, discovered by R. Cohen in [27]. Although we should note that while Mahowald's η_j family is detected in algebraic filtration 2 in the mod 2 Adams spectral sequence, Cohen's families are detected in algebraic filtration 3 in the mod p Adams spectral sequence for p odd. Finally there is an analogue of the Hill-Hopkins-Ravenel theorem at odd primes, proved much earlier by Ravenel [131]. The analogue of h_j^2 at an odd prime is an indecomposable element often called $b_j \in Ext^{2,2(p-1)p^{j+1}}_{\mathcal{A}_p}(\mathbb{Z}/p,\mathbb{Z}/p)$ which Ravenel showed is not an infinite cycle in the mod p Adams spectral sequence for $p \geq 5$ and $j \geq 1$.

11

Cobordism theory

Cobordism is a theory coming out of work of Pontrjagin and Thom which gives one of the most important connections between differential topology and stable homotopy theory. The Pontrjagin-Thom theorem basically says that to classify smooth manifolds up to "cobordism", perhaps with structure (eg an orientation, almost complex structure, framing), one needs to study the homotopy type of a corresponding spectrum. This theorem supplied a tremendously important computational tool in differential topology, while at the same time served as an important stimulus for the development of stable homotopy theory. In this chapter we prove the Pontrjagin-Thom theorem, and use it as Thom did [150], to compute the unoriented cobordism ring. Along the way we show that the unoriented cobordism Thom spectrum, traditionally denoted $\mathbb{MO}$, is built out of mod 2 Eilenberg-MacLane spectra (the spectrum corresponding to ordinary mod 2 cohomology). These spectra and the algebra of natural transformations between them, namely the Steenrod algebra $\mathcal{A}_2$, were introduced and studied in the last chapter. We use this study to show that the unoriented cobordism ring turns out to be a polynomial algebra, and we give explicit examples of manifolds representing generators of this algebra. We will also discuss other cobordism rings (oriented cobordism, almost complex cobordism, framed cobordism). We describe Milnor's famous and beautiful calculation of the almost complex cobordism ring. This uses the "*Adams spectral sequence*" which we will also discuss in this chapter.

A theory such as framed cobordism is much more difficult to compute, but homotopy theoretic techniques lead to fascinating geometric consequences. In particular we will introduce the seminal work of Kervaire and Milnor on homotopy spheres and surgery theory, and study the Kervaire invariant one problem. We discuss its history as well as the dramatic recent work of Hill, Hopkins, and Ravenel [70] that (nearly) completes its solution using modern, sophisticated homotopy theoretic techniques.

Pontrjagin and Thom's theory studies cobordism classes of closed manifolds. This is an equivalence classes based on the following equivalence relation: two closed n-manifolds M^n and N^n are *cobordant* if there is an $(n+1)$-dimensional manifold with boundary W^{n+1} whose boundary is the disjoint union,

$$\partial W^{n+1} = M^n \sqcup N^n.$$

In recent years, an exciting area of research has developed around the study of *"cobordism categories"*. In such a category the objects are closed

n-manifolds, and the morphisms between, M^n and W^n are all possible cobordisms between them. Work of Madsen and Weiss [96], and Galatius, Madsen, Tillmann, and Weiss [54] has lead to work of Galatius and Randal-Williams [55] which uses cobordism categories to study the topology of diffeomorphisms of manifolds in a stable sense, that we will make precise. We give an overview of this exciting area of current research toward the end of this chapter.

11.1 Studying cobordism via stable homotopy: the Pontrjagin-Thom Theorem

In the last chapter we presented Theorem 10.32 which describes the "Thom functor", which is a monoidal functor from the category of spaces over BO to the category of symmetric spectra,

$$Th : \mathcal{C}_{BO} \to Sp^{\Sigma}$$

that takes a map $X \to BO$ to its Thom spectrum X^f. We mentioned the example of the Thom spectrum of the identity map

$$BO \to BO$$

which is denoted $\mathbb{MO}$. Since the Thom functor is monoidal, and since the identity map of BO obviously preserves its multiplicative structure, $\mathbb{MO}$ is a ring spectrum. It can be viewed as being built out of the spaces $\{MO(n),\, n \geq 0\}$, which are the Thom spaces of the universal vector bundles over the classifying spaces $\{BO(n),\, n \geq 0\}$. The structure maps of this spectrum are maps

$$\epsilon_n : \Sigma MO(n) \to MO(n+1)$$

which are maps of Thom spaces induced by the usual inclusion maps $BO(n) \to BO(n+1)$.

The critical feature of the Thom spectrum $\mathbb{MO}$ is that by the following remarkable theorem of Thom, its homotopy type describes cobordism classes of manifolds.

Theorem 11.1. *(Thom, [150] (1954)) There is an isomorphism between the homotopy groups of the Thom spectrum,*

$$\pi_n(\mathbb{MO}) = \lim_{k \to \infty} \pi_{n+k}(MO(k))$$

and the set of cobordism classes of closed n-manifolds, η_n. This is defined to be the set of equivalence classes of n-dimensional closed manifolds defined by saying that M_1^n is cobordant to M_2^n if there is an $(n+1)$ dimensional manifold with boundary, W^{n+1}, with

$$\partial W^{n+1} = M_1^n \sqcup M_2^n.$$

The abelian group structure on η_n corresponding to the group structure on stable homotopy groups is simply induced by disjoint union of manifolds. The identity element in this group is the empty set $\emptyset$ (by convention $\emptyset$ can be viewed as a manifold of any dimension). Notice that this group consists entirely of elements of order 2. One sees this fact by observing that for any closed n-manifold M^n, the disjoint union $M^n \sqcup M^n$ is cobordant to the empty set $\emptyset$ since it is the boundary of $W^{n+1} = M^n \times [0,1]$. Furthermore, the graded abelian groups $\eta_* \cong \pi_*^s(\mathbb{MO})$ is a graded ring (and hence an algebra over $\mathbb{Z}/2$), since $\mathbb{MO}$ is a ring spectrum, with the induced product on $\eta_* = \oplus_n \eta_n$ given by cartesian product of manifolds.

The main goal of this section is to give a proof of this fundamental theorem and its natural generalizations. In particular we will describe the analogue of this theorem in the setting of oriented, (stably almost) complex, and (stably) framed cobordism. The example of framed cobordism was actually proved considerably earlier by Pontrjagin, and so the generalization of this theorem that we will prove is often referred to as the *Pontrjagin - Thom Theorem.*

We begin with Pontrjagin's construction.

Definition 11.1. *Let M^n be a closed, smooth manifold, and N^{n+k} be a smooth $(n+k)$-dimensional manifold (not necessarily closed). Suppose $e : M^n \hookrightarrow N^{n+k}$ is an embedding. A* **framing** *of this embedding is a an extension of e to an embedding $\tilde{e} : M^n \times D^k \hookrightarrow N^{n+k}$ that is a diffeomorphism onto its image. Here D^k denotes the unit open disk in $\mathbb{R}^k$.*

Exercises

1. Show that an embedding $e : M^n \hookrightarrow N^{n+k}$ has a framing if and only if the normal bundle $\nu_e \to M^n$ is a trivial k-dimensional vector bundle.

2. Show that a framing $\tilde{e} : M^n \times D^k \hookrightarrow N^{n+k}$ determines, and is determined by a vector bundle isomorphism

$$\Phi : \nu_e \xrightarrow{\cong} M^n \times \mathbb{R}^k.$$

3. Show that the standard embedding $e : S^n \hookrightarrow \mathbb{R}^{n+1}$ as the unit sphere, has a framing.

4. Show that the inclusion embedding $e : \mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ does *not* have a framing.

Given a framed embedding $\tilde{e} : M^n \hookrightarrow \mathbb{R}^{n+k}$ one can perform the "Pontrjagin- Thom construction" to define a map $\alpha_{\tilde{e}} : S^{n+k} \to S^k$.

$$\begin{aligned}\alpha_{\tilde{e}} : S^{n+k} = \mathbb{R}^{n+k} \cup \infty &\to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \tilde{e}(M \times D^k)) \\ &\cong (M^n \times D^k) \cup \infty \xrightarrow{project} D^k \cup \infty = S^k\end{aligned}$$

Here, when we write $\cup\infty$ we mean the one-point compactification. The map $\alpha_{\tilde{e}} : S^{n+k} \to S^k$ determines a class in the stable homotopy groups of spheres,

$$[\alpha_{\tilde{e}}] \in \pi_n(\mathbb{S}).$$

Conversely, suppose $[\alpha] \in \pi_n\mathbb{S}$ is represented by a smooth, basepoint preserving map $\alpha : S^{n+k} \to S^k$ for some k sufficiently large. Think of S^k as the one-point compactification $S^k = \mathbb{R}^k \cup \infty$ and assume $0 \in \mathbb{R}^k \subset S^k$ is a regular point of α. One loses no generality in this assumption since if this were not the case, then choose a regular value $x_0 \in S^k$ of α near 0, and then by composing α with a degree one map of the sphere that sends x_0 to the origin and keeps $\infty \in S^k$ fixed, one produces a map that is homotopic to α for which the origin is a regular value. So we continue with this assumption. Recall that the preimage $\alpha^{-1}(0) \subset S^{n+k}$ is a closed manifold of dimension n. Indeed by compactness we know that it lies in $(S^{n+k} - \infty) = \mathbb{R}^{n+k}$. So one has a manifold

$$M^n = \alpha^{-1}(0) \subset \mathbb{R}^{n+k}.$$

Notice that this embedding is in fact framed. To see this notice that if $\epsilon > 0$ is sufficiently small, and $B_\epsilon(0) \subset \mathbb{R}^k$ is the ball of radius ϵ, then

$$M^n \times \mathbb{R}^k = \alpha^{-1}(0) \times \mathbb{R}^k \cong \alpha^{-1}(B_\epsilon(0)) \subset \mathbb{R}^{n+k}$$

is a framed embedding.

Exercise. Prove this assertion. Namely, show that $M^n \times \mathbb{R}^k = \alpha^{-1}(0) \times \mathbb{R}^k \cong \alpha^{-1}(B_\epsilon(0))$.

These constructions lead to the famous result giving a correspondence between the stable homotopy group of spheres, $\pi_n(\mathbb{S})$ and the group of "framed cobordism classes of n-dimensional closed, (stably) framed manifolds", η_n^{fr}. Rather than immediately make this precise and provide a proof, we will first prove Thom's Theorem 11.1, and then show how it generalizes to describe the cobordism groups of manifolds with any type of stable normal structure (which we define), including a framing, in terms of the homotopy groups of a corresponding Thom spectrum.

We now proceed with a proof of the Pontrjagin-Thom Theorem 11.1.

Proof. By differentiating, we get for every $x \in M^n$, two subspaces, $De_x(T_xM^n) \subset \mathbb{R}^{n+k}$ and $De_x(T_xM^n)^\perp \subset \mathbb{R}^{n+k}$. Of course the first of these is the tangent space T_xM^n linearly embedded in $\mathbb{R}^{n+k}$, and the second is the normal space $\nu_x^k \subset \mathbb{R}^{n+k}$. Letting x vary over M^n defines continuous maps to Grassmannians,

$$\tau_e : M^n \to Gr_n(\mathbb{R}^{n+k}) \quad \text{and} \quad \nu_e : M^n \to Gr_k(\mathbb{R}^{n+k}). \tag{11.1}$$

Allowing the ambient vector spaces to get large we get maps

$$\tau_{M^n} : M^n \to Gr_n(\mathbb{R}^\infty) \simeq BO(n) \quad \text{and} \quad \nu_{M^n,e} : M^n \to Gr_k(\mathbb{R}^\infty) \simeq BO(k)$$

that classify the tangent bundle and the normal bundle respectively. Notice that the homotopy class of the map $\nu_{M^n,e}$ depends on the embedding e, but, as we have seen earlier, by taking the colimit we obtain the stable normal bundle map, $\nu_{M^n} : M^n \to BO$ whose homotopy type is an invariant of the smooth manifold M^n.

Going back to the original embedding, $e : M^n \hookrightarrow \mathbb{R}^{n+k}$, we may let η_e be a tubular neighborhood. The we can perform the "Thom collapse map"

$$\pi_e : S^{n+k} = \mathbb{R}^{n+k} \cup \infty \to S^{n+k}/(S^{n+k} - \eta_e) = \eta_e \cup \infty \cong T(\nu_e) \qquad (11.2)$$

where, as earlier $T(\nu_e)$ denotes Thom space of the normal bundle ν_e. Recall that we have seen this construction earlier, as well as similar constructions when we discussed Alexander duality (Theorem 10.41) and Atiyah duality (Theorem 10.46).

Composing with the map of Thom spaces induced by the classifying map of the normal bundle,

$$T(\nu_e) \to MO(k)$$

we get the map

$$\alpha_e : S^{n+k} \xrightarrow{\pi_e} T(\nu_e) \to MO(k). \qquad (11.3)$$

This map is known as the *"Pontrjagin - Thom construction"* on the embedding $e : M^n \hookrightarrow \mathbb{R}^{n+k}$. The resulting homotopy class $[\alpha_e] \in \pi_{n+k}(MO(k))$ depends on the embedding e. However if we let the codimension of the embedding get large, we get an element in the homotopy group of the spectrum $\mathbb{MO}$

$$\alpha_{M^n} = [\alpha_e] \in \lim_{k\to\infty} \pi_{n+k} MO(k) = \pi_n(\mathbb{MO}) \qquad (11.4)$$

which does not depend on the embedding, basically because all embeddings are isotopic in sufficiently large codimensions. The next result shows that Pontrjagin - Thom class gives a well-defined homomorphism from the cobordism group to the homotopy groups of $\mathbb{MO}$.

Proposition 11.2. *1. A cobordism W^{n+1} between two closed manifolds M^n and N^n defnes a homotopy between their Pontrjagin-Thom constructions*

$$\alpha_{M^n} : S^{n+k} \to MO(k) \quad \text{and} \quad \alpha_{N^n} : S^{n+k} \to MO(k)$$

for k sufficiently large.

2. The Pontrjagin-Thom construction for a disjoint union of closed n-manifolds, $M_1^n \sqcup M_2^n$ is homotopic to the sum of the Pontrjagin-Thom constructions of each component

$$\alpha_{M_1^n \sqcup M_2^n} = \alpha_{M_1^n} + \alpha_{M_2^n} \in \pi_n(\mathbb{MO}).$$

Proof. For part (1), assume W^{n+1} is a smooth, compact manifold with boundary and $\partial W^{n+1} = M^n \sqcup N^n$. By a relative version of Whitney's embedding theorem, we can find an embedding of manifolds with boundary,

$$e : W^{n+1} \hookrightarrow \mathbb{R}^{n+k} \times [0,1]$$

where

$$e(W^{n+1}) \cap (\mathbb{R}^{n+k} \times \{0\}) = e(M^n) \quad \text{and} \quad e(W^{n+1}) \cap (\mathbb{R}^{n+k} \times \{1\}) = e(N^n).$$

We now do the Pontrjagin-Thom construction for the embedding e:

$$\rho_e : \big((\mathbb{R}^{n+k} \times [0,1]) \cup \infty\big) \to (\mathbb{R}^{n+k} \times [0,1]) / \big((\mathbb{R}^{n+k} \times [0,1]) - \eta_e\big) \to MO(k) \tag{11.5}$$

where η_e is a tubular neighborhood of the embedding e. The relative Tubular Neighborhood Theorem states that this neighborhood is homeomorphic to a k-dimensional normal bundle $\nu_e^k \to W^{n+1}$, and the last map in this composition is the induced map on Thom spaces of the classifying map of ν_e^k. Also observe that the tubular neighborhood η_e has the property that

$$\eta_e^0 = \eta_e \cap (\mathbb{R}^{n+k} \times \{0\}) \quad \text{and} \quad \eta_e^1 = \eta_e \cap (\mathbb{R}^{n+k} \times \{1\})$$

are tubular neighborhoods of the restrictions of the embedding e to M^n and N^n respectively.

Notice that the one point compactification of $\mathbb{R}^{n+k} \times [0,1]$ is given by

$$(\mathbb{R}^{n+k} \times [0,1]) \cup \infty = \big(S^{n+k} \times [0,1]\big) / (\infty \times [0,1]),$$

where, as usual, we are thinking of S^{n+k} as $\mathbb{R}^{n+k} \cup \infty$. We can therefore think of the Pontrjagin-Thom construction ρ_e as a (base point preserving) homotopy between it's restrictions

$$\rho_e^0 : \big(\mathbb{R}^{n+k} \times \{0\}\big) \cup \infty \to \big(\mathbb{R}^{n+k} \times \{0\}\big) \cup \infty / \big((\mathbb{R}^{n+k} \times \{0\}) - \eta_e^0\big) \to MO(k)$$

and

$$\rho_e^1 : \big(\mathbb{R}^{n+k} \times \{1\}\big) \cup \infty \to \big(\mathbb{R}^{n+k} \times \{0\}\big) \cup \infty / \big((\mathbb{R}^{n+k} \times \{1\}) - \eta_e^1\big) \to MO(k)$$

But these maps are Pontrjagin-Thom constructions for the embeddings of the boundary components $e_{|_{M^n}} : M^n \hookrightarrow \mathbb{R}^{n+k} \times \{0\}$ and $e_{|_{N^n}} : M^n \hookrightarrow \mathbb{R}^{n+k} \times \{1\}$. Thus for k-sufficiently large these represent α_{M^n} and α_{N^n} respectively. Part (1) of this proposition now follows.

To prove Part (2) of the proposition, assume we have an embedding of the disjoint union

$$e_{M^n} = e_1 \sqcup e_2 : M_1^n \sqcup M_2^n \hookrightarrow \mathbb{R}^{n+k}.$$

Clearly the images of the components are disjoint. We may assume that the tubular neighborhood $\eta_e = \eta_1 \sqcup \eta_2$ where η_i, $i = 1, 2$, are tubular neighborhoods of the embeddings e_1 and e_2 respectively, and $\eta_1 \cap \eta_2 = \emptyset$. Furthermore, by translating one of these tubular neighborhoods if necessary, we can assume there are disjoint open balls, B_1^{n+k} and B_2^{n+k} in $\mathbb{R}^{n+k}$ containing the tubular neighborhoods $\eta_!$ and η_2 respectively.

Therefore the Thom collapse map

$$\pi_e : \mathbb{R}^{n+k} \cup \infty \to S^{n+k} / \left(S^{n+k} - \eta_e\right) \cong T(\nu_e)$$

factors up to homotopy as the composition

$$\pi_e : \mathbb{R}^{n+k} \cup \infty \to (B_1^{n+k} \cup \infty) \vee (B_2^{n+k} \cup \infty) \to$$
$$(B_1^{n+k} \cup \infty) / \left((B_1^{n+k} \cup \infty) - \eta_{e_1}\right) \vee (B_2^{n+k} \cup \infty) / \left((B_2^{n+k} \cup \infty) - \eta_{e_2}\right) \cong$$
$$T(\nu_{e_1}) \vee T(\nu_{e_2}) \cong T(\nu_e)$$

Notice that the first map in this composition is homotopic to the pinch map $p : S^{n+k} \to S^{n+k} \vee S^{n+k}$, and the second map in this composition is homotopic to the wedge of the Thom collapse maps

$$\pi_{e_1} \vee \pi_{e_2} : S^{n+k} \vee S^{n+k} \to T(\nu_{e_1}) \vee T(\nu_{e_2}).$$

Thus the Pontrjagin-Thom construction α_e is homotopic to the composition

$$\alpha_e : S^{n+k} \xrightarrow{p} S^{n+k} \vee S^{n+k} \xrightarrow{\pi_{e_1} \vee \pi_{e_2}} T(\nu_{e_1}) \vee T(\nu_{e_2}) \to MO(k) \vee MO(k) \to MO(k)$$

where the last map in this composition is the fold map (which exists for any space, $X \vee X \to X$). But this composition represents the sum of the homotopy classes

$$\alpha_{e_1} + \alpha_{e_2} \in \pi_{n+k}(MO(k)).$$

Thus

$$\alpha_{M_1^n \sqcup M_2^n} = \alpha_{M_1^n} + \alpha_{M_2^n} \in \pi_n(\mathbb{MO}).$$

□

We now have a well defined homomorphism

$$\alpha : \eta_n \to \pi_n(\mathbb{MO}).$$

In order to prove that it is an isomorphism, we exhibit an inverse homomorphism, $\rho : \pi_n(\mathbb{MO}) \to \eta_n$. We describe its construction, but our description will not contain full details. It would be a valuable exercise for the reader to fill in the details, or (s)he may consult one of many references that give complete proofs, for example Thom's original paper [150] or the book by R. Stong [146].

Let $\theta \in \pi_n(\mathbb{MO})$ be represented by a basepoint preserving map $f_\theta : S^{n+k} \to MO(k)$. Here we are using the infinite Grassmannian $Gr_k(\mathbb{R}^\infty)$ to represent the classifying space $BO(k)$, and $MO(k)$ is the Thom space of the canonical bundle $\gamma_k \to Gr_k(\mathbb{R}^\infty)$. By the compactness of S^{n+k}, for sufficiently large $N > 0$, f_θ factors through a map, which by abuse of notation we also call

$$f_\theta : S^{n+k} \to T(\gamma_{k,N})$$

where $T(\gamma_{k,N})$ is the Thom space of the canonical bundle $\gamma_{k,N} \to Gr_k(\mathbb{R}^N)$, which we take to be the one-point compactification

$$T(\gamma_{k,N}) = \gamma_{k,N} \cup \infty.$$

Since f_θ must be a basepoint preserving map, we can take the basepoints of both $S^{n+k} = \mathbb{R}^{n+k} \cup \infty$ and $\gamma_{k,N} \cup \infty$ to be ∞. Notice that the total space of $\gamma_{k,N} \subset \gamma_{k,N} \cup \infty = T(\gamma_{k,N})$ is an open subspace which is a smooth manifold. Notice also that the inverse image $f_\theta^{-1}(\gamma_{k,N}) \subset S^{n+k}$ is an open submanifold. We may then assume that the restriction of f_θ to that inverse image,

$$f_\theta : f_\theta^{-1}(\gamma_{k,N}) \to \gamma_{k,N}$$

is a smooth map which is transverse to the zero section $Gr_k(\mathbb{R}^N) \hookrightarrow \gamma_{k,N}$. Notice that this zero section is a codimension k-submanifold of $\gamma_{k,N}$, and so its inverse image, $f_\theta^{-1}(Gr_k(\mathbb{R}^N)) \subset \mathbb{R}^{n+k} \subset S^{n+k}$ is a closed, codimension k-submanifold of $\mathbb{R}^{n+k} \subset S^{n+k}$. We call this n-dimensional manifold

$$M_\theta^n \subset \mathbb{R}^{n+k}.$$

We will define $\rho(\theta) = [M_\theta^n] \in \eta_n$.

Of course we need to show that $\rho : \pi_n(\mathbb{MO}) \to \eta_n$ is well-defined. But first we observe that if $e : M^n \hookrightarrow \mathbb{R}^{n+k} \subset S^{n+k}$ represents a class in the cobordism group η_n, then the Pontrjagin-Thom construction,

$$\alpha_{M^n} : S^{n+k} \xrightarrow{\pi_e} T(\nu_{M^n}) \xrightarrow{T(\nu_e)} T(\gamma_{k,n+k})$$

has the property that $\alpha_{M^n}^{-1}(M^n) \subset S^{n+k}$ is equal to $M^n \subset \mathbb{R}^{n+k}$. This is because, since $T(\nu_e) : T(\nu_{M^n}) \to T(\gamma_{k,n+k})$ is induced by a map of vector bundles which induces a vector space isomorphism along each fiber, the inverse image of the zero section of $T(\gamma_{k,n+k})$ is the zero section of $T(\nu_{M^n})$. That is, $T(\nu_{M^n})^{-1}(Gr_k(\mathbb{R}^{n+k})) = M^n$. Also, clearly the inverse image under the Thom collapse map π_e of M^n is $M^n \subset S^{n+k}$.

This observation tells us that

$$\rho \circ \alpha = identity_{\eta_n}. \tag{11.6}$$

Similarly if $\theta \in \pi_n(\mathbb{MO})$ is represented by $f_\theta : S^{n+k} \to T(\gamma_{k,N})$ as above, and $e : M^n \hookrightarrow \mathbb{R}^{n+k}$ is $f_\theta^{-1}(Gr_k(\mathbb{R}^N))$, then the Pontrjagin - Thom construction

$$\alpha_e : S^{n+k} \to T(\nu_e) \to T(\gamma_{k,N})$$

is clearly homotopic to f_θ. Thus $\alpha \circ \rho = identity_{\pi_n(\mathbb{MO})}$.

Thus once we know that the map $\rho : \pi_n(\mathbb{MO}) \to \eta_n$ is well-defined, we would know that it is an inverse to the Pontrjagin-Thom construction $\alpha : \eta_n \to \pi_n(\mathbb{MO})$ thus proving that α is an isomorphism.

The main step in showing that ρ is well defined, is showing that if f^0_θ and $f^1_\theta : S^{n+k} \to T(\gamma_{k,N})$ are homotopic maps, transverse to $Gr_k(\mathbb{R}^N)$, then their inverse images, $M_0^n = (f^0_\theta)^{-1}(Gr_k(\mathbb{R}^N))$ and $M_1^n = (f^1_\theta)^{-1}(Gr_k(\mathbb{R}^N))$ are cobordant. To see this, suppose

$$F_\theta : S^n \times [0,1] \to T(\gamma_{k,N})$$

be a homotopy between f^0_θ and f^1_θ. Again, assuming F_θ is transverse to the zero section $Gr_k(\mathbb{R}^N)$, its inverse image would be a $(n+1)$-dimensional manifold with boundary, W^{n+1} embedded in $\mathbb{R}^{n+k} \times [0,1]$, that would be a cobordism between $M_0^n \subset \mathbb{R}^{n+k} \times \{0\}$ and $M_1^n \subset \mathbb{R}^{n+k} \times \{1\}$. We leave it to the reader to fill in the details of this sketch of the proof that $\rho : \pi_n(\mathbb{MO}) \to \eta_n$ is well-defined. Once done, this will complete the proof of the Pontrjagin-Thom Theorem (11.1). □

Notice that Theorem 11.1 gives an isomorphism between each homotopy group $\pi_n(\mathbb{MO})$ and the corresponding cobordism group η_n. But recall that since $\mathbb{MO}$ is a homotopy commutative ring spectrum, its homotopy groups, $\pi_*(\mathbb{MO})$ form a graded commutative ring. Similarly the cobordism groups $\{\eta_n,\, n \geq 0\}$. fit together to give a graded ring

$$\eta_* = \bigoplus_{n=0}^{\infty} \eta_n$$

where the product structure is represented by the cartesian product of manifolds.

Exercise: Show that the cartesian product of manifolds induces a well-defined product structure on η_*, That is, show that if M_1 is cobordant to M_1', and M_2 is cobordant to M_2', then $M_1 \times M_2$ is cobordant to $M_1' \times M_2'$.

Proposition 11.3. *The Pontrjagin-Thom construction*

$$\alpha : \eta_* \to \pi_*(\mathbb{MO})$$

is an isomorphism of graded rings.

Proof. . Suppose $e_1 : M_1^n \hookrightarrow \mathbb{R}^{n+k}$ and $e_2 : N_2^m \hookrightarrow \mathbb{R}^{m+s}$ are smooth embeddings. Consider the Thom collapse maps,

$$\rho_1 : S^{n+k} = \mathbb{R}^{n+k} \cup \infty \to \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_{e_1}) = T(\nu_{e_1}) \quad \text{and}$$
$$\rho_2 : S^{m+s} = \mathbb{R}^{m+s} \cup \infty \to \mathbb{R}^{m+s}/(\mathbb{R}^{m+s} - \eta_{e_1}) = T(\nu_{e_2})$$

The Thom collapse map for the product $e_1 \times e_2 : M_1^n \times M_2^m \hookrightarrow \mathbb{R}^{n+k} \times \mathbb{R}^{m+s}$ makes the following diagram commute:

$$\begin{array}{ccccc}
(\mathbb{R}^{n+k} \times \mathbb{R}^{m+s}) \cup \infty & \xrightarrow{\rho_{e_1 \times e_2}} & (\mathbb{R}^{n+k} \times \mathbb{R}^{m+s})/((\mathbb{R}^{n+k} \times \mathbb{R}^{m+s}) - (\eta_{e_1} \times \eta_{e_2})) & \xrightarrow{=} & T(\nu_{e_1 \times e_2}) \\
= \downarrow & & \downarrow = & & \downarrow = \\
(\mathbb{R}^{n+k} \cup \infty) \wedge (\mathbb{R}^{m+s} \cup \infty) & \xrightarrow[\rho_{e_1} \wedge \rho_{e_2}]{} & \mathbb{R}^{n+k}/(\mathbb{R}^{n+k} - \eta_{e_1}) \wedge \mathbb{R}^{m+s}/(\mathbb{R}^{m+s} - \eta_{e_2}) & \xrightarrow[=]{} & T(\nu_{e_1}) \wedge T(\nu_{e_2}
\end{array}$$

Notice also that the classifying map of the normal bundle of $e_1 \times e_2$, $\nu_{e_1 \times e_2}$ is given by the composition

$$M_1^n \times M_2^m \xrightarrow{\nu_{e_1} \times \nu_{e_2}} BO(k) \times BO(s) \xrightarrow{\mu} BO(k+s)$$

where μ represents the Whitney sum map. Therefore the following diagram of Thom spaces commutes:

$$\begin{array}{ccccc}
T(\nu_{e_1}) \wedge T(\nu_{e_2}) & \xrightarrow{t\nu_{e_1} \wedge t\nu_{e_2}} & MO(k) \wedge MO(s) & \xrightarrow{t\mu} & MO(k+s) \\
= \downarrow & & & & \downarrow = \\
T(\nu_{e_1 \times e_2}) & & \longrightarrow & & MO(k+s)
\end{array}$$

In this diagram when $\nu : X \to BO(n)$ is a classifying map, then $t\nu : T(\nu) \to MO(n)$ represents the induced map of Thom spaces.

Putting these two diagrams together means we have a commutative diagram of Pontrjagin-Thom constructions:

$$\begin{array}{ccccc}
\alpha_{M_1 \times M_2} : S^{n+m+k+s} & \xrightarrow{\rho_{e_1 \times e_2}} & T(\nu_{e_1 \times e_2}) & \longrightarrow & MO(k+s) \\
= \downarrow & & \downarrow = & & \downarrow = \\
\alpha_{M_1} \cdot \alpha_{M_2} : S^{n+k} \wedge S^{m+s} & \xrightarrow[\rho_{e_1} \wedge \rho_{e_2}]{} & T(\nu_{e_1}) \wedge T(\nu_{e_2}) & \longrightarrow & MO(k+s)
\end{array}$$

That is, the map $\alpha : \eta_* \to \pi_*(\mathbb{MO})$ satisfies

$$\alpha([M_1 \times M_2]) = \alpha([M_1]) \cdot \alpha([M_2])$$

thus proving that α is a ring homomorphism. Combining this with Theorem 11.1 implies that the Pontrjagin-Thom map α is an isomorphism of graded rings. $\square$

11.2 Unoriented cobordism: Thom's calculation

Thom also did a complete calculation of this graded ring.

Theorem 11.4. *[150]*

$$\eta_* \cong \mathbb{Z}_2[b_2, b_4, b_5, \cdots, b_r, \cdots : r \neq 2^k - 1].$$

In other words, η_* is a polynomial algebra over the field $\mathbb{Z}/2$ with one generator b_r of dimension $r > 0$ so long as r is not of the form $2^k - 1$ for any integer $k > 0$.

In fact Thom gave a complete description of the homotopy type of the spectrum $\mathbb{MO}$.

Theorem 11.5. *[150] The spectrum $\mathbb{MO}$ has the homotopy type of a wedge of Eilenberg-MacLane spectra,*

$$\mathbb{MO} \simeq \bigvee_{\omega \in I} \Sigma^{|\omega|} \mathbb{HZ}/2$$

where the indexing set I consists of all monomials in $\mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$. The notation $|\omega|$ refers to the dimension of the monomial $b_\omega \in \mathbb{Z}/2[b_2, b_4, \cdots, b_r \cdots, : r \neq 2^k - 1]$.

The main step in proving both Theorems 11.4 and 11.5 is to compute the cohomology $H^*(\mathbb{MO}; \mathbb{Z}/2)$ as a module over the Steenrod algebra $\mathcal{A}_2$.

Proposition 11.6. *$H^*(\mathbb{MO}; \mathbb{Z}/2)$ is a free module over $\mathcal{A}_2$.*

Proof. We begin this proof with a basic algebraic lemma below about Hopf algebras and coalgebras.

Lemma 11.7. *Let $\mathcal{A}$ be a connected Hopf algebra over a field k. Let $\mathcal{P}$ be a connected coalgebra over k which is a left $\mathcal{A}$-module and such that its coproduct map $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is a map of $\mathcal{A}$-modules. Let $u \in \mathcal{P}$ be the unique class of degree zero mapping to $1 \in k$ under the counit $\epsilon : \mathcal{P} \to k$. Consider the map*

$$\begin{aligned} \mu : \mathcal{A} &\to \mathcal{P} \\ a &\to a \cdot u. \end{aligned}$$

If the map μ is injective, then $\mathcal{P}$ is a free $\mathcal{A}$-module.

Proof. Let $\mathcal{A}^+$ be the submodule of $\mathcal{A}$ consisting of elements of positive degree. Let $Q = \mathcal{P}/\mathcal{A}^+\mathcal{P}$. Consider a splitting of k-vector spaces $\iota : Q \to \mathcal{P}$ of the natural projection map $\pi : \mathcal{P} \to Q$. Define

$$\begin{aligned} \phi : \mathcal{A} \otimes Q &\to \mathcal{P} \\ a \otimes q &\to a \cdot \iota(q). \end{aligned}$$

Clearly ϕ is a map of $\mathcal{A}$-modules. To prove the lemma we show that ϕ is an isomorphism.

We first show that ϕ is surjective. Notice that in degree zero ϕ is the identity map. Inductively assume that ϕ is surjective in all degrees less than k. Let $\alpha \in \mathcal{P}$ have degree k.

$$\begin{aligned}\pi(\alpha - \phi(1 \otimes \pi(\alpha)) &= \pi(\alpha - \iota(\pi(\alpha))) \\ &= \pi(\alpha) - \pi(\iota(\pi(\alpha))) \\ &= \pi(\alpha) - \pi(\alpha) \\ &= 0\end{aligned}$$

So one can write
$$\alpha - \phi(1 \otimes \pi(\alpha)) = \sum a_i \alpha_i$$
where $a_i \in \mathcal{A}^+$ and $\alpha_i \in \mathcal{P}$. Notice that all of the α_i's have degree less than the degree of α, which is k. So by the inductive hypothesis we can find $x_i \in \mathcal{A} \otimes Q$ with $\phi(x_i) = \alpha_i$. This implies that

$$\alpha = \phi\left(1 \otimes \pi(\alpha) + \sum a_i x_i\right)$$

which proves surjectivity. To see that ϕ is injective, consider the sequence of $\mathcal{A}$-module maps:

$$\mathcal{A} \otimes Q \xrightarrow{1 \otimes \iota} \mathcal{A} \otimes \mathcal{P} \to \mathcal{P} \xrightarrow{\Delta} \mathcal{P} \otimes \mathcal{P} \xrightarrow{1 \otimes \pi} \mathcal{P} \otimes Q.$$

By tracing through these maps, one sees that the image of a class $a \otimes q$ is of the form $a \cdot u \otimes q$ plus elements of different bidegrees. So since the map

$$\begin{aligned}\mu : \mathcal{A} &\to \mathcal{P} \\ a &\to a \cdot u.\end{aligned}$$

is injective, then this composition is injective. But since $\phi : \mathcal{A} \otimes Q \to \mathcal{P}$ is the composition of the first two maps, it also is injective. This establishes that $\phi : \mathcal{A} \otimes Q \to \mathcal{P}$ is an isomorphism of $\mathcal{A}$-modules and hence $\mathcal{P}$ is a free $\mathcal{A}$-module. □

We want to make use of Lemma 11.7, by applying it to $H^*(\mathbb{MO}; \mathbb{Z}/2)$. We first observe that since $\mathbb{MO}$ is an associative, homotopy commutative ring spectrum, the multiplication map

$$\mu : \mathbb{MO} \wedge \mathbb{MO} \to \mathbb{MO}$$

induces a commutative algebra structure on it's homology, $\mu_* : H_*(\mathbb{MO}; \mathbb{Z}/2) \otimes H_*(\mathbb{MO}; \mathbb{Z}/2) \to H_*(\mathbb{MO}; \mathbb{Z}/2)$, and a cocommutative coalgebra structure on it's cohomology

$$\mu^* : H^*(\mathbb{MO}; \mathbb{Z}/2) \to H^*(\mathbb{MO}; \mathbb{Z}/2) \otimes H^*(\mathbb{MO}; \mathbb{Z}/2).$$

Notice furthermore that the comultiplication map μ^* is a map of $\mathcal{A}_2$-modules, since it is induced by a map of spectra. Furthermore $H^*(\mathbb{MO};\mathbb{Z}/2)$ is obviously a connected coalgebra because $\mathbb{MO}$ is a connected spectrum. Also, since $H^0(\mathbb{MO};\mathbb{Z}/2) = \mathbb{Z}/2$ and is generated by the Thom class $u \in H^0(\mathbb{MO};\mathbb{Z}/2)$, then by Lemma 11.7, in order to prove Proposition 11.6 it suffices to prove that the map

$$\begin{aligned} \phi : \mathcal{A}_2 &\to H^*(\mathbb{MO};\mathbb{Z}/2) \\ a &\to a \cdot u \end{aligned} \tag{11.7}$$

is injective.

To do this it suffices to work on the space level, to show that the map

$$\begin{aligned} \phi : \mathcal{A}_2 &\to H^*(MO(k);\mathbb{Z}/2) \\ a &\to a \cdot u_k \end{aligned}$$

is injective for $deg\, a \leq \rho(k)$ where $\rho : \mathbb{Z} \to \mathbb{Z}$ is some strictly increasing function of k.

Now consider the multiplication map $\mu_k : MO(1) \wedge \cdots \wedge MO(1) \to MO(k)$ where there are k-copies of the Thom space $MO(1)$ in this wedge product. This map is the induced map on Thom spaces of the product map on classifying spaces

$$BO(1) \times \cdots \times BO(1) \to BO(k).$$

The induced map in cohomology,

$$\mu_k^* : H^*(MO(k);\mathbb{Z}/2) \to H^*(MO(1)\wedge\cdots\wedge MO(1);\mathbb{Z}/2) \cong \tilde{H}^*(MO(1);\mathbb{Z}/2)^{\otimes k}$$

preserves Thom classes, so it suffices to show that the map

$$\begin{aligned} \phi : \mathcal{A}_2 &\to H^*(MO(1) \wedge \cdots \wedge MO(1);\mathbb{Z}/2) \cong \tilde{H}^*(MO(1);\mathbb{Z}/2)^{\otimes k} \\ a &\to a \cdot u_k \end{aligned} \tag{11.8}$$

is injective for $deg\, a \leq \rho(k)$ where $\rho(k)$ is an increasing function.

Now consider the homotopy type of the Thom space $MO(1)$. By definition,

$$MO(1) = D(\gamma^1)/S(\gamma^1)$$

where $\gamma^1 \to BO(1) = \mathbb{RP}^\infty$ is the universal line bundle and $D(\gamma^1)$ and $S(\gamma^1)$ are the associated unit disk and sphere bundles respectively. Clearly $D(\gamma^1)$ has the base space $\mathbb{RP}^\infty$ as a neighborhood deformation retract, so it is homotopy equivalent to $\mathbb{RP}^\infty$. On the other hand

$$\begin{aligned} S(\gamma^1) = \{(L, u)\ &: L \subset \mathbb{R}^\infty \text{is a one dimensional subspace,} \\ &\text{and}\, u \in L\, \text{has}\, \|u\| = 1\}. \end{aligned}$$

But this is just the infinite dimensional sphere $S^\infty \subset \mathbb{R}^\infty$, which is contractible. Therefore we have a homotopy equivalence

$$MO(1) = D(\gamma^1)/S(\gamma^1) \simeq D(\gamma(1)) \simeq \mathbb{RP}^\infty. \tag{11.9}$$

With respect to this homotopy equivalence, the Thom class $u_1 \in H^1(MO(1); \mathbb{Z}/2)$ corresponds to the generator $a_1 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$. Therefore under the product map

$$\mu_k^* : H^k(MO(k); \mathbb{Z}/2) \to H^k(MO(1)^{\wedge k}; \mathbb{Z}/2) \cong H^k((\mathbb{RP}^\infty)^{\wedge k}; \mathbb{Z}/2)$$

the Thom class u_k maps to $a_1^{\otimes k}$.

We therefore can complete the proof of Proposition 11.7 by proving the following.

Lemma 11.8. *The map*

$$\begin{aligned}\phi_k : \mathcal{A}_2 &\to H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)^{\otimes k} \\ \alpha &\to \alpha \cdot (a_1 \otimes \cdots \otimes a_1)\end{aligned}$$

is injective for $\deg \alpha \leq k$.

This lemma follows because we know explicitly how the Steenrod algebra acts on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ (see Proposition 10.51 above), the Cartan product formula (which tells us how $\mathcal{A}_2$ acts on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)^{\otimes k}$) and induction on n. This argument is carried out in the proof of Proposition 3.2 in [144]. □

We can now turn back to the proof of Theorem 11.5.

Proof. In our proof in the last chapter of Theorem 10.65, we showed that if $\mathbb{E}$ is a spectrum whose mod 2 cohomology is a free module over $\mathcal{A}_2$ with basis $\mathcal{B}$, then there is a map

$$\phi : \mathbb{E} \to \bigvee_{b_\omega \in \mathcal{B}} \Sigma^{|\omega|} \mathbb{H}\mathbb{Z}/2 \tag{11.10}$$

that induces an isomorphism in cohomology with $\mathbb{Z}/2$-coefficients. From Proposition 11.6 we can let $\mathbb{E} = \mathbb{MO}$. In order to know that ϕ is a weak homotopy equivalence, we can appeal to Theorem 10.65 once we know that the cohomology of $\mathbb{MO}$ is zero with $\mathbb{Z}/p$-coeffiencients for p an odd prime, and with rational coefficients. For this, the first thing to recall is that since, by Thom's Theorem 11.1, the homotopy groups $\pi_*(\mathbb{MO}) \cong \eta_*$ is a vector space over $\mathbb{Z}/2$. As was observed after the statement of Theorem 11.1, this is because any cobordism class represented by a manifold M^n is 2-torsion, since twice this class $2[M^n]$ is represented by the disjoint union $M^n \sqcup M^n$ which is the boundary of $M^n \times I$, and therefore is zero as a cobordism class.

Next, in order to apply Theorem 10.65 to $\mathbb{MO}$, we need to prove the following lemma.

Lemma 11.9. *Let $\mathbb{E}$ be a spectrum such that $\pi_*(\mathbb{E})$ is finitely generated* 2-*torsion. That is,*

$$\pi_*(\mathbb{E}) \otimes \mathbb{Z}/p = 0 \quad \textit{for } p \textit{ any odd prime, and } \pi_*(\mathbb{E}) \otimes \mathbb{Q} = 0.$$

Then, the same is true of homology, That is,

$$H_*(\mathbb{E};\mathbb{Z}/p) = 0 \quad \textit{for } p \textit{ any odd prime, and } H_*(\mathbb{E};\mathbb{Q}) = 0.$$

Proof. We first discuss $H_*(\mathbb{E};\mathbb{Z}/p)$, where p is an odd prime. Consider the self map of the sphere spectrum $\times p : \mathbb{S} \to \mathbb{S}$ representing $p \in \mathbb{Z} = \pi_0(\mathbb{S})$. Taking the smash produce with the spectrum $\mathbb{E}$ gives us a self map,

$$\times p : \mathbb{E} \to \mathbb{E}$$

defined to be

$$\mathbb{E} \wedge \mathbb{S} \xrightarrow{1\wedge(\times p)} \mathbb{E} \wedge \mathbb{S}.$$

Notice that the map $\times p : \mathbb{E} \to \mathbb{E}$ induces multiplication by the prime p in homotopy groups, since, by definition, it does so on the sphere spectrum. That means, since $\pi_*(\mathbb{E})$ is 2-torsion, then

$$(\times p)_* : \pi_*(\mathbb{E}) \to \pi_*(\mathbb{E})$$

is an isomorphism. Therefore $\times p : \mathbb{E} \to \mathbb{E}$ is a weak homotopy equivalence. So if $\mathbb{X}$ is a spectrum representing a generalized homology theory, then

$$\times p : \mathbb{E} \wedge \mathbb{X} \xrightarrow{(\times p)\wedge 1} \mathbb{E} \wedge \mathbb{X}$$

is a weak homotopy equivalence that represents multiplication by the prime p in homotopy groups. But when $\mathbb{X}$ is the Eilenberg-MacLane spectrum $H\mathbb{Z}/p$, this map in homotopy groups is given by

$$(\times p)_* : H_*(\mathbb{E};\mathbb{Z}/p) \xrightarrow{\cong} H_*(\mathbb{E};\mathbb{Z}/p).$$

Since $\tilde{H}_*(\mathbb{E};\mathbb{Z}/p)$ is a $\mathbb{Z}/p$- vector space, multiplication by p must be zero, so we must conclude that $\tilde{H}_*(\mathbb{E};\mathbb{Z}/p) = 0$.

We now turn our attention to $H_*(\mathbb{E};\mathbb{Q})$. By Proposition 10.22, we know that

$$H_*(\mathbb{E};\mathbb{Q}) \cong \pi_*(\mathbb{E}) \otimes \mathbb{Q} = 0$$

since each $\pi_q(\mathbb{E})$ is assumed to be a finitely generated abelian 2-torsion group. □

□

Thus we know that $\mathbb{E} = \mathbb{MO}$ satisfies the hypotheses of Theorem 10.65, and so we may conclude that there is a weak homotopy equivalence

$$\mathbb{MO} \simeq \bigvee_{\omega \in \mathcal{B}} \Sigma^{|\omega|}\mathbb{HZ}/2, \tag{11.11}$$

where $\mathcal{B}$ forms a basis for $H^*(\mathbb{MO})$ as a module over the Steenrod algebra, $\mathcal{A}_2$. Notice that the homotopy groups of the right hand side of this equivalence, and hence of $\mathbb{MO}$, is the $\mathbb{Z}/2$-vector space spanned by $\mathcal{B}$. That is, the basis generating $H^*(\mathbb{MO})$ as an $\mathcal{A}_2$-module can be identified with a basis for its homotopy groups. Comparing the homology of both sides of this equivalence we have the following:

Corollary 11.10. *We again assume all (co)homology is taken with $\mathbb{Z}/2$-coefficients. There is an isomorphism*

$$\begin{aligned} H_*(\mathbb{MO}) &\cong \pi_*(\mathbb{MO}) \otimes H_*(\mathbb{HZ}/2) \\ &\cong \pi_*(\mathbb{MO}) \otimes \mathcal{A}_2^* \end{aligned}$$

where $\mathcal{A}_2^*$ is the dual of the Steenrod algebra.

Before we compute the cobordism ring $\eta_* \cong \pi_*(\mathbb{MO})$, we draw some immediate geometric conclusions from what we've shown so far.

Corollary 11.11. *Two closed n-manifolds M^n and N^n are cobordant if and only if the images of their fundamental classes under their stable normal bundle homomorphisms are equal. That is, if $\nu_M : M^n \to BO$ and $\nu_N : N^n \to BO$ are the stable normal bundle maps for M^n and N^n respectively, then these manifolds are cobordant if and only if*

$$(\nu_M)_*([M^n]) = (\nu_N)_*([N^n]) \in H_n(BO; \mathbb{Z}/2).$$

Proof. By the Pontrjagin-Thom theorem, we need to know when the classes

$$\alpha_{M^n} \quad \text{and} \quad \alpha_{N^n} \in \pi_n(\mathbb{MO})$$

are equal. But as one sees from Corollary 11.10, the Hurewicz homomorphism

$$h : \pi_n(\mathbb{MO}) \to H_n(\mathbb{MO}; \mathbb{Z}/2)$$

is injective, so this is equivalent to knowing that $h(\alpha_{M^n}) = h(\alpha_{N^n}) \in H_n(\mathbb{MO}; \mathbb{Z}/2)$. But by the Thom isomorphism theorem, this is equivalent to knowing that

$$u \cap h(\alpha_{M^n}) = u \cap h(\alpha_{N^n}) \in H_n(BO; \mathbb{Z}/2).$$

Here $u \in H^0(\mathbb{MO}; \mathbb{Z}/2)$ is the Thom class. The corollary then follows from the following straightforward exercise.

Exercise. For any closed manifold M^n,

$$u \cap h(\alpha_{M^n}) = (\nu_{M^n})_*[M^n] \in H_n(BO; \mathbb{Z}/2).$$

□

The result of Corollary 11.11 is often stated in a different way. Recall that $H^*(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \cdots, w_i, \cdots]$ where w_i is the i^{th} Stiefel-Whitney class.

Definition 11.2. *Let M^n be a closed n- manifold, and $f : M^n \to BO$, be a map, which we think of as classifying a stable vector bundle over M^n. We define "Stiefel-Whitney numbers" of f as follows. Let $p(w)$ be a polynomial in the Stiefel-Whitney classes, that is an element of $H^*(BO; \mathbb{Z}/2)$. $p(w)$ determines a "Stiefel-Whitney number" by the rule*

$$\langle f^*(p(w));\, [M^n]\rangle \in \mathbb{Z}/2.$$

Note. This evaluation map is to be interpreted as follows. The polynomial $p(w)$ is a sum of monomials of varying dimensions. The evaluation $\langle f^*(p(w));\, [M^n]\rangle = \langle p(w), \alpha\rangle$ on an n-dimensional homology class α is the sum of the evaluations of its monomials on $f_*(\alpha)$. A monomial having dimension other than n has, by convention, zero evaluation on an n-dimensional homology class.

If $\tau_{M^n} : M^n \to BO$ classifies the stable tangent bundle of M^n we call its Stiefel-Whitney numbers the *"tangential Stiefel-Whitney numbers"* of M^n, or sometimes just the Stiefel-Whitney numbers of M^n. Similarly, if $\nu_{M^n} : M^n \to BO$ classifies the stable normal bundle, we call its Stiefel-Whitney numbers, the *"normal Stiefel Whitney numbers"* of M^n.

Notice that the tangential and normal Stiefel-Whitney numbers of a manifold are invariants of the manifold. We can now interpret Corollary 11.11 as follows:

Corollary 11.12. *Two manifolds are cobordant if and only if they have the same normal Stiefel-Whitney numbers.*

The inverse relation between the stable tangent and stable normal bundle maps of a manifold will allow us to quickly prove the following.

Corollary 11.13. *Two manifolds are cobordant if and only if they have the same tangential Stiefel-Whitney numbers.*

Proof. Recall that for any space X the fact that BO is an infinite-loop space implies that the set of homotopy classes, $[X, BO]$ is an abelian group (equal to it's reduced K-theory). Let $-\iota : BO \to BO$ represent the homotopy class in $[BO, BO]$ that is inverse to the class represented by the identity map. Since the sum of the tangent bundle of a manifold with the normal bundle of any embedding of the manifold in Euclidean space is a trivial bundle, that means that the classes in $[M^n, BO]$ represented by τ_{M^n} and ν_{M^n} are inverse to each other in this group structure. In other words, the composition

$$M^n \xrightarrow{\tau_{M^n}} BO \xrightarrow{-\iota} BO$$

is homotopic to ν_{M^n}, and similarly the composition

$$M^n \xrightarrow{\nu_{M^n}} BO \xrightarrow{-\iota} BO$$

is homotopic to τ_{M^n}. Because $(-\iota) \circ (-\iota) \simeq id : BO \to BO$, $-\iota : BO \to BO$ is a homotopy equivalence.

Let $p(w)$ be a polynomial in the Stiefel-Whitney classes. Then

$$(\tau_{M^n})^*(p(w)) = (\nu_{M^n})^*((-\iota)^*(p(w)) \quad \text{and} \quad (\nu_{M^n})^*(p(w)) = (\tau_{M^n})^*((-\iota)^*(p(w)).$$

So the tangential Stiefel-Whitney number determined by $p(w)$ is the normal Stiefel-Whitney number determined by $(-\iota)^*(p(w)))$, and vice-versa. Since $(-\iota)^*$ is an isomorphism, we can conclude that two manifolds have the same normal Stiefel-Whitney numbers if and only if they have the same tangential Stiefel-Whitney numbers. The corollary now follows from Corollary 11.12. □

Corollary 11.14. *A manifold that can be* **stably framed,** *i.e a manifold M^n whose stable normal bundle map $\nu_{M^n} : M^n \to BO$ is null homotopic, is the boundary of an $(n+1)$-dimensional manifold.*

Proof. If M^n is a stably framed manifold, all of its Stiefel-Whitney numbers are zero. The sphere S^n is stably frameable, since the standard embedding in $\mathbb{R}^{n+1}$ has a trivial normal bundle. So by Corollary 11.12 M^n is cobordant to S^n, which is null-cobordant since it is the boundary of D^{n+1}. Therefore M^n is null-cobordant. □

We remark that the fact that the Stiefel-Whitney numbers of a manifold are cobordism invariants is not very difficult, as we will see below. What was difficult, and was a major achievement of Thom, is that the Stiefel-Whitney numbers of a manifold are a *complete* cobordism invariant as stated in Corollaries 11.12 and 11.13.

We now give an elementary proof of the following fact.

Proposition 11.15. . *If M^n is a closed manifold that is the boundary of W^{n+1}, then all of the tangential Stiefel-Whitney numbers of M^n are zero.*

Proof. Pick a metric on W^{n+1}. Then there is a unique outward normal vector field along $\partial W^{n+1} = M^n$, spanning a trivial line bundle ϵ^1. Therefore, the restriction of the tangent bundle to its boundary

$$T(W^{n+1})_{|_{M^n}} = T(M) \oplus \epsilon^1.$$

Hence, the Stiefel-Whitney classes of M^n are the restriction of Stiefel-Whitney classes of W^{n+1}. By the long exact sequence,

$$\cdots \to H^n(W^{n+1}; \mathbb{Z}/2)) \to H^n(\partial W^{n+1}; \mathbb{Z}/2) \xrightarrow{\delta} H^{n+1}(W, \partial W); \mathbb{Z}/2) \to \cdots$$

this implies that $\delta(w) = 0$ for every tangential Stiefel-Whitney class w. The natural map $\partial : H_{n+1}(W, \partial W; \mathbb{Z}/2) \to H_n(\partial W; \mathbb{Z}/2)$ takes the fundamental class $[W, \partial W]$ to the fundamental class $[\partial W] = [M^n]$.

Therefore if $p(w)$ is any polynomial in the Stiefel-Whitney classes,

$$\langle (\tau_{M^n})^* p(w), [M^n] \rangle = \langle \delta((\tau_{M^n})^* p(w)), [W^{n+}, \partial W^{n+1}] \rangle = 0.$$

$\square$

Exercise. Show that this proposition implies that if M_1 and M_2 are cobordant manifolds, they have the same tangential Stiefel-Whitney numbers.

We now turn our attention back to computing the cobordism ring $\eta_* \cong \pi_*(\mathbb{MO})$.

Since we know the homology $H_*(\mathbb{MO})$ we will be able to calculate $\pi_*(\mathbb{MO}) = \eta_*$, using our knowledge of $\mathcal{A}_2^*$ (Theorem 10.59). and their relation given by Corollary 11.10.

To understand $\pi_*(\mathbb{MO})$ as a graded $\mathbb{Z}/2$ vector space, it suffices to compute the dimension of $\pi_n(\mathbb{MO})$ for each n. For this we use a little combinatorics as is done in [157].

Recall that a *partition* of a positive integer n is an unordered sequence $(i_i, \cdots, i_k)$ of positive integers whose sum equals n. Let $p(n)$ be the number of partitions of n. Notice that in the polynomial ring $\mathbb{Z}/2[e_1, e_2, \cdots, e_k, \cdots]$ where $|e_i| = i$ (recall from Theorem 10.36 that this is $H_*(\mathbb{MO}; \mathbb{Z}/2)$), a monomial in degree n determines, and is determined by, a partition of n. Let $p(n)$ denote the number of partitions of n. This is then the dimension of $H_n(\mathbb{MO})$. (Here, as above, when we don't specify coefficients in (co)homology we mean $\mathbb{Z}/2$-coefficients.)

Now by Theorem 10.59, $\mathcal{A}_2^* \cong \mathbb{Z}/2[\xi_j,\, j \geq 1]$, where the degree $|\xi_j| = 2^j - 1$. A monomial in this ring of degree n is also a partition, but a very special one. Namely it is a *"dyadic partition"*, meaning a partition $(i_i, \cdots, i_k)$ where each i_j is of the form $2^m - 1$ for some m. We write $p_d(n)$ to be the number of dyadic partitions of n. By convention we let $p_d(0) = 1$. Notice that $p_d(n)$ is the dimension of $\mathcal{A}_2^*$ in degree n.

Finally we say that a partition $(i_i, \cdots, i_k)$ of n is *nondyadic* if none of the i_j's are of the form $2^m - 1$ for any m. We let $p_{nd}(n)$ be the number of nondyadic partitions of n, with the convention that $p_{nd}(0) = 1$. The following gives a calculation of the cobordism groups:

Theorem 11.16. *The cobordism group of n-dimensional closed manifolds, $\eta_n \cong \pi_n(\mathbb{MO})$ is a $\mathbb{Z}/2$-vector space of dimension $p_{nd}(n)$.*

Proof. The main step in the proof of this theorem is the following fact from combinatorics.

Lemma 11.17. *For every positive integer n,*

$$p(n) = \sum_{i=0}^{n} p_d(i) p_{nd}(i).$$

Furthermore, if $f(n)$ is an function defined for nonnegative integers n satisfying

$$p(n) = \sum_{i=0}^{n} p_d(i) f(n-i)$$

then $f(m) = p_{nd}(m)$ for all $m \geq 0$.

Proof. If $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_r)$ are partitions of n and m respectively, write $IJ = (i_1, \cdots, i_k, j_1, \cdots, j_r)$. This is a partition of $n+m$. Notice that if I is any partition of n, the we can uniquely write

$$I = I_d I_{nd}$$

where I_d is the dyadic partition obtained by taking all the entries of I of the form $2^k - 1$ for some k, and I_{nd} is the nondyadic partition obtained by taking the remaining entries of I (i.e those not of the form $2^k - 1$ for any k). Conversely, for every i such that $0 \leq i \leq n$, if I_d is a dyadic partition of i and I_{nd} is a nondyadic partition of $n-i$, the $I_d I_{nd}$ is a partition of n. Notice that there are $p_d(i) p_{nd}(n-i)$ ways of making such a partition of n for each i. This verifies the formula $p(n) = \sum_{i=0}^{n} p_d(i) f(n-i)$. The second statement is proved by an easy induction on n. □

We now complete the proof of Theorem 11.16. By Corollary 11.10, we know that

$$H_*(\mathbb{MO}) \cong \pi_*(\mathbb{MO}) \otimes \mathcal{A}_2^*. \tag{11.12}$$

We then have

$$dim\, H_n(\mathbb{MO}) = \sum_{i=0}^{n} dim\, \pi_i(\mathbb{MO}) \cdot dim\, (\mathcal{A}_2^*)_{n-i}.$$

As observed above, $dim\, H_n(\mathbb{MO}) = p(n)$ and $dim\, (\mathcal{A}_2^*)_k = p_d(k)$. If we let $f(m) = dim\, \pi_m(\mathbb{MO})$ then the result follows from Lemma 11.17. □

We can now draw an immediate geometric consequence of Theorem 11.16, which is much more difficult to prove without Thom-Pontrjagin theory,

Corollary 11.18. *Every closed 3-dimensional manifold is the boundary of a 4-dimensional manifold.*

Proof. By Theorem 11.16 η_3 is a $\mathbb{Z}/2$-vector space of dimension $p_{nd}(3)$. But there are no nondyadic partitions of 3, so $p_{nd}(3) = 0$. □

We now observe that we can restate Theorem 11.16 in the following way.

Corollary 11.19. *Consider the polynomial algebra $\mathbb{Z}/2[b_2, b_4, \cdots, b_i, \cdots]$ such that $|b_i| = i$ and i is not of the from $2^k - 1$ for any k. Then there is a graded $\mathbb{Z}/2$ vector space isomorphism between this algebra and $\pi_*(\mathbb{MO}) \cong \eta_*$.*

Proof. Notice that the monomials of degree n in this polynomial algebra are in bijective correspondence with nondyadic partitions of n, The result then follows from Theorem 11.16. □

Our goal is to strengthen this corollary to show that there is an isomorphism of algebras between the cobordism ring and this polynomial algebra. This would establish Theorem 11.4.

Proof. Recall from Theorem 6.20 that the splitting principle gives us an alternative description of $H^*(BO(n))$ as the ring of symmetric polynomials $\mathbb{Z}/2[\sigma_1, \cdots, \sigma_n]$, where σ_i is the i^{th} elementary symmetric polynomial in m-variables, say $x_1, \cdots, x_m$, all of which have degree one. Here we are choosing $m > n$ so that the elementary symmetric polynomials are algebraically independent.

Definition 11.3. *We say that two monomials in $x_1, \ldots, x_m$ are equivalent if there is a permutation of $x_1, \ldots, x_m$ that takes one to the other. Define $\sum x_1^{a_1} \cdots x_r^{a_r}$ to be the sum of all monomials in $x_1, \ldots, x_m$ which are equivalent to $x_1^{a_1} \cdots x_r^{a_r}$.*

Exercise. (See Lemma 16.1 of [121]) Show that an additive basis for $\mathcal{S}^k$, the group of homogeneous symmetric polynomials of degree k in $x_1, \ldots, x_m$ is given by the polynomials $\sum x_1^{a_1} \cdots x_r^{a_r}$, where $(a_1, \ldots, a_r)$ range through all partitions of k of length $r \leq m$.

Now let $I = (i_1, \cdots i_r)$ be a partition of n, and let $s_I = s_{(i_1, \cdots, i_r)}$ be the unique polynomial satisfying

$$s_I(\sigma_1, \cdots \sigma_n) = \sum x_1^{i_1} \cdots x_r^{i_r}.$$

For $m \geq n$ the $p(n)$ polynomials $s_I(\sigma_1, \cdots \sigma_n)$ are linearly independent and form a basis of $\mathcal{S}^n$. See Milnor and Stasheff's book [121] for a more complete discussion of these symmetric polynomials.

Given a vector bundle ξ over a closed n-manifold M^n, by recalling that the i^{th} Stiefel-Whitney class $w_i \in H^*(BO)$ can be identified with the i^{th} elementary symmetric polynomial σ_i, then I is a partition of n we may write

$$S_I(w(\xi)) = s_I(w_1(\xi), \ldots, w_n(\xi)) \in H^n(M).$$

(We recall that unless otherwise stated, all (co)homology is taken with $\mathbb{Z}/2$-coefficients.).

Now since the symmetric polynomials s_I form an additive basis for the ring of symmetric polynomials, which, by the splitting principle is isomorphic to $H^*(BO)$, then by Corollary 11.13 the cobordism type of any closed n-manifold M^n is completely determined by the collection of numbers

$$S_I[M^n] = \langle (s_I(\tau_{M^n}), [M^n] \rangle \in \mathbb{Z}/2,$$

where τ_{M^n} is the tangent bundle and I ranges over all partitions of n. A quick calculation using the Cartan product rule for Stiefel-Whitney classes that is done in [157], verifies the following:

Lemma 11.20. *Let ξ and ζ be any two vector bundles over a closed n-manifold M^n. Then for any partition I*

$$S_I(w(\xi \oplus \zeta)) = \sum_{I_1 I_2 = I} s_{I_1}(w(\xi)s_{I_2}(w(\zeta)),$$

and if N^m is another closed manifold,

$$S_I(M^n \times N^m) = \sum_{I_1 I_2 = I} S_{I_1}[M^n] S_{I_2}[N^m].$$

Recall the fact from Theorem 6.22 that the total Stiefel-Whitney class of the projective space is given by

$$w(\tau_{\mathbb{RP}^n}) = (1+a)^{n+1}$$

where $a \in Hq(\mathbb{RP}^n)$ is the nonzero class. This will immediately imply the following:

Lemma 11.21. *Consider the length-one partition (n) of n. Then*

$$S_{(n)}[\mathbb{RP}^n] = n+1 \in \mathbb{Z}/2.$$

As we will see below, these projective spaces can be taken to be generators of the cobordism ring η_* when n is even. To construct other generators, Thom considered hypersurfaces $H_{m,n}$ defined as follows.

Definition 11.4. *Let m and n be positive integers with $m \leq n$. Let $\mathbb{RP}^m$ have homogeneous coordinates $[x_0, \ldots, x_m]$ and $\mathbb{RP}^n$ have homogeneous coordinates $[y_0, \ldots, y_n]$. Let $H_{m,n} \subset \mathbb{RP}^m \times \mathbb{RP}^n$ be the subset defined by coordinates $([x_0, \ldots, x_m], [y_0, \ldots, y_n])$ satisfying the equation*

$$\sum_{i=0}^{m} x_i y_i = 0.$$

Exercise Show that $H_{m,n}$ is a smooth manifold of dimension $m+n-1$.

Using Lemma 11.20 Weston gave a direct calculational proof of the following (see Proposition 11.4 of [157]).

Lemma 11.22.

$$S_{(m+n-1)}[H_{m,n}] = \binom{m+n}{m} \in \mathbb{Z}/2.$$

We are now ready to complete the proof of Theorem 11.4. Define manifolds B_n that we show will generate the cobordism ring, as follows.

Definition 11.5. *Let* $B_i = \begin{cases} \mathbb{RP}^i & \textit{if } i \textit{ is even} \\ H_{2^p,2^{p+1}q} & \textit{if } i = 2^p(2q+1)-1 \textit{ and not of the form } 2^m-1 \end{cases}$

Let $b_i \in \eta_i$ be the cobordism class represented by the manifold B_i. We will show that η_* is the polynomial algebra generated by the classes $\{b_i \text{ such that } i \text{ is not of the form } 2^k-1\}$.

First of all notice that

$$S_{(i)}[B_i] = 1$$

by Lemmas 11.21 and 11.22. Let $I = (i_1, \cdots, i_k)$ be a nondyadic partitiion of n. We define the n-manifold

$$M_I = M_{i_1} \times \cdots \times M_{i_k}.$$

We will show that the set $\{[M_I] : I \quad \text{is a nondyadic partition of } n\}$ is a $\mathbb{Z}/2$-vector space basis for the cobordism group η_n. For this we follow the argument in [157] (Theorem 13.4). Since by Theorem 11.16 we know the dimension of this vector space is $p_{nd}(n)$, which is the number of nondyadic partitions of n, we need only show that this set of cobordism classes is linearly independent.

To do this we put a partial ordering on the set of partitions of n using the notion of *"refinement"*. Let I' be another partition of n. We say that I' is a ' refinement of the partition $I = (i_1, \ldots, i_k)$ if we can write $I' = I_1 \cdots I_k$, where each I_j is a partition of the coordinate i_j of I. This gives a partial order to the set of partitions of n by saying that $I \leq I'$ is I' refines I. In particular it gives a partial ordering to the set of nondyadic partitions of n.

Let I and J be nondyadic partitions of n with $I = (i_1, \cdots, i_k)$. Then by Lemma 11.20, we have

$$S_J[M_I] = \sum_{I_i \cdots I_k = J} S_{I_1}([M_{I_1}]) \cdots S_{I_k}([M_{i_k}]). \tag{11.13}$$

So in particular if J does not refine I, $S_J[M_I]$ must be zero. Also this equation says that $S_I([M_I]) = 1$ since in this case there is exactly one choice of $I_1, \cdots I_k$ giving a nonzero contribution to this sum.

Now choose a total ordering of the partitions of n compatible by the partial ordering given by refinement. Then we can form a $(p_{nd}(n) \times p_{nd}(n))$- dimensional matrix whose rows and columns are indexed by nondyadic partitions of n according to our ordering, and the entry indexed by (I, J) is given by $S_J([M_I])$. Then these calculations tell us that this is a triangular matrix with one's along the diagonal. Therefore the $p_{nd}(n)$ columns of this matrix are linearly independent. Each column is indexed by a nondyadic partition J of n, and its I^{th} coordinate is $S_J([M_I])$. Again, as J varies, these columns are linearly independent. Now since the polynomials s_J are a basis of the symmetric functions in degree n which can be viewed as the cohomology group

$H^n(BO)$, this says that the tangential Stiefel-Whitney numbers of the M_I's also give linearly p_{nd} linearly independent vectors. By Corollary 11.13 this says that the M_I's are linearly independent vectors in η_n, which as mentioned above, implies that they form a basis for η_n. But the set of $[M_I]$'s in η_n constitutes the set of monomials in the b_j's of degree n, and therefore this describes $\eta_* = \oplus_n \eta_n$ as the polynomial algebra $\mathbb{Z}/2[b_i$: i is not of the form $2^m - 1]$. □

11.3 Cobordism groups of manifolds with stable normal structure

We now record a natural generalization of the Pontrjagin-Thom Theorem 11.1 to cobordism groups of manifolds endowed with structures on their stable normal bundles. Such structures include orientations, stable almost complex structures, and stable framings. These normal structures are defined in terms of liftings of the classifying maps of the normal bundles to embeddings, to classifying spaces that determine the structure. For example, an orientation structure is determined by a lifting to $BSO(n)$ (for the appropriate n), an almost complex structure is determined by a lifting to $BU(m)$, etc. We now give a more precise definition.

Definition 11.6. *. Let $p_m : B_m \to BO(m)$ be a fibration. Let M^n be a closed n-dimensional manifold and let $\phi : M^n \to BO(m)$ classify the m-dimensional vector bundle $\xi \to M^n$. A (B_m, p_m) structure on the bundle ξ is defined to be a homotopy class of liftings $\tilde{\phi} : M^n \to B_m$. By a "lifting" we mean that $p_m \circ \tilde{\phi} = \phi : M^n \to BO(m)$, and by a "homotopy class of liftings" we mean an equivalence class of liftings, under the equivalence relation that two liftings $\tilde{\phi}_0$ and $\tilde{\phi}_1$ are equivalent if there is a continuous one parameter family of liftings, $\tilde{\phi}_t : M^n \to B_m$, $t \in [0, 1]$ connecting $\tilde{\phi}_0$ and $\tilde{\phi}_1$.*

From the point of view of cobordism theory, the most important type of bundle structure is one on the normal bundle of an embedding. Suppose $e : M^n \hookrightarrow \mathbb{R}^{n+k}$ is an embedding. Recall that the classifying map of the tangent bundle of M^n can be given the following explicit form in terms of the Grassmannian model for $BO(n)$:

Consider the map

$$\tau(e) : M^n \to Gr_n(\mathbb{R}^{n+k}) \hookrightarrow Gr_n(\mathbb{R}^\infty) = BO(n)$$

where the first map is defined by sending a point $x \in M^n$ to the image of the tangent space under the derivative of the embedding e, $D_x(e)(T_x M^n) \subset \mathbb{R}^{n+k}$. The second map is simply the one induced by the linear inclusion of vector spaces, $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k \subset \mathbb{R}^n \times \mathbb{R}^\infty = \mathbb{R}^\infty$. Clearly the map $\tau(e)$ classifies the tangent bundle τ_{M^n}.

The classifying map to the normal bundle of the embedding e also has a natural, explicit description.

$$\nu(e) : M^n \to Gr_k(\mathbb{R}^{n+k}) \to Gr_k(\mathbb{R}^\infty) = BO(k)$$

Here the first map in the composition sends a point $x \in M^n$ to the image of the orthogonal complement of the tangent space under the derivative of the embedding e, $D_x(e)(T_x M^n)^\perp \subset \mathbb{R}^{n+k}$, and the second map is again induced by the linear embedding $\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^\infty$.

Now since, for large k, all embeddings of a closed n manifold are isotopic, then, as observed in Chapter 7, all normal bundles are isomorphic, and hence have homotopic classifying maps. The following puts these facts in the context of normal structures. We leave the details of its proof as an exercise for the reader.

Lemma 11.23. *Let M^n be a closed n-dimensional manifold, and $p_m : B_m \to BO(m)$ a fibration. Then if $m > 0$ is sufficiently large then there is a bijective correspondence between the (B_m, p_m) structures on the normal bundle of any two embeddings, $e_1, e_2 : M^n \hookrightarrow \mathbb{R}^{n+m}$.*

This leads us to the following definition. Suppose we have a sequence of fibrations $p_m : B_m \to BO(m)$ together with maps of fibrations

$$\begin{array}{ccc} B_m & \xrightarrow{j_m} & B_{m+1} \\ {\scriptstyle p_m}\downarrow & & \downarrow{\scriptstyle p_{m+1}} \\ BO(m) & \xrightarrow[\iota_m]{} & BO(m+1) \end{array}$$

Now suppose we have a (B_m, p_m) structure on the normal bundle to an embedding $e : M^n \hookrightarrow \mathbb{R}^{n+m}$. Then we have an induced (B_{m+1}, p_{m+1}) structure on the normal bundle of the embedding $M^n \xrightarrow{e} \mathbb{R}^{n+m} \times \{0\} \subset \mathbb{R}^{n+m+1}$ given by the composition

$$p_{m+1} : M^n \xrightarrow{p_m} B_m \xrightarrow{j_m} B_{m+1}.$$

We then say that the normal bundle structures p_m and p_{m+1} are *compatible*.

Definition 11.7. *A (B, p) (stable normal) structure on a manifold M^n is an equivalence class of compatible (B_m, p_m) structures on the normal bundles of M^n, where the equivalence class is given by homotopy of liftings for sufficiently large m.*

Notice that if M^n is a compact manifold, we can think of this definition as saying that a (stable normal) structure on a manifold M^n is a homotopy class of lifting

$$\tilde{\nu}_{M^n} \to B$$

of its stable normal bundle map $\nu_{M^n} : M^n \to BO$. Here $B = \lim_{m\to\infty} B_m$ where the limit is taken with respect to the maps of fibrations $j_m : B_m \to B_{m+1}$ and $p : B \to BO$ is the limiting fibration induced by the fibrations $p_m : B_m \to BO(m)$. That is why such a structure is often referred to as a "stable normal" structure on the manifold M^n.

Standard examples of (B,p) structures are given by the fibrations $p_m : BSO(m) \to BO(m)$ and $p_{2m} : BU(m) \to BO(2m)$. Another important example is given by the universal bundles $EO(m) \to BO(m)$. Recall that in this case each $EO(m)$ is contractible.

Exercises.
Show that in these examples, a (stable normal) structure on a manifold M^n is given by an orientation, an almost complex structure on its stable normal bundle, and a framing (trivialization) on its stable normal bundle respectively.

Notice that if we have a compatible collection of fibrations $p_m : B_m \to BO(m)$ and maps of fibrations

$$\begin{array}{ccc} B_m & \xrightarrow{j_m} & B_{m+1} \\ {\scriptstyle p_m}\downarrow & & \downarrow{\scriptstyle p_{m+1}} \\ BO(m) & \xrightarrow[\iota_m]{} & BO(m+1), \end{array}$$

Then we can take the Thom space TB_m of the bundle classified by $p_m : B_m \to BO(m)$, and we have induced maps on the Thom spaces

$$T(j_m) : \Sigma TB_m \to B_{m+1}.$$

This data defines a spectrum that we call $\mathbb{M}B$. The following is a generalization of the Pontrjagin-Thom Theorem 11.1. Its proof follows the same structure and detail as that of Theorem 11.1. We leave the details as an exercise to the reader.

Theorem 11.24. *Suppose we have a compatible collection of fibrations $p_m : B_m \to BO(m)$ and maps of fibrations $j_m : B_m \to B_{m+1}$ as above. Then there is an isomorphism between the homotopy groups of the induced Thom spectrum,*

$$\pi_n(\mathbb{M}B)$$

and the set of cobordism classes of closed n-manifolds, endowed with (stable normal) (B,p) structures, η_n^B. This is defined to be the set of equivalence classes of n-dimensional closed manifolds with (B,p) structure, $(M^n, \tilde{\nu}_{M^n})$, *defined by saying* $(M_1^n, \tilde{\nu}_{M_1^n})$ *is cobordant to* $(M_2^n, \tilde{\nu}_{M_2^n})$ *if there is an* $(n+1)$ *dimensional manifold with boundary,* W^{n+1}, *endowed with a (stable normal)* (B,p)*-structure* $\{\tilde{\nu}_{W_m^{n+1}} : W^{n+1} \to B_m\}$ *such that*

$$\partial W^{n+1} = M_1^n \sqcup M_2^n$$

and the stable normal structure $\tilde{\nu}_{W^{n+1}}$ *restricts to* $\tilde{\nu}_{M_1^n}$ *and* $\tilde{\nu}_{M_2^n}$ *on its boundary.*

In the examples of normal structures given above, the relevant Thom spectra are denoted by $\mathbb{MSO}$, $\mathbb{MU}$, and $\mathbb{S}$, respectively. (Check that the Thom spectrum of the universal bundles $EO(m) \to BO(m)$ is the sphere spectrum $\mathbb{S}$.). We will describe some calculations of the corresponding oriented, almost complex, and framed cobordism rings in the following sections.

11.4 Oriented Cobordism

In this section we sketch a calculation, due to Thom [150], of rational oriented cobordism. That is, we will describe his calculation of

$$\eta_*^{SO} \otimes \mathbb{Q} \cong \pi_*(\mathbb{MSO}) \otimes \mathbb{Q}.$$

This calculation will be similar to Thom's calculation of $\eta_* \cong \pi_*(\mathbb{MO})$ given above, except the role of Stiefel-Whitney classes will be taken by Pontrjagin classes. We suggest that the reader review section 5.5 above for a discussion of Pontrjagin classes and their properties.

Let $I = (i_1, \cdots, i_k)$ be a partition of a positive integer n, and suppose that M^n is a closed, oriented n-dimensional manifold. Like we did with Stiefel-Whitney numbers we define the Pontrjagin number of a vector bundle $\xi \to M^n$ as

$$P_I(\xi) = \langle p_{i_1}(\xi) \cdot p_{i_2}(\xi) \cdots p_{i_k}(\xi); [M^n] \rangle \tag{11.14}$$

where $[M^n] \in H_n(M^n; \mathbb{Z})$ is the fundamental class. Similarly we define

$$S_I(p(\xi)) = \langle s_I(p_1(\xi), \cdots, p_n(\xi)); [M^n] \rangle.$$

Also, like we did when we used Stiefel-Whitney classes, we write

$$P_I[M^n] = P_I(\tau_{M^n}) \quad \text{and} \quad S_I[M^n] = S_I(\tau_{M^n}).$$

Lemma 11.25. *Let* ξ *and* ζ *be any two vector bundles over a closed* n*-manifold* M^n*. Then for any partition* I

$$2s_I(p(\xi \oplus \zeta)) = 2 \sum_{I_1 I_2 = I} s_{I_1}(p(\xi)) s_{I_2}(p(\zeta)),$$

and if N^m is another manifold,

$$S_I[M^n \times N^m] = \sum_{I_1 I_2 = I} S_{I_1}[M^n] S_{I_2}[N^m].$$

Proof. . These statements are completely analogous to those made about Stiefel-Whitney numbers in Lemma 11.20. The 2's are present in the first statement to eliminate torson. They are not needed in the second statement because the Kronecker index of any torsion class evaluated on the fundamental class of a manifold is zero. We leave completing the proof as an exercise to the reader. □

Before we can compute the oriented cobordism ring rationally, $\eta_*^{SO} \otimes \mathbb{Q}$, we need to record the following lemma, which is a direct consequence of our calculation of $H^*(BSO(n); R)$, when R is an integral domain containing $1/2$, done in Chapter 5.

Corollary 11.26. *$H^i(BSO(n); \mathbb{Z})$ is finite if i is not divisible by 4, and has rank $p(i/4)$ if i is divisible by 4. Recall that $p(m)$ is the number of partitions of m.*

Proof. This follows immediately from Theorem 6.45 and the Universal Coefficient Theorem. The rank can be computed from this theorem using the coefficient ring $R = \mathbb{Q}$. □

We can now deduce the following calculation of the oriented cobordism groups:

Theorem 11.27. *The oriented cobordism group η_n^{SO} is finite for n not divisible by 4, and has rank $p(n/4)$ for n-divisible by 4.*

Proof. To prove this theorem we need to consider

$$\eta_n^{SO} \otimes \mathbb{Q} \cong \pi_n(\mathbb{MSO}) \otimes \mathbb{Q} \cong H_*(\mathbb{MSO}; \mathbb{Q}),$$

where the last isomorphism is given by the equivalence of stable rational homotopy groups and rational homology given in Proposition 10.22. Now by the Thom isomorphism, $H_*(MSO; \mathbb{Q}) \cong H_*(BSO; \mathbb{Q})$. Thus the statements in the theorem regarding the ranks of these groups follows from Corollary 11.26. □

Like the unoriented cobordism spectrum $\mathbb{MO}$, the oriented cobordism spectrum $\mathbb{MSO}$ is a commutative ring spectrum induced by the pairings

$$MSO(n) \wedge MSO(m) \to MSO(n+m),$$

which are in turn induced on the Thom space level by the Whitney sum pairings,

$$BSO(n) \times BSO(m) \to BSO(n+m).$$

This ring spectrum structure on $\mathbb{MSO}$ defines a graded commutative ring structure on $\pi_*(\mathbb{MSO}) \cong \eta_*^{SO}$. As we saw above in the unoriented cobordism setting, the ring structure on η_*^{SO} is induced by Cartesian products of manifolds.

We are now able to compute the ring structure on $\eta_*^{SO} \otimes \mathbb{Q}$ explicitly, and give manifold representatives of the generators.

Theorem 11.28. *$\eta_*^{SO} \otimes \mathbb{Q}$ is isomorphic to the polynomial algebra generated by classes represented by even (complex) dimensional complex projective spaces. That is,*

$$\eta_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}\left[[\mathbb{CP}^2], [\mathbb{CP}^4], \dots, [\mathbb{CP}^{2n}], \quad \textit{for} \quad n \geq 1\right].$$

Proof. Recall from Corollary 6.42 that the total Pontrjagin class of $\mathbb{CP}^{2n}$ is given by

$$p(\mathbb{CP}^n) = (1 + a^2)^{2n+1}$$

where $a \in H^2(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$ is the generator given by the first Chern class of the tangent bundle. This immediate will tell us that the Pontrjagin number

$$P_{(n)}[\mathbb{CP}^{2n}] = 2n + 1.$$

Now like the argument in the last part of the proof of Theorem 11.4, we can conclude from this and Lemma 11.25 that the $[\mathbb{CP}^{2n}]$'s are algebraically independent as elements of $\eta_*^{SO} \otimes \mathbb{Q}$. This means that the map of algebras

$$j : \mathbb{Q}\left[[\mathbb{CP}^2], [\mathbb{CP}^4], \dots, [\mathbb{CP}^{2n}], \quad \text{for} \quad n \geq 1\right] \to \eta_*^{SO} \otimes \mathbb{Q}$$

given by associating to a sum of products of these projective spaces the oriented cobordism class it represents, is a monomorphism of graded algebras. But by Theorem 11.27 these algebras have the same rank in every dimension. Therefore this map is an isomorphism of graded algebras. □

The following is an immediate geometric consequence of this calculation:

Exercise. Show that Theorem 10.37 implies that for every closed, oriented manifold M^n, there is a number k such that the disjoint union of k copies of M^n is (oriented) cobordant to a disjont union of products of complex projective spaces.

As one might guess, the Pontrjagin numbers of a $4n$-dimensional oriented manifold are oriented cobordism invariants. That is, we have the following result.

Theorem 11.29. *Let M^{4n} be a closed, oriented $4n$-dimensional oriented manifold. If M^{4n} is the boundary of an oriented $(4n+1)$-dimensional manifold W^{4n+1}, then all of the Pontrjagin numbers of its tangent bundle are zero.*

Remark. Notice that this does not say that the vanishing of the Pontrjagin numbers are a *sufficient* condition for an oriented manifold to be a boundary. Contrast this to the role of Stiefel-Whitney numbers in unoriented cobordism (see Corollary 11.13).

Proof. (Taken from [157]) Consider the following portions of the exact sequence for the homology and cohomology of the pair (W^{4n+1}, M^{4n}). All coefficients are assumed to be $\mathbb{Z}$.

$$H_{4n+1}((W^{4n+1}, M^{4n})) \xrightarrow{\partial} H_{4n}(M^{4n}) \xrightarrow{i_*} H_{4n}(W^{4n+1})$$

and

$$H^{4n}(W^{4n+1}) \xrightarrow{i^*} H^{4n}(M^{4n}) \xrightarrow{\delta} H^{4n+1}((W^{4n+1}, M^{4n})).$$

Notice that $\partial([(W^{4n+1}, M^{4n})] = [M^{4n}]$, where these are the fundamental homology classes.

Since there is a unique "outward pointing" normal vector along $M^{4n} \subset W^{4n+1}$ the normal bundle is a trivial line bundle and therefore we have an bundle equation

$$\tau(W^{4n+1})_{|_{M^4n}} \cong \tau(M^n) \oplus \epsilon^1.$$

Therefore

$$p(\tau(W^{4n+1}) = p(\tau(M^{4n})).$$

So for any partition I of $4n$, we have

$$\begin{aligned} P_I[M^{4n}] &= P_I(\tau(W^{4n+1})_{|_{M^{4n}}}) \\ &= \langle p_I(\tau(W^{4n+1})_{|_{M^{4n}}}), [M^{4n}]\rangle \\ &= \langle i^* p_I(\tau(W^{4n+1})), [M^{4n}]\rangle \\ &= \langle i^* p_I(\tau(W^{4n+1})), \partial[W^{4n+1}, M^{4n}]\rangle \\ &= \langle \delta i^* p_I(\tau(W^{4n+1}), [W^{4n+1}, M^{4n}]\rangle \\ &= \langle 0, [M^{4n}]\rangle \\ &= 0 \end{aligned} \tag{11.15}$$

by exactness. □

A complete calculation of η_*^{SO} is very difficult, but was obtained by C.T.C Wall in [155]. We state his result without proof.

Theorem 11.30. *(Wall [155]) Two oriented closed manifolds are cobordant if and only if their tangent bundles have the same Pontrjagin numbers and Stiefel-Whitney numbers. η_*^{SO} is the algebra over $\mathbb{Z}$ generated by manifolds X_{4i} of dimension $4i$ for all $i \geq 1$, and by manifolds $Y_{2i-1,j}$ of dimension $2i-1$, subject only to the relations $2Y_{2i-1,j} = 0$. There is one torsion generator $Y_{2i-1,j}$ for each partition of i into distinct positive integers, none of which is a power of 2.*

11.5 Complex Cobordism: Milnor's calculation using the Adams spectral sequence

The complex cobordism group η_n^U is the group of cobordism classes of stably almost complex n-dimensional closed manifolds. As was the case with unoriented and oriented cobordism, the groups η_*^U form a graded commutative ring, where the multiplication is given by cartesian products of manifolds. By the Pontrjagin Thom Theorem 11.24 this is isomorphic to the ring $\pi_*(\mathbb{MU})$. These homotopy groups form a graded ring because $\mathbb{MU}$ is a homotopy commutative ring spectrum. $\pi_*(\mathbb{MU})$ was computed explicitly by Milnor [118] in one of the early, dramatic applications of the Adams spectral sequence. The ring structure was determined by Novikov [126]. We will describe Milnor's calculation of $\eta_*^U \cong \pi_*(\mathbb{MU})$ which constitutes a beautiful example of how knowledge of the Steenrod algebra module structure of cohomology and the Adams spectral sequence constitute powerful computational tools.

As the reader will recall, one of the reasons that the calculation of the unoriented cobordism groups $\eta_* \cong \pi_*(\mathbb{MO})$ were tractable, was that the cohomology $H^*(\mathbb{MO};\mathbb{Z}/2)$ was shown to be a free module over the Steenrod algebra, $\mathcal{A}_2$. This quickly implied that $\mathbb{MO}$ was a wedge of mod 2 Eilenberg-MacLane spectra. Furthermore, since $\pi_*(\mathbb{HZ}/2) = \mathbb{Z}/2$, where the nontrivial element lies in dimension zero, then if $\mathbb{E}$ is any spectrum of the homotopy type of a wedge of mod 2 Eilenberg-MacLane spectra, then

$$\pi_*(\mathbb{E}) \cong Hom^*_{\mathcal{A}_1}(H^*(\mathbb{E};\mathbb{Z}/2);\mathbb{Z}/2).$$

where on the right side we are considering homomorphisms as modules over $\mathcal{A}_2$, and the indexing reflects the degree shift of the homomorphism in cohomology.

Milnor's strategy for the calculation of the complex cobordism ring $\pi_*(\mathbb{MU})$ is to first compute $H^*(\mathbb{MU};\mathbb{Z}/p)$ as a module over the mod p - Steenrod algebra for any prime p. His calculations follow similar lines as Thom's calculation of $H^*(\mathbb{MO};\mathbb{Z}/2)$. In the case of $\mathbb{MU}$, its cohomology will turn out not to be a free $\mathcal{A}_p$-module, but as we will see, it is an explicitly describable module. Milnor then appealed to the Adams spectral sequence, which was quite new at the time.

Recall from Theorem 10.37 that $H_*(\mathbb{MU}) = \mathbb{Z}[H_*(\mathbb{MU}(1))] = \mathbb{Z}[t_1, \dots, t_i, \dots, |t_i| = 2i]$. (Here we are taking integral coefficients.). We can conclude the following facts from this observation.

1. The homology and cohomology of $\mathbb{MU}$ is torsion-free, and is nonzero only in even dimensions.

2. The product map $\bigwedge_N \mathbb{MU}(1) \to \mathbb{MU}(N) \to \mathbb{MU}$ induces and injection in cohomology $H^q(\mathbb{MU}; k) \to H^q(\bigwedge_N \mathbb{MU}(1); k)$ for N large with respect to q, where k represents any coefficient group.

Now consider the mod p Steenrod algebra $\mathcal{A}_p$, where p is any prime. Let $\beta \in \mathcal{A}_p$ be the Bockstein operator. When $p = 2$, $\beta = Sq^1$. Dividing out by the 2-sided ideal generated by the Bockstein β, we can consider the algebra $\mathcal{A}_p/(\beta)$. Because the cohomology of $\mathbb{MU}$ is only nonzero in even dimensions, and since applying the Bockstein operator β increases the dimension by one, the next result follows immediately.

Lemma 11.31. *The action of the Steenrod algebra on $H^*(\mathbb{MU}; \mathbb{Z}/p)$ factors through an action of the quotient algebra, $\mathcal{A}_p/(\beta)$.*

Now recall that $\mathbb{MU}(1) = \Sigma^{-2}\Sigma^\infty\mathbb{CP}^\infty$. The action of the Steenrod algebra $\mathcal{A}_p$ on $H^*(\mathbb{CP}^\infty; \mathbb{Z}/p)$ is computed in much the same way as we computed the action of $\mathcal{A}_2$ on $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ in chapter 10. One therefore implicitly knows the action of $\mathcal{A}_p$ and hence of $\mathcal{A}_p/(\beta)$ on $H^*(\mathbb{MU}(1); \mathbb{Z}/p)^{\otimes N}$. Using property (2) of (11.5) above, Milnor was able to prove the following in much the same way as we described Thom's proof that $H^*(\mathbb{MO}; \mathbb{Z}/2)$ is a free module over $\mathcal{A}_2$ (Proposition 11.6).

Theorem 11.32. *(Milnor [118]) For every prime p, $H^*(\mathbb{MU}; \mathbb{Z}/p)$ is a free module over $\mathcal{A}_p/(\beta)$, with even dimensional generators.*

To get an idea of why the generators of $H^*(\mathbb{MU}; \mathbb{Z}/p)$ as a module over $\mathcal{A}_p/(\beta)$ are even dimensional, we consider homology. For ease of notation we will consider the case when p is an odd prime. The case $p = 2$ is similar, and we leave considering that case as an exercise to the reader. We claim that the dual, $(\mathcal{A}_p/(\beta))^* \subset \mathcal{A}_p^*$ is given by

$$(\mathcal{A}_p/(\beta))^* \cong \mathbb{Z}/p[\xi_1, \xi_2, \cdots] \subset \mathcal{A}_p^* = \mathbb{Z}/p[\xi_1, \xi_2, \cdots] \otimes E(\tau_1, \tau_2, \dots) \cong \mathcal{A}_p^*.$$

Exercise. Prove this claim.

Since $H_*(\mathbb{MU}; \mathbb{Z}/p) = \mathbb{Z}/p[t_1, t_2, \dots,$ such that $|t_i| = 2i]$, then there is an abstract algebra isomorphism,

$$H_*(\mathbb{MU}; \mathbb{Z}/p) \cong (\mathcal{A}_p/(\beta))^* \otimes \mathbb{Z}/p[t_j, \text{ such that } |t_j| = 2j \text{ and } j \neq p^k - 1 \text{ for any k}].$$

This tells us that, given the fact that $H^*(\mathbb{MU}; \mathbb{Z}/p)$ is a free module over $\mathcal{A}_p/(\beta)$, then it would be generated by classes of the same dimension as monomials in $\mathbb{Z}/p[t_j$, such that $j \neq p^k - 1]$. These are all even dimensional classes.

The significance of this theorem lies in the following result, whose proof makes use of the Adams spectral sequence.

Theorem 11.33. *If $\mathbb{Y}$ is a spectrum of finite type and $H^*(\mathbb{Y};\mathbb{Z}/p)$ is a free $\mathcal{A}_p/(\beta)$-module on even dimensional generators, then $\pi_*(\mathbb{Y})$ contains no p-torsion.*

Before we describe Milnor's proof of this theorem we describe how it yields an effective calculation of $\pi_*(\mathbb{MU}) = \eta_*^U$. An immediate consequence of Theorems 11.32 and 11.33 is the following.

Corollary 11.34. $\eta_*^U = \pi_*(\mathbb{MU})$ *is torsion free.*

Notice that this corollary tells us that for each n, $\pi_n(\mathbb{MU})$ is a free $\mathbb{Z}$-module of the same dimension as the dimension of $\pi_n(\mathbb{MU}) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space. But notice that since $\pi_*(\mathbb{MU}) \otimes \mathbb{Q} \cong H_*(\mathbb{MU};\mathbb{Q})$, by the equivalence of rational stable homotopy and rational homology (Proposition 10.22), we have that the composition

$$\pi_*(\mathbb{MU}) \to H_*(\mathbb{MU};\mathbb{Z}) \to H_*(\mathbb{MU};\mathbb{Q})$$

is a monomorphism in every dimension, between a finitely generated free abelian group and a rational vector space, where the rank of the abelian group is equal to the dimension of the vector space. (Here the first map in this composition is the Hurewicz map, and the second map is simply rationalization.) The second map in this composition also is, in each dimension, a monomorphism between a free abelian group and a rational vector space of the same dimension. Therefore we can conclude that the Hurewicz map

$$\eta_*^U = \pi_*(\mathbb{MU}) \to H_*(\mathbb{MU};\mathbb{Z})$$

is, in every dimension, a monomorphism between finitely generated free abelian group of the same rank. Since the rank of $H_*(\mathbb{MU};\mathbb{Z})$ in every dimension is well understood (Theorem 10.37), this gives us an effective calculation of $\eta_*^U = \pi_*(\mathbb{MU})$ in every dimension.

We now describe Milnor's proof of Theorem 11.33.

Proof. Recall that the Adams spectral sequence for computing the p-primary component of $\pi_*(\mathbb{Y})$ has E_2-term given by

$$E_2^{s,t} = Ext_{\mathcal{A}_p}^{s,t}(H^*(\mathbb{Y};\mathbb{Z}/p), \mathbb{Z}/p).$$

Since we are assuming that $H^*(\mathbb{Y};\mathbb{Z}/p)$ is a free module over $\mathcal{A}_p/(\beta)$, we therefore need to understand $Ext_{\mathcal{A}_p}^{*,*}(\mathcal{A}_p/(\beta), \mathbb{Z}/p)$. Milnor achieves this by proving the following proposition.

Proposition 11.35. *There is a free resolution of $\mathcal{A}_p/(\beta)$ as a module over $\mathcal{A}_p$*

$$\cdots \to \cdots \to F_s \to F_{s-1} \to \cdots \to F_0 \to \mathcal{A}_p/(\beta)$$

where F_s has a basis over $\mathcal{A}_p$ consisting of elements $b_{(r_0,\dots,r_i,\dots)}$ such that $\sum_i r_i = s$ (i.e the r_i's form a partition of s), and where the dimension of $b_{(r_0,\dots,r_i,\dots)}$ is $\sum_i 2r_i(p^i-1)+s$.

Proof. (Sketch). Define operations $Q_i \in \mathcal{A}_p$ inductively by

$$Q_0 = \beta, \quad Q_{i+1} = P^{p^i}Q_i - Q_iP^{p^i}.$$

Notice that the dimension $|Q_i| = 2p^i - 1$. Let $\mathcal{A}_0 \subset \mathcal{A}_p$ be the subalgebra generated by the Q_i's. Verifying the following properties of $\mathcal{A}_0$ is straightforward and is done in Milnor [118]. We leave them as exercises for the reader.

Exercises. 1. Show that $Q_iQ_j = -Q_jQ_i$ for all i, j and in particular $Q_i^2 = 0$. So $\mathcal{A}_0$ is an exterior (or Grasmann) algebra.

2. Show that $\mathcal{A}_p$ is a free $\mathcal{A}_0$-module with basis $\{P^J\}$ of admissible monomials that involve no Bocksteins. In other words they are admissible monomials in the reduced power operations P^i. Show that this means that

$$\mathcal{A}_p \otimes_{\mathcal{A}_0} \mathbb{Z}/p = \mathcal{A}_p/(\beta),$$

where $\mathbb{Z}/p$ has the trivial left $\mathcal{A}_0$-module structure.

Now as Milnor points out, since $\mathcal{A}_0$ is an exterior algebra, there is a polynomial algebra P and a differential operator d on $\mathcal{A}_0 \otimes P$ such that $(\mathcal{A}_0, d)$ gives a free, acyclic resolution of $\mathbb{Z}/p$ over $\mathcal{A}_0$. Explicitly,

$$P = \mathbb{Z}/p[b_i : |b_i| = |Q_i| + 1],$$

and so it has a $\mathbb{Z}/p$-basis given by monomials $b_{(r_0,\dots,r_i,\dots)}$ of dimension $\sum_i r_i|b_i|$. The differential is given by

$$d(a \otimes b_{(r_0,\dots,r_i,\dots)}) = \sum_i aQ_i \otimes b_{(r_0,\dots,r_{i-1},r_i-1,r_{i+1},\dots)}$$

where the sum is taken over all i for which $r_i > 0$.

We then let $\tilde{F}_s$ be the free $\mathcal{A}_0$-module generated by symbols (monomials) $b_{(r_0,\dots,r_i,\dots)}$ such that $\sum r_i = s$. We then let

$$F_s = \mathcal{A}_p \otimes \tilde{F}_s.$$

The modules F_s together with the induced differential operator d give a free $\mathcal{A}_p$- resolution of $\mathcal{A}_p \otimes_{\mathcal{A}_0} \mathbb{Z}/p = \mathcal{A}_p/(\beta)$. ☐

To prove Theorem 11.33 Milnor does not use this resolution and the Adams spectral sequence to directly compute $\pi_*(\mathbb{Y}) = [\mathbb{S}, \mathbb{Y}]_*$, but rather he uses it to first understand the Adams spectral sequence for maps from a *"Moore spectrum"* to $\mathbb{Y}$. The $\mathbb{Z}/p$-Moore spectrum is defined to be

$$\mathcal{M} = \mathbb{S} \cup_p D^1. \tag{11.16}$$

To clarify, consider the map of degree p from the circle to itself

$$S^1 \xrightarrow{\times p} S^1$$
$$z \to z^p$$

where we are thinking of $S^1 \subset \mathbb{C}$ as the unit circle. We denote the mapping cone of this map by $S^1 \cup_p D^2$, and we define the spectrum $\mathcal{M}$ to be

$$\mathcal{M} = \mathbb{S} \cup_p D^1 = \Sigma^{-1}\Sigma^\infty(S^1 \cup_p D^2).$$

Milnor's goal is to use the following lemma, which he attributes to F.P. Peterson.

Lemma 11.36. *If $\mathbb{Y}$ is a spectrum of finite type, then if $\pi_*(\mathbb{Y})$ contains p-torsion, then*

$$[\mathcal{M}, \mathbb{Y}]_m = [\Sigma^m \mathcal{M}, \mathbb{Y}]$$

must be nonzero for two consecutive values of m.

This lemma follows from an observation of Peterson that the cofibration sequence $\mathbb{S} \to \mathcal{M} \to \Sigma^\infty S^1$ induces a "universal coefficient" - type exact sequence

$$0 \to [\Sigma^\infty S^1, \mathbb{Y}]_n \otimes \mathbb{Z}/p \to [\mathcal{M}, \mathbb{Y}]_n \to Tor([\Sigma^\infty S^1, \mathbb{Y}]_{n-1}, \mathbb{Z}/p) \to 0.$$

Milnor's strategy is to then use the Adams spectral sequence for computing $[\mathcal{M}, \mathbb{Y}]_*$ to show that for $\mathbb{Y}$ satisfying the hypotheses of Theorem 11.33, these groups cannot be nonzero in two consecutive dimensions. We sketch that argument now.

To compute $[\mathcal{M}, \mathbb{Y}]_*$ using the Adams spectral sequence, we first need to consider the E_2 term, which is $Ext^{*,*}_{\mathcal{A}_p}(H^*(\mathbb{Y}), H^*(\mathcal{M}))$. (We are now, of course taking $\mathbb{Z}/p$ coefficients for our homology and cohomology groups.) Now $H^*(\mathbb{Y})$ is assumed to be a free module over $\mathcal{A}_p/(\beta)$, on even dimensional generators, say $\{y_\alpha\}$, to compute this we can take

$$F'_s = \bigoplus_\alpha \Sigma^{|y_\alpha|} F_s,$$

where $\{F_s, d_s\}$ is the resolution of $\mathcal{A}_p/(\beta)$ constructed in the proof of Proposition 11.35. We then have a resulting free resolution of $H^*(\mathbb{Y})$ over $\mathcal{A}_p$. The

E_2-term of the Adams spectral sequence for computing $[\mathcal{M}, \mathbb{Y}]_*$ is then given by the cohomology of the resulting cochain complex,

$$\cdots \xrightarrow{d^*_{s-1}} Hom_{\mathcal{A}_p}(F'_s; H^*(\mathcal{M})) \xrightarrow{d^*_s} \cdots$$

We now compute $Hom_{\mathcal{A}_p}(F'_s; H^*(\mathcal{M}))$. By the definition F_s in the proof of Proposition 11.35 , F'_s has a basis consisting of symbols $b_{(\alpha, r_0, \dots)}$ of dimension $= dim\, y_\alpha + \sum_i 2r_i(p^i - 1) + s$. Then it is straightforward to check that $E_1^{s,t} = Hom^t_{\mathcal{A}_p}(F'_s; H^*(\mathcal{M}))$ has as a basis consisting of the following homomorphisms:

- For each basis element $b_{(\alpha, r_0, \dots)}$ of dimension t, there is a homomorphism $h_{(\alpha, r_0, \dots)}$ taking $b_{(\alpha, r_0, \dots)}$ to $x \in H^0(\mathcal{M})$, and all other basis elements to zero, and
- there is a homomorphism $h'_{(\alpha, r_0, \dots)}$ taking $b_{(\alpha, r_0, \dots)}$ to $\beta x \in H^1(\mathcal{M})$ and all other basis elements to zero.

The boundary operator $d^*_s : Hom^t_{\mathcal{A}_p}(F'_s; H^*(\mathcal{M})) \to Hom_{\mathcal{A}_p}(F'_{s+1}; H^*(\mathcal{M}))$ is easily see to be given by

$$d^*_s(h_{(\alpha, r_0, \dots)}) = h'_{(\alpha, r_0+1, r_1, \dots)}, \text{and}$$
$$d^*_s(h'_{(\alpha, r_0, \dots)}) = 0$$

We encourage the reader to prove these identities as an exercise, or to consult Milnor [118] for the details.

This means that the cohomology of this cochain complex, which gives us the E_2-term of the Adams spectral sequence, has as a basis for $E_2^{s,t}$ the set of elements $h'_{(\alpha, r_0, \dots)}$ of total dimension $t - s = dim\, y_\alpha + \sum_i 2r_i(p^i - 1) - 1$. Since the dimension of y_α is assumed to be even, this is an odd number. This means that $E_2^{s,t}$ is zero for $t - s$ even. Because all the differentials in the Adams spectral sequence change total dimension $(t - s)$ by one, this implies that they all must be zero and the Adams spectral sequence collapses at the E_2 level (i.e $E_2 = E_\infty$). This in particular implies that $[\mathcal{M}, Y]_m$ can only be nonzero when m is odd. By lemma 11.36, this means that $\pi_*(\mathbb{Y})$ has no p-torsion. Since p was arbitrary, the proof of Theorem 11.33 is now complete. □

As described earlier, this theorem results in an effective calculation of $\pi_*(\mathbb{MU})$. We end this section by stating a few further results.

The following two results were verified by Milnor in [118]

1. $\eta^U_* \otimes \mathbb{Q} = \mathbb{Q}\left[[\mathbb{CP}^1], [\mathbb{CP}^2], \cdots, \right]$

2. Two stably almost complex closed manifolds are cobordant if and only if they have the same Chern numbers. (The reader should prove this as an exercise, using the fact that, as proved above, the Hurewicz homomorphism $\pi_*(\mathbb{MU}) \to H_*(\mathbb{MU})$ is injective.)

Also, as mentioned earlier, Novikov computed the algebra structure of η_*^U [126]. In particular he proved the following:

Theorem 11.37. *(Novikov [126]) The complex cobordism ring η_*^U is a polynomial algebra over $\mathbb{Z}$ with one generator a_i in dimension $2i$ for each $i > 0$.*

We refer the reader to Stong's book [146] for a geometric proof of this result.

11.6 Framed cobordism and the Kervaire Invariant

In this section we consider the ring of "framed cobordism" classes of manifolds, η_*^{fr}. As pointed out in Chapter 11, framed cobordism is a particular type of cobordism of manifolds with stable normal structures. As is the case with all such cobordism theories, one studies normal structures with respect to fibrations $p : B_k \to BO(k)$ (see Definition 11.7). In the case of framed cobordism, the total spaces B_k are assumed to be contractible. They therefore would represent universal principle bundles, $p_n : EO(k) \to BO(k)$.

Since $EO(k)$ is contractible, a lift of the classifying map of a normal bundle, $\tilde{\nu}_M^k : M^n \to EO(k)$ is a trivialization of the normal bundle classified by $\nu_M^k : M^n \to BO(k)$. Therefore a manifold M^n has a (stable, normal) framing, if and only if its normal bundle to a large codimension embedding in Euclidean space is trivial. This is equivalent to the stable normal bundle map $\nu_M : M^n \to BO$ being null homotopic, which in turn is equivalent to the tangent bundle being stably trivial. Recall that this means that there is a bundle isomorphism

$$\tau(M^n) \times \mathbb{R}^L \cong M^n \times \mathbb{R}^{n+L}$$

for L sufficiently large.

Exercise. Show that a closed manifold M^n has a stable normal framing if and only if its tangent bundle is trivial after adding one line to it. That is,

$$\tau(M^n) \times \mathbb{R} \cong M^n \times \mathbb{R}^{n+1}.$$

You may use the fact that a closed n- manifold has the homotopy type of an n-dimensional CW-complex. This will be proved in Chapter 12.

Since the normal structures that we are considering are with respect to bundles $p_k : EO(k) \to BO(k)$ where $EO(k)$ is contractible, the pullback of the universal bundle over $\gamma_k \to BO(k)$ to $EO(k)$ is trivial.

$$\begin{array}{ccc} p_k^*(\gamma_k) & \xrightarrow{\cong} & EO(k) \times \mathbb{R}^k \\ \downarrow & & \downarrow \\ EO(k) & \xrightarrow[=]{} & EO(k). \end{array}$$

The Thom space of this trivial bundle is then $EO(k)_+ \wedge S^k$, which is homotopy equivalent to the sphere S^k since $EO(k)$ is contractible. Therefore the Thom spectrum corresponding to these framed normal structures is the sphere spectrum $\mathbb{S}$, and we have the following consequence of the Pontrjagin-Thom Theorem 11.24:

Corollary 11.38. *There is an isomorphism of graded rings,*

$$\alpha : \eta_*^{fr} \cong \pi_*(\mathbb{S}).$$

As with other cobordism theories, the ring structure in η_*^{fr} is induced by cartesian products of manifolds and their framings, and the ring structure of $\pi_*(\mathbb{S})$ comes from the commutative ring spectrum structure of the sphere spectrum $\mathbb{S}$.

Notice that this result gives yet another reason why the study of the stable homotopy groups of spheres is of such importance. Clearly they are of central importance in homotopy theory, and therefore in algebraic topology more generally. But this result shows how they give play a tremendously important role in differential topology as well. As we will see in the remainder of this section, the study of cobordism classes of framed manifolds has yielded quite important results in manifold theory during the last 60 years.

We begin by considering basic facts laid out in the following exercise.

Exercises.

1. Let M^n be a closed, connected n-dimensional manifold that has a stable normal framing. Show that the equivalence classes of such framings are in bijective correspondence with the set of homotopy classes of maps, $[M^n, SO(N)]$ for N sufficiently large. Indeed show, by using the fibration sequences $SO(N) \to SO(N+1) \to S^N$ that it suffices to take $N > n$.

2. Using Exercise 1, show that $\eta_1^{fr} = \pi_1(\mathbb{S}) = \mathbb{Z}/2$. Describe explicitly a nontrivial stable normal framing on manifold S^1. That is, describe a stable normal framing on S^1 that does not extend to a framing of any surface with boundary equal to S^1.

In Chapter 3 Theorem 3.22, we proved that every Lie group G is parallelizable, (i.e has a trivial tangent bundle). Indeed we gave an explicit trivialization, which identified the tangent bundle τG with $G \times T_1 G$. This in turn induces a stable normal framing on G (**Exercise:** Why?). This implies that every compact Lie group represents an element of $\eta_*^{fr} = \pi_*(\mathbb{S})$. It is not known precisely which elements of $\pi_*(\mathbb{S})$ can be represented by compact Lie groups. Of course S^1 is a compact Lie group in that it is the unit sphere of the complex numbers, $\mathbb{C}$. Similarly S^3 is a noncommutative compact Lie group as it is the unit sphere in the quaternions, $\mathbb{H}$. S^7 is a nonassociative compact Lie group, as seen because it is the unit sphere in the octonians. Even though it is nonassociative, the multiplicative structure suffices to give it a trivialized tangent bundle, and thus a stable normal framing. As you might guess, these framed spheres represent the Hopf maps in their respective dimensions.

11.6.1 The h-cobordism theorem, and the Poincaré conjecture

Perhaps the most interesting and far-reaching early investigation of framed cobordism from a differential topology perspective was the work of Kervaire and Milnor [85]. In this section we describe some of their work, some consequences, and some work that followed that eventually led to one of the most dramatic theorems in algebraic topology in the last 60 years [70].

As the reader may already know, one of the most surprising results about manifolds in the 1950's was a result by Milnor [116] saying that the 7-dimensional sphere, S^7, has "exotic differentiable structures". This means that there are smooth, closed 7-dimensional manifolds that are homeomorphic to the unit sphere $S^7 \subset \mathbb{R}^8$, but are not *diffeomorphic* to S^7.

More generally, a closed, oriented, smooth manifold M^n is said to be a *"homotopy n-sphere"* if it is homotopy equivalent to S^n. Kervaire and Milnor put the following equivalence relation on the collection of homotopy n-spheres.

Definition 11.8. *Two closed, oriented, smooth manifolds M_1^n and M_2^n are said to be "h-cobordant" if the disjoint union $M_1^n \sqcup -M_2^n$ is the boundary of an oriented $(n+1)$-dimensional manifold W^{n+1} where both inclusions $M_1 \hookrightarrow W^{n+1}$ and $M_2 \hookrightarrow W^{n+1}$ are homotopy equivalences. Here $-M_2^n$ denotes the manifold M_2^n with the opposite orientation.*

In a deep, far-reaching, and beautifully written paper [85], Kervaire and Milnor investigated the groups Θ_m of h-cobordism classes of closed, oriented manifolds that are homotopy equivalent to the sphere S^m. The group structure in Θ_n is given by the connected sum operation.

The relevance of h-cobordisms in this setting is due to the powerful h-cobordism theorem of S. Smale [140].

Theorem 11.39. *(The "h-cobordism theorem"). Let W be a compact $(n+1)$-dimensional smooth h-cobordism between two smooth closed simply connected n-manifolds M^n and N^n, where $n \geq 5$. Then $W^{n+1}$is diffeomorphic to $M^n \times [0,1]$ (or, equivalently, $N^n \times [0,1]$). In particular, M^n and W^n are diffeomorphic.*

As a nearly immediate consequence, Smale proved the analogue the generalized Poincare conjecture in dimensions 5 and greater:

Corollary 11.40. *("Generalized Poincaré conjecture") Assume $n \geq 6$. If a closed, smooth manifold Σ^n is homotopy equivalent to the sphere S^n, it is homeomorphic to S^n.*

So in particular we know that for $n \geq 6$, every homotopy sphere is homeomorphic to S^n. Furthermore it follows from the h-cobordism theorem that two homotopy spheres in these dimensions are h-cobordant if and only if they are diffeomorphic. So for $n \geq 6$ the group Θ_n can be described as the set of diffeomorphism classes of differentiable structures on the topological n-sphere. It is one of the remarkable achievements of Milnor [116] and Kervaire-Milnor [85] that these groups, for $n \geq 5$ are not necessarily trivial, but are finite.

We now sketch how the generalized Poincaré conjecture follows from Smale's h-cobordism theorem.

Proof. We first prove the following lemma.

Lemma 11.41. *Suppose W^n is a compact, smooth, contractible manifold of dimension $n \geq 6$ with a simply connected boundary. Then W^n is diffeomorphic to the disk D^n.*

Proof. Let $D_0 \subset W^n$ be a smooth disk embedded in the interior of W^n. Then notice that $W - Int\, D_0$ is an h-cobordism between ∂W^n and $\partial D_0 = S^{n-1}$. Therefore, by the h-cobordism theorem, there is a diffeomorphism

$$W^n - Int\, D_0 \simeq \partial W^n \times [0,1] \cong S^{n-1} \times [0,1].$$

Thus

$$W^n \cong (W^n - Int\, D_0) \cup Int\, D_0 \cong S^{n-1} \times [0,1] \cup_{\partial D_0 \cong S^{n-1}} D_0 \cong D^n.$$

□

Note. In certain dimensions, Kervaire and Milnor proved more. In particular the proved they following result.

Theorem 11.42. *(Kervaire and Milnor [85]). Suppose M^n is a closed, simply connected, smooth manifold that is homotopy equivalent to S^n. Then if $n = 4, 5,$ or 6, M^n bounds a smooth, compact, contractible manifold.*

Notice that this, together with Lemma 11.41 implies that for $n = 5$ or 6, M^n is a actually diffeomorphic to S^n. In particular the generalized Poincaré conjecture that was proved by Smale actually holds in dimension 5 as well. We will discuss what is known in lower dimensions below.

We now sketch the proof of Corollary 11.40.

Let $D_0 \subset M^n$ be a smoothly embedded disk. Then $W^n = M^n - Int\, D_0$ satisfies the hypotheses of Lemma 11.41 and so is diffeomorphic to a disk D^n. Therefore $M^n = (M^n - Int\, D_0) \cup D_0$ is diffeomorphic to two copies D_1^n, D_2^n of the n-disk with boundaries identified under a diffeomorphism $h : \partial D_1^n \xrightarrow{\cong}$

∂D_2^n. Such a manifold is called a "twisted sphere", and the proof is completed by showing that any twisted sphere,

$$M^n = D_1^n \cup_h D_2^n$$

is homeomorphic to S^n. In [113] Milnor describes such a homeomorphism explicitly.

Let $g_1 : D_1^n \to S^n$ be a diffeomorphism onto the lower hemisphere of $S^n \subset \mathbb{R}^{n+1}$ equal to $\{(x_1, \dots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1} \text{ such that } x_{n+1} \leq 0\}$.

Each point of D_2^n may be written as $t \cdot v$, $0 \leq t \leq 1$, $v \in \partial D_2$. Then define $g : M^n \to S^n$ by

$$g(u) = \begin{cases} g_1(u) \quad \text{if } u \in D_1^n, \quad \text{and} \\ sin\frac{\pi t}{2} g_1(h^{-1}(v)) + cos\frac{\pi t}{2} e_{n+1} \text{ where } e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}, \\ \text{for all points } t \cdot v \in D_2^n. \end{cases}$$

We leave it for the reader to check that g is a well defined, one-to-one continuous map onto S^n, and hence is a homeomorphism. □

11.6.1.1 Some comments on low dimensions

Much of the work on differential topology in the 1960's and 1970's made use of Smale's h-cobordism theorem (Theorem 11.39). Since this theorem only applies in "high dimensions", i.e where one is studying h- cobordisms between two n-manifolds for $n \geq 5$, much of the work about the topology of smooth manifolds done during this period was about manifolds of dimension 5 and greater. This all changed in the 1980's, when the study of low-dimensional topology went through some revolutionary developments. While not the focus of this book, the study of the topology of 3 and 4 dimensional manifolds has been a major focus of research for the last 40 years. We now mention a few of the early results during this period.

In 1982 M. Freedman proved the four-dimensional Poincaré conjecture: Every closed, smooth, 4-dimensional manifold that is homotopy equivalent to S^4 is in fact homeomorphic to S^4. He actually proved that every smooth, simply connected, closed 4-dimensional manifold is completely determined up to homeomorphism by its intersection form [50].

The situation is very different for classifying smooth manifolds up to diffeomorphism. S.K. Donaldson showed that there are severe restrictions on the intersection forms that can be realized by smooth 4-manifolds [39] thereby showing that many topological manifolds do not have any smooth structure. He then showed that there are nontrivial *smooth* h-cobordisms between smooth 4-dimensional manifolds [40]. (By "nontrivial" we mean smooth h-cobordisms that are not diffeomorphic to a cylinder $M^4 \times [0, 1]$.) Using this, Freedman showed that $\mathbb{R}^4$ has an exotic differentiable structure [51], and Taubes proved that there are uncountably many such exotic structures [149]. The question of whether every four-dimensional manifold that is homotopy equivalent to S^4 is actually diffeomorphic to S^4 remains a difficult, open queston, and is referred to as the "smooth 4-dimensional Poincaré conjecture".

The study of the smooth topology of 4-dimensional manifolds that was pioneered by Donaldson uses spaces of solutions of partial differential equations coming from physics (Yang-Mills equations, Seiberg-Witten equations) or from symplectic geometry (Gromov-Witten equations). This area has been one of the most active areas of research in topology in the last 40 years, and continues to produce surprising results of a very different nature than that which occurs in dimension ≥ 5.

In dimension three the history goes back to the early days of topology and the work of H. Poincaré. The famous 3-dimensional Poincaré conjecture is that every closed, simply connected 3-dimensional manifold is homeomorphic to S^3.

Exercise. Show that a closed, simply connected 3-dimensional manifold is homotopy equivalent to S^3.

Hint. Use Poincaré duality.

In the 1930's, the great British topologist J.H.C. Whitehead discovered a noncompact contractible 3-dimensional manifold that is not homeomorphic to $\mathbb{R}^3$. Using this he gave what turned out to be a false proof of Poincaré's conjecture. Since that time many many unsuccessful attempts were made by many mathematicians to prove Poincaré's conjecture. Three dimensional topology was revolutionized in the 1970's by William Thurston who made a general conjecture about the classification of 3-manifolds having to do with the geometric structures they possess. This was known as Thurston's "Geometrization Conjecture" and it, among other things, implied the Poincaré Conjecture. The interplay between the topology and geometry of 3-manifolds revolutionized the way topologists studied 3-manifolds, and it continues to be a tremendously important direction of research today.

In 2003, G. Perelman wrote a series of unpublished papers that gave a proof of Thurston's Geometrization Conjecture, and therefore of the 3-dimensional Poincaré Conjecture. It made use of Ricci curvature flow in the study of the topology of 3-manifolds initiated by Richard Hamilton [64] in 1982. Morgan and Tian eventually wrote a book giving full details of Perelman's proof [123]. The proofs of the conjectures of Poincaré and Thurston were among the greatest mathematical achievements of the early part of the 21st century. A nice description of this work, with a nonlinear PDE perspective, was given by T. Tao [148].

11.6.2 The work of Kervaire-Milnor on homotopy spheres, surgery, and framed cobordism

We now continue our discussion on the work of Kervaire and Milnor [85] and its implications to the understanding of framed cobordism.

As mentioned above, the main goal of [85] was to understand the groups Θ_n of homotopy n-spheres, which, as seen above, for $n \geq 5$, can be described as the group of diffeomorphism classes of differentiable structures on the topological n-sphere S^n. As noted above, the group structure comes from the connected sum operation. The relevance of this group to framed cobordism is the following result in [85], whose proof we sketch.

Theorem 11.43. *(Kervaire and Milnor [85]). Every homotopy n-sphere is stably parallelizable. That is, it has a stable normal framing, and therefore a choice of such a framing defines an element of the framed cobordism group* η_n^{fr}.

Proof. (Sketch). We sketch the proof that appears in [85]. As you will see it involves some homotopy theoretic calculations of Adams [5] that were new at the time.

Let Σ be a homotopy n-sphere. Let $\nu_\Sigma : \Sigma \to BO$ be the classifying map of its stable normal bundle. ν_Σ is well-defined up to homotopy. To show that Σ is stably parallelizable, we need to show that ν_Σ is null homotopic.

Since Σ is a homotopy sphere, we have a bijection between homotopy classes of maps

$$[\Sigma, BO] \cong \pi_n(BO) \cong \pi_{n-1}(O).$$

Thus the obstruction to Σ being stably parallelizable is a well-defined class

$$\nu_\Sigma \in \pi_{n-1}(O). \tag{11.17}$$

Now the homotopy groups of O were computed by Bott [14] when he proved what is known as "Real Bott periodicity". These groups are 8-periodic in the sense that $\pi_k(O) \cong \pi_{k+8}(O)$, for $k \geq 0$. These groups are given by the following:

Congruence class of n mod 8:	0	1	2	3	4	5	6	7
$\pi_{n-1}(O)$:	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

So one can conclude immediately that if n is congruent to 3, 5, 6, or 7 mod 8, then Σ is stably parallelizable, because the obstruction ν_Σ lies in the zero group.

Now assume n is congruent to 0 or 4 mod 8. Write $n = 4k$. Then the Hurewicz map

$$\mathbb{Z} \cong \pi_{4k-1}(O) \cong \pi_{4k}(BO) \to H_{4k}(BO) \cong Hom(H^{4k}(BO); \mathbb{Z})$$

is injective, and the image of the generator is is determined by its value on the k^{th} Pontrjagin class $p_k(\Sigma)$. But by another major result of that era, the Hirzebruch signature theorem [41], $p_k(\Sigma) = 0$ because it can be identified with an invariant (the "signature") of a quadratic form defined on $H^{2k}(\Sigma)$. But $H^{2k}(\Sigma) = 0$. Thus $\nu_\Sigma = 0$.

We are therefore left to consider the case when n is congruent to 1 or 2 mod 8, in which case $\pi_{n-1}(O) \cong \mathbb{Z}/2$.

The argument in this case makes use of calculations of J.F. Adams of the image of the so-called "J-homomorphism" in homotopy theory. The J-homomorphism is a map from the homotopy groups of the (infinite) orthogonal group, which are known by real Bott periodicity, to the stable homotopy groups of spheres.

$$J : \pi_k(O) \to \pi_k(\mathbb{S}).$$

This homomorphism is defined as follows. An element $A \in O(n)$ is a linear orthogonal homomorphism, $A : \mathbb{R}^n \to \mathbb{R}^n$. Since A is metric preserving, it extends to a well defined map of the one point compactification, sending ∞ to ∞.

$$A : S^n = \mathbb{R}^n \cup \infty \to \mathbb{R}^n \cup \infty = S^n.$$

If we view $\infty \in S^n$ as the basepoint, we can think of A as a basepoint-preserving map between S^n and itself, and therefore is an element of the n-fold loop space, $\Omega^n S^n$. This association defines a map

$$J_n : O(n) \to \Omega^n S^n$$

and therefore an induced homomorphism on homotopy groups,

$$(J_n)_* : \pi_k(O(n)) \to \pi_k(\Omega^n S^n) = \pi_{k+n}(S^n).$$

One easily checks that these homomorphisms are compatible as n increases, and so we get a homomorphism

$$J : \pi_k(O) = \lim_{n\to\infty} \pi_k(O(n)) \to \lim_{n\to\infty} \pi_{k+n}(S^n) = \pi_k(\mathbb{S}). \tag{11.18}$$

There is another, more geometric way to view the J-homomorphism. An element $\alpha \in \pi_k(O)$ determines a (stable) normal framing on the standard k sphere.

Exercise. Show this is true. That is, show how elements of $\pi_k(O)$ determine stable normal framings of S^k.

Let $\phi(\alpha)$ represent the stable normal framing on S^k induced by by $\alpha \in \pi_k(O)$. Let $[\phi(\alpha)] \in \eta_k^{fr}$ be the corresponding framed cobordism class.

Exercise. Show that the class $[\phi(\alpha)] \in \eta_k^{fr} \cong \pi_k(\mathbb{S})$ corresponds, up to sign, with the image of the J-homomorphism $J(\alpha) \in \pi_k(\mathbb{S})$ as described above.

Now consider again a homotopy n- sphere Σ and the obstruction class to it being stably parallelizable, $\nu_\Sigma \in \pi_{n-1}(O)$. We note that the definition of $\gamma_n(\Sigma)$ can be described slightly differently as follows.

Consider an embedding of Σ into a high codimension Euclidean space, $e : \Sigma \hookrightarrow \mathbb{R}^{n+L}$. The normal bundle to e then defines a map to the Grassmannian,

$$\nu_e : \Sigma \to Gr_L(\mathbb{R}^{n+L})$$
$$x \to (De(T_x\Sigma))^{\perp} \subset \mathbb{R}^{n+L}$$

For each $x \in \Sigma$, $(De(T_x\Sigma))^{\perp}$ is the L-dimensional normal space to the tangent space $T_x\Sigma$ linearly embedded via the differential De in $\mathbb{R}^{n+L}$. For L sufficiently large, we can view this map as classifying the stable normal bundle map $\nu_\Sigma : \Sigma \to BO$.

Now let $\tilde{\Sigma}$ be the n-dimensional manifold with boundary obtained by removing a small n-dimensional ball around a basepoint $x_0 \in \Sigma$:

$$\tilde{\Sigma} = \Sigma - B^n(x_0).$$

Notice that the boundary is a sphere, $\partial\tilde{\Sigma} \cong S^n$. Notice furthermore that $\tilde{\Sigma}$ is contractible since

$$\tilde{\Sigma} \cup B^n(x_0) = \Sigma$$

which is homotopy equivalent to S^n. Therefore the normal bundle ν_e, when restricted to $\tilde{\Sigma}$ is trivial, and up to homotopy, there is a unique trivialization. This information can be encoded as follows. Let $Fr(\gamma^L) \to Gr_L(\mathbb{R}^{n+L})$ be the frame bundle. That is a point in $Fr(\gamma^L)$ lying over an L-dimensional subspace $P \subset \mathbb{R}^{n+L}$ is the space of orthogonal bases of P. This is a principal $O(L)$ - bundle, and it is the pullback of the universal principal $O(L)$ - bundle $EO(L) \to BO(L)$ under the inclusion $Gr_L(\mathbb{R}^{n+L}) \hookrightarrow BO(L)$.

Since the normal bundle ν_e, when restricted to $\tilde{\Sigma} \subset \Sigma$ is trivialized, there is a a lifting, well-defined up to homotopy,

$$\tilde{\nu}_e : \tilde{\Sigma} \to Fr(\gamma^L)$$

of the restriction of ν_e to $\tilde{\Sigma}$. Since the restriction of the lifting $\tilde{\nu}_e$ to the boundary $\partial\tilde{\Sigma} \cong S^{n-1}$ has a well-defined extension to $B^n(x_0)$ when projected to $Gr_L(\mathbb{R}^{n+L})$, one has a well-defined (up to homotopy) lifting of this restriction to the fiber over the basepoint,

$$S^{n-1} \cong \partial\tilde{\Sigma} \to O(L). \tag{11.19}$$

By the construction of the classifying map to the normal bundle ν_e, one sees that this map is homotopic to the classifying map $\nu_\Sigma : S^{n-1} \to O$ described above 11.17.

Now notice that $\tilde{\Sigma}$, viewed as a framed manifold with boundary S^3 defines a null-cobordism of its boundary together with its framing. In other words, the class

$$\phi(\nu_\Sigma) \in \eta_n^{fr} \quad \text{which corresponds to} \quad J(\nu_\Sigma) \in \pi_n(\mathbb{S})$$

is zero.

Now soon before Kervaire and Milnor wrote their paper [85], J.F Adams proved that when n is congruent to 1 or 2 mod 8,

$$J_{n-1} : \pi_{n-1}(O) \to \pi_{n-1}(\mathbb{S})$$

is a monomorphism [5]. Adams's calculation of the J-homomorphism was extremely important in algebraic topology, and in this case it allowed Kervaire and Milnor to conclude that the obstruction class $\nu_\Sigma \in \pi_{n-1}(O)$ is zero. This implies that Σ has a stable normal framing.

□

We now know that every homotopy n- sphere $\Sigma \in \Theta_n$ can be stably framed. Thus if we define

$$\Theta_n^{fr} = \{(\Sigma, \phi), \quad \text{where} \quad \phi \quad \text{is a stable normal framing of} \quad \Sigma\} \tag{11.20}$$

then by taking framed bordism classes, we get a map

$$\psi : \Theta_n^{fr} \to \eta_n^{fr}. \tag{11.21}$$

Another, more homotopy theoretic way of considering this relationship is the following. As we've seen, by choosing a framing on $\Sigma \in \Theta_n$ and then passing to the framed bordism class, one obtains an element of $\eta_n^{fr} \cong \pi_n(\mathbb{S})$. This does not produce a well-defined map from the group of homotopy spheres Θ_n to the stable homotopy group of spheres $\pi_n(\mathbb{S})$, because different choices of framings could yield different cobordism classes. However, as we saw in the proof of Theorem 11.43 two different choices of stable framings on a homotopy n- sphere Σ produce, when passing to the framed bordism classes, two elements of $\pi_n(\mathbb{S})$ whose difference lies in the image of the J-homomorphism, $J : \pi_n(O) \to \pi_n(\mathbb{S})$. So if *Coker J* is the cokernel of the J homomorphism one has a well defined homomorphism

$$\psi : \Theta_n \to Coker\ J. \tag{11.22}$$

This map was the main object of study in Kervaire and Milnor's seminal paper [85]. In particular one can ask about the image of this map, and in particular ask the following geometric question:

Question.
Is

$$\psi : \Theta_n^{fr} \to \eta_n^{fr} \tag{11.23}$$

surjective? In other words, is every framed manifold frame cobordant to a framed homotopy sphere?

In studying this question, differential and algebraic topology have been tremendously developed and advanced over the last fifty years. We continue this section by discussing these developments.

Kervaire and Milnor, in considering this question, described how to alter the homotopy groups of a manifold and remain in the same cobordism class. This technique is known widely as "surgery", although Kervaire and Milnor referred to it as "spherical modification". Here is the basic construction.

Definition 11.9. *Let M^n be a smooth manifold. Suppose*

$$\phi : S^p \times D^{q+1} \to M^n$$

is a smooth embedding, where $p + q + 1 = n$. Define a new manifold $M' = \chi(M^n, \phi)$ to be formed from the disjoint union

$$M' = (M^n - \phi(S^p \times 0)) \sqcup \left(D^{p+1} \times S^q\right) / \sim$$

where " $\sim$ " denotes the identification of $\phi(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$, and $0 < t \leq 1$.

One says that M' is obtained by "doing surgery" along ϕ. Kervaire and Milnor refer to M' as being the "spherical modification" $\chi(\phi)$. Notice that the boundary of M' is equal to the boundary of M.

The following lemma, which is straightforward, describes how the homotopy group changed after doing surgery.

Let $\lambda \in \pi_p(M^n)$ denote the homotopy class of $\phi_{|S^p \times 0} : S^p \to M^n$.

Lemma 11.44. *The homotopy groups of M' are given by*

$$\pi_i(M') \cong \pi_i(M^n) \quad \text{for } i < min\,(p, q),$$

and

$$\pi_p M' \cong \pi_p(M^n)/\Lambda$$

provided that $p < q$; where Λ denotes a certain subgroup of $\pi_p(M^n)$ containing λ.

Notice that this lemma says that if $p < q$ the effect of this surgery was to "kill" the homotopy class λ.

We now observe that one can kill any homotopy class λ whose dimension is less than half the dimension of M^n.

Lemma 11.45. *Suppose M^n is a stably framed manifold and $\lambda \in \pi_p(M^n)$ with $n \geq 2p + 1$. Then the class λ is represented by an embedding $\phi : S^p \times D^{n-p} \hookrightarrow M^n$.*

Proof. . Since $n \geq 2p+1$, Whitney's embedding theorem says that $\lambda \in \pi_p(M^n)$ can be represented by an embedding, $\phi_0 : S^p \hookrightarrow M^n$. The normal bundle of this embedding ν_{ϕ_0} satisfies the bundle equation

$$TS^p \oplus \nu_{\phi_0} \cong \phi_0^*(TM^n)$$

and therefore we can stabilize this equation

$$TS^p \oplus \epsilon^L \oplus \nu_{\phi_0} \cong \phi_0^*(TM^n \oplus \epsilon^L) \tag{11.24}$$

where ϵ^L denotes the trivial L-dimensional bundle. Now since M^n is assumed to have a trivial stable normal bundle, and hence a trivial stable tangent bundle, then for L sufficiently large, $TM^n \oplus \epsilon^L$ is trivial, i.e isomorphic to ϵ^{L+n}. On the other hand, we know that the sphere S^p has a stably trivial tangent bundle, in fact $TS^p \oplus \epsilon^1 \cong \epsilon^{p+1}$, and hence bundle equation (11.24) simplifies to the equation

$$\epsilon^{p+L} \oplus \nu_{\phi_0} \cong \epsilon^{n+L}.$$

In other words, the normal bundle ν_{ϕ_0} is stably trivial. But since the fiber dimension of the normal bundle ($= n-p$) is larger than the dimension of the base space S^p, this means the normal bundle must be trivial without stabilizing. That is,

$$\nu_{\phi_0} \cong \epsilon^{n-p}.$$

Now since the normal bundle of the embedding $\phi_0 : S^p \to M^n$ is trivial, then its tubular neighborhood is diffeomorphic to $S^p \times D^{n-p}$. This proves the lemma. □

So we now know we can "kill" any homotopy class $\lambda \in \pi_p(M^n)$ where M^n is a stably framed manifold and $n \geq 2p+1$. However it is not clear, and it may not be true, that the resulting manifold M' will be stably framed. However the following was proven by Milnor in [119].

Lemma 11.46. *[119](Milnor) Under the same hypothesis of Lemma 11.45, the embedding $\phi : S^p \times D^{n-p} \hookrightarrow M^n$ can be chosen within its homotopy class so that the modified manifold M' will also be stably framed.*

We now observe that when M^n is closed and one does surgery on a homotopy class $\lambda \in \pi_p(M^n)$ where M^n is stably framed and $n \geq 2p+1$, one obtains a manifold M' that is cobordant to M^n. Indeed this can be taken to be a framed cobordism if the embedding $\phi : S^p \times D^{n-p} \to M^n$ is chosen as in Lemma 11.46. We can take the cobordism to be

$$W_\phi^{n+1} = (M^n \times [0,1]) \sqcup \left(D^{p+1} \times D^{q+1}\right) / \sim$$

where $(x,y) \in S^p \times D^{q+1} \subset \partial(D^{p+1} \times D^{q+1}$ is identified with $(\phi(x,y),1) \in M^n \times \{1\}$. Clearly

$$\partial W_\phi^{n+1} = M^n \sqcup M'.$$

Unfortunately W_ϕ^{n+1} is not smooth as it stands, but has "corners", i.e points where the coordinate neighborhoods naturally look like one quadrant of the plane $\times \mathbb{R}^n$ instead of a Euclidean half space. However there is a canonical way of making it a smooth manifold with boundary, by a process called "straightening the angles", which is described, for example, in [37]. Furthermore, as in Lemma 11.46 the cobordism can be taken to be a framed cobordism. By iterating the surgery procedure one then has the following result:

Corollary 11.47. *If M^n is a closed, stably framed n-dimensional manifold and p is an integer satisfying $2p+1 \leq n$, then M^n is frame cobordant to a closed, stably framed n-manifold M' that is p-connected.*

We are now ready to answer Question 11.23 for odd dimensional closed, stably framed manifolds.

Theorem 11.48. *Let M^{2m+1} be a connected, closed $(2m+1)$-dimensional stably framed manifold. Then M^{2m+1} is frame cobordant to a homotopy sphere.*

Proof. First consider the case of a one dimensional manifold. Then Corollary 11.47 implies that every closed, stably framed 1-manifold is frame cobordant to a framed circle.

Now suppose the dimension of the manifold is odd and larger than one. Say $n = 2m+1$, for $m \geq 1$. Then Corollary 11.47 says that a closed, stably framed manifold M^{2m+1} is frame cobordant to an m-connected stably framed manifold M'. In particular this implies that $H_q(M') = 0$ for $1 \leq q \leq m$ and $H_0(M') \cong \mathbb{Z}$ since M' is connected. By the Universal Coefficient Theorem this implies that $H^q(M') = 0$ for $1 \leq q \leq m$ as well. Now since M' is simply connected, it is orientable, and hence it satisfies Poincaré duality. This implies that $H_{2m+1-q}(M') = 0$ for $1 \leq q \leq m$ and $H_{2m+1}(M') \cong \mathbb{Z}$.

Putting these observations together we see that M^{2m+1} is frame cobordant to a manifold M' whose homology is given by

$$H_r(M') \cong \begin{cases} 0 & \text{if } 1 \leq r \leq 2n \quad \text{and} \\ \mathbb{Z} & \text{if } r = 0 \text{ or } 2m+1 \end{cases}$$

In other words, M' has the homology of the sphere S^{2m+1}. Now consider a map $M' \to S^{2m+1}$ defined as follows. Let $x_0 \in M'$ be basepoint, with a neighborhood B which is diffeomorphic to the open disk D^{2m+1}. Then the "pinch map"

$$p : M' \to M'/(M' - B) \cong \bar{D}^{2m+1}/\partial\bar{D}^{2m+1} = S^{2m+1}$$

is a degree one map and therefore induces an isomorphism in H_{2m+1}, and therefore in H_r for all r. Since both M' and S^{2m+1} are simply connected we conclude that $p : M' \to S^{2m+1}$ is a homotopy equivalence. In other words, M' is a homotopy $(2m+1)$-sphere □

The situation for even dimensional framed manifolds is much more difficult and interesting. Consider a closed framed manifold of dimension $n = 2m$, M^{2m}.

Exercise. Show that the same Poincaré duality argument used to prove Theorem 11.48 proves the following theorem.

Theorem 11.49. *Let M^{2m} be a connected, closed, stably framed manifold of dimension $2m$. Then M^{2m} is frame cobordant to a manifold M' with the following properties:*

1. *M' is $(m-1)$-connected.*

2. $$H_q(M') = \begin{cases} \mathbb{Z} & \text{if } q = 0 \quad \text{or} \quad 2m \\ 0 & \text{if } 1 \leq q \leq m-1 \quad \text{and if} \quad m+1 \leq q \leq 2m-1 \\ \text{a finitely generated free abelian group, if } q = m. \end{cases}$$

Observation. Notice that property (2) of $H_q(M')$ in this theorem follows from property (1).

The next question to be addressed is the following.

Question. Can one do surgery in a closed stably framed manifold of dimension $2m$ that is $(m-1)$-connected, so as to remove the middle dimension homology? If not, what are the obstructions to doing so?

This question was addressed by Kervaire and Milnor in [85] and by Kervaire in [83].

An important tool in the description of obstructions to doing framed surgery to remove the middle dimension homology is the intersection pairing, as was described and studied in Chapter 9, §2

$$H_m(M^{2m}) \times H_m(M^{2m}) \to H_0(M^{2m}) \cong \mathbb{Z} \tag{11.25}$$
$$\alpha \times \beta \to \alpha \cdot \beta.$$

This pairing is nondegenerate by Poincaré duality, and is symmetric if m is even and skew-symmetric if m is odd.

The following was an important technical result that allowed Kervaire and Milnor to address the above surgery question.

Lemma 11.50. *(Kervaire and Milnor, see Lemma 7.1 of [85]), Let M^{2m} be an $(m-1)$-connected, $2m$-dimensional closed manifold for $m \geq 3$. Suppose $H_m(M^{2m})$ is free abelian with basis $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r$ satisfying*

$$\lambda_i \cdot \lambda_j = 0, \quad \lambda_k \cdot \mu_j = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta function. Suppose further that every embedded sphere in M which represents a homology class in the subgroup generated by $\{\lambda_1, \dots, \lambda_r\}$ has trivial normal bundle. Then $H_m(M^{2m})$ can be killed by a sequence of surgeries.

Remark

1. The assumption that $m \geq 3$ is necessary because as was shown by Kervaire and Milnor in [84], there exist homology classes in $H_2(M^4)$ that are not represented by embedded spheres, but Milnor showed in [119] that any homology class in $H_m(M^{2m})$ can by represented by an embedded sphere if $m \geq 3$.

2. The proof of this lemma is a fairly straightforward, nice argument examining the effect of doing surgery on a specific homotopy (homology) class. We refer the reader to [85] for the proof.

11.6.3 The signature and the Kervaire invariant as surgery obstructions

Our next step is to use Lemma 11.50 to give another partial answer to Question 11.23.

Consider a closed, simply connected, stably framed manifold of dimension divisible by 4, M^{4k}. Assume furthermore that M^{4k} is $(2k-1)$-connected. (We know from the above discussion that any such manifold is frame cobordant to a $(2k-1)$-connected manifold.) Then $H_{2k}(M^{4k})$ is a finitely generated free abelian group and its intersection form

$$\langle\, , \rangle : H_{2k}(M^{4k}) \times H_{2k}(M^{4k}) \to \mathbb{Z}$$

is a nonsingular, symmetric bilinear form. Given a basis $\{\alpha_q\}$ for $H_{2k}(M^{4k})$ then this form has a matrix representation, where the $(i,j)^{th}$ entry is the value of $\langle \alpha_i, \alpha_j \rangle$. This matrix is symmetric and nonsingular, and therefore can be diagonalized over $\mathbb{R}$. Recall that the *index* of this bilinear form is the rank of the positive eigenspace minus the rank of the negative eigenspace of this matrix. The index of this intersection matrix is called the *signature* of M^{4k}, written $\sigma(M^{4k})$.

Theorem 11.51. *A closed, simply connected, stably framed manifold of dimension* $4k$*,* M^{4k} *is frame cobordant to a homotopy sphere if an only if its signature is zero*

$$\sigma(M^{4k}) = 0.$$

Proof. We prove this result in several steps. We refer the readers to [85] and in [119] for details.

Theorem 11.52. . *The signature of an oriented* $(4k)$*-dimensional manifold is an oriented cobordism invariant. That is, if* M^{4k} *is oriented cobordant to* N^{4k}*, then*

$$\sigma(M^{4k}) = \sigma(N^{4k}).$$

Proof. We first consider some basic properties of the signature of a $4k$-dimensional oriented, closed manifold.

Exercises.

1. Show that the signature of a disjoint union is the sum of the signatures:

$$\sigma(M_1^{4k} \sqcup M_2^{4k}) = \sigma(M_1^{4k}) + \sigma(M_2^{4k}).$$

2. Show that the signature of a product of two manifolds is the product of their signatures:

$$\sigma(M_1^{4k} \times M_2^{4q}) = \sigma(M_1^{4k})\sigma(M_2^{4q}).$$

The next result, taken together with the result of the first exercise, will imply Theorem 11.52:

Lemma 11.53. *Lt $n = 4k$, and suppose M^n is an oriented, closed manifold that is the boundary of an oriented manifold W^{n+1},*

$$M^n = \partial W^{n+1}.$$

Then $\sigma(M^n) = 0$.

Proof. Let $[M] \in H_n(M^n)$, and $[W, M] \in H_{n+1}(W, M)$ denoted the fundamental (orientation) classes. Let $\iota : M^n = \partial W^{n+1} \hookrightarrow W^{n+1}$ be the inclusion mapping. Then by Lefschetz duality we have a commutative diagram

$$\begin{array}{ccccccc}
\cdots \to H^r(W) & \xrightarrow{\iota^*} & H^r(M) & \xrightarrow{\delta} & H^{r+1}(W,M) & \longrightarrow & H^{r+1}(W) \to \cdots \\
\cong \big\downarrow \cap[W,M] & & \cong \big\downarrow \cap[M] & & \cong \big\downarrow \cap[W,M] & & \cong \big\downarrow \cap[W,M] \\
\cdots \to H_{n+1-r}(W,M) & \xrightarrow[\partial]{} & H_{n-r}(M) & \xrightarrow[\iota_*]{} & H_{n-r}(W) & \longrightarrow & H_{n-r}(W,M) \to \cdots
\end{array}$$

Let $Im^r = Image(\iota^*)^r$ be the image of the cohomology homomorphism ι^* in degree r, and similarly let $K_q = Ker((\iota_*)_q)$ be the kernel of the homology homomorphism ι_* in degree q. By exactness we have an isomorphism

$$\cap[M] : Im^r \xrightarrow{\cong} K_{n-r}.$$

Now if $a \in Im^r$ and $b \in Im^{n-r}$, then the pairing

$$\langle a \cup b, [M] \rangle = 0.$$

To see this, notice that the above diagram yields

$$\begin{aligned}
\langle a \cup b, [M] \rangle &= \langle \iota^*(\alpha \cup \beta), \partial[W, M] \rangle \quad \text{for some } \alpha \in H^r(W),\ \beta \in H^{n-r}(W) \\
&= \langle \delta\iota^*(\alpha \cup \beta), [W, M] \rangle \\
&= 0 \quad \text{by exactness.}
\end{aligned}$$

Now since the coefficients of the (co)homology groups above are a field, the universal coefficient theorem says that $H^i(M) \cong H_i(M)$ and that ι^* is dual to ι_*. That is, the following diagram commutes:

$$\begin{array}{ccc} H_{n-p}(W) & \xleftarrow[\iota_*]{} & H_{n-p}(M) \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ H^{n-p}(W) & \xrightarrow[\iota^*]{} & H^{n-p}(M). \end{array}$$

So we have that $H_{n-p}(M)/K_{n-p}$ is the dual of Im^{n-p}. Thus Im^p is precisely the annihilator of Im^{n-p}.

Since M^n is $4k$-dimensional, we can write

$$H^{2k}(M) \cong Im^{2k} \oplus B^{2k}$$

where Im^{2k} and B^{2k} are dually paired with bases $\{v_i\}$ and $[u_j\}$ respectively, with

$$\langle v_i \cup u_j, [M] \rangle = \delta_{i,j} \quad \langle v_i \cup v_j, [M] \rangle = \langle u_i \cup u_j, [M] \rangle = 0.$$

Ordering the bases $v_1, u_1, v_2, u_2, \cdots$, the matrix of the intersection pairing consists of 2×2 blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ along the diagonal, with zeros elsewhere. One then computes the index of matrix to be zero.

□

□

Notice now that this theorem together with the result of Exercise 2 above implies that the signature defines a ring homomorphism

$$\sigma : \eta_*^{SO} \to \mathbb{Z} \tag{11.26}$$

which is defined to be zero in dimensions other than those congruent to zero mod 4, and dimensions divisible by 4 it is defined to be the signature.

The following is an interesting characterization of this homomorphism.

Corollary 11.54. *The signature ring homomorphism*

$$\sigma : \eta_*^{SO} \to \mathbb{Z}$$

is the unique ring homomorphism whose value on each $\mathbb{CP}^{2n}$ *is one.*

Proof. This result follows from Theorem 11.28 and the fact that any homomorphism from a finitely generated abelian group to the integers is determined by its values on its maximal torsion free subgroup. □

We now return to the sketch of a proof of Theorem 11.51. Notice that by Theorem 11.52, if M^{4k} is frame cobordant to a homotopy sphere, its signature must be zero. We now consider the converse of this statement. So suppose M^{4k} is a connected closed framed manifold whose signature is zero. Then we can do frame surgery to obtain a manifold M', frame cobordant to M^{4k} whose nonzero homology lies only in dimensions zero, $4k$ and $2k$.

The quadratic form $\lambda \to \lambda \cdot \lambda$ had determinant ± 1 by Poincaré duality. Since it has signature zero, it is possible to choose a basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ for $H_{2m}(M')$ so that

$$\lambda_i \cdot \lambda_j = 0 \quad \lambda_i \cdot \mu_j = \delta_{i,j}.$$

For any embedded sphere representing a homology class of the form $\lambda = \sum_{i=1}^r n_i \lambda_i$, the self intersection $\lambda \cdot \lambda = 0$ which implies its normal bundle is trivial (see [119], Lemma 7). Then by Lemma 11.50 one can do frame surgery to kill $H_{2k}(M')$ and therefore obtain a homotopy sphere. Thus, in the setting when $\sigma(M^{4k}) = 0$, M^{4k} is frame cobordant to a homotopy sphere.

□

We now take stock of our study of Question 11.23, asking when closed (stably) framed manifolds are frame cobordant to framed homotopy spheres. By Theorem 11.48 we know that if the dimension of the manifold is odd, then the answer to the question is yes: every closed, odd dimensional framed manifold is frame cobordant to a homotopy sphere. If the dimension of the manifold is divisible by 4, then Theorem 11.51 gives us an answer to this question: a closed framed manifold of dimension divisible by 4 is frame cobordant to a homotopy sphere if and only if its signature is zero. What is left is to study the case of framed manifolds of dimension congruent to 2 mod 4. This case is the most difficult and the most interesting. It has been the topic of much research in both differential and algebraic topology since the time of Kervaire and Milnor's seminal paper [85]. We will sketch some of these developments below. As we will see, the obstruction to a closed framed manifold of dimension $4k+2$ being frame cobordant to a homotopy sphere is a $\mathbb{Z}/2$-valued invariant called the "Kervaire invariant". When there exist such manifolds having Kervaire invariant one has been a question of great interest for all these years. As we will see, this question has a formulation as a question about the Adams spectral sequence about which much has been studied over these years, with a nearly complete answer recently obtained in a dramatic paper of Hill, Hopkins, and Ravenel [70]. We now sketch some of these developments.

Let M^{2n} be a closed, (stably) framed manifold of dimension $2n$, where $n = 2k+1$. Using framed surgery we can assume that M^{2n} is $(n-1)$-connected. As mentioned earlier, by modifying Whitney's proof of his embedding theorem, Milnor in [119] (Lemma 6) showed that every class in $[\alpha] \in H_n(M^{2n})$ can be represented by an embedded sphere

$$\alpha : S^n \hookrightarrow M^{2n}.$$

We've seen that we can do surgery on α and reduce the dimension of $H_n(M^{2n})$ if and only if we can extend the embedding to an embedding

$$S^n \times D^n \hookrightarrow M^{2n}.$$

In other words, we can do surgery on α if and only if $\alpha : S^n \hookrightarrow M^{2n}$ has a trivial normal bundle, $\nu_\alpha \cong S^n \times \mathbb{R}^n$.

Define a function

$$q_M : H_n M \to \mathbb{Z}/2 \tag{11.27}$$

as follows:

Represent $\alpha \in H_n(M)$ by an embedded sphere $\alpha : S^n \hookrightarrow M^{2n}$.

Let

$$q_M(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ has a trivial normal bundle } \nu_\alpha \\ 1 & \text{if } \nu_\alpha \text{ is nontrivial} \end{cases}$$

Theorem 11.55. *(Kervaire and Milnor [85])* $q_M : H_n M \to \mathbb{Z}/2$ *is a well defined quadratic function with respect to the intersection product. Namely,*

$$q_M(\alpha + \beta) = q(\alpha) + q(\beta) + \langle \alpha, \beta \rangle.$$

We will describe the ideas behind a proof of this theorem below, but first we show how this leads to the definition of the "Kervaire invariant".

In general one can consider a finite dimensional $\mathbb{Z}/2$-vector space V equipped with a nonsingular, symmetric bilinear form $<\, ,\, >$.

A function

$$q : V \to \mathbb{Z}/2$$

is *quadratic* with respect to $<\, ,\, >$, or a *quadratic refinement* of $<\, ,\, >$, if $q(0) = 0$ and

$$q(x + y) = q(x) + q(y) + < x, y > .$$

Note. Let $x = y$. Since $q(0) = 0$, and we are working over $\mathbb{Z}/2$, this implies $< x, x >= 0$. In other words, $<\, ,\, >$ is a *symplectic* form.

This implies that there is a *symplectic basis* $\{a_i, b_i,\ i = 1, \cdots, p\}$ of V such that

$$< a_i, b_j >= \delta_{i,j} \quad \text{ and } < a_i, a_j >=< b_i, b_j >= 0.$$

One can then define the *Arf invariant*

$$A(q) = \sum_{i=1}^{p} q(a_i) q(b_i) \tag{11.28}$$

Exercise. Show that A is a "democratic" invariant. That is, $A(q) = 0$ if and only if $\#\{q^{-1}(0)\} \geq \#\{q^{-1}(1)\}$. (Note. Referring to A as a "democratic invariant" is an idea due to W. Browder [18].)

The significance of this invariant was proved by Arf in the 1940's. He showed the following.

Theorem 11.56. *The Arf invariant completely determines the quadratic form up to isomorphism. That is $A(q_1) = A(q_2)$ if and only if there is a $<,>$ - preserving isomorphism $\phi : V \xrightarrow{\cong} V$ with $q_2 = q_1 \circ \phi$.*

Definition 11.10. *Given an $(n-1)$-connected framed manifold M^{2n}, $n = 2k+1$, define the Kervaire invariant*

$$\kappa(M) = A(q_M).$$

The following theorem describes the Kervaire invariant as a surgery invariant, and is due to Kervaire and Milnor in [85]. We give a rough sketch of a generalization of this result below, but we refer the interested reader to the seminal paper of Kervaire and Milnor for its original proof.

Theorem 11.57. *(Kervaire, Milnor [85])*
1. For M^{2n} as above, M can be surgered to a framed homotopy sphere if and only if $\kappa(M) = 0$.
2. $\kappa(M)$ is an invariant of the framed cobordism class of M.

11.6.4 The Kervaire invariant one problem

Notice that Theorem 11.57 completes our answer to Question 11.23, at least in terms of the yet-to-be computed Kervaire invariant. In order to prove Theorem 11.55 and the resulting Theorem 11.57, Kervaire and Milnor defined the function q_M (11.27) in terms of a cohomology operation that detects whether the normal bundle of an embedded sphere is trivial. We describe a variation of that approach due to Brown [21] which allows for a generalization of the Kervaire invariant that has proven quite useful.

We want to begin by describing the Kervaire invariant as a map

$$\kappa : \eta_{4k+2}^{fr} \to \mathbb{Z}/2.$$

More generally, following Brown [21] we will describe a map

$$\kappa : \eta_{4k+2}^{\xi} \to W(\xi)$$

where η_*^ξ is an appropriate cobordism theory arising from a fibration $B_\xi \to BO$. $W(\xi)$ is a "Witt" group, which is a group that classifies certain quadratic forms. Here is how this invariant is defined.

Recall that a manifold has a ξ-structure if the map classifying its stable normal bundle

$$\nu_M : M \to BO$$

has a lifting $\phi : M \to B_\xi$. Let η_*^ξ be the group of cobordism classes of manifolds with ξ-structures, (M, ϕ). By Pontrjagin-Thom theory,

$$\eta_*^\xi \cong \pi_*(\mathbb{M}\xi)$$

where $\mathbb{M}\xi$ is the Thom spectrum of $B_\xi \to BO$.

Pontrjagin-Thom theory says that the generalized homology theory induced by the spectrum $\mathbb{M}\xi$ is the ξ-bordism groups, where the ξ-bordism group of a space X, $\eta_m^\xi(X)$, is given by cobordism classes of triples (M^m, ϕ, f), where (M, ϕ) is a closed manifold with ξ-structure ϕ, and $f : M \to X$ is a continuous map. Then

$$\eta_m^\xi(X) \cong \pi_m(\mathbb{M}\xi \wedge X_+).$$

We also will make use of the reduced theory,

$$\tilde{\eta}_m^\xi(X) \cong \pi_m(\mathbb{M}\xi \wedge X).$$

Let K_n be an Eilenberg-MacLane space of type $(\mathbb{Z}/2, n)$, and let (M, ϕ) be a $2n$-dimensional manifold with ξ-structure. Consider the function

$$q : H^n(M; \mathbb{Z}/2) \to \tilde{\eta}_{2n}^\xi(K_n)$$

defined as follows:

Represent $\gamma \in H^n(M; \mathbb{Z}/2)$ by a map, which by abuse of notation we also call $\gamma : M \to K_n$. Then

$$q(\gamma) = [(M, \phi, \gamma)] - [(M, \phi)] \in \tilde{\eta}_{2n}^\xi(K_n).$$

The following is proved in [21],[33].

Theorem 11.58. *Consider the case $\eta^\xi = \eta^{fr}$. Then*

1. $\tilde{\eta}_{2n}^{fr}(K_n) = \pi_{2n}^s(K_n) \cong \mathbb{Z}/2$.
2. *Let $(M^{2n}, \phi))$ be a $2n$-dimensional, framed manifold, $n = 2k+1$. Then*

$$q : H^n(M; \mathbb{Z}/2) \to \tilde{\eta}_{2n}^\xi(K_n) \cong \mathbb{Z}/2$$

is a quadratic function. That is $q(x+y) = q(x) + q(y) + < x \cup y; [M] >$.

The following relates this definition of $q : \eta_{2n}^{fr} \to \mathbb{Z}/2$ to the previous definition (11.27).

Theorem 11.59. *Let M^{2n}, $n = 2k+1$, be a (stably) framed closed manifold that is $(n-1)$-connected. Then the following diagram commutes:*

$$\begin{array}{ccc} H^n(M) & \xrightarrow{q} & \tilde{\eta}_{2n}^{fr}(K_n) \\ \cap[M]\downarrow\cong & & \downarrow\cong \\ H_n(M) & \xrightarrow[q_M]{} & \mathbb{Z}/2. \end{array}$$

In other words, if M^{2n} is a framed manifold, $n = 2k+1$, $S^n \hookrightarrow M^{2n}$, represents a homology class $\alpha \in H_n(M)$ with Poincare dual $D(\alpha) : M \to K_n$, then $[M, \phi, D(\alpha)] - [M, \phi] \in \tilde{\eta}_{2n}^{fr}(K_n)$ is zero if and only if $S^n \hookrightarrow M^{2n}$ has a trivial stable normal bundle.

Before we indicate the proofs of these theorems, we describe how they generalize. Let (M^{2n}, ϕ) now represent a manifold with ξ-structure, and consider again

$$\begin{aligned} q : H^n(M^{2n}, \mathbb{Z}/2) &\to \tilde{\eta}_{2n}^{\xi}(K_n) \\ (\gamma : M \to K_n) &\to [M, \phi, \gamma] - [M, \phi]. \end{aligned}$$

Also consider the homomorphism

$$\begin{aligned} P : \eta_{2n}^{\xi}(K_n) &\to H_{2n}(B_\xi; \mathbb{Z}/2) \\ [M, \phi, f] &\to \phi_*([M]) \end{aligned}$$

(Recall $\phi : M \to B_\xi$ is a lift of the stable normal bundle map $\nu_M : M \to BO$.)

For the statement of the next result, we need the notion of the "Wu class" of a vector bundle. Let $u \in H^0(\mathbb{MO}; \mathbb{Z}/2)$ be the Thom class. Recall the canonical antiautomorphism of the Steenrod algebra $\chi : \mathcal{A}_2 \to \mathcal{A}_2$ discussed in Chapter 10 (10.41). Then we can consider the class $\chi(Sq^i)u \in H^i(\mathbb{MO}; \mathbb{Z}/2)$. The Thom isomorphism then defines a unique class $v_i \in H^i(BO; \mathbb{Z}/2)$ which maps to $\chi(Sq^i)u$ under the Thom isomorphism

$$\cup u : H^*(BO; \mathbb{Z}/2) \xrightarrow{\cong} H^*(\mathbb{MO}; \mathbb{Z}/2).$$

In other words, $v_i \in H^i(BO; \mathbb{Z}/2)$ is the unique class such that

$$v_i \cup u = \chi(Sq^i)u \in H^i(\mathbb{MO}; \mathbb{Z}/2). \tag{11.29}$$

$v_i \in H^i(BO; \mathbb{Z}/2)$ is called the i^{th} *Wu class.* If $f_\xi : B_\xi \to BO$ classifies the stable vector bundle ξ, then the i^{th} Wu class of ξ, written $v_i(\xi)$, is defined to be $f_\xi^*(v_i) \in H^i(X; \mathbb{Z}/2)$.

Theorem 11.60. *(Brown [21]) Suppose $v_{n+1}(\xi) = 0$. Then there is an exact sequence*

$$0 \to \mathbb{Z}/2 \xrightarrow{\iota} \tilde{\eta}^{\xi}_{2n}(K_n) \xrightarrow{P} H_{2n}(B_\xi; \mathbb{Z}/2) \to 0.$$

Furthermore, if M^{2n} as a ξ-orientation, i $q : H^n(M^{2n}, \mathbb{Z}/2) \to \tilde{\eta}^{\xi}_{2n}(K_n)$ is quadratic in the sense that $q(a+b) = q(a) + q(b) + \iota\langle a, b\rangle$.

We now operate under the assumption that $v_{n+1}(\xi) = 0$. In this case the *Kervaire invariant* of a $2n$- manifold with ξ- structure, $\kappa(M, \phi)$ is defined to be the *Witt group* classification of the quadratic form $q : H^n(M; \mathbb{Z}/2) \to \tilde{\eta}^{\xi}_{2n}(K_n)$. This classification works as follows:

Let G be an abelian group equipped with an injection $i : \mathbb{Z}/2 \to G$. Let V be a finite dimensional $\mathbb{Z}/2$- vector space equipped with a nonsingular bilinear pairing $\langle , \rangle : V \times V \to \mathbb{Z}/2$. One can then talk about quadratic refinements $q : V \to G$ of this pairing, as above.

Definition 11.11.

- *The form (V, q) is said to be Witt equivalent to zero if there is a subspace $Q \subset V$ with $2 \cdot dim Q = dim V$ and $q(v) = 0$ for all $v \in Q$.*
- *Two forms (V_1, q_1) and (V_2, q_2) are said to be Witt equivalent if the form $(V_1 \oplus V_2, q_1 - q_2)$ is Witt equivalent to zero.*
- *The Witt group of G, $W(G)$ is defined to be the set of Witt equivalence classes of such forms. Addition is induced by direct sum.*

Remark. What Arf showed is that the Arf invariant $A : W(\mathbb{Z}/2) \to \mathbb{Z}/2$ is an isomorphism.

The proofs of these theorems can be found in [21] and [33]. We outline their proofs, while concentrating mostly on the framed manifold setting. (**Note.** The above results show that one can define the Kervaire invariant in any cobordism theory in which the appropriate Wu class is zero.)

The following are the results that need verification.

1. $\tilde{\eta}^{fr}_{2n}(K_n) = \pi^s_{2n}(K_n) \cong \mathbb{Z}/2$,
2. If (M^{2n}, ϕ) is a framed manifold, then $q : H^n(M; \mathbb{Z}/2) \to \tilde{\eta}^{fr}_{2n}(K_n) \cong \mathbb{Z}/2$ is a quadratic refinement of the intersection pairing,
3. Let $\alpha : S^n \hookrightarrow M^{2n}$ be an embedding of a sphere into a framed, closed manifold of twice its dimension. Call its Poincaré dual cohomology class $D(\alpha) \in H^n(M^{2n}; \mathbb{Z}/2)$. Then α has a trivial normal bundle if and only if $q(D(\alpha)) = 0$.

The proofs of the first two of these statements are purely homotopy theoretic, using a classical technique known as the "EHP - sequence". We refer the reader to [21] and [33] for these arguments.

For the third of these statements, notice that since M^{2n} and S^n both have stably trivial tangent bundles, the normal bundle $\nu(\alpha)$ to the embedding $\alpha : S^n \hookrightarrow M^{2n}$ must be stably trivial. In fact we know, for dimension reasons, that $\nu(\alpha) \oplus \epsilon^1 \cong \epsilon^{n+1}$, where ϵ^k denotes the trivial k dimensional bundle. Since the kernel of $\pi_n(BO(n)) \to \pi_n(BO(n+1))$ is known to be $\mathbb{Z}/2$, generated by the tangent bundle TS^n, we can conclude that either $\nu(\alpha)$ is trivial, or $\nu(\alpha) \cong TS^n$.

Exercise Show that if the normal bundle $\nu(\alpha)$ is trivial, then $q(D(\alpha)) = 0$.

What remains to show is that if $\nu(\alpha) \cong TS^n$, then $q(D(\alpha)) = 1 \in \mathbb{Z}/2$. This is done by first verifying that that $D(\alpha)$ is given by the composition

$$M \xrightarrow{\tau} (S^n)^{\nu(\alpha)} \xrightarrow{u} K_n$$

where τ is the Thom collapse map onto the Thom space, and u is the Thom cohomology class. An explicit example of when $\nu(\alpha)$ is isomorphic to the tangent bundle, which allows one to do an explicit calculation, is when $M^{2n} = S^n \times S^n$, and the embedding $\alpha : S^n \to S^n \times S^n$ is the diagonal map. We refer the reader to [21] for details of this argument.

We are therefore naturally left with the question of the existence of framed manifolds of dimension $4k+2$ having Kervaire invariant one. We state some results motivated by this question.

The first major result along these lines after the Kervaire-Milnor paper [85] was due to Brown and Peterson. In [24] the authors settled half of the cases by showing that in dimensions $8k+2$ all framed manifolds have Kervaire invariant zero. They did this by considering the generalized Kervaire invariant defined by Brown [21] and applying it to Spin manifolds.

Now as we've seen and used extensively, frame cobordism classes of manifolds, correspond, by Pontrjagin's original work, to the stable homotopy groups of spheres $\pi_*(\mathbb{S})$, which can be computed using the Adams spectral sequence (see section 10.10). In 1969 W. Browder [17] used this correspondence to translate the existence of stably framed manifolds of Kervaire invariant one into a problem about the Adams spectral sequence.

Theorem 11.61. *(Browder [17]) A cobordism class of a stably framed manifold $[M] \in \eta^{fr}_{4k+2}$ has Kervaire invariant zero unless the dimension of the manifold is of the form $2^{j+1}-2$, in which case it has Kervaire invariant one if and only if as an element of $\pi_{2^{j+1}-2}(\mathbb{S})$ it is represented by*

$$h_j^2 \in Ext^{2,2^{j+1}}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$$

in the E_2 term of the mod 2 Adams spectral sequence.

For many years after Browder's theorem was proven, the conjecture that $h_j^2 \in E_2^{2,2^{j+1}}$ is an infinite cycle in the Adams spectral sequence and therefore represents a class in $\pi_{2^{j+1}-2}(\mathbb{S})$, was one of the major conjectures in homotopy theory. A hypothetical element of $\pi_{2^{j+1}-2}(\mathbb{S})$ represented by h_j^2 was even given a name by M. Mahowald, who called such an element θ_j. We now review what is known about the existence of the θ_j's, and therefore about Kervaire invariant conjecture, and ultimately the answer the Question 11.23 about whether framed manifolds are frame cobordant to homotopy spheres.

First recall that by Adams's work [3], the only elements of positive dimension $\pi_*(\mathbb{S})$ of Hopf invariant one are $\eta \in \pi_1(\mathbb{S})$, $\nu \in \pi_3(\mathbb{S})$, and $\sigma \in \pi_7(\mathbb{S})$. Recall that these elements are represented by $h_1 \in E_2^{1,2}$, $h_2 \in E_2^{1,4}$, and $h_3 \in E_2^{1,8}$ respectively, in the Adams spectral sequence. As framed manifolds, these elements correspond to S^1, S^3, and S^7 given framings coming from the fact that they are Lie groups and have corresponding trivializations of their tangent bundles. (Note that S^7 viewed as the unit Octonians, is a *nonassociative* Lie group. Its tangent bundle is canonically trivialized nonetheless.) Therefore $h_1^2 \in E_2^{2,4}$, $h_2^2 \in E_2^{2,8}$, and $h_3^2 \in E_2^{2,16}$ are all infinite cycles in the Adams spectral sequence, and represent elements $\theta_1 \in \pi_2(\mathbb{S})$, $\theta_2 \in \pi_6(\mathbb{S})$, and $\theta_3 \in \pi_{14}(\mathbb{S})$ which correspond to (stably) framed manifolds having Kervaire invariant one.

θ_4 is known to exist by work of Mahowald and Tangora [99]. That is, they proved that $h_4^2 \in E_2^{2,32}$ is an infinite cycle in the Adams spectral sequence and represents an element $\theta_4 \in \pi_{30}(\mathbb{S})$, which in turn corresponds to a (stably) framed manifold of dimension 30 that has Kervaire invariant one. An explicit 30-dimensional framed manifold of Kervaire invariant one was later constructed by Jones in [82].

$\theta_5 \in \pi_{62}(\mathbb{S})$ is also known to exist by a complicated homotopy theory argument of Barratt, Jones, and Mahowald [12] done in 1985. Their work was substantially simplified by Z. Xu [166] in 2016. However, as of the writing of this book, there has yet to be constructed an explicit 62-dimensional (stably) framed manifold of Kervaire invariant one, even though by these results and Pontrjagin-Thom theory, one knows such manifolds exist.

The existence of θ_6, which would lie in $\pi_{126}(\mathbb{S})$ is not known, as of the writing of this book. So it is not known whether there exists a stably framed manifold of dimension 126 with Kervaire invariant one.

In a stunning piece of work that involves the Adams spectral sequence coupled to generalized cohomology theories, as well as equivariant and chromatic stable homotopy, Hill, Hopkins, and Ravenel [70] proved that for $j \geq 7$, *θ_j does not exist*. That is for $j \geq 7$ the Kervaire invariant is zero on all (stably) framed manifolds of diimension $2^{j+1}-2$. Their work uses a panorama of techniques which are surely to became essential in every homotopy theorist's tool bag for years to come. These techniques are developed and beautifully explained in their book [71]. We encourage the interested reader to study that book to learn the techniques and the details of this tremendous advance in algebraic topology.

Comments.

1. We now know that in dimensions of the form $4k+2$, the obstruction to a framed manifold being frame cobordant to a homotopy sphere, the Kervaire invariant, is usually zero. Namely, in every dimension of the form $4k+2$, except dimensions 2, 6, 14, 30, 62, and possibly 126, every framed manifold is frame cobordant to a homotopy sphere. We know that there exist manifolds of dimensions 2, 6, 14, 30, and 62 that are not frame cobordant to homotopy spheres, even though at this time no explicit such framed manifold in dimension 62 has been produced. In dimension 126 the Kervaire invariant question has yet to be resolved, so this is still an unsolved problem.

2. By the work of Hill, Hopkins, and Ravenel [70] one knows that for $j \geq 7$, $h_j^2 \in E_2^{2,2^{j+1}}$ carries a nonzero differential in the Adams spectral sequence. As of the writing of these notes, it is an interesting open question what the values of these differentials are.

11.6.5 The Kervaire invariant for unframed manifolds

As described by Brown in [21], there is a generalization of the Kervaire invariant that can be defined on manifolds that are not necessarily framed.

Let $p : B_\xi \to BO$ be a fibration with the property that the $(n+1)^{st}$ Wu class is zero.

$$v_{n+1}(\xi) = p^*(v_{n+1}) = 0.$$

Then, as above, let η_*^ξ be the associated cobordism theory, so that by the Pontrjagin-Thom theorem,

$$\tilde{\eta}_*^\xi \cong \pi_*(\mathbb{M}\xi)$$

where $\mathbb{M}\xi$ is the Thom spectrum of the map $p : B\xi \to BO$. Then, by Theorem 11.60, if M^{2n} is a closed manifold equipped with a normal ξ -structure,

$$q : H^n(M^{2n}, \mathbb{Z}/2) \to \tilde{\eta}_{2n}^\xi(K_n)$$

is quadratic in the sense that $q(a+b) = q(a) + q(b) + \iota\langle a, b\rangle$, where

$$0 \to \mathbb{Z}/2 \xrightarrow{\iota} \tilde{\eta}_{2n}^\xi(K_n) \xrightarrow{P} H_{2n}(B_\xi; \mathbb{Z}/2) \to 0.$$

is the exact sequence described in Theorem 11.60.

Notice from this exact sequence that if $G = \tilde{\eta}_{2n}^\xi(K_n)$, then $4G = 0$. Using this, Brown proceeded as follows to construct his generalized Kervaire invariant.

Let G be any abelian group such that $4G = 0$, equipped with an injective homomorphism $\iota : \mathbb{Z}/2 \hookrightarrow G$. Now let (V, q) be a G-valued quadratic form in the sense that $q(a+b) = q(a) + q(b) + \iota(a \cdot b)$. Let $h : G \to \mathbb{Z}/4$ be a homomorphism such that the composition

$$h \circ \iota : \mathbb{Z}/2 \to G \to \mathbb{Z}/4$$

is injective. In [21] Brown showed there is an isomorphism of the Witt group $W(\mathbb{Z}/4)$ with $\mathbb{Z}/8$:

$$\sigma : W(\mathbb{Z}/4) \xrightarrow{\cong} \mathbb{Z}/8. \tag{11.30}$$

He also proved the following.

Lemma 11.62. *[21] Let $\oplus_h \mathbb{Z}/8$ be a direct sum of copies of $\mathbb{Z}/8$ indexed by all homomorphisms $h : G \to \mathbb{Z}/4$ with $h \circ \iota : \mathbb{Z}/2 \to \mathbb{Z}/4$ injective. Then the homomorphism*

$$\sigma = \oplus \sigma_h : W(G) \to \oplus_h \mathbb{Z}/8$$

is injective.

Brown's generalized Kervaire invariant can now be described as follows.

Definition 11.12. *[21] We define the Kervaire invariant for a bordism theory η^{ξ}_{2n} with $v_{n+1}(\xi) = 0$,*

$$\kappa^{\xi} : \eta^{\xi}_{2n} \to \oplus_h \mathbb{Z}/8$$

where the direct sum is taken over all homomorphisms $h : \tilde{\eta}^{\xi}_{2n}(K_n) \to \mathbb{Z}/4$ such that $h \circ \iota : \mathbb{Z}/2 \to \mathbb{Z}/4$ is injective. Let $[M^{2n}, \phi] \in \eta^{\xi}_{2n}$. Then $\kappa^{\xi}([M^{2n}, \phi])$ is defined to be the Witt classification of the quadratic form

$$H^n(M^{2n}, \mathbb{Z}/2) \xrightarrow{q} \tilde{\eta}^{\xi}_{2n}(K_n) \xrightarrow{\oplus h} \oplus_h \mathbb{Z}/4$$

which lives in the Witt group $W(\oplus_h \mathbb{Z}/4)$, which by (11.30) is equal to $\oplus_h \mathbb{Z}/8$.

Exercise. Show that in the case of framed cobordism, this generalized Kervaire invariant takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/8$ and is the usual Kervaire invariant for framed manifolds.

In [33] Cohen, Jones, and Mahowald calculated the generalized Kervaire invariant in the case of certain cobordism groups of immersions, which generalize the notion of stable normal framing. We describe some of their results now.

Let G be one of the Lie groups $O(1)$ or $SO(2)$. We consider the natural map $BG \to BO$ and we call the resulting bordism theories $\eta^{O(1)}_*$ and $\eta^{SO(2)}_*$. By Smale-Hirsch theory as described in Chapter 7, $\eta^{O(1)}_m$ represents cobordism classes of m-dimensional manifolds, immersed in codimension one Euclidean space. More specifically, an element of $\eta^{O(1)}_m$ is an equivalence class of pairs $[M^m, \phi]$ where $\phi : M^m \looparrowright \mathbb{R}^{m+1}$ is an immersion. Two such, $[M^m_1, \phi_1]$

and $[M_2^m, \phi_2]$ are cobordant in this theory if there exists an $(m+1)$ dimensional manifold with boundary W^{m+1} whose boundary is the disjoint union $\partial W^{m+1} = M_1^m \sqcup M_2^m$, together with an immersion $\Phi : W^{m+1} \looparrowright \mathbb{R}^{m+1} \times [0,1]$ with the following properties.

1. The restriction of the immersion Φ to the interior of W^{m+1} is an immersion into the open space $\mathbb{R}^{m+1} \times (0,1)$,
$$\Phi : Int\, W^{m+1} \looparrowright \mathbb{R}^{m+1} \times (0,1).$$
2. The restriction of the immersion Φ to $M_1^m \subset \partial W^{m+1}$ equals ϕ_1,
$$\Phi_{|_{M_1}} = \phi_1 : M_1^m \looparrowright \mathbb{R}^{m+1} \times \{0\}.$$
3. The restriction of the immersion Φ to $M_2^m \subset \partial W^{m+1}$ equals ϕ_2,
$$\Phi_{|_{M_2}} = \phi_2 : M_2^m \looparrowright \mathbb{R}^{m+1} \times \{1\}.$$

$\eta_*^{SO(2)}$ has a similar interpretation as cobordism classes of oriented manifolds immersed in codimension two Euclidean space.

Exercise. Show that frame cobordism, η_*^{fr} can be interpreted as cobordism classes of oriented manifolds immersed in codimension one Euclidean space. **Hint.** Notice that the group $SO(1)$ is the trivial group.

Notice that by the Pontrjagin-Thom theorem we have that

$$\begin{aligned} \eta_*^{O(1)} &\cong \pi_*(\mathbb{MO}(1)) = \pi_{*+1}^s(\mathbb{RP}^\infty) \quad \text{and} \\ \eta_*^{SO(2)} &\cong \pi_*(\mathbb{M}SO(2)) = \pi_{*+2}^s(\mathbb{CP}^\infty). \end{aligned} \tag{11.31}$$

We first discuss the results of [33] for $\eta_*^{O(1)}$.

First, they computed the Witt groups in this case, and found

$$W_{2n}^{O(1)} = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \neq 2^k - 1 \\ \mathbb{Z}/8 & \text{if } n = 2^k - 1 \end{cases} \tag{11.32}$$

Let $\kappa_{O(1)} : \eta_{2n}^{O(1)} \to W^{O(1)}$ be the Brown-Kervaire generalized invariant in this theory. In [33] the authors proved the following.

Theorem 11.63. *[33]*

1. *$\kappa_{O(1)} : \eta_{2n}^{O(1)} \to W^{O(1)}$ is zero if $2n \neq 2^{k+1} + 2^m - 4$.*
2. *$\kappa_{O(1)} : \eta_2^{O(1)} \to \mathbb{Z}/8$ is an isomorphism.*

3. $\kappa_{O(1)}: \eta_6^{O(1)} \to \mathbb{Z}/8$ *is surjective.*

4. *If* $2n \neq 2, 6$, *then* $2\kappa_{O(1)}(x) = 0$ *for all* $x \in \eta_{2n}^{O(1)}$

5. *If* $2n = 2^{k+1} + 2^m - 4$ *but* $2n \neq 2, 6$, *then* $x \in \eta_{2n}^{O(1)} = \pi_{2n}(\mathbb{MO}(1))$ *has nonzero Brown-Kervaire invariant if and only if it is represented by a class of the form* $e_{2^m-2}h_k h_k$ *for* $m \leq 3$ *or* $k \leq 2$, $e_{2^{k+1}-2}h_k h_k \in Ext^{2,*}_{\mathcal{A}_2}(H^*(\mathbb{MO}(2), \mathbb{Z}/2)$ *in the* E_2*-term of the Adams spectral sequence. Here* $e_q \in H_q(\mathbb{MO}(1)) = H_{q+1}(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ *is the generator. It determines an element in* $Ext^{0,q}_{\mathcal{A}_2}(H^*(\mathbb{MO}(1)), \mathbb{Z}/2)$ *if and only if* $q = 2^r - 2$ *for some* r.

Notice that the qualitative statements of this theorem are the same as those in Browder's Theorem 11.61[18]. Namely, unless $n = 1, 3$, the only elements in $\eta_{2n}^{O(1)}$ that can have nonzero Kervaire invariant are those elements in $\pi_{2n}(\mathbb{MO}(1))$ detected by classes in $Ext^{2,*}(H^*(\mathbb{MO}(1)), \mathbb{Z}/2)$ in the image of multiplication by h_j^2 under the pairing

$$Ext^{0,*}_{\mathcal{A}_2}(H^*(\mathbb{MO}(1)), \mathbb{Z}/2) \otimes Ext^{2,*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2) \to Ext^{2,*}_{\mathcal{A}_2}(H^*(\mathbb{MO}(1)), \mathbb{Z}/2).$$

Using techniques of Mahowald [97], the authors do show that there exist nonzero elements $\rho_j \in \pi_{2^j}(\mathbb{MO}(1))$ represented by $e_{2^j-2}h_1h_1 \in Ext^{2,2^{j+2}}_{\mathcal{A}_2}(H^*\mathbb{MO}(1)), \mathbb{Z}/2)$ that have nonzero Kervaire invariant. However these elements are of only limited interest since by Browder's result, they are *not* in the image of framed bordism (or equivalently of bordisms of *oriented* manifolds immersed in codimension one Euclidean space. (See the above exercise.))

Cohen, Jones, and Mahowald then proved an analogous theorem to Theorem 11.63 for the bordism group of oriented codimension two immersed manifolds, $\eta_*^{SO(2)}$. And they then showed that a far more interesting collection of manifolds in this cobordism theory have nonzero Kervaire invariant. More specifically, they prove the following two results about this cobordism theory.

Theorem 11.64. *1. The Wu class* $v_q(SO(2)) = 0$ *if and only if* $q \neq 2^k - 2$ *for any* k, *and therefore there is a Kervaire invariant for* $\eta_{2n}^{SO(2)}$ *so long as* n *is not of the form* $2^k - 3$. *The Witt groups are given by* $W_{2n}^{SO(2)} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

2. $\kappa_{SO(2)}: \eta_{2n}^{SO(2)} \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *is zero if* $2n \neq 2^{k+1} + 2^m - 6$.

3. *If* $2n = 2^{k+1} + 2^m - 6$, $x \in \eta_{2n}^{SO(2)} = \pi_{2n}(\mathbb{M}SO(2))$ *has nonzero Kervaire invariant if and only if* x *is represented by a class of the form* $e_{2^m-4}h_k h_k$ *for* $m = 2, 3$, *or if* $k \leq 2$, $e_{2^k-4}h_k h_k$, *or* $e_{2^{k-1}-4}h_k h_k$ *in* $Ext^{2,2n+2}_{\mathcal{A}_2}(H^*(\mathbb{M}SO(2)), \mathbb{Z}/2)$ *in the Adams spectral sequence.*

Here $e_{2q} \in H_{2q}(\mathbb{M}SO(2)\mathbb{Z}/2) = H_{2q+2}(\mathbb{CP}^\infty;\mathbb{Z}/2)$ *is the generator. It determines an element of* $Ext^{0,2q}_{\mathcal{A}_2}(H^*(\mathbb{M}SO(2)),\mathbb{Z}/2)$ *if and only if* q *is of the form* $2^r - 2$.

Remark. Let $x \in \eta^{SO(2)}_{2n}$ be in the image of the natural map from frame cobordism,

$$\eta^{fr}_* = \eta^{SO(1)}_* \to \eta^{SO(2)}_*.$$

Then by Browder's Theorem 11.61, x has nonzero Kervaire invariant if and only if x is represented by $e_0 h_j h_j \in Ext^{2,*}_{\mathcal{A}_2}(H^*(\mathbb{M}SO(2)),\mathbb{Z}/2)$ in the Adams spectral sequence. The main positive result of [33] was to show that these classes are in fact infinite cycles and therefore represent classes in $\pi_*(\mathbb{M}SO(2)) \cong \eta^{SO(2)}_*$ that have nonzero Kervaire invariant.

Theorem 11.65. *[33] For every* $j \geq 1$ *there exists an element* $\theta_j(SO(2)) \in \eta^{SO(2)}_{2^{j+1}-2}$ *which is represented by* $e_0 h_j h_j \in Ext^{2,2^{j+1}}(H^*(\mathbb{M}SO(2)),\mathbb{Z}/2)$ *in the Adams spectral sequence. Hence* $\theta_j(SO(2))$ *represents a* $2^{j+1}-2$ *dimensional, oriented manifold immersed in codimension* 2 *Euclidean space that has nonzero Kervaire invariant.*

Remark. When this theorem was proved in the mid 1980's, it was hoped that there would be a representative of $\theta_j(SO(2)) \in \eta^{SO(2)}_{2^{j+1}-2}$, which would be an oriented manifold of dimension $2^{j+1}-2$ that is immersed in $\mathbb{R}^{2^{j+1}}$, that could actually be immersed in codimension one Euclidean space, i.e $\mathbb{R}^{2^{j+1}-1}$. If that were true, that manifold would represent an element of $\eta^{SO(1)}_{2^{j+1}-2} = \eta^{fr}_{2^{j+1}-2}$ with Kervaire invariant one. By the theorem of Hill, Hopkins, and Ravenel [70], proven more than twenty years later, for $j \geq 7$ no such manifold exists. That is, for $j \geq 7$, no manifold representing $\theta_j(SO(2)) \in \eta^{SO(2)}_{2^{j+1}-2}$ which, by definition is immersed in codimension two Euclidean space, can be immersed in codimension one.

11.6.6 Open Questions

We end this section on the Kervaire invariant with two fascinating questions which, at the time of this writing are very much open.

Question 1. Theorem 11.65 says that there are elements $\theta_j(SO(2)) \in \pi_{2^{j+1}-2}(\mathbb{M}SO(2)) = \pi^s_{2^{j+2}}(\mathbb{CP}^\infty)$ represented by $e_0 h_j^2$ in the Adams spectral sequence, and therefore represent manifolds in $\eta^{SO(2)}_*$ having nonzero Kervaire invariant. One can ask what is the smallest n, such that there exists a class $\theta_j(SO(2))^{\langle n\rangle} \in \pi^s_{2^{j+1}}(\mathbb{CP}^n)$ represented by $e_0 h_j^2$ in the Adams spectral sequence? One knows that for $j \geq 7$, that the answer must be n larger than one because if there were an element $\theta_j(SO(2))^{\langle 1\rangle} \in \pi^s_{2^{j+1}}(\mathbb{CP}^1) = \pi^s_{2^{j+1}}(S^2) =$

$\pi_{2^{j+1}-2}(\mathbb{S})$ represented by $e_0 h_j^2$, then by Browder's Theorem 11.61 it would represent a framed manifold of Kervaire invariant one. But by the theorem of Hill, Hopkins, and Ravenel [70], no such manifolds exist for $j \geq 7$.

Notice that if the answer to this question is n, that is, there exists an element in $\theta_j^{\langle n \rangle} \in \pi_{2^{j+1}}^s(\mathbb{CP}^n)$ represented by $e_0 h_j^2$, but there is no analogous class in $\pi_{2^{j+1}}^s(\mathbb{CP}^{n-1})$, then the projection of $\theta_j^{\langle n \rangle}$ to the homotopy groups of the top cell,

$$\pi_{2^{j+1}}^s(\mathbb{CP}^n) \to \pi_{2^{j+1}}^s(S^{2n}) = \pi_{2^{j+1}-2n}(\mathbb{S})$$

is a nonzero class. So one might ask, in addition, how this class is represented in the E_2-term Adams spectral sequence for $\pi_*(\mathbb{S})$? The answer to this question might shed light on the question of the explicit nonzero differential of h_j^2 that is known to exist for $j \geq 7$ by the theorem of Hill, Hopkins, and Ravenel [70], but, as of now, it is not known what that nontrivial differential is.

Question 2. A classical question in homotopy theory is, "How many cells must a CW-complex have so that the Steenrod operation Sq^n acts nontrivially on its (mod 2) cohomology?" Since Sq^n is a stable operation (i.e it commutes with suspension and desuspension) one can ask this question in the category of spectra. In fact we can more explicitly consider CW-spectra of the form

$$\mathbb{X} = \mathbb{S} \cup D^{n_1} \cup D^{n_2} \cup \cdots \cup D^{n_k},$$

with each $n_i > 0$, and we can ask what is the minimum k such that there exists a spectrum $\mathbb{X}$ of this form and so that $Sq^n(\sigma_0) \in H^n(\mathbb{X}; \mathbb{Z}/2)$ is nonzero? Here $\sigma_0 \in H^0(\mathbb{X}; \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator.

One can effectively reduce this question to asking for the minimum k that Sq^{2^j} can act nontrivially, since these are the indecomposables in $\mathcal{A}_2$. We can begin by asking if it is possible that $k = 1$. In other words, for which j can Sq^{2^j} act nontrivially on a CW-spectrum of the form $\mathbb{S} \cup D^{2^j}$? As we have seen in Chapter 10, this is equivalent to the Hopf invariant one question which was solved by Adams [3]. Namely, Sq^{2^j} can act nontrivially on a complex of the form $\mathbb{S} \cup D^{2^j}$ if and only if $j = 0, 1, 2, 3$.

One can then ask if it is possible that $k = 2$. In other words, for which j can Sq^{2^j} act nontrivially on a CW-spectrum of the form $\mathbb{S} \cup D^{n_1} \cup D^{2^j}$? In Adams's solution of the Hopf invariant one problem, he showed if Sq^{2^j} acts nontrivially on such a CW-spectrum for $j > 3$, then it has to act nontrivially on a CW-spectrum of the form

$$\mathbb{X}_j = \mathbb{S} \cup D^{2^j - 1} \cup D^{2^j},$$

where if one considers the subcomplex $\hat{\mathbb{X}}_j = \mathbb{S} \cup D^{2^j-1} \subset \mathbb{X}_j$, then $\hat{\mathbb{X}}_j$ is the mapping cone of a map

$$\theta_{j-1} : S^{2^j - 2} \to \mathbb{S}$$

where $\theta_{j-1} \in \pi_{2^j-2}(\mathbb{S})$ is represented by $h_{j-1}^2 \in Ext_{\mathcal{A}_2}^{2,2^j}(\mathbb{Z}/2, \mathbb{Z}/2)$ in the

Adams spectral sequence. Therefore by the solution of the Kervaire invariant problem by Hill, Hopkins, and Ravenel, this is only possible if $j \leq 7$.

Remark. Adams actually described his result in terms of factorizing Sq^{2^j} into "secondary cohomology operations". However what he proved is equivalent to the formulation given here.

So by the solutions to both the Hopf invariant one and the Kervaire invariant one questions, which were done approximately 56 years apart, for most j's the minimum k so that Sq^{2^j} acts nontrivially on a CW-spectrum $\mathbb{X}$ of the form

$$\mathbb{X} = \mathbb{S} \cup D^{n_1} \cup D^{n_2} \cup \cdots \cup D^{n_k}$$

is at least $k = 2$. So an important open question is, what is that minimum k?

11.7 Cobordism categories and stable diffeomorphisms of manifolds

Until now, our study of cobordism theory has involved the study of cobordism classes of manifolds, perhaps with stable normal structure. In other words, we have put an equivalence relation on manifolds, saying two manifolds are equivalent if they are cobordant. By the Pontrjagin-Thom theorem the cobordism classes of manifolds can be computed in terms of the homotopy groups of a Thom spectrum, which has proven quite valuable.

More recently there has been much progress in a deeper study of cobordism theory. In this study one considers a topological category, the "cobordism category", in which the objects are closed manifolds of a given dimension, perhaps with a specified tangential structure, and the morphisms are cobordisms between them. In an important piece of work, Galatius, Madsen, Tillmann, and Weiss [54], building on work of Madsen and Weiss [96], showed how to compute the homotopy type of the classifying space of this category. This work has led to much progress in the study of the homotopy type of classifying spaces of diffeomorphism group of manifolds by Galatius, Randal-Williams, and others. In this section we describe some of these results.

11.7.1 The homotopy type of the cobordism category. The work of Galatius, Madsen, Tillmann, and Weiss

One problem that one encounters immediately when constructing a cobordism category of n-dimensional cobordisms, $\mathcal{C}_n$, whose classifying space can be studied, is that if one takes the objects to be all closed $(n-1)$-dimensional manifolds, and the morphisms n-dimensional cobordisms between them, then neither the objects nor the morphisms form sets, and so one will not have a "small category". The way one deals with this situation is to let the objects consist of all closed, smooth submanifolds M^{n-1} of $\mathbb{R}^\infty$. This is indeed a set and has a natural topology defined as follows.

For a fixed closed, $(n-1)$-dimensional smooth manifold M^{n-1}, consider the embedding space $Emb(M^{n-1}, \mathbb{R}^{n-1+k})$ of smooth embeddings into codimension k Euclidean space. Recall that Whitney's Embedding Theorem says that for $k \geq n-1$ this space is nonempty, and for $k \geq n$ this space is connected since any two embeddings in this dimension are isotopic. Continuing in this way, Whitney's theorem says the connectivity of the embedding space $Emb(M^n, \mathbb{R}^{n-1+k})$ increases with k. In particular one has the following consequence of Whitney's theorem.

Theorem 11.66. *The embedding space $Emb(M^{n-1}, \mathbb{R}^\infty)$ is contractible.*

Now notice that the group of diffeomorphisms, $Diff(M^{n-1})$ acts freely on the embedding spaces $Emb(M^{n-1}, \mathbb{R}^{n-1+k})$ and therefore on $Emb(M^{n-1}, \mathbb{R}^\infty)$ by precomposition. Furthermore as is described in [86] the projection map $Emb(M^{n-1}, \mathbb{R}^\infty) \to Emb(M^{n-1}, \mathbb{R}^\infty)/Diff(M^{n-1})$ is a smooth fiber bundle, and so we may conclude the following:

Corollary 11.67. *$Emb(M^{n-1}, \mathbb{R}^\infty)/Diff(M^{n-1})$ is a model for the classifying space $BDiff(M^{n-1})$.*

Notice that as a set, the space $Emb(M^{n-1}, \mathbb{R}^\infty)/Diff(M^{n-1})$ consists of all submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to M^{n-1}. We can therefore take, as our object space of our cobordism category $\mathcal{C}_n$ to be

$$Ob\,(\mathcal{C}_n) = \coprod_{M^{n-1}} Emb(M^{n-1}, \mathbb{R}^\infty)/Diff(M^{n-1}) \times \mathbb{R} \simeq \coprod_{M^{n-1}} BDiff(M^{n-1}) \tag{11.33}$$

where the disjoint union is taken over all diffeomorphism classes of closed, smooth $(n-1)$-dimensional manifolds M^{n-1}. The $\mathbb{R}$-factor in the definition of our object space will be explained below, when we define morphisms. Notice that it does not affect the homotopy type of the space of objects.

Again we stress that as a set, $Ob\,(\mathcal{C}_n)$ consists of pairs (M^{n-1}, a) where M^{n-1} is a smoothly embedded closed submanifold of $\mathbb{R}^\infty$ and $a \in \mathbb{R}$ is a real number.

The morphism space of the cobordism category $\mathcal{C}_n$ consists of cobordisms between these embedded submanifolds. More precisely, a morphism between objects (M_1^{n-1}, a) and (M_2^{n-1}, b) for $a < b$ is a triple (W^n, a, b) where W^n is a compact submanifold

$$W^n \subset \mathbb{R}^\infty \times [a, b]$$

such that for some $\epsilon > 0$, we have

1. $W^n \cap \mathbb{R}^\infty \times [a, a+\epsilon] = M_1^{n-1} \times [a, a+\epsilon]$,
2. $W^n \cap \mathbb{R}^\infty \times [b-\epsilon, b] = M_2^{n-1} \times [b-\epsilon, b]$,
3. $\partial W^n = W^n \cap (\{a, b\} \times \mathbb{R}^\infty) = M_1^{n-1} \sqcup M_2^{n-1}$

In other words, morphisms are embedded cobordisms, which, in an appropriate sense are "flat" near their boundaries. This allows for the composition of a morphism $(W_1^n, [a, b])$ from $(M_1^{n_1}, a)$ to (M_2^{n-1}, b) with a morphism $(W_2^n, [b, c])$ from (M_2^{n-1}, b) to (M_3^{n-1}, c) to be given by

$$(W_1^n \times [a, b]) \cup_{(M_2, b)} W_2^n \times [b, c].$$

This defines the cobordism category $\mathcal{C}_n$ as a category of sets. We've already defined the topology on the space of objects of $\mathcal{C}_n$. The topology on the space of morphisms is defined similarly as follows.

Consider cobordisms between $(n-1)$ dimensional manifolds M_0 and M_1 consisting of triples (W, h_0, h_1) where W is a compact n-manifold with boundary, and h_0 and h_1 are "collars" in that they are embeddings

$$h_0 : [0, 1) \times M_0 \hookrightarrow W \quad \text{and} \quad h_1 : (0, 1] \times M_1 \hookrightarrow W$$

such that ∂W is the disjoint union of the two spaces $h_j(\{j\} \times M_j)$, for $j = 0$, 1.

Let $Emb(W, [0, 1]] \times \mathbb{R}^\infty)$ be the space of embeddings

$$\phi : W \hookrightarrow [0, 1] \times \mathbb{R}^\infty$$

for which there exist embeddings $\phi_j : M_j \hookrightarrow \mathbb{R}^\infty$, $j = 0, 1$ such that there exists an $\epsilon > 0$ such that

$$\phi \circ h_0(t_0, x_0) = (t_0, \phi_0(x_0)) \quad \text{and} \quad \phi \circ h_1(t_1, x_1) = (t_1, \phi_1(x_1))$$

for all $t_0 \in [0, \epsilon)$, $t_1 \in (1-\epsilon, 1]$ and $x_j \in M_j$, $j = 0$, 1.

Let $Diff_\epsilon^\partial(W)$ be the group of diffeomorphisms of W that restrict to the product diffeomorphism on the ϵ-collars, and let $Diff^\partial(W) = colim_{\epsilon \to 0} Diff_\epsilon^\partial(W)$. As before, the quotient space $Emb(W, [0, 1]] \times \mathbb{R}^\infty)/Diff^\partial(W)$ is a model for the classifying space $BDiff^\partial(W)$.

Topologize the space of morphisms, $Mor\, \mathcal{C}_n$ by

$$Mor\, \mathcal{C}_n = Ob\, \mathcal{C}_n \sqcup \coprod_W (\mathbb{R}_+^2 \times Emb(W, [0, 1] \times \mathbb{R}^\infty)/Diff(W))$$

where $\mathbb{R}_+^2$ is the open half plane $a_0 < a_1$, and W varies over cobordisms $W = (W, h_0, h_1)$, one in each diffeomorphism class.

We can say this a little differently as follows. If we call the "incoming boundary", $\partial_{in} W = (\{a_0\} \times \mathbb{R}^\infty) \cap W$ and the "outgoing boundary" $\partial_{out} W = (\{a_1\} \times \mathbb{R}^\infty) \cap W$, then

$$Emb(W, [0, 1]] \times \mathbb{R}^\infty)/Diff(W)) \simeq BDiff(W; \{\partial_{in} W\}, \{\partial_{out} W\}),$$

where if $A, B \subset W$, $Diff(W; A, B)$ denotes the group of diffeomorphisms of W that leave A and B pointwise fixed.

To summarize, we have homotopy equivalences,

$$Ob\,(\mathcal{C}_n) \simeq \coprod_M BDiff(M) \quad \text{and} \tag{11.34}$$
$$Mor\,(\mathcal{C}_n) \simeq \coprod_W BDiff(W; \{\partial_{in}W\}, \{\partial_{out}W)$$

The main theorem in [54] about the cobordism category $\mathcal{C}_n$ is the identification of the homotopy type of the classifying space, $B\mathcal{C}_n$. As we will see, it can be viewed as a generalization of R. Thom's theorem about how groups of cobordism classes of manifolds can be identified with the homotopy groups of a Thom spectrum. However the generalization given in [54] tells us much more. In some sense it is a statement about the topology of the space of cobordisms, as opposed to only cobordism equivalence classes of manifolds. We will see that in some cases this yields new, deep information about the topology of the classifying spaces of diffeomorphism groups. In fact their theorem is a generalization of the theorem of Madsen and Weiss [96] that proved an old conjecture of Mumford about the topology of the classifying spaces of diffeomorphisms of surfaces. We will now explain these results.

In order to describe the main results of [54], as above, let $Gr_d(\mathbb{R}^{n+d})$ be the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^{n+d}$. There are two important vector bundles, $\gamma_{d,n}$ and $\gamma_{d,n}^{\perp}$ over $Gr_d(\mathbb{R}^{n+d})$. Recall that

$$\gamma_{d,n} = \{(V, v) \in Gr_d(\mathbb{R}^{n+d}) \,:\, v \in V\} \quad \text{and}$$
$$\gamma_{d,n}^{\perp} = \{(V, v) \in Gr_d(\mathbb{R}^{n+d}) \,:\, v \perp V\}.$$

As we've seen before, given an embedded submanifold $W^d \subset [a_0, a_1] \times \mathbb{R}^{n+d-1}$ one can assign to every point of W^d its tangent space as a subspace of the tangent space of $[a_0, a_1] \times \mathbb{R}^{n+d-1}$, viewed as an element of $Gr_d(\mathbb{R}^{n+d})$. This defines a map

$$\tau_W : W^d \to Gr_d(\mathbb{R}^{n+d}).$$

Again, as earlier observed, the pull-back bundle $\tau_W^*(\gamma_{n,d})$ is the tangent bundle τ_{W^d}, and the pullback bundle $\tau_W^*(\gamma_{d,n}^{\perp})$ is the normal bundle ν_{W^d}.

The bundle of main interest in [54] is $\gamma_{d,n}^{\perp} \to Gr_d(\mathbb{R}^{n+d})$. Let $T(\gamma_{d,n}^{\perp})$ be the Thom space. As n varies, these Thom spaces fit together to give a spectrum having structure maps

$$\epsilon_n : \Sigma T(\gamma_{d,n}^{\perp}) \to T(\gamma_{d,n+1}^{\perp})$$

defined as follows. Recall that there is a natural homeomorphism $\Sigma T(\gamma_{d,n}^{\perp}) \cong$

$T(\gamma_{d,n}^{\perp} \times \mathbb{R})$. Now notice that there is a natural map of vector bundles

$$\begin{array}{ccc} \gamma_{d,n}^{\perp} \times \mathbb{R} & \xrightarrow{\bar{\iota}} & \gamma_{d,n+1}^{\perp} \\ \downarrow & & \downarrow \\ Gr_d(\mathbb{R}^{n+d}) & \xrightarrow[\iota]{} & Gr_d(\mathbb{R}^{n+d+1}) \end{array} \tag{11.35}$$

where $\iota : Gr_d(\mathbb{R}^{n+d}) \to Gr_d(\mathbb{R}^{n+d+1})$ is defined via the inclusion of the ambient spaces,

$$\mathbb{R}^{n+d} = \mathbb{R}^{n+d} \times \{0\} \hookrightarrow \mathbb{R}^{n+d} \times \mathbb{R} = \mathbb{R}^{n+d+1},$$

and on total space $\bar{\iota} : \gamma_{d,n}^{\perp} \times \mathbb{R} \to \gamma_{d,n+1}$ is defined by

$$\bar{\iota}((V, v), t)) = (V \times \{0\}, (v, t))$$

where $V \times \{0\}$ is a subspace of $\mathbb{R}^{n+d} \times \mathbb{R} = \mathbb{R}^{n+d+1}$. Clearly $(v, t) \in (V \times \{0\})^{\perp}$.

The resulting spectrum is called $\mathbb{M}TO(d)$. The zero space of the corresponding ω-spectrum (also called $\mathbb{M}TO(d)$) is the infinite loop space

$$\Omega^{\infty}\mathbb{M}TO(d) = \mathrm{colim}_{n\to\infty}\, \Omega^{n+d} T(\gamma_{d,n}^{\perp}),$$

and a space that is most relevant to the main theorem of [54] is the next space in this ω-spectrum, that the authors call

$$\Omega^{\infty-1}\mathbb{M}TO(d) = \mathrm{colim}_{n\to\infty}\, \Omega^{n+d-1} T(\gamma_{d,n}^{\perp}). \tag{11.36}$$

Given a morphism in $\mathcal{C}_d$, $W \subset [a_0, a_1] \times \mathbb{R}^{n+d-1}$, the Pontrjagin-Thom collapse map onto the Thom space of the normal bundle gives a map $[a_0, a_1]_+ \wedge S^{n+d-1} \to T(\nu)$, where ν is the normal bundle, and by composing with map on Thom spaces induced by the map $\tau_{W^d} : W^d \to Gr_d(\mathbb{R}^{n+d})$ induced by this embedding, we get a map

$$[a_0, a_1]_+ \wedge S^{n+d-1} \to T(\gamma_{d,n}^{\perp}),$$

whose adjoint determines a path in $\Omega^{n+d-1}T(\gamma_{d,n}^{\perp})$. Taking a (co)limit as $n \to \infty$, one gets a path in $\Omega^{\infty-1}\mathbb{M}TO(d)$. By using more care, the authors of [54] show how this Pontrjagin-Thom construction defines a functor from the cobordism category $\mathcal{C}_d$ to the (unbased) path category of $\Omega^{\infty-1}\mathbb{M}TO(d)$, $(\Omega^{\infty-1}\mathbb{M}TO(d))^I$. (Recall from Definition 5.14 that the path category of a space X is the topological category whose objects are the points of X and whose morphisms are paths between the objects.) Now Corollary 5.31 tells us that the classifying space of the path category of any space X, $B(X^I)$, is weakly homotopy equivalent to X. So by passing to classifying spaces, this construction defines a map

$$\alpha_d : B(\mathcal{C}_d) \to B\left((\Omega^{\infty-1}\mathbb{M}TO(d))^I\right) \simeq \Omega^{\infty-1}\mathbb{M}TO(d). \tag{11.37}$$

The main theorem of [54] is the following.

Theorem 11.68. *([54]) The map*

$$\alpha_d : B\mathcal{C}_d \to \Omega^{\infty-1}\mathbb{M}TO(d)$$

is a weak homotopy equivalence.

For $d = 2$ this theorem was proved by Madsen and Weiss in [96]. We will discuss this important special case in more detail below.

Notice that for any topological category $\mathcal{C}$, the set of path components $\pi_0 B\mathcal{C}$ can be thought of as the set of equivalence classes of path components of the space of objects, $\pi_0 Ob_{\mathcal{C}}$, where the equivalence relation is given by two (path components of) objects are equivalent if there is a morphism connecting them.

Exercise. Prove this fact.

So for the cobordism category $\mathcal{C}_d$, this says that $\pi_0 B\mathcal{C}_d$ is the group of cobordism classes of closed manifolds of dimension $d-1$. But classical Pontrjagin-Thom theory says that this group is isomorphic to the homotopy group $\pi_{d-1}(\mathbb{MO})$. One therefore has the following.

Theorem 11.69. *There is an isomorphism*

$$\pi_0(\Omega^{\infty-1}\mathbb{M}TO(d)) \cong \pi_{d-1}(\mathbb{MO}).$$

We show that this result can be proven directly as follows.

Proof. We begin with a lemma.

Lemma 11.70. *There is a homotopy cofibration sequence of spectra*

$$\mathbb{M}TO(d) \to \Sigma^\infty(BO(d)_+) \xrightarrow{\partial} \mathbb{M}TO(d-1).$$

Proof. This lemma will be a consequence of the following result, which we will leave as an exercise.

Exercise.
Suppose $\zeta \to X$ and $\xi \to X$ are vector bundles over the same base space X. Let $p : S(\zeta) \to X$ be the associated sphere bundle of ζ. Prove that there is a (homotopy) cofibration sequence of Thom spaces,

$$T(p^*(\xi)) \to T(\xi) \to T(\xi \oplus \zeta).$$

Now apply the result of this exercise to $X = Gr_d(\mathbb{R}^{n+d})$, $\zeta = \gamma_{d,n} \to Gr_d(\mathbb{R}^{n+d})$ and $\xi = \gamma_{d,n}^\perp \to Gr_d(\mathbb{R}^{n+d})$. Recall that the Grassmannian can be described by

$$Gr_d(\mathbb{R}^{n+d}) = O(n+d)/\left(O(n) \times O(d)\right)$$

and that the bundle $p : \gamma_{d,n} \to Gr_d(\mathbb{R}^{n+d})$ is given by the projection

$$p : (O(n+d)/O(n)) \times_{O(d)} \mathbb{R}^d \to O(n+d)/\left(O(n) \times O(d)\right).$$

One then sees that the sphere bundle $p : S(\zeta) \to Gr_d(\mathbb{R}^{n+d})$ is given by the projection

$$p : S(\zeta) = (O(n+d)/O(n)) \times_{O(d)} S^{d-1} \to O(n+d)/\left(O(n) \times O(d)\right).$$

But since $S^{d-1} = O(d)/O(d-1)$, we have that

$$S(\zeta) = (O(n+d)/O(n)) \times_{O(d)} O(d)/O(d-1) = O(n+d)/\left(O(n) \times O(d-1)\right).$$

Now since $Gr_{d-1}(\mathbb{R}^{n+d-1}) = O(n+d-1)/\left(O(n) \times O(d-1)\right)$, the inclusion $O(n+d-1) \hookrightarrow O(n+d)$ defines a map

$$\begin{gathered} O(n+d-1)/\left(O(n) \times O(d-1)\right) \to O(n+d)/\left(O(n) \times O(d-1)\right) \\ Gr_{d-1}(\mathbb{R}^{n+d-1}) \to S(\zeta). \end{gathered}$$

This map has the same connectivity as the inclusion map $O(n+d-1) \to O(n+d)$ which is $(n+d-2)$-connected. The bundle $p^*(\gamma_{d,n}^{\perp})$ over $S(\zeta)$ is easily seen to restrict to $\gamma_{d-1,n}^{\perp}$ over $Gr_{d-1}(\mathbb{R}^{n+d-1})$, so on the level of Thom spaces the map

$$T(\gamma_{d-1,n}^{\perp}) \to T(p^*(\gamma_{d,n}^{\perp}))$$

is $(2n+d-2)$-connected.

Now the right hand term in the cofibration sequence of Thom spaces in the exercise is the Thom space of the trivial $(n+d)$-dimensional bundle $\gamma_{d,n} \oplus \gamma_{d,n}^{\perp}$ over $Gr_d(\mathbb{R}^{d+n})$ and is therefore equal to $Gr_d(\mathbb{R}^{d+n})_+ \wedge S^{n+d}$. Moreover the map $Gr_d(\mathbb{R}^{n+d}) \to Gr_d(\mathbb{R}^{\infty}) = BO(d)$ is $(n-1)$-connected. Therefore the cofibration sequence in the exercise gives a cofibration sequence of spectra

$$\Sigma^{-1}\mathbb{M}TO(d-1) \to \mathbb{M}TO(d) \to \Sigma^{\infty}(BO(d)_+) \to \mathbb{M}TO(d-1) \to \cdots \quad (11.38)$$

which proves the lemma. □

We now complete the proof of Theorem 11.69. Notice that the connecting maps in the cofibration sequences in the statement of Lemma 11.70 defines a directed sequence of maps of spectra

$$\begin{aligned} \mathbb{M}TO(0) \to \Sigma\mathbb{M}TO(1) \to \Sigma^2\mathbb{M}TO(2) \to \cdots \to \Sigma^{d-1}\mathbb{M}TO(d-1) \\ \to \Sigma^d\mathbb{M}TO(d) \to \cdots \end{aligned} \quad (11.39)$$

We can call the direct limit the spectrum $\mathbb{M}TO$. The following is an interesting fact proved in [54].

Lemma 11.71. *There is a weak homotopy equivalence of spectra,*

$$\mathbb{M}TO \simeq \mathbb{M}\mathbb{O}.$$

Proof. Recall that there is a homeomorphism

$$h_{d,n} : Gr_d(\mathbb{R}^{d+n}) \xrightarrow{\cong} Gr_n(\mathbb{R}^{d+n})$$

sending a d-dimensional subspace $V \subset \mathbb{R}^{d+n}$ to the n-dimensional subspace $V^\perp \subset \mathbb{R}^{d+n}$. Notice that the pull back bundle

$$h^*_{d,n}(\gamma_{n,d}) = \gamma^\perp_{d,n}.$$

Therefore we have maps of Thom spaces, which by abuse of notation we still call $h_{d,n}$,

$$h_{d,n} : T(\gamma^\perp_{d,n}) \xrightarrow{\cong} T(\gamma_{n,d}) \to T(\gamma_{n,\infty}) = MO(n).$$

which induces an isomorphism in homotopy groups in dimensions less that $d+n$, and a surjection in dimension $d+n$. The maps $h_{d,n}$ fit together to give a maps of spectra,

$$h_d : \Sigma^d \mathbb{M}TO(d) \to \mathbb{MO}$$

which therefore induces an isomorphism in homotopy groups in dimensions less than d and a surjection in dimension d. The lemma follows. □

We can complete the proof of Theorem 11.69 quickly now. We have an isomorphism

$$\pi_{-1}\mathbb{M}TO(d) \cong \pi_{d-1}\Sigma^d\mathbb{M}TO(d) \xrightarrow{\cong} \pi_{d-1}\mathbb{MO}.$$

On the level of infinite loop spaces, this is an isomorphism

$$\pi_0(\Omega^{\infty-1}\mathbb{M}TO(d)) \xrightarrow{\cong} \pi_{d-1}(\Omega^\infty\mathbb{MO}).$$

□

Just to emphasize the point previously made, Theorem 11.69 implies that the main theorem of [54] that $B\mathcal{C}_d \cong \Omega^{\infty-1}\mathbb{M}TO(d))$, recovers, on the level of path components, the classical Pontrjagin-Thom theorem that

$$\eta^O_{d-1} \cong \pi_0(B\mathcal{C}_d) \cong \pi_0(\Omega^{\infty-1}\mathbb{M}TO(d)) \cong \pi_{d-1}\mathbb{MO}.$$

However the GMTW-theorem gives us much more information about cobordisms of manifolds, as well as diffeomorphism groups, as we will point out below. Before we do, however, we remark that the GMTW-theorem generalizes to other cobordism theories and their corresponding cobordism categories. It does so using the notion of a "tangential structure on a manifold".

Let $\theta : B \to BO(d)$ be a fibration. Recall from Definition 11.6 of a θ-structure on a bundle classified by a map $f : X \to BO(d)$ is a lifting $\tilde{f} : X \to B$ of f. Namely $\theta \circ \tilde{f} = f : X \to BO(d)$. Here our focus is on θ-structures on the tangent bundle of a manifold M^d, classified by a map $f : M^d \to BO(d)$. This is referred to as a "tangential structure" on the manifold M^d.

Now suppose $\xi \to X$ and $\zeta \to Y$ are vector bundles. We consider the space $Bun(\xi, \zeta)$ of bundle maps between them. These are maps of vector bundles

$$\begin{array}{ccc} \xi & \longrightarrow & \zeta \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where the diagram commutes, and the induced map on fibers are linear isomorphisms. $Bun(\xi, \zeta)$ is topologized using the compact-open topology. The following result is standard, but we give the presentation of it found in [54].

Lemma 11.72. *. Let $\xi \to X$ be a d-dimensional vector bundle. Let $\gamma_d \to BO(d)$ be the universal bundle. Then the space $Bun(\xi, \gamma_d)$ is contractible.*

Proof. Recall that $\gamma_d = \{(V, v) : V \subset \mathbb{R}^\infty \quad \text{is a } d\text{-dimensional subspace, and } v \in V\}$. Therefore $\gamma_d \subset BO(d) \times \mathbb{R}^\infty$. From this perspective we can think of

$$Bun(\xi, \gamma_d) \subset Map(\xi, \mathbb{R}^\infty)$$

as the subspace of maps that restrict to linear monomorphisms on each fiber of $\xi \to X$. Now define linear monomorphisms $\mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$\begin{aligned} \iota_1(x) &= (x_0, 0, x_1, 0, x_2, 0, \dots), \\ \iota_t(x) &= (1-t)x + \iota_1(x), \quad 0 \le t \le 1 \\ j(x) &= (0, x_0, 0, x_1, 0, x_2, \dots). \end{aligned} \tag{11.40}$$

Notice that there is a homotopy

$$\begin{aligned} [0,1] \times Map(\xi, \mathbb{R}^\infty) &\to Map(\xi, \mathbb{R}^\infty) \\ (t, f) &\to (1-t)(\iota_t \circ f) \end{aligned} \tag{11.41}$$

that restricts to a homotopy of self-maps of $Bun(\xi, \gamma_d)$ starting at the identity. Now pick $g \in Bun(\xi, \gamma_d)$ and define a homotopy

$$\begin{aligned} [0,1] \times Map(\xi, \mathbb{R}^\infty) &\to Map(\xi, \mathbb{R}^\infty) \\ (t, f) &\to (1-t)(\iota_1 \circ f) + t(j \circ g). \end{aligned}$$

This restricts to a homotopy of self-maps of $Bun(\xi, \gamma_d)$ that start at $f \to \iota_1 \circ f$. Combining this with homotopy (11.41) yields a homotopy of self maps of $Bun(\xi, \gamma_d)$ which starts at the identity and ends at the constant map $j \circ g$. □

Given a fibration $\theta : B \to BO(d)$ we can now define the cobordism category $\mathcal{C}_d^\theta$ of manifolds with tangential θ-structures.

Recall that a morphism in $\mathcal{C}_d$ is a d-dimensional submanifold with boundary $W^d \subset [a_0, a_1] \times \mathbb{R}^{d-1+n}$ for n large, where $a_0 < a_1 \in \mathbb{R}$. Given a point $x \in W^d$ we can take the tangent space $T_x W^d$ to define a map

$$\tau_W : W^d \to Gr_d(\mathbb{R}^{d+n}) \to Gr_d(\mathbb{R}^\infty) = BO(d)$$

which classifies the tangent bundle of W^d.

Definition 11.13. *Let $\mathcal{C}_d^\theta$ be the topological category with morphisms*

$$(W^d, a_0, a_1, \ell)$$

where $(W^d, a_0, a_1) \in Mor(\mathcal{C}_d)$, and $\ell : W^d \to B$ is a lifting of τ_W. $Mor\,\mathcal{C}_d^\theta$ is topologized in the similar manner to $Mor\,\mathcal{C}_d$ but with $B_\infty(W^d)$ replaced by $B_\infty^\theta = Emb^\theta(W^d; [0,1] \times \mathbb{R}^\infty)/Diff(W^d)$, where Emb^θ is defined by the pull-back square,

$$\begin{array}{ccc} Emb^\theta(W^d; [0,1] \times \mathbb{R}^\infty) & \longrightarrow & Bun(TW, \theta^*(\gamma_d)) \\ \big\downarrow & & \big\downarrow \\ Emb(W^d; [0,1] \times \mathbb{R}^\infty) & \longrightarrow & Bun(TW, \gamma_d). \end{array}$$

The objects of $\mathcal{C}_d^\theta$ are $(d-1)$-dimensional closed submanifolds of $\mathbb{R}^\infty$ with tangental θ structures. $Ob\,(\mathcal{C}_d^\theta)$ are topologized similarly.

We now define the spectrum $\mathbb{M}T\theta$ as follows.

Definition 11.14. *We define $\mathbb{M}T\theta$ to be the Thom spectrum of the map*

$$B \xrightarrow{\theta} BO(d) \to BO$$

Notice that zero space of the ω-spectrum associated to $\mathbb{M}T\theta$ is

$$\Omega^\infty \mathbb{M}T\theta = \operatorname{colim}_{n\to\infty} \Omega^{d+n} T(\theta_{d,n}^*(\gamma_{d,n}^\perp)),$$

and similarly, the next space in the spectrum is given by

$$\Omega^{\infty-1} \mathbb{M}T\theta = \operatorname{colim}_{n\to\infty} \Omega^{d+n-1} T(\theta_{d,n}^*(\gamma_{d,n}^\perp)).$$

The generalization of Theorem 11.68 to tangential structures that was proved in [54] is the following:

Theorem 11.73. *[54] Let $\theta : B \to BO(d)$ be a fibration. Then the weak homotopy equivalence*

$$\alpha_d : B\mathcal{C}_d \to \Omega^{\infty-1}\mathbb{M}TO(d)$$

lifts to a weak homotopy equivalence

$$\alpha_d^\theta : B\mathcal{C}_d^\theta \to \Omega^{\infty-1}\mathbb{M}T\theta.$$

For example $\Omega^{\infty-1}\mathbb{M}TSO(d)$ is equivalent to the classifying space of the cobordism category of oriented manifolds where the cobordisms have dimension d, and $\Omega^{\infty-1}\mathbb{M}TU(d)$ is equivalent to the classifying space of the cobordism category of manifold with tangential almost complex structure, where cobordisms have dimension $2d$.

Remark. The fact that the structures used to study the cobordism categories are defined on the tangent bundles, as opposed to the stable normal bundles used in Thom's theory, has suggested the notation $\mathbb{M}TO$ and $\mathbb{M}T\theta$ for the corresponding Thom spectra, as opposed to the $\mathbb{MO}$ and $\mathbb{M}\theta$ notation in Thom's theory (the "T" standing for tangent). This notation was suggested by M.J. Hopkins, when he also pointed out that this $\mathbb{M}T$ notation can also be thought of as referencing Madsen and Tillmann who originally introduced these spectra in [123] as a way of studying cobordism categories. These spectra are now often referred to as the "Madsen-Tillmann" spectra.

11.7.2 The surface cobordism category and Madsen and Weiss's solution of the Mumford Conjecture

An important special case of Theorem 11.73 is when $d = 2$ and the tangential structure θ is an orientation. This case was originally proved by Madsen and Weiss in [96]. In other words, the Madsen-Weiss theorem which motivated Theorem 11.73 is the following.

Theorem 11.74. *[96](Madsen and Weiss). There is a weak homotopy equivalence*

$$\alpha^{SO(2)} : B\mathcal{C}^{SO(2)} \xrightarrow{\simeq} \Omega^{\infty-1}\mathbb{M}TSO(2).$$

We now discuss the important implications of this theorem, including a proof of a famous conjecture of Mumford on the stable cohomology of groups of diffeomorphisms of surfaces.

We first observe that in the cobordism category $\mathcal{C}^{SO(2)}$, the morphisms are compact oriented surfaces, which are classified up to diffeomorphism by their genus and the number of their boundary components. As observed above, the morphism space $Mor_{\mathcal{C}^{SO(2)}}$ is topologized in terms of classifying spaces of diffeomorphisms of these surfaces. So it is not surprising that the Madsen-Weiss Theorem 11.74 yields important information about these classifying spaces. In order to make this more explicit, we first discuss some background about diffeomorphisms of surfaces and their classifying spaces.

Let $\Sigma_{g,b}$ be an oriented surface of genus g with b-boundary components. For $g, b \geq 1$ let $Diff^+(\Sigma_{g,b}, \partial)$ be the group of orientation preserving diffeomorphisms of $\Sigma_{g,b}$ that fix the boundary pointwise. Let $Diff_1^+(\Sigma_{g,b}, \partial)$ denote the connected oomponent of the identity diffeomorphism. The following is an important result of Earle and Eells from the late 1960's.

Theorem 11.75. *(Earle and Eells) For $g, b \geq 1$ the topological group $Diff_1^+(\Sigma_{g,b}, \partial)$ is contractible.*

The following is a straightforward exercise.

Exercise. Show that every path component of a topological group has the same homotopy type.

This implies the following important corollary. Let $\Gamma_{g,b} = \pi_0(Diff^+(\Sigma_{g,b}, \partial))$ be the discrete group of path components. This group is called the *"mapping class group"* of the surface $\Sigma_{g,b}$.

Corollary 11.76. *For $g, b \geq 1$, the group homomorphism given by passing to path components,*

$$Diff^+(\Sigma_{g,b}, \partial) \to \pi_0(Diff^+(\Sigma_{g,b}, \partial)) = \Gamma_{g,b}$$

is a homotopy equivalence. In particular they have homotopy equivalent classifying spaces,

$$BDiff^+(\Sigma_{g,b}, \partial) \simeq B\Gamma_{g,b}.$$

So the (co)homology of $BDiff^+(\Sigma_{g,b}, \partial)$ is the same as the (co)homology of the classifying space of the mapping class group $B\Gamma_{g,b}$. The cohomology of the mapping class group is important in algebraic geometry, just as $H^*(BDiff^+(\Sigma_{g,b}, \partial))$ is important in topology. We will explain a bit more about the importance of the cohomology of mapping class groups in algebraic geometry below. An important advance in this study was the "Harer Stability Theorem".

Theorem 11.77. *(J. Harer [65], and improved by N. Ivanov [81]) $H^k(B\Gamma_{g,b}); \mathbb{Z})$ is independent of the genus g and the number of boundary components b in the range $k < g/2 - 1$, so long as $b \geq 1$.*

This result can be restated in the following way. Consider a surface $\Sigma_{g,b}$ of genus g with b-boundary components with $b \geq 1$. By choosing a distinguished boundary component one can "glue" on a surface $\Sigma_{1,2}$ of genus one with two boundary components along one of the boundary components, to obtain a surface $\Sigma_{g+1,b}$ with the same number of boundary components as $\Sigma_{g,b}$, but whose genus has increased by one. See the first of the diagrams below. Furthermore, given a diffeomorphism $\phi : \Sigma_{g,b} \xrightarrow{\cong} \Sigma_{g,b}$ that is the identity on all the boundary components, one gets a diffeomorphism $\sigma(\phi) : \Sigma_{g+1,b} \xrightarrow{\cong} \Sigma_{g+1,b}$ by letting $\sigma(\phi) = \phi$ on $\Sigma_{g,b} \subset \Sigma_{g+1,b}$, and the identity on the glued surface of genus one. This defines homomorphisms

$$\sigma : Diff^+(\Sigma_{g,b}, \partial) \to Diff^+(\Sigma_{g+1,b}, \partial) \quad \text{and} \quad \sigma : \Gamma_{g,b} \to \Gamma_{g+1,b}$$

and the resulting maps on classifying spaces, which by abuse of notation we still call σ,

$$\sigma : BDiff^+(\Sigma_{g,b}, \partial) \to BDiff^+(\Sigma_{g+1.b}, \partial) \quad \text{and} \quad \sigma : B\Gamma_{g,b} \to B\Gamma_{g+1,b}. \tag{11.42}$$

We call these maps "stabilization maps".

Similarly, one can "cap off" a boundary component of $\Sigma_{g,b}$ by attaching a disk along its boundary to a designated boundary component of $\Sigma_{g,b}$ to produce a surface $\Sigma_{g,b-1}$ of the same genus with one fewer boundary components. See the second of the diagrams below. Furthermore one can define homomorphisms

$$\gamma : Diff^+(\Sigma_{g,b}, \partial) \to Diff^+(\Sigma_{g,b-1}, \partial) \quad \text{and} \quad \gamma : \Gamma_{g,b} \to \Gamma_{g,b-1}$$

by extending a diffeomorphism of $\Sigma_{g,b}$ that is the identity on the boundary to a diffeomorphism of $\Sigma_{g,b-1}$ by letting it be the identity on the glued disk.

The following is an alternative statement of the Harer Stability Theorem 11.77.

Theorem 11.78. *The induced map in homology (or cohomology),*

$$\sigma_* : H_k(B\Gamma_{g,b}; \mathbb{Z}) \to H_k(B\Gamma_{g+1,b}; \mathbb{Z})$$

and

$$\gamma_* : H_k(B\Gamma_{g,b}; \mathbb{Z}) \to H_k(B\Gamma_{g,b-1}; \mathbb{Z})$$

are isomorphisms if $k < g/2 - 1$ and in the case of σ_*, $b \geq 1$, and for γ_*, b must be at least 2.

This theorem says that for g sufficiently large, $H^*(B\Gamma_{g,b})$ does not depend on the number of boundary components b, so long as $b \geq 1$. The cohomology in this range of dimensions is referred to as the *"stable cohomology"* of the mapping class groups. We can think about this stable cohomology in the following way. Since the number of boundary components is irrelevant in stable cohomology, we might as well focus on surfaces with two boundary components, one "incoming", and one "outgoing" $\Sigma_{g,2}$. By gluing in order to increase the genus we get a sequence of surfaces,

$$\Sigma_{g,2} \subset \Sigma_{g+1,2} \subset \Sigma_{g+2,2} \subset \cdots \subset \Sigma_{g+L,2} \subset \cdots$$

By taking the colimit of the corresponding mapping class group, we get the *"stable mapping class group"* Γ_∞

$$\Gamma_\infty = \operatorname{colim}_L \Gamma_{g+L,2}. \tag{11.43}$$

See the second diagram below for this stabilization process.

D. Mumford's famous conjecture is about the rational cohomology of the stable mapping class group.

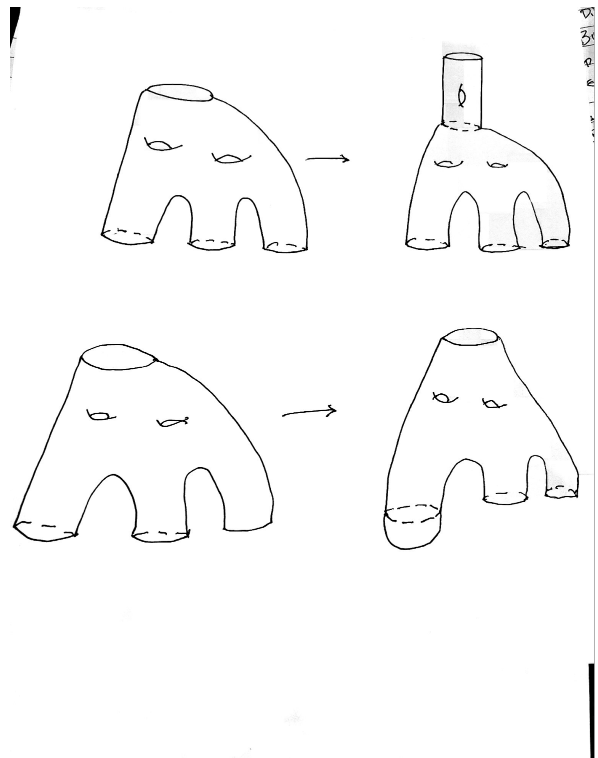
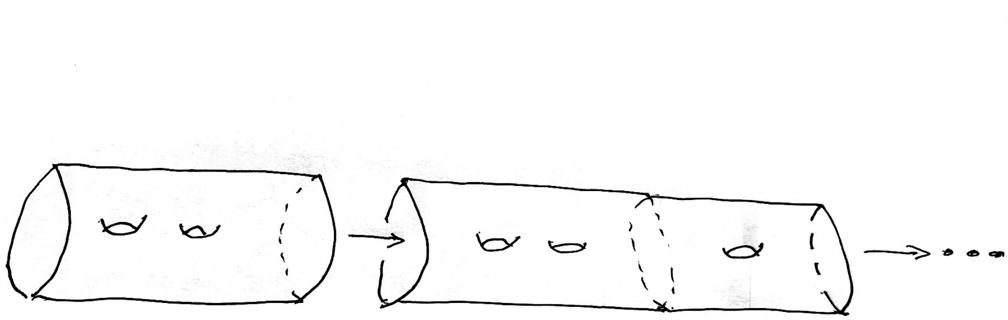


FIGURE 11.1
Increasing the genus and decreasing the number of boundary components



Conjecture 11.79. *("The Mumford Conjecture" [125]). The rational cohomology ring of the stable mapping class group is a polynomial algebra,*

$$H^*(B\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_i, \ldots]$$

where the generators κ_i have dimension $2i$.

Remark. The generating classes κ_i were constructed explicitly by Mumford, as well as by Morita and Miller, so they are often referred to as the "*MMM - classes*". It was also shown that the polynomial algebra generated by these classes injects into the rational cohomology of $B\Gamma_\infty$. The content of the Mumford conjecture was therefore that the entire rational cohomology of the stable mapping class group can be expressed in terms of polynomials in the MMM - classes.

Now consider again the cobordism category, $\mathcal{C}_2^{SO(2)}$. The objects are disjoint unions of circles, embedded in high dimensional Euclidean space, and the morphisms between them are oriented surface cobordisms (also embedded in high dimensional Euclidean space). If one fixes a circle C embedded in $\mathbb{R}^\infty$ then by the definition of the morphisms in the category $\mathcal{C}_2^{SO(2)}$ and the classification of oriented, compact surfaces, one sees that the morphism space from C to itself has the homotopy type

$$Mor_{\mathcal{C}_2^{SO(2)}}(C,C) \simeq \coprod_{g\geq 0} BDiff^+(\Sigma_{g,2},\partial) \simeq \coprod_{g\geq 0} B\Gamma_{g,2}.$$

In this setting we are thinking of $\Sigma_{g,2}$ as a surface with one "incoming" and one "outgoing" boundary component, corresponding to a cobordism form C to C.

By composing morphisms, $Mor_{\mathcal{C}_2^{SO(2)}}(C,C)$ is endowed with a monoid structure, and as such it is a subcategory of $\mathcal{C}_2^{SO(2)}$. With respect to this equivalence, the product structure in this monoid is given by pairings

$$B\Gamma_{g,2} \times B\Gamma_{k,2} \to B\Gamma_{g+k,2}$$

that are induced by the gluing of surfaces $\Sigma_{g,2} \times \Sigma_{k,2} \to \Sigma_{g+k,2}$ and their resulting diffeomorphism groups. These gluings are described above.

Now recall that a morphism in a category defines a path in its classifying space, where the starting and ending points of the path are the points in the classifying space represented by the source and the target of the morphism. if we take the basepoint in $B\mathcal{C}_2^{SO(2)}$ to be the point represented by the object $C \in Ob_{\mathcal{C}_2^{SO(2)}}$, then we have a map

$$\coprod_{g\geq 0} B\Gamma_{g,2} \simeq Mor_{\mathcal{C}_2^{SO(2)}}(C,C) \to \Omega B\mathcal{C}_2^{SO(2)}.$$

In fact this map factors as the composition

$$\coprod_{g\geq 0} B\Gamma_{g,2} \to \Omega B\left(\coprod_{g\geq 0} B\Gamma_{g,2}\right) \to \Omega B\mathcal{C}_2^{SO(2)},$$

where the middle space is the (based) loop space of the classifying space of the monoid $\coprod_{g\geq 0} B\Gamma_{g,2}$ under the gluing operation described above, and the map to the right hand space is induced by thinking of this monoid as a subcategory with the single object C, of the category $\mathcal{C}_2^{SO(2)}$.

Now given a monoid $\mathcal{M}$, the map $\mathcal{M} \to \Omega B\mathcal{M}$ is known as the *"group completion"* of $\mathcal{M}$. This map has been studied a great deal beginning with the seminal work of Quillen [129] and including the important homological understanding of the group completion construction due to McDuff and Segal [110].

The groups $\Gamma_{g,2}$ are known to be "perfect" for $g \geq 2$. A group being perfect means that it equals its own commutator subgroup. Then according to Quillen [129] there is a construction which is now known as the "Quillen plus construction",

$$B\Gamma_\infty \to B\Gamma_\infty^+$$

that has the following properties:

1. $B\Gamma_\infty \to B\Gamma_\infty^+$ induces an isomorphism in homology
2. The group completion map $\coprod_{g\geq 0} B\Gamma_{g,2} \to \Omega B\left(\coprod_{g\geq 0} B\Gamma_{g,2}\right)$ factors as a composition

$$\coprod_{g\geq 0} B\Gamma_{g,2} \to \mathbb{Z}\times B\Gamma_\infty \to \mathbb{Z}\times B\Gamma_\infty^+ \xrightarrow{\simeq} \Omega B\left(\coprod_{g\geq 0} B\Gamma_{g,2}\right)$$

   where the last map is a homotopy equivalence.

In particular we have the following result.

Proposition 11.80. *There is a map*

$$\mathbb{Z}\times B\Gamma_\infty \to \Omega B\left(\coprod_{g\geq 0} B\Gamma_{g,2}\right)$$

that induces an isomorphism in (co)homology.

As described above, the monoid $\coprod_{g\geq 0} B\Gamma_{g,2}$ is the subcategory of $\mathcal{C}_2^{SO(2)}$ where one restricts to the single object C. Now the object C is represented by a single circle embedded in high dimensional Euclidean space, and as a subcategory, the morphisms are cobordisms whose incoming and outgoing boundaries are both the single circle C. By Harer's stability theorem 11.78, the space of cobordisms between any two objects has the same homology in the stable range as the space of cobordisms between C and itself. In [96] Madsen and Weiss use this to show that induced map of group completions,

$$\Omega B \left(\coprod_{g \geq 0} B\Gamma_{g,2} \right) \xrightarrow{\simeq} \Omega B\mathcal{C}_2^{SO(2)} \tag{11.44}$$

is a homotopy equivalence.

Combining this result with Proposition 11.80 we have the following:

Theorem 11.81. *There is a map*

$$\tilde{\alpha} : \mathbb{Z} \times B\Gamma_\infty \to \Omega B\mathcal{C}_2^{SO(2)}$$

that induces an isomorphism in homology.

Furthermore, combining this with Madsen and Weiss's Theorem 11.74 we have the following important result.

Theorem 11.82. *(Madsen and Weiss [96]) There is a map*

$$\alpha : \mathbb{Z} \times B\Gamma_\infty \to \Omega^\infty \mathbb{M}TSO(2)$$

that induces an isomorphism in (co)homology.

Remark. The fact that $\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space, so that $\mathbb{Z} \times B\Gamma_\infty$ has the homology of an infinite loop space was proved by Tillmann in [151]. The Madsen-Weiss theorem identifies the homotopy type of this infinite loop space.

We end this subsection by showing how Theorem 11.82 implies the truth of Mumford's Conjecture 11.79.

Proof. We begin with the following exercise.

Exercise. We know that the classifying space $BSO(1)$ is contractible since $SO(1)$ is the trivial group. This means that the Thom spectrum $MSO(1)$ is the sphere spectrum $\mathbb{S}$. Show that the spectrum $\mathbb{M}TSO(1) \simeq \Sigma^{-1}\mathbb{S}$.

Now consider the cofibration sequence of spectra given by Lemma 11.70:

$$\mathbb{M}TSO(2) \to \Sigma^\infty(\mathbb{CP}^\infty_+) \to \mathbb{M}TSO(1),$$

which by the exercise is

$$\mathbb{M}TSO(2) \to \Sigma^\infty(\mathbb{CP}^\infty_+) \to \Sigma^{-1}\mathbb{S}.$$

Therefore we have a homotopy fibration sequence of the zero spaces of the corresponding ω-spectra

$$\Omega^\infty \mathbb{M}TSO(2) \to \Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+) \to \Omega(\Omega^\infty S^\infty).$$

Now we have an isomorphism of homotopy groups

$$\pi_k(\Omega(\Omega^\infty S^\infty)) \cong \pi_{k+1}(\Omega^\infty S^\infty) = \pi_{k+1}(\mathbb{S}).$$

Since these groups are all finite, we have that

$$\pi_k(\Omega(\Omega^\infty S^\infty)) \otimes \mathbb{Q} = 0$$

for all k. By the homotopy exact sequence of a fibration, we have that

$$\pi_*(\Omega^\infty \mathbb{M}TSO(2)) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(\Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+) \otimes \mathbb{Q}$$

and therefore

$$H_*(\Omega^\infty \mathbb{M}TSO(2)); \mathbb{Q}) \xrightarrow{\cong} H_*(\Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+); \mathbb{Q}).$$

This implies an isomorphism in rational cohomology,

$$H^*(\Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+); \mathbb{Q}) \xrightarrow{\cong} H^*(\Omega^\infty \mathbb{M}TSO(2)); \mathbb{Q}). \tag{11.45}$$

To determine this rational cohomology, first recall the natural adjunction maps

$$\Sigma^n \Omega^n Y \to Y$$

for any space Y defined by

$$t \wedge \alpha \to \alpha(t) \in Y.$$

By applying Ω^n we have a map

$$\Omega^n \Sigma^n \Omega^n Y \to \Omega^n Y.$$

In other words, if X is an n-fold loop space, $X \simeq \Omega^n Y$, then there is a natural map

$$\gamma : \Omega^n \Sigma^n X \to X.$$

Allowing n to get arbitrarily large, we see that if X is an infinite loop space, there is a natural map

$$\gamma : \Omega^\infty \Sigma^\infty X \to X.$$

In particular, by Bott periodicity, $\mathbb{Z} \times BU$ is an infinite loop space, so it comes equipped with a map

$$\gamma : \Omega^\infty \Sigma^\infty (\mathbb{Z} \times BU) \to \mathbb{Z} \times BU.$$

Now consider the basepoint preserving map $\tilde{\phi} : \mathbb{CP}^\infty_+ \to \mathbb{Z} \times BU$ defined by sending the disjoint basepoint to the basepoint in $\{0\} \times BU$ and sending $\mathbb{CP}^\infty = BU(1)$ to $\{1\} \times BU(1) \hookrightarrow \{1\} \times BU \subset \mathbb{Z} \times BU$. Using the infinite loop structure of $\mathbb{Z} \times BU$ given by Bott periodicity we then get a map

$$\phi : \Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+) \xrightarrow{\Omega^\infty \Sigma^\infty \tilde{\phi}} \Omega^\infty \Sigma^\infty (\mathbb{Z} \times BU) \xrightarrow{\gamma} \mathbb{Z} \times BU.$$

We claim that ϕ induces an isomorphism in rational (co)homology. This will follow if we can show it induces an isomorphism in rational homotopy groups. Now we know that

$$\pi_q(\Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+)) \otimes \mathbb{Q} = \pi^s_q(\mathbb{CP}^\infty_+) \otimes \mathbb{Q} = H_q(\mathbb{CP}^\infty_+; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \text{ is even, and} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

Of course by Bott periodicity these are the same as the rational homotopy groups of $\mathbb{Z} \times BU$. Furthermore, we know that the map in homology

$$H_*(\mathbb{CP}^\infty_+; \mathbb{Z}) \to H_*(\mathbb{Z} \times BU; \mathbb{Z})$$

is injective. Therefore the map

$$\pi_q(\Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+)) \otimes \mathbb{Q} = \pi^s_q(\mathbb{CP}^\infty_+) \otimes \mathbb{Q} = H_q(\mathbb{CP}^\infty_+; \mathbb{Q}) \to H_q(\mathbb{Z} \times BU; \mathbb{Q})$$

is injective. Hence

$$\pi_q(\Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+)) \otimes \mathbb{Q} \to \pi_q(\mathbb{Z} \times BU) \otimes \mathbb{Q}$$

is a monomorphism between isomorphic rational vector spaces. Hence it is an isomorphism.

So we may conclude that the map

$$\phi : \Omega^\infty \Sigma^\infty (\mathbb{CP}^\infty_+) \to \mathbb{Z} \times BU$$

induces an isomorphism in rational (co)homology. Therefore the composition

$$\Omega^\infty \mathbb{M}TSO(2) \to \Omega^\infty \Sigma^\infty (\mathbb{CP}_+) \xrightarrow{\phi} \mathbb{Z} \times BU$$

induces an isomorphism in rational cohomology. Hence the composition

$$H^*(\mathbb{Z} \times BU; \mathbb{Q}) \to H^*(\Omega^\infty \mathbb{M}TSO(2); \mathbb{Q}) \xrightarrow{\cong} H^*(\mathbb{Z} \times B\Gamma_\infty; \mathbb{Q})$$

is an isomorphism of rational cohomology rings, where the second map in this composition is the isomorphism induced by the Madsen-Weiss Theorem 11.82. Since we know the cohomology of BU, the Mumford conjecture follows. □

Remark. The mod p (co)homology of the infinite loop space $\Omega^\infty \mathbb{M}TSO(2)$ and therefore of the stable mapping class groups $B\Gamma_\infty$ have been computed by Galatius [53] for every prime p. The computations are explicit, although quite complicated.

11.7.3 The work of Galatius and Randal-Williams on moduli spaces of manifolds

In an important series of work, most notably, [55], [56], and [57], Galatius and Randal-Williams took the works of Madsen and Weiss [96] and of Galatius, Madsen, Tillmann, and Weiss [54], and expanded upon them greatly to develop and study the theory of "moduli spaces of manifolds".

Coming from ideas in algebraic geometry, a "moduli space" is a way of classifying a "representable functor". (Recall that this term was used in our study of Brown's Representablity Theorem 10.3 above.). In the setting studied by Galatius and Randal-Williams, the functor to be studied are concordance classes of smooth fiber bundles over a manifold.

Definition 11.15. *A "smooth fiber bundle" of dimension d consists of smooth manifolds E and X (without boundary) and a smooth proper map*

$$\pi : E \to X$$

such that the derivative $D\pi : \tau E \to \tau X$ is surjective and the vector bundle $\tau_\pi E = Ker\,(D\pi)$ has d-dimensional fibers. The bundle $\tau_\pi E$ is called the "vertical tangent bundle".

Definition 11.16. *Let $\pi_0 : E_0 \to X$ and $\pi_1 : E_{\to} X$ be smooth bundles over X.*

- *An isomorphism beween π_0 and π_1 is a diffeomorphism $\phi : E_0 \xrightarrow{\cong} E_1$ living over the identity of X.*
- *A concordance between π_0 and π_1 is a smooth fiber bundle $\pi : E \to \mathbb{R} \times X$ together with isomorphisms from π_0 and π_1 to the pullbacks of π along the two embeddings $X \cong \{0\} \times X$ and $X \cong \{1\} \times X \subset \mathbb{R} \times X$, respectively.*

The basic representability theorem for smooth bundles was proved by Galatius and Randal-Williams is the following:

For a smooth manifold X without boundary, let $\mathcal{F}[X]$ denote the set of concordance classes of smooth fiber bundles $\pi : E \to X$

Theorem 11.83. *The functor $X \to \mathcal{F}[X]$ is representable in the sense that there exists a space $\mathcal{M}$ and a natural bijection*

$$\mathcal{F}[X] \cong [X, \mathcal{M}].$$

The space $\mathcal{M}$ in this theorem is not very useful itself because it is highly disconnected. It is preferable, therefore, to study one path component of $\mathcal{M}$ at a time, which corresponds to fixing the concordance class of the fibers.

To do this, let W be a closed d-dimensional manifold. We can consider W to be a smooth fiber bundle over a point, so it represents a path component $[W] \in \pi_0(\mathcal{M})$. We write $\mathcal{M}(W) \subset \mathcal{M}$ for the path component of $\mathcal{M}$ containing W. Notice that $\mathcal{M}(W)$ is the classifying space (or "moduli space") for smooth fiber bundles $\pi : E \to X$ whose restriction to any point $x \in X$ is concordant to W.

Exercise. Show that the homotopy type of the moduli space $\mathcal{M}(W)$ is the classifying space $BDiff(W)$.

Notice that as a model for the space $\mathcal{M}(W)$ we can take the space of all closed submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to W, as we did in the last subsection.

To describe one of the main results of Galatius and Randal-Williams, we need a slight variation and generalization of the notion of "tangential structure" used in [54] that applies nicely to the setting of smooth fiber bundles.

Definition 11.17. *If Θ is a space with a continuous $GL_d(\mathbb{R})$-action, a smooth fiber bundle with Θ-structure consists of a smooth fiber bundle $\pi : E \to X$, together with a continuous $GL_d(\mathbb{R})$- equivariant map $\rho : Fr(\tau_\pi E) \to \Theta$. Here $Fr(\tau_\pi E)$ is the frame bundle of of the vertical tangent bundle $\tau_\pi E$. It is a principal $GL_d(\mathbb{R})$-bundle over E whose fiber at a point $e \in E$ is the space of linear bases of the vector space given by the fiber $\tau_\pi E(e) = KerD\pi : \tau_e E \to \tau_{\pi(e)} X$ over $e \in E$.*

The relation to our previous notion of a structure on a vector bundle is the following. Given a space Θ as in this definition, we can take the homotopy orbit space, $EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta$, and there is a natural map which we can assume is a fibration,

$$EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta \to EGL_d(\mathbb{R})/GL_d(\mathbb{R}) = BGL_d(\mathbb{R}) \simeq BO(d).$$

A Θ - structure on a fiber bundle $\pi : E \to X$ as in the above definition yields a lift of the classifying map of the vertical tangent bundle $E \to BO(d)$ to $EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta$.

Using this Galatius and Randal-Williams generalized Theorem 11.83 as follows. For X a smooth closed manifold, let $\mathcal{F}^\Theta[X]$ denote the set of concordance classes of pairs (π, ρ) of a smooth fiber bundle $\pi : E \to X$ with Θ-structure $\rho : Fr(\tau_\pi E) \to \Theta$. (Notice that X is fixed but the fiber bundle E is allowed to vary.)

Theorem 11.84. *The functor $X \to \mathcal{F}^\Theta[X]$ is representable in the sense that there exists a space $\mathcal{M}^\Theta$ and a natural bijection*

$$\mathcal{F}^\Theta[X] \cong [X, \mathcal{M}^\Theta].$$

One of the main theorems of Galatius and Randal-Williams is identifying the homology type of the various path components of these moduli spaces, at least through a range of dimensions. To make this more precise we need a few more definitions.

Definition 11.18. *. Let Θ be a space with a $GL_d(\mathbb{R})$ action, W a closed d-manifold, and $\rho_W : Fr(\tau W) \to \Theta$ an equivariant map. Consider this as a structure on the trivial fiber bundle $W \to point$. This defines a class $[(W, \rho_W)] \in \pi_0(\mathcal{M}^\Theta)$, and we write $\mathcal{M}^\Theta(W, \rho_W) \subset \mathcal{M}^\Theta$ for the path component containing (W, ρ_W). This path component is a classifying space for smooth fiber bundles $\pi : E \to X$ with structure $\rho : Fr(\tau_\pi E) \to \Theta$, whose restriction to an point $x \in X$ is concordant to (W, ρ_W).*

A main theorem in [57] describes the homology of the moduli spaces $\mathcal{M}^\Theta(W, \rho_W)$ through a range of dimensions, depending on a generalized notion of "genus", which we now describe.

Notice that the classical notion of the genus of a closed, oriented, connected surface Σ can be described in terms of embeddings of the punctured torus $S^1 \times S^1 - \{*\} \hookrightarrow \Sigma$. Namely, the genus g of Σ is the maximal number of disjoint embeddings of the punctured torus into Σ. The generalized definition of Galatius-Randal-Williams used a similar notion.

Consider the "higher dimensional punctured torus" $S^n \times S^n - \{*\}$. We need the notion of an "admissible" Θ-structure on this manifold. Notice that $S^n \times S^n - \{*\}$ is diffeomorphic to the pushout of the two embeddings

$$S^n \times \mathbb{R}^n \hookleftarrow \mathbb{R}^n \times \mathbb{R}^n \hookrightarrow \mathbb{R}^n \times S^n \tag{11.46}$$

where we are thinking of S^n as the one-point compactification of $\mathbb{R}^n$, and the above embeddings are induced by a choice of coordinate chart $\mathbb{R}^n \subset S^n$. A Θ-structure on $S^n \times \mathbb{R}^n$ is called *admissible* if is equivariantly homotopic to a structure that extends over some embedding $S^n \times \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}$. A Θ-structure on $S^n \times S^n - \{*\}$ is admissible if the restriction to each piece of the gluing (11.46) is admissible.

Definition 11.19. *Assume $d = 2n > 0$ and that W is a a connected, closed d-dimensional manifold. The genus* $g(W, \rho_W)$ of a Θ-manifold (W, ρ_W) is the maximal number of disjoint embeddings of the open manifold $S^n \times S^n - \{*\} \hookrightarrow W$ such that $j^*\rho_W$ is admissible.

Now consider again the homotopy orbit space $EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta$ and the map

$$EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta \to EGL_d(\mathbb{R})/GL_d(\mathbb{R}) = BGL_d(\mathbb{R}) \simeq BO(d) \to BO.$$

Similar to the notation we used in the last subsection we call the Thom spectrum of this map $MT\Theta$.

One of the main theorems of [57] is the following:

Theorem 11.85. *Let $d = 2n > 4$, W a closed, simply connected d-manifold, and let $\rho_W : Fr(\tau W) \to \Theta$ be an n-connected $GL_d(\mathbb{R})$-equivariant map. Then there is a map*

$$\alpha : \mathcal{M}^{\Theta}(W, \rho_W) \to \Omega^{\infty} MT\Theta$$

inducing an isomorphism in integral homology onto the path component that it hits, in dimensions $\leq (g(W, \rho_W) - 4)/3$.

At first glance, this notion of genus, while geometrically straightforward, may seem very difficult to compute. To address this Galatius and Randal-Williams gave both an upper and lower bound for the genus defined in terms of middle dimensional Betti numbers. (Recall that the k^{th} Betti number of a space X, $b_k(X)$, is the dimension of the rational vector space $H_k(X; \mathbb{Q})$.)

Since W is assumed to be a closed, simply connected $d = 2n$ dimensional manifold, the intersection form

$$\langle\, , \,\rangle : H_n(W; \mathbb{Q}) \times H_n(W; \mathbb{Q}) \to \mathbb{Q}$$

is nondegenerate, and either symmetric, if n is even, or antisymmetric if n is odd. In the case when n is even, this nonsingular symmetric bilinear form splits the vector space $H_n(W; \mathbb{Q})$ into its positive and negative eigenspaces. Common notation is to let b_n^+ denote the dimension of the positive eigenspace, and b_n^- the dimension of the negative eigenspace. So in particular, in this case

$$b_n = b_n^+ + b_n^-.$$

The following result of Galatius and Randal-Williams says that the genus can be estimated in terms of calculable Betti-numbers.

Theorem 11.86. *([57]) Assume $d = 2n > 4$, that the homotopy orbit space, $B = EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta$ is simply connected, and that $\rho_A W : Fr(\tau W) \to \Theta$ is n-connected. Write $g^a(W) = min(b_n^+, b_n^-)$ if n is even, and $g^a(W) = b_n/2$ if n is odd. Then*

$$g^q(W) - c \leq g(W; \rho_W) \leq g^a(W),$$

With $c = 1 + e$, where e is the minimal numbers of generators of the abelian group $H_n(B; \mathbb{Z})$. If n is even one may take $c = e$ and get a slightly better lower bound.

Finally, in an argument similar to, but more general than the argument given in the previous section showing that the Mumford Conjecture 11.79 follows from the Madsen-Weiss Theorem 11.82, Galatius and Randal-Williams showed that Theorem 11.85 implies the following calculation of the rational cohomology, in a stable range, of the moduli space of manifolds.

As above, let B be the homotopy orbit space, $B = EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta$.

For a smooth bundle $\pi : E \to X$ with Θ-structure $\rho : Fr(\tau_\pi E) \to \Theta$, consider the map (well-defined up to homotopy)

$$\ell : E \simeq EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} Fr(\tau_\pi E) \xrightarrow{1\times\rho} EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta = B.$$

Now let $\mathbb{Z}^\omega$ denote the coefficient system on B arising from the nontrivial action of $\pi_0(GL_d(\mathbb{R})) = \mathbb{Z}^\times = \{\pm 1\}$ on $\mathbb{Z}$, and let $A^\omega = A \otimes \mathbb{Z}^\omega$ for any abelian group A. Given any class $c \in H^{d+k}(B; A^\omega)$, Galatius and Randal-Williams defined a generalized Miller-Morita-Mumford class

$$\kappa_c(\pi) \in H^k(\mathcal{M}^\Theta(W, \rho_W; A)).$$

Galatius and Randal-Williams then showed that their Theorem 11.85 implies the following rational calculation:

Theorem 11.87. *Let $d = 2n > 4$, W a closed simply-connected d-manifold, and $\rho_W : Fr(W) \to \Theta$ be a Θ structure which is n-connected. Equip $B = EGL_d(\mathbb{R}) \times_{GL_d(\mathbb{R})} \Theta$ with the local coefficient system $\mathbb{Q}^\omega$ as above, and assume that $H^{k+d}(B; \mathbb{Q}^\omega)$ is finite dimensional for each $k \geq 1$. Then the ring homomorphism*

$$\mathbb{Q}[\kappa_c \,|\, c \in \textit{basis of}\, H^d(B; \mathbb{Q}^\omega)] \to H^*(\mathcal{M}^\Theta(W, \rho_W); \mathbb{Q})$$

is an isomorphism in cohomological degrees $\leq (g(W, \rho_W) - 4)/3$.

12

Classical Morse Theory

In this chapter we discuss the traditional, "classical" approach to Morse theory. An approach based on moduli spaces of flows will be discussed in the next chapter. The best reference to this classical approach is Milnor's well known book [112]. We encourage the reader to study that book, not only for the details of the foundations of the subject, but also for applications that are still quite relevant more than 50 years after its publication.

12.1 The Hessian and the index of a critical point

Let M be a manifold, and $f : M \longrightarrow \mathbb{R}$ a C^2 function. As explained earlier, a point $p \in M$ is called a *critical point* of f if $df_p = 0$. $f : M \to \mathbb{R}$ is a **Morse function** if all of its critical points are nondegenerate. To understand what it means for a critical point $p \in M$ to be nondegenerate, we may work in a coordinate chart around p, with respect to which we may think of $f : \mathbb{R}^n \to \mathbb{R}$, with p corresponding to the origin in $\mathbb{R}^n$. In such coordinates we can think of the derivative as a map

$$\begin{aligned} Df : \mathbb{R}^n &\to (\mathbb{R}^n)^* \\ x &\to df_x. \end{aligned}$$

A critical point is then a zero of Df, and $0 \in \mathbb{R}^n$ is a nondegenerate critical point precisely if it is a *regular point* of Df. Notice $0 \in \mathbb{R}^n$ being a nondegenerate critical point is equivalent to the linear map

$$D(Df)_0 : \mathbb{R}^n \to (\mathbb{R}^n)^*$$

being an isomorphism. This in turn is equivalent to the $n \times n$ Hessian matrix,

$$\mathrm{Hess}_0\, f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$$

being nonsingular.

We now make this into a formal definition. Let $p \in M$ be a critical point o $f : M \to \mathbb{R}$, and let $(U, \phi : U \longrightarrow \mathbb{R}^n)$ be a coordinate chart around p, so that $\phi(p) = 0$. Write ϕ as $(x_1, \ldots, x_n)$. Write tangent vectors v and w in T_pM as

$(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$, respectively (specifically, $d\phi_p(v) = (v_1, \ldots, v_n)$ and similarly for w).

Definition 12.1. *Using the coordinate chart (U, ϕ), The Hessian of f at p, is the quadratic form* $\mathrm{Hess}_p f$ *defined by the formula*

$$\mathrm{Hess}_p(f)(v) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} v_i v_j$$

Proposition 12.1. *When p is a critical point for $f : M \longrightarrow \mathbb{R}$, the Hessian at p is independent of the coordinate chart.*

Proof. One can do this directly by a straightforward calculation which we leave to the reader. But more generally, with respect to local coordinates, we may consider an open set $V \subset \mathbb{R}^n$ containing $0 \in \mathbb{R}^n$, and a C^2-map $g : V \to \mathbb{R}$ having 0 as a critical point. Let $h : V \xrightarrow{\cong} U$ be a C^2 diffeomorphism of open sets in $\mathbb{R}^n$ taking $0 \in \mathbb{R}^n$ to itself. Then the reader should verify that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathrm{Hess}_0(gh)} & \mathbb{R} \\ {\scriptstyle Dh_0}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{R}^n & \xrightarrow[Hess_0 g]{} & \mathbb{R}. \end{array}$$

This gives an invariance of the Hessian under local diffeomorphisms, which is to say, an invariance of the Hessian under changes of coordinate charts around a critical point. □

Remark. If p is not a critical point of f, then the Hessian at p is not well-defined, in that using the above notation, it would depend on the coordinate chart. However there are ways to extend the Hessian to all of M: by patching together coordinate charts and using partitions of unity; by choosing a metric on M, then using the Levi–Civita connection corresponding to this metric to take the covariant derivative of df at p, and so on. But these approaches all require extra data (namely the choice of metric or connection). In these notes, however, we will primarily be concerned with the Hessian at critical points.

12.2 Morse Functions

Definition 12.2. *If $p \in M$ is a critical point for a C^2 function $f : M \longrightarrow \mathbb{R}$, then we call p* nondegenerate *if the quadratic form $\mathrm{Hess}_p(f)$ is nonsingular. If all critical points of M are nondegenerate, we say that f is* Morse.

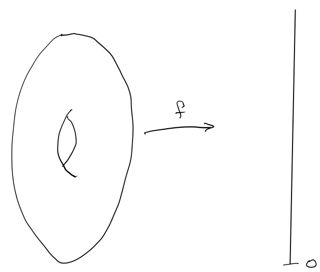


FIGURE 12.1
f is the "height function" given by projecting the torus onto the vertical line. This is probably the archetypical example of a Morse function.

We will show in Section 12.5 that every manifold M admits a Morse function, and in fact the set of Morse functions is dense in the set of smooth functions.

An important property of Morse functions on closed, Riemannian manifolds, is that they lead to a CW complex description of the manifold, with a cell of dimension λ for each critical point of index λ of f. In this section, we prove this statement up to homotopy. That is, we construct a homotopy equivalence of the manifold to a CW complex of the kind just described. We follow the approach of Milnor [112] in this chapter.

Throughout this chapter, we will assume M is a closed manifold and $f : M \longrightarrow \mathbb{R}$ is a smooth Morse function. We will also consider the following spaces, which are often manifolds (with boundary):

$$M^a = f^{-1}(-\infty, a] = \{x \in M \mid f(x) \leq a\}.$$

where a is any real number. If a is less than the minimum value of f, then M^a is the empty set. If a is larger than the maximum value of f, then M^a is M. The values of a in between will provide, up to homotopy, the necessary cell decomposition.

There are a number of technical details, but the intuition is simple: Let M

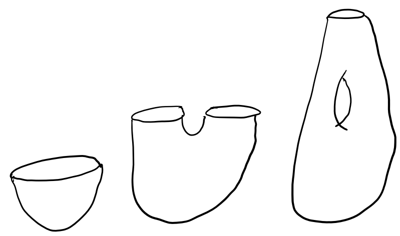


FIGURE 12.2
M^a for different values of a

be a surface embedded in $\mathbb{R}^3$, and f be the vertical coordinate z. We initially let a be less than the minimum value of f so that $M^a = \emptyset$, and gradually increase a (see Figure 12.2). This is analogous to gradually filling the surface with water, so that M^a is the part of the surface that is under water. Now if a increases from a_1 to a_2 without passing through critical values, then M^{a_1} and M^{a_2} are diffeomorphic.

But if, by increasing from a_1 to a_2, we pass through one critical point, then at that point the water may do something more interesting. Up to homotopy, this turns out to be an attaching of a cell of dimension λ, where λ is the index of the critical point (see Figure 12.4).

So as we pass critical points one by one, the manifold is created by successively attaching cells (up to homotopy type). This demonstrates that the manifold is homotopy equivalent to a CW complex of the type described above.

In this chapter we prove the details of the above intuition. First we prove that nothing happens to the homotopy type (and even to the diffeomorphism type) if there is no critical point between two levels, using the results of gra-

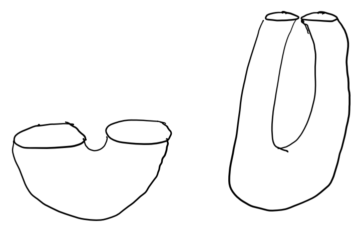


FIGURE 12.3
M^{a_1} and M^{a_2} are diffeomorphic if there are no critical values between a_1 and a_2.

dient flow lines from chapter 12.1. Then we show that if there is one critical point between the two levels, the homotopy type changes by adding a cell. We prove this via the Morse Lemma (Theorem 12.4), which studies the behavior of f near a critical point. We conclude by producing the homotopy equivalence between the manifold and the CW complex, and giving some interesting applications to topology.

Exercise:
Let M be a manifold and let $f : M \longrightarrow \mathbb{R}$ be a Morse function. Prove that $f^{-1}(\{a\})$, the boundary of M^a, is a manifold if a is a regular value of f.

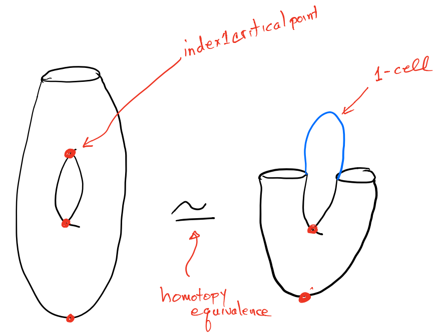


FIGURE 12.4
When there is one critical value between a_1 and a_2, M^{a_2} is homotopy equivalent to M^{a_1} with a cell attached.

12.3 The Regular Interval Theorem

We first show that if we increase M^a from M^{a_1} to M^{a_2}, and there are no critical values between a_1 and a_2, then M^{a_1} and M^{a_2} are diffeomorphic.

The main point is the following theorem:

Theorem 12.2 (Regular interval theorem)**.** *Let $f : M \longrightarrow [a, b]$ be a smooth map on a compact Riemannian manifold with boundary. Suppose that f has no critical points and that $f(\partial M) = \{a, b\}$. Then there is a diffeomorphism*

$$F : f^{-1}(a) \times [a, b] \longrightarrow M$$

making the following diagram commute:

$$\begin{array}{ccc} f^{-1}(a) \times [a,b] & \xrightarrow{F} & M \\ {\scriptstyle proj.}\downarrow & & \downarrow{\scriptstyle f} \\ [a,b] & \xrightarrow[=]{} & [a,b]. \end{array}$$

In particular all the level surfaces are diffeomorphic.

In the proof of this theorem we will make use of the **gradient vector field** $\nabla(f)$ of the function $f : M \to \mathbb{R}$, when M has a Riemannian metric. The definition of $\nabla(f)$ depends on the metric in the following way. Recall that a Riemannian metric g defines a nonsingular, symmetric biinear pairing on the tangent bundle,

$$< , >_g : TM \times TM \to \mathbb{R}.$$

Equivalently, by taking the adjoint of this pairing we may think of the metric g as defining an isomorphism of the tangent bundle with the cotangent bundle,

$$g : TM \xrightarrow{\cong} T^*M.$$

The differential df is a section of the cotangent bundle, $df(x) \in T^*_x M$ for every $x \in M$, and its definition *does not* depend on the metric. The *gradient vector field* $\nabla_x(f) \in T_x M$ is defined to be the section of TM determined by df, using the metric g. Said more explicitly, the gradient is the unique vector field (section of TM) that satisfies

$$< \nabla_x f \, , v >_g = df(x)(v) \tag{12.1}$$

for every $x \in M$ and $v \in T_x M$. We notice that the zeros of the gradient $\nabla(f)$ are the same as the zeros of the differential df and are exactly the critical points of $f : M \to \mathbb{R}$.

With this definition we are now ready to prove this theorem.

Proof. Since f has no critical points we may consider the vector field

$$X(x) = \frac{\nabla_x(f)}{|\nabla_x(f)|^2}.$$

Let $\eta_x(t)$ be a curve through x satisfying

$$\frac{d}{dt}\eta_x(t) = X(\eta_x(t))$$

and $f(\eta_x(t)) = t$.

Let I be a maximal interval on which η_x is defined. We wish to show that $I = [a, b]$. First, since M is compact, $f(\eta_x(I)) = I$ is bounded.

Let d be the supremum, $d = \sup(I)$. Then by the compactness of M, there is a point $x \in M$ that is a limit point of $\eta_x(d - 1/n)$. Since $\eta'_x(t) = X(\eta_x(t))$ is bounded, this limit point is unique, and $\lim_{t \to d^-} \eta_x(t) = x$. We can extend η_x to d by making $\eta_x(d) = x$.

Now $\lim_{t \to d} \eta'_x(t) = \lim_{t \to d} X(\eta_x(t)) \to X(\eta_x(d))$, and let v be this limit. We will now show that $\eta'_x(d) = v$. In particular, we will show that for every $\epsilon > 0$, there exists a $\delta > 0$ so that for all h with $0 < h < \delta$,

$$\left| \frac{\eta_x(d) - \eta_x(d-h)}{h} - v \right| < \epsilon.$$

Note that a coordinate chart is chosen near $\eta_x(d)$ to allow the subtraction here.

So let $\epsilon > 0$ be given. By the definition of v, there exists a δ_1 so that for all w with $0 < h < \delta_1$,

$$|\eta'_x(d-h) - v| < \epsilon$$

By the fundamental theorem of calculus,

$$\begin{aligned}
\eta_x(d-h) - \eta_x(d) &= \int_{d-h}^{d} \eta'_x(t)\, dt \\
\eta_x(d-h) - \eta_x(d) + vh &= \int_{d-h}^{d} (\eta'_x(t) - v)\, dt \\
|\eta_x(d-h) - \eta_x(d) + vh| &\leq \int_{d-h}^{d} |\eta'_x(t) - v|\, dt \\
&\leq \int_{d-h}^{d} \epsilon\, dt \\
&\leq \epsilon h \\
\left| \frac{\eta_x(d-h) - \eta_x(d)}{h} + v \right| &\leq \epsilon \\
\left| \frac{\eta_x(d-h) - \eta_x(d)}{-h} - v \right| &\leq \epsilon
\end{aligned}$$

Therefore $\eta'_x(d) = v$, and since $v = X(\eta_x(d))$, the flow equation is satisfied by η_x at d.

By maximality of I, $d \in I$. Similarly with $c = \inf(I)$, we see that $c \in I$. Therefore I is closed.

If $\eta_x(s) \notin \partial M$, then by the existence of solutions of ODEs, there is an interval $(s - \epsilon, s + \epsilon)$ around s on which η_x satisfies the differential equation $\eta'_x(t) = X(\eta_x(t))$. Therefore $\eta_x(c)$ and $\eta_x(d)$ are in ∂M. Thus $c = f(\eta_x(c))$ and $d = f(\eta_x(d))$ may be either a or b. Since the derivative of $f \circ \eta_x$ is one, we see that $c = a$ and $d = b$. Therefore $I = [a, b]$.

Since $x \in M$ was arbitrary, and $a \leq f(x) \leq b$, we see that $f(M) = [a, b]$.

Furthermore, if $x \notin \partial M$, then by the existence of solutions to ODEs, as above, we have η_x defined in a small neighborhood of $t = f(x)$, so that $a < f(x) < b$. Therefore $f^{-1}(a)$ and $f^{-1}(b)$ are unions of boundary components.

Define a map

$$F : f^{-1}(a) \times [a, b] \longrightarrow M$$

by the formula

$$F(x, t) = \eta_x(t).$$

The differentiability of F follows from the same argument as in Theorem 13.2 to prove the differentiability of T, but with η_x instead of γ_x.

Define

$$G : M \longrightarrow f^{-1}(a) \times [a, b]$$

as

$$G(x) = (\eta_x(a), f(x)).$$

The differentiability of G follows in the same way as the differentiability of F. We claim that F and G are inverses. To prove this, note that the integral curves through x and $\eta_x(t)$ are the same, that $f(\eta_x(t)) = t$ and by uniqueness of solutions to ODEs, we have $F(G(x)) = x$ and $G(F(x, t)) = (x, t)$. This proves that F is a diffeomorphism. □

Corollary 12.3. *Let M be a compact manifold, and $f : M \longrightarrow \mathbb{R}$ a smooth Morse function. Let $a < b$ and suppose that $f^{-1}[a, b] \subset M$ contains no critical points. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.*

Proof. First we prove that M^a is a deformation retract of M^b. By the regular interval theorem (Theorem 12.2), there is a natural diffeomorphism F from $f^{-1}([a, b])$ to $f^{-1}(a) \times [a, b]$. Since $f^{-1}(a) \times \{a\}$ is a deformation retract of $f^{-1}(a) \times [a, b]$, we see that $f^{-1}(a)$ is a deformation retract of $f^{-1}([a, b])$. We can now paste this deformation retraction with the identity on M_a to obtain the deformation retracton from M_b to M_a.

To prove that M^a is diffeomorphic to M^b we apply the same principle, but we need to be more careful to preserve smoothness during the patching process.

Since the set of critical points of f is a closed subset of the compact set M (and hence is compact), the set of critical values of f is compact. Therefore there are real numbers c and d with $c < d < a$ so that there are no critical values in $[c, b]$.

By Theorem 12.2 there is a natural diffeomorphism F from $f^{-1}([c, b])$ to $f^{-1}(c) \times [c, b]$, that maps $f^{-1}([c, a])$ diffeomorphically onto $f^{-1}(c) \times [c, a]$. There is also a diffeomorphism $H : f^{-1}(c) \times [c, b] \longrightarrow f^{-1}(c) \times [c, a]$, and we can insist that it be the identity on $f^{-1}(c) \times [c, d]$ (finding this function is an easy exercise in one-variable analysis, and in case you are interested, is listed as an exercise below). Thus

$$F^{-1} \circ H \circ F : f^{-1}([c, b]) \longrightarrow f^{-1}([c, a])$$

is a diffeomorphism that is the identity on $f^{-1}([c,d])$, and thus we can patch it together with the identity on M_d to create a diffeomorphism from M_b to M_a. □

This corollary says that the topology of the submanifolds M^a does not change with $a \in \mathbb{R}$ so long as a does not pass through a critical value.

Exercise Fill in the detail of the proof of Corollary 12.3 that finds a diffeomorphism $H : f^{-1}(c) \times [c,b] \longrightarrow f^{-1}(c) \times [c,a]$ that is the identity on $f^{-1}(c) \times [c,d]$.

12.4 Passing through a critical value

We now examine what happens to the topology of these submanifolds when one does pass through a critical value. For this, we will need to understand the function f in the neighborhood of a critical point. This is what the Morse lemma provides us:

Theorem 12.4 (Morse Lemma)**.** *Let p be a nondegenerate critical point of a smooth function $f : M \longrightarrow \mathbb{R}$, where M is an n-dimensional manifold. Then there is a local coordinate system $(x_1, \ldots, x_n)$ in a neighborhood U of p with $x_i(p) = 0$ with respect to which*

$$f(x_1, \ldots, x_n) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2.$$

λ is called the index of the critical point p.

The proof given here is essentially that in Milnor's famous book on Morse theory [112].

Proof. Since this is a local theorem we might as well assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with a critical point at the origin, $p = 0$. We may also assume without loss of generality that $f(0) = 0$. Given any coordinate system for $\mathbb{R}^n$ we can therefore write

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j g_j(x_1, \ldots, x_n)$$

for $(x_1, \ldots, x_n)$ in a neighborhood of the origin. In this expression we have

$$g_j(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_j}(tx_1, \ldots, tx_n)dt.$$

Now since 0 is a critical point of f, each $g_j(0) = 0$, and hence we may write it in the form

$$g_j(x_1,\ldots,x_n) = \sum_{i=0}^{n} x_i h_{i,j}(x_1,\ldots,x_n).$$

Let $\phi_{i,j} = (h_{i,j} + h_{j,i})/2$. Hence we can combine these equations and write

$$f(x_1,\ldots,x_n) = \sum_{i,j=1}^{n} x_i x_j \phi_{i,j}(x_1,\ldots,x_n)$$

where $(\phi_{i,j})$ is a symmetric matrix of functions. By doing a straightforward calculation one sees furthermore that the matrix

$$(\phi_{i,j}(0)) = \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right)$$

and hence by the nondegeneracy assumption is nonsingular. From linear algebra we know that symmetric matrices can be diagonalized. The Morse lemma will be proved by going through the diagonalization process with the representation of f as $\sum x_i x_j \phi_{i,j}$.

Assume inductively that there is a neighborhood U_k of the origin and coordinates $\{u_1,\ldots,u_n\}$ with respect to which

$$f = \pm(u_1)^2 \pm \cdots \pm (u_k)^2 + \sum_{i,j\geq k+1} u_i u_j \psi_{i,j}(u_1,\ldots,u_n)$$

where $(\psi_{i,j})$ is a symmetric, $n{-}k{\times}n{-}k$ matrix of functions. By a linear change in the last $n{-}k$ coordinates if necessary, we may assume that $\psi_{k+1,k+1}(0) \neq 0$.

Let

$$\sigma(u_1,\ldots,u_n) = \sqrt{|\psi_{k+1,k+1}(u_1,\ldots,u_n)|}$$

in perhaps a smaller neighborhood $V \subset U_k$ of the origin. Now define new coordinates

$$v_i = u_i \quad \text{for } i \neq k+1$$

and

$$v_{k+1}(u_1,\ldots u_n) = \sigma(u_1,\ldots,u_n)\left[u_{k+1} + \sum_{i=k+2}^{n} u_i \frac{\psi_{i,k+1}(u_1,\ldots,u_n)}{\psi_{k+1,k+1}(u_1,\ldots,u_n)}\right].$$

The v_i's give a coordinate system in a sufficiently small neighborhood U_{k+1} of the origin. Furthermore a direct calculation verifies that with respect to this coordinate system

$$f = \sum_{i=1}^{k+1} \pm(v_i)^2 + \sum_{i,j=k+2}^{n} v_i v_j \theta_{i,j}(v_1,\ldots,v_n)$$

where $(\theta_{i,j})$ is a symmetric matrix of functions. This completes the inductive step. The only remaining point in the theorem is to observe that the number of negative signs occuring in the expression for f as a sum and difference of squares is equal to the number of negative eigenvalues (counted with multiplicity) of $Hess_0(f)$ which does not depend on the particular coordinate system used. This is the *index* of the critical point. □

Remark. The Morse Lemma describes the behavior of the function f near a critical point, but it does not describe the behavior of the gradient near the critical point. The reason for this is that the gradient vector field depends on the Riemannian metric, and if we use the coordinate system given by the Morse Lemma, we do not know how this metric behaves.

Corollary 12.5. *If M is a manifold and $f : M \longrightarrow \mathbb{R}$ is Morse, then the set of critical points of f is a discrete subset of M.*

Proof. Suppose there were a sequence of critical points x_n converging to some point $a \in M$. Since df is a continuous one-form on M, we know that a is a critical point of f. Then apply the Morse Lemma above to a, which gives a formula for f in a neighborhood of a. But there are no critical points in this neighborhood as can be seen directly by calculating df in these coordinates. This is a contradiction. □

Exercise.

Prove the converse of Exercise 12.2; that is, if M is a compact manifold and $f : M \longrightarrow \mathbb{R}$ is a Morse function, and if a is not a regular value of f, then $f^{-1}(\{a\})$ is not a manifold.

Definition 12.3. *Let $f : M \longrightarrow [a, b]$ be a Morse function on a compact manifold. We say that f is* admissible *if $\partial M = f^{-1}(a) \cup f^{-1}(b)$, where a and b are regular values. This implies that each of $f^{-1}(a)$ and $f^{-1}(b)$ are unions of connected components of ∂M.*

Theorem 12.6. *Let $f : M \longrightarrow \mathbb{R}$ be an admissible Morse function on a compact manifold. Suppose f has a unique critical point z of index λ. Say $f(z) = c$. Then there exists a λ - dimensional cell D^λ in the interior of M with $D^\lambda \cap f^{-1}(c) = \partial D^\lambda$, and there is a deformation retraction of M onto $f^{-1}(c) \cup D^\lambda$.*

Proof, following [72]. By replacing f by $f(x) - c$ we can assume that $f(z) = 0$. Notice that by the regular interval theorem Theorem 12.2 it is sufficient to prove the theorem for the restriction of f to the inverse image of any closed subinterval of $[a, b]$ around $c = 0$.

Let (ϕ, U) be an chart around z with respect to which the Morse lemma is satisfied. Write $\mathbb{R}^n = \mathbb{R}^\Lambda \times \mathbb{R}^{n-\Lambda}$. ϕ maps U diffeomorphically onto an open set $V \subset \mathbb{R}^\Lambda \times \mathbb{R}^{n-\Lambda}$, and

$$f \circ \phi^{-1}(x, y) = -|x|^2 + |y|^2.$$

Notice that $\phi(z) = (0,0)$. Put $g(x,y) = -|x|^2 + |y|^2$.

We will use gradient flows, which depend on the metric on M. We choose a metric for M by pulling back the Euclidean metric on $\mathbb{R}^n$ by ϕ, and extending the metric arbitrarily to the rest of M. In this way, ϕ will be a local isometry, and

$$D\phi(u)(\nabla_u(f)) = \nabla_v(g),$$

for any $u \in U$ such that $\phi(u) = v \in V$.

Let $0 < \delta < 1$ be such that V contains $\Lambda = B^\Lambda(\delta) \times B^{n-\Lambda}(\delta)$ where

$$B^i(\delta) = \{x \in \mathbb{R}^i \mid \sum_{j=1}^{n} x_j^2 \leq \delta\}$$

is the closed coordinate ball around the origin of radius δ.

Let $\epsilon > 0$ be small enough that $\sqrt{4\epsilon} < \delta$, and let

$$c^\Lambda = B^\Lambda\left(\sqrt{\epsilon}\right) \times \{0\} \subset V$$

and we define

$$D^\Lambda = \phi^{-1}(c^\Lambda) \subset M.$$

A deformation of $f^{-1}[-\epsilon,\epsilon]$ to $f^{-1}(\epsilon) \cup D^\Lambda$ is made by patching together two deformations. First consider the set

$$\Lambda_1 = B^\Lambda\left(\sqrt{\epsilon}\right) \times B^{n-\Lambda}\left(\sqrt{2\epsilon}\right).$$

Consider the following figure for the case $\Lambda = 1$, $n = 2$.

Note that inside Λ_1, $f(x,y) = -|x|^2 + |y|^2 > -\epsilon + |y|^2 > -\epsilon$. Furthermore, since $x \in B^\Lambda\left(\sqrt{\epsilon}\right)$, we have that $(x,0) \in c^\Lambda$.

In $\Lambda_1 \cap g^{-1}[\epsilon,\epsilon]$ a deformation is obtained by moving (x,y) at constant speed along the interval joining (x,y) to the point $(x,0) \in g^{-1}(-\epsilon) \cup B^\Lambda$, by $(x,(1-t)y)$. This deformation then induces a deformation of $\phi^{-1}(\Lambda_1)$.

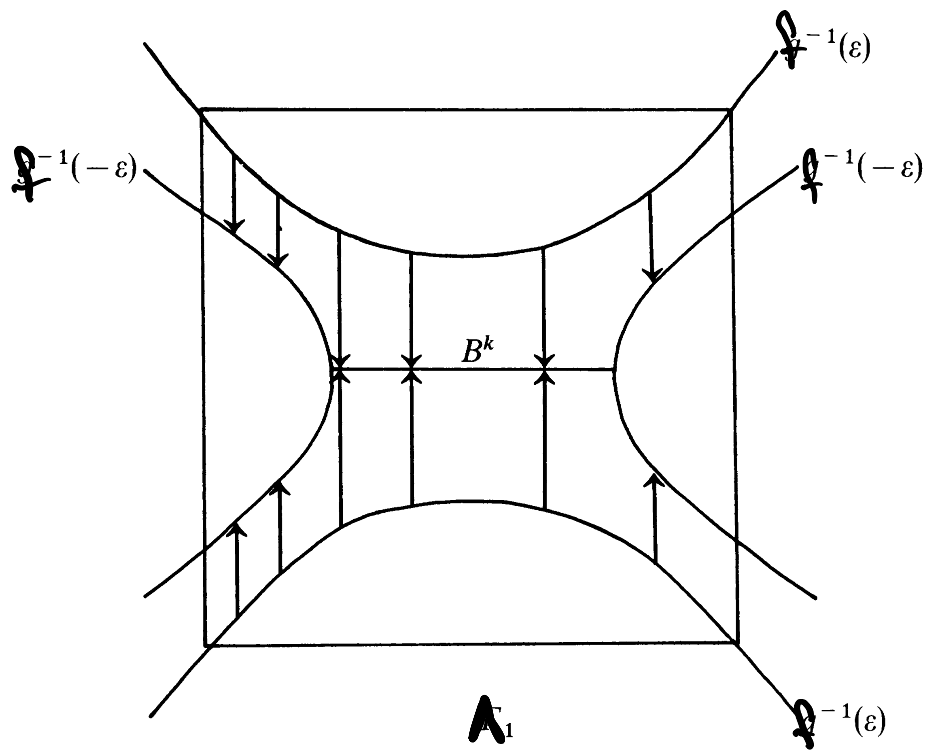
Outside the set

$$\Lambda_2 = B^\Lambda(\sqrt{2\epsilon}) \times B^{n-\Lambda}(\sqrt{3\epsilon})$$

the deformation moves each point along the vector field $-\nabla(g)$ so that it reaches $g^{-1}(-\epsilon)$ in unit time. (The speed of each point is chosen to equal the length of its path under the deformation.) See the following figure for a pictorial description of this deformation.

This deformation is transported to $U - \phi^{-1}(\Lambda_2)$ by ϕ, and is then extended over $M - \phi^{-1}(\Lambda_2)$ by following the gradient flow lines of f.

Now if such a flow enters V, we now show it may not enter Λ_2: Suppose we have a flow that enters V from the outside at time t. Then since the closure of Λ_2 is in V, there is a time arbitrarily close to t where the point is (x,y) which is not in Λ_2. Then at this time either $|x|^2 > 2\epsilon$ or $|y|^2 > 3\epsilon$. But if $|y|^2 > 3\epsilon$ then because for$g^{-1}([-\epsilon,\epsilon])$, we have $\epsilon > -|x|^2 + |y|^2 > -|x|^2 + 3\epsilon$ so that $|x|^2 > 2\epsilon$. Therefore, either way, $|x|^2 > 2\epsilon$. But for x non-zero, $|x|$ increases



along flow lines. Therefore (x, y) will not be in Λ_2 for any later time until it leaves V (and by repeating the argument for future visits to V, it never enters Λ_2).

In $f^{-1}([-\epsilon, \epsilon]) - \phi^{-1}(\Lambda_2)$, then, the downward gradient flow is defined, and since we assume there are no other critical points than z, the methods of the proof of Theorem 12.2 show that the flows defined there flow downward to $f^{-1}(-\epsilon)$.

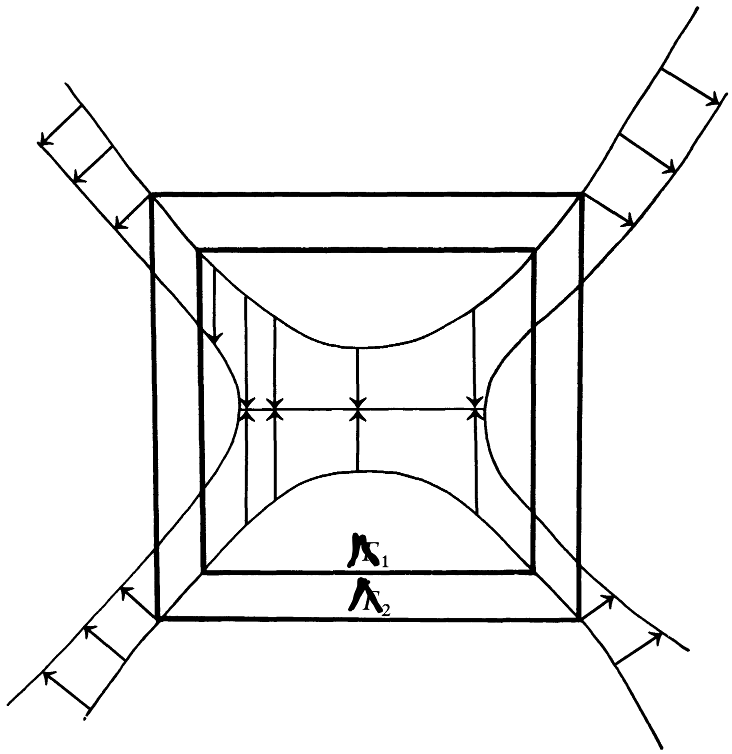
On $f^{-1}([-\epsilon, \epsilon]) - \phi^{-1}(\Lambda_2)$, then, we can define the deformation to flow along the gradient flow with constant speed, with speed equal to the length of the flow line from the point to its destination on $f^{-1}(-\epsilon)$. In this way, after unit time, everything in $f^{-1}([-\epsilon, \epsilon]) - \phi^{-1}(\Lambda_2)$ is deformed into $f^{-1}(-\epsilon)$.

To extend the deformation to points of $\Lambda_2 - \Lambda_1$ it suffices to find a vector field on Λ which agrees with X in Λ_1 and with $-\nabla(g)$ in $\Lambda - \Lambda_2$. Such a vector field is

$$Y(x, y) = 2(\mu(x, y)x, -y)$$

where the map $\mu : \mathbb{R}^\Lambda \times \mathbb{R}^{n-\Lambda} \longrightarrow [0, 1]$ vanishes in Λ_1 and equals 1 outside Λ_2. The fact that each integral curve of Y which starts at a point of

$$(\Lambda_2 - \Lambda_1) \cap g^{-1}[-\epsilon, \epsilon]$$



must reach $g^{-1}(-\epsilon)$ because $|x|$ is nondecreasing along integral curves.

The global deformation of $f^{-1}[-\epsilon, \epsilon]$ into $f^{-1}(-\epsilon) \cup D^\Lambda$ is obtained by moving each point of Λ at constant speed along the flow line of Y until it reaches $g^{-1}(-\epsilon) \cup B^\Lambda$ in unit time and transporting this motion to M via ϕ; while each point of $M - \phi^{-1}(\Lambda)$ moves at constant speed along the flow line of $\nabla(f)$ until it reaches $f^{-1}(-\epsilon)$ in unit time. Points on $f^{-1}(-\epsilon) \cup D^\Lambda$ stay fixed. □

12.5 Homotopy equivalence to a *CW* complex and the Morse inequalities

Theorem 12.7. *Let M be a closed manifold, and $f : M \longrightarrow \mathbb{R}$ a Morse function on M. Then M has the homotopy type of a CW complex, with one cell of dimension Λ for each critical point of index Λ.*

Proof. Without loss of generality, the critical points of f all have different values under f (if $f(p) = f(q)$ and p and q are critical points, then let $B_1 \subset B_2$

be balls around q small enough that in $B_2 - B_1$, we have $|\nabla f|$ bounded away from zero by some ϵ, and add a small bump function to f supported in B_2 and constant in B_1 whose gradient is bounded above by ϵ, and which does not raise the value of $f(q)$ high enough to reach another critical value of f).

Now let $a_0 < \cdots < a_k$ be a sequence of real numbers so that a_0 is less than the minimum value of f, a_k is greater than the maximum value of f, and between a_i and a_{i+1} there is exactly one critical point. By Theorem 12.6 we have a homotopy equivalence h_i between $M^{a_{i+1}}$ and $M^{a_i} \cup D^{\lambda_i}$ (where the union is via an attaching map as in a CW complex). By composing the h_i's, we obtain a homotopy equivalence from $M = M^{a_k}$ to a union of disks attached by CW attaching maps.

□

Corollary 12.8. *Given $f : M \longrightarrow \mathbb{R}$ as above there is a chain complex referred to as the* Morse–Smale complex

$$\ldots \longrightarrow C_\lambda \xrightarrow{\partial_\Lambda} C_{\lambda-1} \longrightarrow \ldots \xrightarrow{\partial_1} C_0 \tag{12.2}$$

whose homology is $H_*(M;\mathbb{Z})$, where C_λ is the free abelian group generated by the critical points of f of index Λ.

Proof. This is the cellular chain complex coming from the CW complex in Theorem 12.7. □

We can now prove some of the results promised in the introduction, that relate the topology of M to the numbers of critical points of f:

Corollary 12.9 (Morse's Theorem)**.** *Let $f : M \longrightarrow \mathbb{R}$ be a C^∞ function so that all of its critical points are nondegenerate. Then the Euler characteristic $\chi(M)$ can be computed by the following formula:*

$$\chi(M) = \sum (-1)^i c_i(f)$$

where $c_i(f)$ is the number of critical points of f having index i.

Proof. The Euler characteristic $\chi(M)$ can be computed as the alternating sum of the ranks of the chain groups of any CW decomposition of M. □

Corollary 12.10 (Weak Morse Inequalities)**.** *Let c_p be the number of critical points of index p and let β_p be the rank of the homology group $H_p(M)$. Then*

$$\beta_p \leq c_p.$$

Proof. The chain group $C_p \otimes \mathbb{R}$ generated by the c_p cells of dimension p is a vector space of dimension c_p. The group of cycles is of dimension at most c_p. After quotienting by the boundaries, we see that $H_p(M;\mathbb{R})$ is a vector space of dimension at most c_p. □

Corollary 12.11 (Strong Morse Inequalities)**.** *Let M, f, $c_i(f)$, and $b_i(M)$ be as above. Then for all natural numbers i,*

$$\sum_{k=0}^{i}(-1)^{i-k}c_i(f) \geq \sum_{k=0}^{i}(-1)^{i-k}b_i(M).$$

Proof. The proof is similar except we take a closer look at the boundaries. Tensoring the chains with $\mathbb{R}$, so that we write $V_k = C_k \otimes \mathbb{R}$, we get the following chain complex of vector spaces:

$$\ldots \longrightarrow V_i \xrightarrow{\partial_i} V_{i-1} \longrightarrow \ldots \xrightarrow{\partial_1} V_0$$

We write V_k as $Im(\partial_{k+1}) \oplus H_k(M;\mathbb{R}) \oplus (V_k/\ker(\partial_k))$ and note that $Im(\partial_{k+1})$ is of the same dimension as $V_{k+1}/\ker(\partial_{k+1})$. Thus if we define d_k to be the dimension of $V_k/\ker(\partial_k)$, we have

$$c_k = d_{k+1} + b_k + d_k$$

and applying the alternating sum above we get

$$\sum_{k=0}^{i}(-1)^{i-k}c_i(f) = d_{i+1} + \sum_{k=0}^{i}(-1)^{i-k}b_i(M)$$

(where here we need that $d_0 = 0$). This proves the strong Morse inequalities. □

To see that the strong Morse inequalities prove the weak Morse inequalities, write down the strong Morse inequality for i and for $i+1$, and subtract the two inequalities. To see that the strong Morse inequalities imply Morse's theorem, apply the strong Morse inequality for i and for $i+1$ for i larger than the dimension of the manifold M, noting that $c_j = 0$ and $b_j = 0$ for all $j > \dim(M)$.

A typical application of these result is to use homology calculations to deduce critical point data. For example we have the following.

Application

Every Morse function on the complex projective space

$$f : \mathbb{CP}^n \longrightarrow \mathbb{R}$$

has at least one critical point in every even dimension $\leq 2n$.

The following is a historically important application of Morse theory, due to Reeb, that follows from the techniques we have mentioned so far.

Application

Let M^n be a closed manifold admitting a Morse function

$$f : M \longrightarrow \mathbb{R}$$

with only two critical points. Then M is homeomorphic to the sphere S^n.

Remark This theorem does *not* imply that M is diffeomorphic to S^n. In [116] Milnor found an example of a manifold that is homeomorphic, but not diffeomorphic to S^7. Indeed he proved that there are 28 distinct differentiable structures on S^7! Milnor actually used this fact to prove that the manifolds he constructed were homeomorphic to S^7.

Proof of Theorem 12.5. Let S and N be the critical points. By the compactness of M we may assume that S is a minimum and N is a maximum. (Think of them as the eventual south and north poles of the sphere.) Let $f(S) = t_0$ and $f(N) = t_1$. By the Morse lemma there are coordinates $(x_1, \ldots, x_n)$ in a neighborhood U_+ of N with respect to which f has the form

$$-x_1^2 + \cdots + -x_n^2 + t_1.$$

Therefore there is a $b < t_1$ so that if we let $D_+ = f^{-1}[b, t_1]$ then there is a diffeomorphism

$$D_+ \cong D^n$$

with $\partial D_+ = f^{-1}(b) \cong S^{n-1}$. Repeating this process with the minimum point P we obtain a point $a > t_0$ and a diffeomorphism of the space $D_- = f^{-1}[t_1, a]$,

$$D_- \cong D^n$$

with $\partial D_- = f^{-1}(a) \cong S^{n-1}$. By Theorem 12.2 we have that

$$f^{-1}[a, b] \cong f^{-1}(a) \times [a, b] \cong S^{n-1} \times [a, b].$$

Hence we have a decomposition of the manifold

$$\begin{aligned} M = f^{-1}[t_0, t_1] &= f^{-1}[t_0, a] \cup f^{-1}[a, b] \cup f^{-1}[b, t_1] \\ &\cong D^n \cup S^{n-1} \times [a, b] \cup D^n \end{aligned}$$

where the attaching maps are along homeomorphisms of S^{n-1}. We leave it as an exercise to now construct a homeomorphism from this manifold to S^n.

□

Exercise

Finish the proof of Theorem 12.5 by showing that the resulting space

$$D^n \cup S^{n-1} \times [a, b] \cup D^n$$

is homeomorphic to S^n. Hint: Start by embedding one D^n into S^n, then embed

$S^{n-1} \times [a, b]$ into S^n to match the first embedding, then to put the last D^n in, you must think of D^n as the cone on S^{n-1}. This last part is why the proof does not prove that this is diffeomorphic to S^n.

In general, there are many applications of this work to the problem of classifying manifolds of dimensions 5 and higher, leading to the h-cobordism theorem and the s-cobordism theorem, and surgery theory. There are many books that describe these developments of the 1960s and 1970s, the old classics being Milnor's book on the h-cobordism theorem, [113], Wall's book on surgery theory [154], and Browder's book [18].

We now show that the set of Morse functions is open and dense in the set of smooth functions. In particular, every manifold M admits a Morse function $f : M \longrightarrow \mathbb{R}$. In the proof, we will use the transversality theorem, done in Chapter 8.

Theorem 12.12. *Let M be a compact n-manifold. Let $r \geq 2$. The set of C^r Morse functions from M to $\mathbb{R}$ is dense in $C^r(M, \mathbb{R})$.*

Proof. . We refer the reader to [72] for a complete proof. However we describe the proof of a related fact that is a key component of the proof of this theorem. Consider the exterior derivative map

$$d : C^\infty(M; \mathbb{R}) \to \Omega^1(M) = \Gamma_M(T^*M)$$

where $\Gamma_M(T^*M)$ denotes the space of smooth sections of the cotangent bundle.

Let $\zeta \subset T^*M$ be the zero section of T^*M. Inside $\Gamma_M(T^*M)$ we have the space of sections that are transverse to the zero section, which we denote by $\pitchfork (M, T^*M; \zeta) \subset \Gamma_M(T^*M)$. We observe that the space of Morse functions is simply the inverse image under d of $\pitchfork (M, T^*M; \zeta)$. Furthermore, by the transversality theorem (Corollary 8.9), we can conclude that $\pitchfork (M, T^*M; \zeta) \subset \Gamma_M(T^*M)$ a dense subspace.

To see this characterization of Morse functions, observe that df being transverse to the zero section means that whenever $df_p = 0$ (i.e p is a critical point), then $D(df)_p(T_pM) \oplus T_p\zeta(M)(= T_pM) = T_{df_p}T^*M$. But one can easily check that this condition is equivalent to Hess f_p being nonsingular. $\square$

Exercises

(1) Let $M \subset \mathbb{R}^L$ be a closed, smooth submanifold. For each $v \in S^{L-1}$ let $f_v : M \to \mathbb{R}$ be the map $f_v(x) =< v, x >$. (This is essentially orthogonal projection into the line through v.) Show that the set of $v \in S^{L-1}$ such that f_v is a Morse function is open and dense.

(2) Let $M \subset \mathbb{R}^L$ be a closed, smooth submanifold. Show that the set of points $u \in \mathbb{R}^L$ such that the map $x \to |x - u|^2$ is a Morse function on M, is open and dense.

Remark. The functions described in exercise (1) are called "height functions". The functions described in exercise (2) are "distance functions". These are both very common and highly useful examples of Morse functions.

(3). Recall that $\mathbb{RP}^n = \{(x_1, \cdots x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}\}/\sim$ where $(x_1, \cdots x_{n+1}) \sim -(x_1, \cdots x_{n+1})$. We denote an equivalence class using square brackets $[x_1, \cdots x_{n+1}] \in \mathbb{RP}^n$.

Define a smooth function

$$f : \mathbb{RP}^n \to \mathbb{R}$$

by

$$f([x_1, \cdots x_{n+1}]) = \sum_{k=1}^{n+1} k x_k^2$$

(a) Show that the critical points of f are $u_1, \cdots u_{n+1}$, where $u_i = [0, \cdots 0, 1, 0, \cdots 0]$, where the 1 occurs in the i^{th} coordinate.

(**Hint.** First construct charts U_i, $i = 1, \cdots, n+1$, where $U_i = \{[x_1, \cdots x_{n+1}] : x_i \neq 0\}$, by proving that there are diffeomorphisms $\psi_i : U_i \cong B_1^n$, where B_i^n the unit open ball around the origin in $\mathbb{R}^n$. ψ_i given by

$$\psi_i[x_1, \cdots x_{n+1}] = (x_1, \cdots x_{i-1}, x_{i+1}, \cdots, x_n).$$

Then compute the differential of the composition

$$B_i^n \xrightarrow{\psi_i^{-1}} U_i \subset \mathbb{RP}^n \xrightarrow{f} \mathbb{R}.$$

Use this to show that the only critical point of f in U_i is u_i.

(b). Compute the index of each critical point.

(c). Show that $f : \mathbb{RP}^n \to \mathbb{R}$ is a Morse function.

(d). Using parts (a) - (c) to show that the Euler characteristic of $\mathbb{RP}^n$ is 0 if n is odd and 1 if n is even.

(e) Prove that if n is even, $\mathbb{RP}^n$ does not admit a nowhere zero vector field.

13

Spaces of Gradient Flows

13.1 The gradient flow equation

Let M be a manifold, g a Riemannian metric on M, and $f : M \longrightarrow \mathbb{R}$ be a Morse function. A *(gradient) flow line* is a curve

$$\gamma : (a, b) \longrightarrow M$$

that satisfies the differential equation

$$\frac{d\gamma}{dt}(s) + \nabla_{\gamma(s)}(f) = 0 \tag{13.1}$$

for all $s \in (a, b)$. Here $\nabla(f)$ is the gradient vector field as defined in (12.1). If we imagine a particle that travels along γ, with t describing time, the particle travels in the path of steepest descent, with velocity given by the gradient.

Recall from the discussion in the previous chapter, that the gradient vector field $\nabla(f)$, and therefore the gradient flow equation depends on the Riemannian metric g in the following way:

$$< \nabla_x f \, , v >_g = df(x)(v)$$

where $< \, , >_g : T_xM \times T_xM \to \mathbb{R}$ is the nonsingular, symmetric bilinear form on the tangent space at $x \in M$ defined by the metric g. The typical gradient seen in undergraduate calculus classes occurs on $\mathbb{R}^n$ with the standard Euclidean metric.

Exercises

(1). Verify that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable function on $\mathbb{R}^n$, and if we use the Euclidean metric on $\mathbb{R}^n$, then

$$\nabla(f) = \frac{\partial f}{\partial x_1} e_1 + \cdots + \frac{\partial f}{\partial x_n} e_n.$$

(2). Let f be as in the previous exercise, but suppose the metric is given by an arbitrary symmetric matrix g (that is, $< e_i, \, e_j >_g = g_{ij}$).

Find the formula for $\nabla(f)$ in terms of f and g.

Remark. Notice that the property of $p \in M$ being a critical point of f does not depend on the metric. As a bilinear form, the Hessian of f at a critical point $p \in M$ does not depend on the metric either. Therefore the concepts of p being a non-degenerate critical point, and the index of a critical point do not depend on a choice of metric.

Example. If a is a critical point of f, then the constant curve $\gamma(t) = a$ satisfies the flow equations, so γ is a flow line. Conversely, by the uniqueness of solutions of ordinary differential equations, if any flow line contains a critical point $a \in M$, then it must be the constant curve at a.

Example Let $M = \mathbb{R}^2$ with the Euclidean metric, and let $f(x, y) = x^2 + y^2$. Then we can solve the gradient flow equations:

$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = -2y$$

and therefore the gradient flow lines are $(x(t), y(t)) = (ae^{-2t}, be^{-2t})$ for some fixed a and b. For any such flow line, y/x is a constant, so each flow parameterizes an open line in the plane emanating from the origin. See figure 13.1.

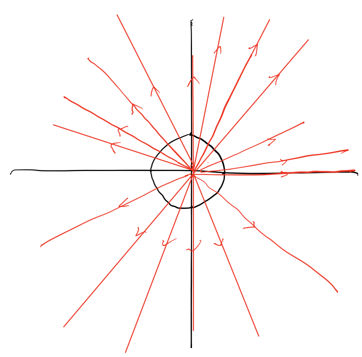


FIGURE 13.1
Flow lines for $f(x, y) = x^2 + y^2$

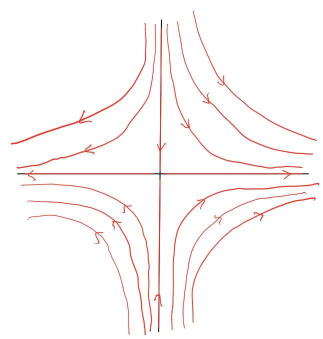


FIGURE 13.2
Flow lines for $f(x, y) = x^2 - y^2$

Example Let $M = \mathbb{R}^2$ with the Euclidean metric, and let $f(x, y) = x^2 - y^2$. Then it turns out that the gradient flow lines are $(x, y) = (ae^{2t}, be^{-2t})$ for some fixed a and b. For any such flow line, xy is a constant, so the gradient flow lines are hyperbolas of the form $xy = c$. See figure 13.1.

Example Let $M = S^2 \subset \mathbb{R}^3$ with the standard round metric, and let $f(x, y, z) = z$ (the so-called "height function" defined by the embedding of S^2 into $\mathbb{R}^3$). Then there are two critical points: one minimum at $(0, 0, -1)$, and one maximum at $(0, 0, 1)$. The flow lines are "lines of longitude". See figure 13.1.

Example Let T^2 be the torus in $\mathbb{R}^3$, embedded as follows:

$$(\theta, \phi) \longrightarrow (b\cos(\phi), (a + b\sin(\phi))\cos(\theta), (a + b\sin(\phi))\sin(\theta)$$

where $0 < b < a$. The picture looks like a donut standing on its edge, as in figure 13.4. Again, take for f the "height function" z. Then there are four critical points: $(\theta, \phi) = (\pm\pi/2, \pm\pi/2)$, as you can check. The index for $(\pi/2, \pi/2)$ is 2, the index for $(\pi/2, -\pi/2)$ and $(-\pi/2, \pi/2)$ is 1, and the index for $(-\pi/2, -\pi/2)$ is 0.

There are two natural choices for a metric on T^2: either the metric induced from the embedding from $\mathbb{R}^3$, or the flat metric defined by $ds^2 = d\theta^2 + d\phi^2$. Although pictorially it may help to ponder the resulting gradient flow lines

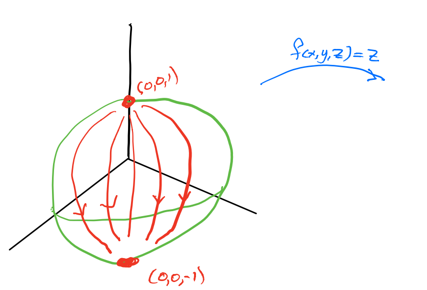


FIGURE 13.3
Flow lines for the height function on S^2

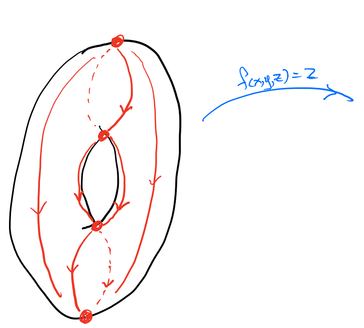


FIGURE 13.4
Flow lines for the height function on the torus

from the metric induced by $\mathbb{R}^3$ (these are the actual flows of steepest descent on a physical donut), it is easier to calculate the flow lines when the flat metric is used. The flow lines can be described explicitly, or else you can verify that there are flows with $\theta = \pm\pi/2$ for which θ is constant, and flows with $\phi = \pm\pi/2$ for which ϕ is constant. These flows give rise to two flows from the index 2 critical point to one of the index 1 critical points, two flows from one index 1 critical point to the other, and two flows from the lower index 1 critical point to the index 0 critical point. The other flows are in a one-parameter family of flows which go from the index 2 critical point to the index 0 critical point.

Exercise Work out the details of the above examples. Find the closed form solutions to the gradient flow equations and find which critical points they connect to.

Lemma 13.1. *A smooth function $f : M \longrightarrow \mathbb{R}$ is nonincreasing along flow lines. f is strictly decreasing along any flow line which does not contain a critical point.*

Proof. Let $\gamma : (a,b) \longrightarrow M$ be a flow line. Consider the composition $f \circ \gamma : (a,b) \longrightarrow \mathbb{R}$. Its derivative is given by

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= \langle \nabla_{\gamma(t)}(f), \frac{d\gamma(t)}{dt} \rangle \\ &= \langle \nabla_{\gamma(t)}(f), -\nabla_{\gamma(t)}(f) \rangle \\ &= -\left|\nabla_{\gamma(t)}(f)\right|^2 \leq 0. \end{aligned}$$

The only way this can be zero is if $\gamma(t)$ is on a critical point of f. In particular, if $\gamma(t)$ does not contain in its image a critical point of f, then $f(\gamma(t))$ is strictly decreasing.

□

Remark In the above proof, we showed

$$\frac{d}{dt} f(\gamma(t)) = -\left|\nabla_{\gamma(t)}(f)\right|^2.$$

We can also show

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= \langle \nabla_{\gamma(t)}(f), \frac{d\gamma(t)}{dt} \rangle \\ &= \langle -\frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt} \rangle \\ &= -\left|\frac{d\gamma(t)}{dt}\right|^2 \leq 0. \end{aligned}$$

and this would also prove that $f(\gamma(t))$ is nonincreasing.

Remark Now if $\gamma(t)$ *does* contain a critical point p, then by Example 13.1 the flow must be a constant flow, and $f(\gamma(t))$ is constant on this flow.

Thus there are two kinds of flow lines: constant flows that stay at a critical point, and flows that descend for all t, and do not contain a critical point.

Theorem 13.2. *Suppose that M is a closed, smooth manifold, and $f : M \to \mathbb{R}$ a smooth map. Then given any $x \in M$ there is a unique flow line defined on entire real line*

$$\gamma_x : \mathbb{R} \longrightarrow M$$

that satisfies the initial condition

$$\gamma_x(0) = x.$$

Furthermore the limits

$$\lim_{t \to -\infty} \gamma_x(t) \quad \text{and} \quad \lim_{t \to +\infty} \gamma_x(t)$$

converge to critical points of f. These are referred to as the starting *and* ending points *of the flow γ_x.*

The flow map

$$T : M \times \mathbb{R} \longrightarrow M$$

defined by $T(x,t) = \gamma_x(t)$ is smooth.

Proof. Let $x \in M$. By the existence and uniqueness of solutions to ordinary differential equations, there is an $\epsilon > 0$ and a unique path

$$\gamma_x : (-\epsilon, \epsilon) \longrightarrow M$$

satisfying the flow equation

$$\frac{d\gamma_x(t)}{dt} + \nabla_{\gamma_x(t)}(f) = 0$$

for all $|t| < \epsilon$, and the initial condition $\gamma_x(0) = x$. By the compactness of M we can choose a uniform ϵ for all $x \in M$. Notice therefore that for $|t| < \epsilon$ we can define a self map of M,

$$\gamma_t : M \longrightarrow M$$

by the formula $\gamma_t(x) = \gamma_x(t)$. Notice that $\gamma_0 = id$, the identity map. By uniqueness it is clear that

$$\gamma_{t+s} = \gamma_t \circ \gamma_s$$

providing that $|t|$, $|s|$, $|t+s| < \epsilon$. Among other things this implies that each γ_t is a *diffeomorphism* of M because $\gamma_t^{-1} = \gamma_{-t}$.

Now suppose that $|t| \geq \epsilon$. Write $t = k(\epsilon/2) + r$ where $k \in \mathbb{Z}$ and $|r| < \epsilon/2$. If $k \geq 0$ we define

$$\gamma_t = \gamma_{\frac{\epsilon}{2}} \circ \gamma_{\frac{\epsilon}{2}} \circ \dots \gamma_{\frac{\epsilon}{2}} \circ \gamma_r$$

where the map $\gamma_{\frac{\epsilon}{2}}$ is repeated k times. If $k < 0$ then replace $\gamma_{\frac{\epsilon}{2}}$ by $\gamma_{\frac{-\epsilon}{2}}$. Thus for every $t \in \mathbb{R}$ we have a map $\gamma_t : M \longrightarrow M$ satisfying $\gamma_t \circ \gamma_s = \gamma_{t+s}$, and hence each γ_t is a diffeomorphism.

The curves

$$\gamma_x : \mathbb{R} \longrightarrow M$$

defined by $\gamma_x(t) = \gamma_t(x)$ clearly satisfy the flow equations and the initial condition $\gamma_x(0) = x$. This means that the gradient flow equations can be solved for all $t \in \mathbb{R}$, and in particular, we will from now on require that gradient flow lines be defined as functions $\gamma : \mathbb{R} \longrightarrow M$ instead of being defined only on an open interval.

Now let γ be a flow line. Consider the composition $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$. By the Fundamental Theorem of Calculus, if $a < b$, then

$$(f \circ \gamma)(b) - (f \circ \gamma)(a) = \int_a^b \frac{d}{dt}(f \circ \gamma)(t)\, dt.$$

Since M is compact $f \circ \gamma$ has bounded image, so the left side is bounded. By Lemma 13.1, $\frac{d}{dt}(f \circ \gamma) < 0$. Therefore

$$\lim_{t \to \pm\infty} \frac{d}{dt}(f \circ \gamma)(t) = 0.$$

By the proof of Lemma 13.1 we know that

$$0 = \lim_{t \to \pm\infty} \frac{d}{dt} f(\gamma(t)) = \lim_{t \to \pm\infty} -\left|\nabla_{\gamma(t)}(f)\right|^2.$$

Let U be any union of small disjoint open balls around the critical points. By the compactness of M, $M - U$ is compact, so $|\nabla_x(f)|^2$ has a minimum value on $M - U$. Since $M - U$ has no critical points, this minimum value is strictly positive. But since the above limit is zero, we know that for sufficiently large $|t|$, $\gamma(t) \in U$. Since the balls are disjoint and $\gamma(t)$ is continuous, there is a critical point p so that for any open ball around p, $\gamma(t)$ is in that ball for sufficiently large t. Therefore $\lim_{t\to\infty} \gamma(t)$ exists and is equal to p; similarly, $\lim_{t\to-\infty} \gamma(t)$ exists and is equal to a critical point.

The differentiability of the flow map $T(x,t) = \gamma_x(t)$ with respect to t follows because $\gamma_x(t)$ satisfies the differential equation. The differentiability of T with respect to x follows from Peano's theorem (the differentiable dependence of solutions to ODEs with respect to initial conditions). This is proved in Hartman's book on ODEs [66] in chapter V, Theorem 3.1. □

Let $\gamma(t)$ be a non-constant gradient flow line from p to q. Then by Lemma 13.1, we know that $h(t) = f(\gamma(t))$ is strictly decreasing, and in particular, is a diffeomorphism from $\mathbb{R}$ to the open interval $(f(q), f(p))$. We can therefore consider the smooth curve $\eta(t) = \gamma(h^{-1}(t))$ from $(f(q), f(p))$ to M. Then it is easy to check that $f(\eta(t)) = t$. So γ and η have the same image, but the parameter in η represents height (that is, the value of f).

Exercise Prove that $f(\eta(t)) = t$ as claimed above.

We can also extend η to a continuous map from the closed interval $[f(q), f(p)]$ to M by defining $\eta(f(q)) = q$ and $\eta(f(p)) = p$.

Exercise Prove that the extension of η to the closed interval $[f(q), f(p)]$ is continuous.

Definition 13.1. *If $\gamma(t)$ is a non-constant gradient flow line for f, and $h(t) = f(\gamma(t))$, then*

$$\eta(t) = \gamma(h^{-1}(t)) : [f(q), f(p)] \longrightarrow \mathbb{R}$$

is the height-reparameterization *of γ, and such a curve is a* height-parameterized gradient flow *of f.*

Remark This reparameterization of γ is a direction-reversing one, since h is strictly decreasing. This is to be expected since $f(\gamma(t))$ is decreasing but $f(\eta(t)) = t$ is increasing.

We now differentiate η.

Exercise Prove

$$\frac{d}{dt}\eta(t) = \frac{\nabla_{\eta(t)}(f)}{\left|\nabla_{\eta(t)}(f)\right|^2}$$

Therefore, $\eta(t)$ is the solution to another differential equation which may be described as follows:

Lemma 13.3. *Away from the critical points of f, we may consider the vector field*

$$X(x) = \frac{\nabla_x(f)}{|\nabla_x(f)|^2}.$$

Then a curve $\zeta : (s_1, s_2) \longrightarrow M$ that satisfies

$$\frac{d}{dt}\zeta(t) = X(\zeta(t))$$

is a height-reparameterized flow line.

Proof. We insist that (s_1, s_2) be maximal. We then can show that $\frac{d}{dt}f(\zeta(t)) = 1$ as usual (do this now if you wish). Pick a number $s \in (s_1, s_2)$, and consider the gradient flow line $\gamma(t)$ so that $\gamma(0) = \zeta(s)$. We do the height-reparameterization to γ to get a height-reparameterized curve η. Now η satisfies the same differential equation as ζ, and $\eta(f(\zeta(s))) = \zeta(s)$, so we translate the domain as follows: $\eta_0(t) = \eta(t + f(\zeta(s)) - s)$ satisfies the same differential equation as ζ and $\eta_0(s) = \zeta(s)$ so by the uniqueness of solutions to ODEs, $\eta_0 = \zeta$.

Therefore solutions to $\frac{d}{dt}\zeta(t) = X(\zeta(t))$ are precisely those that are height-parameterized flows. □

Therefore $X(x)$ and $\nabla(f(x))$ have the same integral curves, although with different parameterizations.

13.2 Stable and unstable manifolds

As before, for any point $x \in M$, let $\gamma_x(t)$ be the flow line through x, i.e. it satisfies the differential equation

$$\frac{d}{dt}\gamma = -\nabla_\gamma(f)$$

with the initial condition $\gamma(0) = x$. We know by Theorem 13.2 that $\gamma_x(t)$ tends to critical points of f as $t \to \pm\infty$. So for any critical point a of f we define the *stable manifold* $W^s(a)$ and the *unstable manifold* $W^u(a)$ as follows:

Definition 13.2. *Let M be a smooth manifold, and f a smooth function on M. Let a be a critical point for f. We define the two subsets of M:*

$$W^s(a) = \{x \in M : \lim_{t \to +\infty} \gamma_x(t) = a\}$$
$$W^u(a) = \{x \in M : \lim_{t \to -\infty} \gamma_x(t) = a\}.$$

and call $W^s(a)$ the stable manifold *of a and $W^u(a)$ the* unstable manifold *of a.*

In other words, $W^s(a)$ is the set of points on M that flow down to a, and $W^u(a)$ is the set of points on M flow out from a. The use of the term "manifold" is justified by the stable manifold theorem:

Theorem 13.4 (Stable Manifold Theorem)**.** *Let M be an n-dimensional smooth manifold, and $f : M \longrightarrow \mathbb{R}$ a Morse function. Let a be a critical point of f of index λ. Then $W^u(a)$ and $W^s(a)$ are smooth submanifolds diffeomorphic to the open disks D^λ and $D^{n-\lambda}$, respectively.*

This will be proved in Section **??** below for a large class of metrics (though it is in general true for all metrics).

Proposition 13.5. *If M is a closed smooth manifold with Riemannian metric g, and $f : M \longrightarrow \mathbb{R}$ is a Morse function, then*

$$M = \bigcup_a W^u(a)$$

is a partition of M into disjoint sets, where the union is taken over all critical points a of f.

Proof. The fact that the union of the $W^u(a)$ is M comes from the fact that every point of M lies on a flow line γ, and we can always find $\lim_{t \to -\infty} \gamma(t)$.

The fact that the $W^u(a)$ and $W^u(b)$ are disjoint when $a \neq b$ is due to the fact that γ is unique. □

Exercise Find the unstable manifolds for each critical point in Example 13.1.

Exercise Find the unstable manifolds for each critical point in Example 13.4.

From these exercises you can see that this decomposition of M makes M look like a CW complex, with one cell of dimension λ for each critical point of index λ. The torus example is problematic because an edge gets attached to the middle of another edge, but consider the following fix:

Consider the torus in $\mathbb{R}^3$ as before, but with a slight perturbation. That is, tilt the torus by pulling it down so it is not quite vertical. Then consider

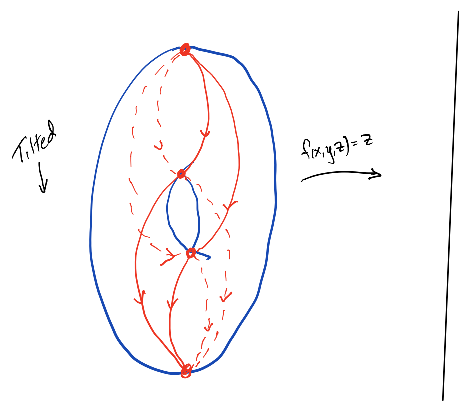


FIGURE 13.5
Flow lines for the height function on the "tilted torus"

the height function $f(x, y, z) = z$. Figure 13.5 is a picture of the resulting flow lines.

The point is that with this example, we have a decomposition of M into cells, with a cell of dimension λ for each critical point of index λ. These are essentially the cells D^λ in Theorem 12.6.

The disks appearing in this result and those appearing in Theorem 12.6 are related in the following way. Suppose that $[t_0, t_1] \subset \mathbb{R}$ has the property that $f^{-1}([t_0, t_1]) \subset M$ has precisely one critical point a of index λ with $f(a) = c \in (t_0, t_1)$. Then by Theorem 12.6 there is a disk $D^\lambda \subset M^{t_1}$ and a homotopy equivalence

$$M^{t_1} \simeq M^{t_0} \cup D^\lambda.$$

Now note that $W^u(a) \cap f^{-1}([t_0, t_1])$ is, under a Euclidean metric defined by the Morse coordinate chart, equal to the D^λ mentioned in the proof of Theorem 12.6.

In Chapter 12 we proved the Morse Lemma (Theorem 12.4), which says that locally, around any nondegenerate critical point, we can choose a coordinate chart so that

$$f(x_1, \ldots, x_n) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2. \tag{13.2}$$

In other words, we have a local explicit formula for f around a critical point, no matter what f is, as long as the critical point is non-degenerate.

What does the gradient vector field look like around such a critical point?

Based on the above equation (13.2), you might expect the gradient to be this:

$$\nabla(f) = (-2x_1, \ldots, -2x_\lambda, 2x_{\lambda+1}, \ldots, 2x_n) \tag{13.3}$$

But because the metric is not prescribed, it is possible (even likely) that the gradient vector field is *not* this at all. Recall that the gradient is obtained by $< v, \nabla(f)) >= df(v)$ and therefore depends on the metric (see the discussion in Chapter 12, and in particular where the gradient was defined 12.1).

Since we are dealing with gradient vector fields, and their corresponding flow lines, it would make sense for us to want to use a metric so that there are local coordinates where (13.3) is true. This is especially the case, since if equation (13.3) is true, then the gradient flow equation

$$\frac{d}{dt}\gamma(t) = -\nabla_{\gamma(t)}(f)$$

would take the form (if we write $\gamma(t) = (x_1(t), \ldots, x_n(t))$):

$$\begin{aligned}
\dot{x}_1 &= 2x_1 \\
&\vdots \\
\dot{x}_\lambda &= 2x_\lambda \\
\dot{x}_{\lambda+1} &= -2x_{\lambda+1} \\
&\vdots \\
\dot{x}_n &= -2x_n
\end{aligned}$$

which is easily solved.

If the metric is anything else, we might still hope to diagonalize this system of differential equations, choosing coordinates $(x_1, \ldots, x_n)$ so that $\dot{\gamma}(t) = -\nabla_{\gamma(t)}(f)$ looks like

$$\dot{x}_i = c_i x_i \tag{13.4}$$

for some non-zero real constants $c_1, \ldots, c_n$. Then the c_i would be negatives of the eigenvalues of the Hessian of f at the critical point, and the corresponding eigenvectors would be the standard basis vectors $\partial/\partial x_i$ in this coordinate chart.

Unfortunately, it is in general impossible to choose coordinates so that (13.4) holds, as the following exercises show:

Exercise Solve the system of differential equations (13.4).

Exercise Solve the system of differential equations

$$\dot{x} = 2x \tag{13.5}$$
$$\dot{y} = -y \tag{13.6}$$
$$\dot{z} = z + xy \tag{13.7}$$
$$\tag{13.8}$$

and show that there is no change of coordinates that transform it into the form (13.4).

Exercise Let $f(x, y, z) = x^2 - y^2 + z^2$. Find a metric $g(x, y, z)$ on a neighborhood of $(0, 0, 0) \in \mathbb{R}^3$ so that the gradient flow equations near the origin are as in equation (13.8). Hence prove that it is in general impossible to choose coordinates so that the gradient flow equations look like equation (13.4) in a neighborhood of the critical point. Note that the metric must be symmetric and positive definite in the neighborhood.

Note that in this exercise, what goes wrong is a kind of "resonance" phenomenon that occurs in ordinary differential equations when two eigenvalues are the same. By analogy, we would expect this kind of problem to be rare, and we might hope that for most situations, we can choose coordinates to put the gradient flow equations in the standard form of equation (13.4), but to address this will take us rather far afield (see [66]).

Instead, we choose to follow Hutchings [80] to modify the given metric, in neighborhoods of the critical points, to the standard metric so that equation (13.2) gives rise to the gradient flow equations in equation (13.4).

This motivates the following definition, due to Hutchings [80]:

Definition 13.3. *Let M be a manifold and f be a Morse function. A metric is said to be* nice *if there exist coordinate neighborhoods around each critical point of f so that for each such neighborhood there are non-zero real numbers $c_1, \dots, c_n$ so that the gradient flow equations are*

$$\dot{x}_i = c_i x_i,$$

as in (13.4).

Proposition 13.6. *Let M be a compact manifold and f a Morse function. There exists a nice metric on (M, f). In fact, these are dense in the L^2 space of metrics.*

Proof. Let g_0 be any smooth metric on M. Consider the set of critical points of f. Apply the Morse lemma (Lemma 12.4), to find nonoverlapping coordinate neighborhoods of each critical point of f in M, each with coordinates $x_1, \dots, x_n$ so that the Morse function in each neighborhood is

$$f(x_1, \dots, x_n) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2.$$

For each critical point a of f, let U_a be the coordinate neighborhood given by the Morse lemma, let B_1 be a coordinate ball around a that is completely inside U_a, and let B_2 be another coordinate ball around a of smaller radius than B_1. (By *coordinate ball* we mean the space whose coordinates $(x_1, \dots, x_n)$ satisfy $x_1^2 + \cdots + x_n^2 < r$ for some r.)

Let $\phi : U_a \longrightarrow \mathbb{R}$ be a smooth function so that ϕ is 1 on B_2 and 0 outside B_1. Let g_E be the standard Euclidean metric with respect to the $x_1, \dots, x_n$ coordinates. Define g to be

$$g = g_0(x)(1 - \phi(x)) + g_E(x)\phi(x).$$

Since the set of symmetric positive definite bilinear forms is a convex set, this convex linear combination of the two metrics will be a metric on U_a. Extend g by setting it equal to g_0 on the rest of M. Then g is a metric for which a is nice.

Now proceed inductively through the other critical points of M. This creates a metric g so that there is a coordinate neighborhood metric ball B around each critical point where both f and the metric are in a standard form. Then the gradient flow equation

$$\frac{d\gamma}{dt} = -\nabla_{\gamma}(f)$$

looks like equation (13.4).

By taking B_2 smaller and smaller, we see that the difference between g and g_0 is supported on an arbitrarily small set, and by the boundedness of the metric on M, we know that this difference is arbitrarily small in L^2. $\square$

We now prove the Stable manifold theorem for nice metrics:

Theorem 13.7 (Stable Manifold Theorem)**.** *Let M be an n-dimensional manifold, with nice metric g, and $f : M \longrightarrow \mathbb{R}$ a Morse function. Let a be a critical point of f of index λ. Then $W^u(a)$ and $W^s(a)$ are smooth submanifolds diffeomorphic to the open disks D^λ and $D^{n-\lambda}$, respectively.*

Remark This theorem is actually true for all metrics (not necessarily "nice"), but the proof of this is considerably more complicated, and for the purposes of this book we won't need the theorem need in this generality. We refer the interested reader to [66] for the proof of this theorem in its most general form.

Proof of Theorem 13.7. If g is a nice metric, then there is a coordinate neighborhood B around each critical point where the gradient flow equations are

$$\frac{d\gamma_i}{dt} = c_i \gamma_i(t)$$

where $\gamma_i(t)$ is the i-th coordinate of γ. Note that the c_i are the negatives of eigenvalues of the Hessian, corresponding to the directions given by the standard basis in the coordinate chart. Reorder the coordinates so that the first λ eigenvalues are the negative ones (so that the first λ values of c_i are positive).

Then explicitly,

$$\gamma_i(t) = \begin{cases} \gamma_i(0)e^{|c_i|t}, & i \leq \lambda \\ \gamma_i(0)e^{-|c_i|t}, & i > \lambda \end{cases} \tag{13.9}$$

inside B.

We prove the theorem for $W^s(a)$. The proof for $W^u(a)$ is exactly analogous, and besides, it follows from the $W^s(a)$ case, applied to the function $-f$. We will first prove that $W^s(a)$ is smooth in a small neighborhood of a.

Let W_0 be the subset of B consisting of those points where $x_1 = x_2 = \cdots = x_\lambda = 0$. Then from the explicit solution (13.9), we see that $W_0 \subset W^s(a)$.

Now W_0 is an open disk of dimension $n - \lambda$ centered on a, and hence is a manifold, and is furthermore a submanifold of M.

Recall from Theorem 13.2 that the flow map defined as

$$T : M \times \mathbb{R} \longrightarrow M$$

$$T(x,t) = \gamma_x(t)$$

is smooth. Apply this flow backward in time by some time t: define $W_t = T(W_0, -t)$. This will be diffeomorphic to W_0 and a subset of $W^s(a)$. As t goes to infinity, we span a larger and larger subset of $W^s(a)$.

Let $x \in W^s(a)$, and γ the corresponding gradient flow line with $\gamma(0) = x$. Since $\lim_{t\to\infty} \gamma(t) = a$, we know that for some $t_0 > 0$, $\gamma(t) \in B$ for all $t \geq t_0$. We will now show that $\gamma(t_0) \in W_0$.

Suppose $\gamma(t_0) \notin W_0$. The translated flow $\eta(t) = \gamma(t+t_0)$ is a gradient flow line, with the property that $\eta(0) \notin W_0$, and $\eta(t) \in B$ for all $t > 0$. Then for some coordinate $i > \lambda$, $\eta_i(0) \neq 0$. By the explicit solution (13.9), $\eta_i(t)$ will grow indefinitely, so that eventually η (and hence γ) leaves the coordinate ball B. This is a contradiction. Therefore, $\gamma(t_0) \in W_0$.

Since every element of $W^s(a)$, when flowed forward, eventually lies in W_0, we know that $\cup_t W_t = W^s(a)$.

Let $\psi : [0,1) \longrightarrow \mathbb{R}$ be a smooth monotonic function with $\psi(0) = 0$ and $\lim_{t\to 1} \psi(t) = +\infty$. Using $|x|$ as $\sqrt{x_1^2 + \cdots + x_n^2}$, and r_0 as the radius of the coordinate ball B, we see that $T(x, \psi(|x|/r_0))$ maps W_0 diffeomorphically onto $W^s(a)$. Recall that W_0 is a submanifold of M which is a disk of dimension $n - \lambda$. Therefore, $W^s(a)$ is a submanifold of M and diffeomorphic to $D^{n-\lambda}$. □

Proposition 13.8. *The tangent space of $W^s(a)$ at a is the positive eigenspace of the Hessian of f at a. Similarly, the tangent space of $W^u(a)$ at a is the negative eigenspace of the Hessian of f at a.*

Proof. Again, for the sake of our proof we are assuming the metric is nice, but this is unnecessary. The result holds in general.

Now $W^s(a)$ is a smooth submanifold of M, so its tangent space at a is well-defined. Define W_0 as in the previous proof, as

$$\{(x_1, \ldots, x_n) \mid x_1 = \cdots = x_\lambda = 0\}.$$

The tangent space to W_0 is therefore the span of $\partial/\partial x_i$ for $i = \lambda + 1$ to n. This is the positive eigenspace of the Hessian.

On the other hand $W_0 \subset W^s(a)$, and since they are of the same dimension, W_0 is an open neighborhood of a in $W^s(a)$. Therefore W_0 and $W^s(a)$ have the same tangent space at a.

The proof for $W^u(a)$ can be done similarly, or if you wish, you may use the result for $W^s(a)$ on $-f$. □

Let a be a critical point of f. Let us consider the function f restricted to $W^u(a)$. Since $W^u(a)$ is defined to be the set of points which in some sense lie "below" a on gradient flow lines, we expect a to be a maximum of f on $W^u(a)$, and level sets to be spheres around a.

Theorem 13.9. *Let (M, g) be a Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ a Morse function. Let a be a critical point of f. Let $h : W^u(a) \longrightarrow \mathbb{R}$ be the restriction of f to $W^u(a)$. Then a is the unique critical point of h, and it is the absolute maximum. If $\epsilon > 0$ is small enough, and $f(a) - \epsilon < c < f(a)$, then $h^{-1}(c)$ is diffeomorphic to a $\lambda - 1$ dimensional sphere in $W^u(a)$ around a.*

Similarly, let $j : W^s(a) \longrightarrow \mathbb{R}$ be the restriction of f to $W^s(a)$. Then a is the unique critical point of j, and it is the absolute minimum. If $\epsilon > 0$ is small enough, and $f(a) < c < f(a) + \epsilon$, then $j^{-1}(c)$ is diffeomorphic to a $n - \lambda - 1$ dimensional sphere in $W^s(a)$ around a.

Proof. We will prove this for $W^u(a)$, and the result for $W^s(a)$ is the same using $-f$ instead of f.

Let $x \in W^u(a)$, and $x \neq a$. Let $\gamma(t)$ be the unique gradient flow line with $\gamma(0) = x$. Since $x \in W^u(a)$, we have that $\lim_{t \to -\infty} \gamma(t) = a$.

According to Lemma 13.1, $f(\gamma(t))$ is strictly decreasing. By the continuity of f, $\lim_{t \to -\infty} f(\gamma(t)) = f(a)$. So $f(a) > f(x)$. Therefore, a is the absolute maximum of h.

Now, $\gamma(t) \in W^u(a)$ for all t, so $\gamma'(0) \in T_x W^u(a)$. Since $f(\gamma(t))$ is strictly decreasing, $\gamma'(0) \neq 0$ (if it were, $\frac{d}{dt} f(\gamma(t)) = \nabla(f) \cdot \gamma'(0)$ would be zero). By the gradient flow equation $\gamma'(t) = -\nabla_{\gamma(t)}(f)$, the $-\nabla_x(f) \neq 0$. Therefore, x is not a critical point of h. Since x was arbitrary, except for not equalling a, there are no critical points of h except for a.

Now we consider the Hessian of h at a. Find a coordinate chart of M around a so that $W^u(a)$ is given by the equations $x_{\lambda+1} = \cdots = x_n = 0$. By the invariance of the Hessian under coordinate change (Proposition 12.1), the Hessian of f can be computed in such a coordinate chart. Since $T_a W^u(a)$ is the negative eigenspace of the Hessian of f (Proposition 13.8) we conclude that the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{ij}$$

is negative definite. Since $W^u(a)$ is given by setting $x_{\lambda+1}, \ldots, x_n$ to be constant (in fact, zero), we see that for $i, j \leq \lambda$, this matrix is the same as

$$\left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)_{ij}.$$

Therefore the Hessian of h at a is negative definite. In particular, a is a non-degenerate critical point of h, and h is Morse.

We now consider the preimages $h^{-1}(c)$.

For this, we use the Morse Lemma (Theorem 12.4) applied to h on the manifold $W^u(a)$. The Morse Lemma states that there exist a coordinate neighborhood U around a with coordinates $x_1, \ldots, x_\lambda$ on $W^u(a)$ so that

$$h(x_1, \ldots, x_\lambda) = f(a) - x_1^2 - \cdots - x_\lambda^2.$$

Let $\epsilon > 0$ be given so that the ball

$$B = \{(x_1, \dots, x_\lambda) | x_1^2 + \cdots + x_\lambda^2 < \epsilon\}$$

is contained in U. Within this ball it is clear that the preimages $h^{-1}(c)$ (when $f(a) - \epsilon < c < f(a)$) are coordinate spheres around a. We will now verify that there are no other parts to $h^{-1}(c)$ which are outside B.

Suppose $x \in W^u(a)$, and $x \notin B$. As earlier in the proof, let $\gamma(t)$ be the gradient flow with $\gamma(0) = x$. As before, $\lim_{t \to -\infty} \gamma(t) = a$. But B is an open set around U. Therefore, for some $t < 0$, $\gamma(t) \in B$. Since $x = \gamma(0)$ is not in B, the generalized Jordan curve theorem says that there exists some $T < 0$ for which $\gamma(T)$ is on the boundary of B. Since $f(\gamma(t))$ is strictly decreasing,

$$f(x) = f(\gamma(0)) < f(\gamma(T)) = f(a) - \epsilon.$$

So $f(x) < f(a) - \epsilon$. Therefore, if $f(a) - \epsilon < c < f(a)$, then $h^{-1}(c)$ is a subset of B, and is therefore the coordinate spheres we found earlier. □

13.3 The Morse–Smale condition

An important, generic condition of a Morse function on a Riemannian manifold, is that the unstable and stable manifolds of the various critical points intersect transversally. This is called the *Morse-Smale transversality* condition, which we study in this subsection.

Definition 13.4. *Suppose $f : M \longrightarrow \mathbb{R}$ is a Morse function on a Riemannian manifold M, that satisfies the extra condition that for any two critical points a and b the unstable and stable manifolds $W^u(a)$ and $W^s(b)$ intersect transversally. This is the* Morse–Smale condition*, and if f satisfies this condition, we call f a* Morse–Smale function.

Smale [139] showed that Morse–Smale functions exist. More specifically, given a metric g and function $f : M \longrightarrow \mathbb{R}$, there exists another metric g' and another function $f' : M \longrightarrow \mathbb{R}$ so that f' is Morse–Smale with respect to g'. His proof also demonstrates that f and f' and g and g' can be made arbitrarily close to each other. Hence the set of configurations of functions and metrics so that the functions are Morse–Smale with respect to that metric is dense.

Actually, more is true: if f is Morse, then for an open, dense set of metrics g, f is Morse–Smale. This can be proved using the same techniques that are used in the proofs in Smale's paper. We will sketch out a proof at the end of this chapter that the set of such metrics is dense. In the meantime we will first study some properties of Morse–Smale functions.

Exercise Let $f : M \to \mathbb{R}$ be a Morse function that is not necessarily Morse–Smale. Suppose b is a critical point of f. Do $W^u(b)$ and $W^s(b)$ always intersect transversally? Why or why not?

The main purpose of the Morse–Smale condition is that it allows us to see how stable and unstable manifolds of different critical points intersect. For every pair of critical points a and b, let

$$W(a,b) = W^u(a) \cap W^s(b).$$

$W(a,b)$ is the space of all points in M that lie on flow lines starting from a and ending at b.

Proposition 13.10. *Let (M,g) be a Riemannian manifold of dimension n, let $f : M \longrightarrow \mathbb{R}$ be Morse–Smale, and a and b be two critical points of f. Then $W(a,b)$ is a smooth manifold of dimension $index(a) - index(b)$.*

Proof. If f is Morse–Smale, then $W^u(a)$ and $W^s(b)$ intersect transversally. Therefore the intersection $W^u(a) \cap W^s(b) = W(a,b)$ is a manifold of dimension $\dim(W^u(a)) + \dim(W^s(b)) - n = index(a) + (n - index(b)) - n = index(a) - index(b)$. □

Corollary 13.11. *Let $f : M \longrightarrow \mathbb{R}$ be a Morse–Smale function, and let a and b be two distinct critical points of f. If $index(a) \leq index(b)$, then $W(a,b) = \emptyset$.*

Proof. If $index(a) < index(b)$, then the previous proposition shows that $W(a,b)$ is a manifold of negative dimension, so it must be empty.

If $index(a) = index(b)$, then similarly $W(a,b)$ must be a manifold of dimension 0, but since the gradient flow acts freely on elements of $W(a,b)$, the dimension of $W(a,b)$ must be at least one. Therefore it must be empty. □

Definition 13.5. *We refer to the number*

$$index(a) - index(b)$$

as the relative index *of a and b.*

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on the smooth, closed Riemannian manifold M. Let a and b be critical points. The fundamental object of study will not usually be $W(a,b)$, which is the space of points lying on flow lines between a and b, but rather a space of flow lines themselves. Notice there is an action of the group $\mathbb{R}$ on $W(a,b)$ by the following. Let $x \in W(a,b)$, and let $\gamma_x : \mathbb{R} \to M$ be the unique gradient flow line satisfying the initial condition

$$\gamma_x(0) = x.$$

Then the action of $\mathbb{R}$ is just the flow. More specifically,

$$\mathbb{R} \times W(a,b) \to W(a,b) \tag{13.10}$$
$$t, x \to \gamma_x(t). \tag{13.11}$$

Notice that this action is free, and we can study the orbit space $W(a,b)/\mathbb{R}$. Notice that two points x and y in $W(a,b)$ are in the same orbit space under this $\mathbb{R}$-action if and only if they lie on the same flow line. Therefore a point in the orbit space $W(a,b)/\mathbb{R}$ can be viewed as simply a flow line. We therefore make the following definition.

Definition 13.6. *Define the "Moduli Space of flow lines" $\mathcal{M}(a,b)$ to be the orbit space,*

$$\mathcal{M}(a,b) = W(a,b)/\mathbb{R}.$$

For good intuition and for practical considerations it is useful to instead pick out a representative of each $\mathbb{R}$ orbit in $W(a,b)$. One way to do this is to select a real number t between $f(a)$ and $f(b)$ and pick the representative in $f^{-1}(t)$. This is the approach will allow us an alternate, but equivalent definition of the moduli space $\mathcal{M}(a.b)$.

Definition 13.7. *Pick a value $t \in \mathbb{R}$ between $f(a)$ and $f(b)$, and let $W(a,b)^t$ to be the set $W(a,b) \cap f^{-1}(t)$.*

Proposition 13.12. *If a and b are distinct critical points of f, then $W(a,b)^t$ is a smooth submanifold of M.*

Proof. First, we see that $f|_{W(a,b)} : W(a,b) \longrightarrow \mathbb{R}$ is transverse to the point $\{t\} \subset \mathbb{R}$. This is because for any point $x \in W(a,b)$ so that $f(x) = t$, $\nabla_x(f)$ is not zero, and so neither is $df_x(\nabla_x(f)) = \|\nabla_x(f)\|^2$. Therefore $t \in \mathbb{R}$ is a regular value of $ff|_{W(a,b)}$, and hence by the Regular Value Theorem, $\left(f|_{W(a,b)}\right)^{-1}(\{t\}) = W(a,b)^t$ is a smooth submanifold of $W(a,b)$ of codimension one. □

Theorem 13.13. *Let a and b be distinct critical points of f. The function*

$$\phi : W(a,b)^t \times \mathbb{R} \longrightarrow W(a,b)$$

defined by

$$\phi(p,s) = T_s(p)$$

is a diffeomorphism.

Proof. We begin by proving ϕ is onto. Let $x \in W(a,b)$. Let γ be the flow line that has $\gamma(0) = x$. Since $\lim_{t\to\infty} f(\gamma(t)) = f(b)$ and $\lim_{t\to-\infty} f(\gamma(t)) = f(a)$, by continuity we have that for some s, $f(\gamma(-s)) = t$. Then $\gamma(-s) = p$ and $T_s(p) = x$.

Now to show ϕ is one-to-one, suppose $x = \phi(p_1, s_1) = \phi(p_2, s_2)$. Then $T_{-s_1}(x) = p_1$ and $T_{-s_2}(x) = p_2$, meaning that the unique flow line γ with $\gamma(0) = x$ also has $\gamma(s_1) = p_1$ and $\gamma(s_2) = p_2$. Since $f(p_1) = t = f(p_2)$, and $\frac{d}{ds} f(\gamma(s)) < 0$, it must be that $s_1 = s_2$ and therefore $p_1 = p_2$.

Therefore ϕ^{-1} is defined as a set map. To show that ϕ^{-1} is continuous, it is necessary to show that if U is an open neighborhood of $(p, s) \in W(a, b)^t \times \mathbb{R}$, then there exists an open neighborhood of $\phi(p, s)$ in $W(a, b)$ that is a subset of $\phi(U)$. It suffices to show this for open neighborhoods U of the form $B_p(\epsilon) \times (s - \epsilon, s + \epsilon)$. Since T_{-s} is a diffeomorphism of M that maps neighborhoods of $\phi(p, s)$ to neighborhoods of $\phi(p, 0)$, it suffices to prove this for $s = 0$.

So what we need to show is if $\epsilon > 0$ is sufficiently small, and $p \in W(a, b)^t$, then there exists a δ so that whenever $d(p, y) < \delta$, then writing $y = \phi(q, r)$ gives us $|r| < \epsilon$ and $d(p, q) < \epsilon$.

Since p is not a critical point, there is a δ_1 so that $B_p(2\delta_1)$ does not contain critical points. In this ball, $m = \inf |\nabla f|^2$ is strictly greater than zero and $\sup |\nabla f|$ is finite. If $\sup |\nabla f| > 1$, then let $M = \sup |\nabla f|$, but otherwise let $M = 1$. By continuity of f there is a δ_2 so that $|f(p) - f(B_p(\delta_2))| < m\epsilon/2M$. Choose δ to be smaller than $\min(\delta_1, \delta_2, \epsilon/2)$.

Now in the proof of Lemma 13.1, we saw that

$$\frac{d}{dt} f(\gamma(t)) = -\left|\nabla(f)\right|^2.$$

Integrating and using the fundamental theorem of calculus, we get

$$|f(\gamma(-r)) - f(\gamma(0))| \geq |r| \inf |\nabla f|^2$$

which leaves us with

$$|r|m = |r| \inf |\nabla f|^2 \leq |f(p) - f(y)| < m\epsilon/2M$$

so that $|r| < \epsilon/2M < \epsilon$.

Now,

$$\begin{aligned} d(q, y) &\leq \int |\gamma'(t)|\, dt \\ &= \int |\nabla(f)|\, dt \\ &\leq Mr < \epsilon/2. \end{aligned}$$

So by the Triangle inequality, $d(p, q) \leq d(p, y) + d(q, y) < \delta + \epsilon/2 < \epsilon$. Therefore ϕ^{-1} is continuous.

To prove ϕ^{-1} is smooth, we estimate $d\phi$ and show it is non-degenerate. Let $(p, s) \in W(a, b)^t \times \mathbb{R}$ and let $v_1, \ldots, v_k$ be a basis for the tangent space of $W(a, b)^t$ at p, and let $\partial/\partial t$ be the tangent vector to $\mathbb{R}$. Now if $d\phi$ is degenerate at (p, s), then $d\phi(v_1), \ldots, d\phi(v_k), d\phi(\partial/\partial t)$ would be linearly dependent. Now since $\phi|_{W(a,b)^t \times \{s\}}$ is just the flow map T_s, and this flow map is a diffeomorphism, we know that $d\phi(v_1), \ldots, d\phi(v_k)$ are linearly independent. Therefore any linear dependence would involve $d\phi(\partial/\partial t)$, so that

$$d\phi(\partial/\partial t) = \sum c_k d\phi(v_k)$$

for some real numbers c_k.

Now since $\phi(p,s) = T_s(p)$, $d\phi(\partial/\partial t)$ at (p,s) is $\frac{\partial}{\partial s}T_s(p) = \gamma'(s)$, where γ is the flow with $\gamma(0) = p$. Then if we compose with T_{-s},

$$\begin{aligned} dT_{-s}d\phi(\partial/\partial t) &= \sum c_k dT_{-s}d\phi(v_k) \\ dT_{-s}\gamma'(s) &= \sum c_k v_k \\ \gamma'(0) &= \sum c_k v_k. \end{aligned}$$

But we know $\gamma'(0)$ is transverse to $TW(a,b)^t$, which is a level set of f. Therefore, we have a contradiction, and $d\phi$ is non-degenerate. Therefore ϕ^{-1} is smooth. □

The following is an immediate consequence of this theorem.

Corollary 13.14. *Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a closed Riemannian manifold M. Let $a, b \in M$ be critical points and $t \in \mathbb{R}$ be a number strictly between $f(a)$ and $f(b)$. Then the composition*

$$W^t(a,b) \hookrightarrow W(a,b) \xrightarrow{project} W(a,b)/\mathbb{R} = \mathcal{M}(a,b)$$

is a diffeomorphism.

We therefore may identify the moduli space of flows $\mathcal{M}(a,b)$ with the level space $W^t(a,b)$.

If we use the notation $+a$ to denote the function $+a : \mathbb{R} \longrightarrow \mathbb{R}$ with $+a(x) = x + a$, then the following diagram commutes:

$$\begin{array}{ccc} W(a,b)^t \times \mathbb{R} & \xrightarrow{\phi} & W(a,b) \\ {\scriptstyle (1,+s)}\downarrow & & {\scriptstyle T_s}\downarrow \\ W(a,b)^t \times \mathbb{R} & \xrightarrow{\phi} & W(a,b) \end{array}$$

We now sketch a proof that the set of metrics for which a Morse function is Morse–Smale is dense.

Theorem 13.15. *Let M be a manifold. Let $f : M \longrightarrow \mathbb{R}$ be a Morse function. For a dense set of metrics g, f is Morse–Smale.*

Proof. (Sketch of proof) We suppose a Riemannian metric g is given, and show that there exists a Riemannian metric g' arbitrarily close to g so that f is Morse–Smale with respect to g'. Recall that ∇_g refers to the gradient using the metric g.

We start by finding a vector field X close to $\nabla_g f$ that agrees with $\nabla_g f$ near the critical points of f but so that the unstable and stable manifolds are transverse (step 1). We then show that for some metric g' close to g, $X = \nabla_{g'}(f)$ (step 2).

Step 1: finding the vector field X

The details of this step are found in Smale's proof of Theorem A in the work just cited above ([139]).

Let the critical values of f be $c_1 < \cdots < c_k$. Choose $\epsilon > 0$ arbitrary, but small enough so that for each i, $c_{i+1} > c_i + 4\epsilon$, and in fact, small enough so that for each critical point p, Theorem 13.9 gives us that $W^s(p) \cap f^{-1}((-\infty, c])$ is a ball for all $f(p) < c < f(p) + 4\epsilon$.

We first let $X = \nabla g$. Then we proceed by induction on i, starting at c_1 and ending at c_k, at each stage altering X in $f^{-1}(c_i + \epsilon, c_i + 3\epsilon)$.

At stage i in the induction, we consider each critical point p so that $f(p) = c_i$. In a neighborhood of p, we consider

$$Q = f^{-1}(c_i + 2\epsilon) \cap W^s(p).$$

Since $-\nabla(f)$ is transverse to level sets of f, the gradient flow can be integrated in a small neighborhood of Q so that there is a coordinate z with $-m \leq z \leq m$ so that $\partial/\partial z$ is $-\nabla(f)$ and $z = 0$ coinciding with Q. Here m is chosen so that this keeps us in $f^{-1}(c_i + \epsilon, c_i + 3\epsilon)$. By the coordinate structure of f near p, a tubular neighborhood U of Q is a trivial λ-disk bundle. So if P is a λ dimensional disk of radius 1, then there is a diffeomorphism sending $[-m, m] \times P \times Q$ onto this tubular neighborhood of Q, so that the first coordinate is the coordinate z, and $0 \times 0 \times Q$ is mapped to Q by the identity function. From now on, we will identify U with $[-m, m] \times P \times Q$ in our notation.

Consider all critical points q with $f(q) > c_i$. Let

$$S = \cup_{q, f(q) > c_i, \nabla_q(f) = 0} (0 \times P \times Q) \cap W^s(q)$$

and let $g : S \longrightarrow P$ be the restriction of $\pi_P : [-m, m] \times P \times Q \longrightarrow 0 \times P \times 0$ to S. By Sard's theorem there exist $v \in P$ arbitrarily close to zero so that $2v$ is a regular value of g.

Now construct $\beta : [-m, m] \longrightarrow \mathbb{R}$ so that $\beta(z) \geq 0$, $\beta(z) = 0$ in a neighborhood of $\partial[-m, m]$, and $\int_0^{\pm m} \beta(z)\, dz = \pm|v|$. If v was chosen small enough, $\beta(z)$ and $|\beta'(z)|$ can be kept smaller than ϵ.

Let $P_0 \subset P$ be a λ-dimensional disk of radius $1/3$.

We also construct a smooth $\gamma : P \longrightarrow \mathbb{R}$ so that $0 \leq \gamma \leq 1$, $\gamma = 0$ in a neighborhood of P, $\gamma = 1$ on P_0, and $|\partial\gamma/\partial x_i| \leq 2$.

Let X' be the vector field on M that equals X outside U, and on $[-m, m] \times$

$P \times Q$ let X' be given by

$$X' = -\frac{\partial}{\partial z} - \beta(z)\gamma(x)\frac{v}{|v|}.$$

We use the bounds on β and γ to ensure that $df(X') > 0$.

To see that the new stable and unstable manifolds $W'^s(p)$ and $W'^u(q)$ intersect transversally, we examine any point of intersection, and flow by X' until it is in $f^{-1}(c_i + 2\epsilon)$. It will then be at a point $\{0\} \times P \times Q \subset [-m, m] \times P \times Q$. The flow X' for time $\pm m$ carries $(0, x, y) \in [-m, m] \times P \times Q$ to $(\pm m, x \pm v, y)$, as can be seen by explicitly integrating out X'.

If q is any critical point with $f(q) > c_i$, then consider the new stable manifold $W'^s(q)$ of q under X'. It agrees with the old stable manifold $W^s(q)$ on $(m, 0, y)$, and after flowing by $-m$ we get to $(0, -v, y)$.

Also, the new unstable manifold $W'^u(p)$ agrees with the old unstable manifold $W^u(p)$ for $z = -m$, and flowing by X' for time m from here shows that $W'^u(p) \cap (0 \times P \times Q)$ is

$$\{(0, x + v, y) | (0, x, y) \in W^u(p)\}.$$

So their intersection is the set

$$\{(0, -v, y) | (0, 2v, y) \in W^u(p)\}$$

and since $2v$ is a regular value of g, this intersection is transverse.

We do this for all the critical points with critical value c_i, and these do not interfere with each other as long as ϵ is small enough that the neighborhoods U do not intersect.

We then proceed with larger and larger i, until we have constructed a new X'.

Step 2: finding the metric g'

Note that X is unchanged (it still equals $\nabla_g f$) near critical points of f. So near critical points of f we define g' to equal g. Outside these neighborhoods we define, at each point $x \in M$, a linear transformation A_x on $T_x M$ that is the identity on the kernel of df, and sends X to

$$\frac{\sqrt{df(X)}}{\|df\|_g}\nabla_g(f).$$

Since $df(X) > 0$, this is invertible, and if X is close to $\nabla_g(f)$, then A_x is close to the identity. Let $g'(v, w) = g(Av, Aw)$. Then g' is close to g.

Now if we write an arbitrary vector $w \in T_x(M)$ as $w = w_0 + aX$ where $df(w_0) = 0$, then it is a matter of computation to verify that $g'(X, w) = df(w)$. By definition of gradient, this means $X = \nabla_{g'}(f)$. □

Corollary 13.16. *Given a Morse function $f : M \longrightarrow \mathbb{R}$, there exists a metric g so that f is Morse–Smale.*

13.4 The moduli space of gradient flows $\mathcal{M}(a,b)$, its compactification, and the flow category of a Morse function

Throughout this section we assume that M is a C^∞ closed, Riemannian metric and that $f : M \to \mathbb{R}$ is a Morse function satisfying the Morse-Smale condition. As seen above, the Morse-Smale condition is generic.

Let a and b be critical points of $f : M \to \mathbb{R}$. As seen above, the moduli space $\mathcal{M}(a,b)$ is a smooth manifold of dimension equal to one less than the relative index,

$$dim\mathcal{M}(a,b) = index(a) - index(b) - 1.$$

The points of $\mathcal{M}(a,b)$ are the gradient flow lines that start at a and end at b. Of course the gradient flow lines in $\mathcal{M}(a,b)$ don't really "start" at a or "end" at b, but rather they satisfy the initial conditions $\lim_{t\to-\infty}\gamma(t) = a$ and $\lim_{t\to+\infty}\gamma(t) = b$. This is a rather clumsy arrangement, especially if we want to "glue" flow lines. That is, if $\alpha \in \mathcal{M}(a,b)$ and $\beta \in \mathcal{M}(b,c)$, then we should be able to describe a ("piecewise") flow $\alpha \circ \beta$ which should "start" at a and "'end" at c. This is most easily done if we reparameterize these curves so that they be "height parameterized gradient flow lines", as defined in Definition 13.1.

13.4.1 The compactified moduli space of flows and the flow category

As above let M be a closed Riemannian manifold and let $f : M \to \mathbb{R}$ be a Morse function satisfying the Morse-Smale condition. Let $\nabla(f)$ be the gradient vector field of f. Consider a flow lines of f which is a curve $\gamma : \mathbb{R} \to M$ satisfying the differential equation

$$\frac{d\gamma}{dt} = -\nabla(f).$$

If γ is a flow-line then $\gamma(t)$ converges to critical points of f as $t \to \pm\infty$ and we define

$$s(\gamma) = \lim_{t\to-\infty}\gamma(t), \qquad e(\gamma) = \lim_{t\to\infty}\gamma(t).$$

Since f is strictly decreasing along flow lines it defines a diffeomorphism of the flow line $\gamma(t)$ with the open interval $(f(b), f(a))$ where $s(\gamma) = a$ and $e(\gamma) = b$. This reparameterises the flow-line as a smooth function

$$\omega : (f(b), f(a)) \to M$$

such that

$$f(\omega(t)) = t.$$

We can extend ω to a smooth function defined on $[f(b), f(a)]$ by setting $\omega(f(b)) = b$ and $\omega(f(a)) = a$. Then as seen above, this extended function satisfies the differential equation

$$\frac{d\omega}{dt} = -\frac{\nabla(f)}{\|\nabla(f)\|^2} \tag{13.12}$$

with boundary conditions

$$\omega(f(b)) = b, \qquad \omega(f(a)) = a. \tag{13.13}$$

It is a "height-parameterized" flow line.

We define $\bar{\mathcal{M}}(a, b)$ to be the space of all continuous curves in M which are smooth on the complement of the critical points of f and satisfy the differential equation (13.12) and boundary condition (13.13). Here, of course, we understand that ω satisfies (13.12) on the complement of the set of critical points of f. This space $\bar{\mathcal{M}}(a, b)$ is topologized as a subspace of the space $Map([f(b), f(a)], M)$, of all continuous maps with the compact open topology. Note that if ω is any solution of (13.12) and (13.13) then if we remove the points where $\omega(t)$ is a critical point of f each component of ω is geometrically a flow-line but it is parameterized so that $f(\omega(t)) = t$. Therefore by an abuse of terminology we refer to a curve in $\bar{\mathcal{M}}(a, b)$ as a **piecewise flow-line** from a to b.

It is rather straightforward to check that $\bar{\mathcal{M}}(a, b)$ is a compact space and it clearly contains $\mathcal{M}(a, b)$. Furthermore, by work of Smale in [139], since $\mathcal{M}(a, b)$ is in fact open and dense in $\bar{\mathcal{M}}(a, b)$ and so $\bar{\mathcal{M}}(a, b)$ is a " *compactification*" of the moduli space of flow lines $\bar{\mathcal{M}}(a, b)$.

There is an obvious associative, continuous composition law

$$\bar{\mathcal{M}}(a, b) \times \bar{\mathcal{M}}(b, c) \to \bar{\mathcal{M}}(a, c)$$

which is denoted by $\gamma_1 \circ \gamma_2$.

Following the work of the author, Jones, and Segal [34] we are now ready to define the *"flow category"* of f, $\mathcal{C}_f$:

Definition 13.8. *The flow category* $\mathcal{C}_f$ is the topological category defined as follows:

- **The objects of $\mathcal{C}_f$:** The objects of $\mathcal{C}_f$ are the critical points of f.
- **The morphisms of $\mathcal{C}_f$:** If a and b are critical points of f then the morphisms from a to b are defined to be

$$\mathcal{C}_f(a, b) = \bar{\mathcal{M}}(a, b).$$

- **The composition law:** The composition law is defined by

$$\begin{aligned} \bar{\mathcal{M}}(a, b) \times \bar{\mathcal{M}}(b, c) &\longrightarrow \bar{\mathcal{M}}(a, c) \\ (\gamma_1, \gamma_2) &\longrightarrow \gamma_1 \circ \gamma_2. \end{aligned} \tag{13.14}$$

In fact $\mathcal{C}_f$ is a topological category in the sense that each of the sets $\mathcal{C}_f(a,b)$ comes equipped with a natural topology and the composition law

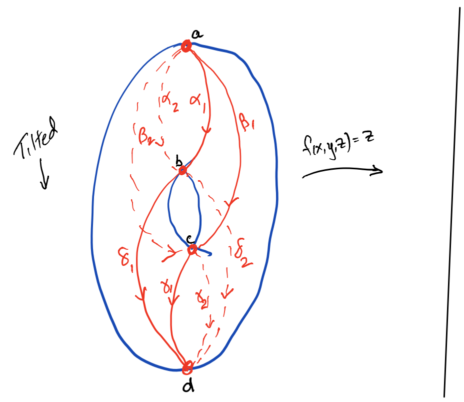
$$\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \longrightarrow \bar{\mathcal{M}}(a,c)$$

is continuous. The topological category $\mathcal{C}_f$ has a simplicial classifying space $B\mathcal{C}_f$. The main result of [34] is the following:

Theorem 13.17. *If M is a closed Riemannian manifold and $f : M \to \mathbb{R}$ is a Morse function satisfying the Morse-Smale condition, then there is a homeomorphism*

$$M \xrightarrow{\cong} B\mathcal{C}_f.$$

We now illustrate this theorem by considering the example of the height function on the "tilted torus". Recall that for this we view the torus as embedded in ordinary three-space, standing on one of its ends with the hole facing the reader, but tilted slightly toward the reader. We let f be the height function.



There are four critical points; a has index 2, b and c have index 1, and d has index 0. As the figure depicts, the moduli spaces $\mathcal{M}(a,b)$, $\mathcal{M}(a,c)$, $\mathcal{M}(b,d)$, and $\mathcal{M}(c,d)$ are all spaces consisting of two distinct points each. We will denote these flows by α_i, β_i, γ_i, and δ_i respectively. All points on the torus not lying on any of these flows is on a flow in $\mathcal{M}(a,d)$. This moduli space is one dimensional, and indeed is the disjoint union of four open intervals. Furthermore the compactification $\bar{\mathcal{M}}(a,d)$ is the disjoint union of four closed intervals.

Now consider the simplicial description in the classifying space $B\mathcal{C}_f$. The vertices correspond to the objects of the category $\mathcal{C}_f$, that is the critical points.

Thus there are four vertices. There is one one simplex (interval) for each morphism (flow line), glued to the vertices corresponding to the starting and endpoints of the flows. Notice that the points in $\bar{\mathcal{M}}(a,d)$ index a one parameter family of one simplices attached to the vertices labelled by a and d. Finally observe that there is a two-simplex for every pair of composable flows.

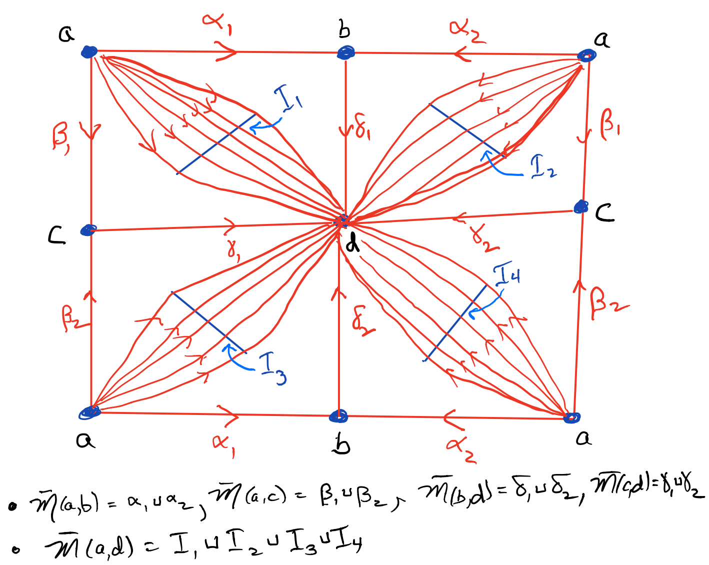


FIGURE 13.6
Simplicial decomposition of $B\mathcal{C}_f$, where f is the height function on the tilted torus

There are eight such pairs (coming from the four points in each of the product moduli spaces $\mathcal{M}(a,b) \times \mathcal{M}(b,d)$ and $\mathcal{M}(a,c) \times \mathcal{M}(c,d)$.) A two-simplex labelled by a pair of flows, say (α, β) will have its three faces identified with the one simplices labelled by α, β, and $\alpha \circ_1 \beta$ respectively. Notice that all higher dimensional simplices in the nerve $\mathcal{N}(\mathcal{C}_f)$ are degenerate and so do not contribute to the geometric realization. The figure depicts the resulting simplicial structure of the classifying space and illustrates Theorem 13.17 that this space is homeomorphic to the underlying manifold.

Remark. The manuscript [34] was never published, primarily because the proof of the main theorem relied on knowing that, assuming $f : M \to \mathbb{R}$ satisfies the Morse-Smale condition, then the compactified moduli spaces, $\bar{\mathcal{M}}(a,b)$ are manifolds with corners and that the corner structure is appropriately preserved under the composition of piecewise flow lines. At the time that manuscript was written, the authors thought that this was a "folk theorem". However upon further inspection, the authors realized that although experts in the community believed that this was true, there was no proof in the literature, and that the issues involved in proving this result were more complicated than the authors originally imagined. Therefore the manuscript was never submitted for publication. In any case, the required manifold with corners properties were eventually proved [130] [156], and the proof of Theorem 13.17 can now be completed using these results. A discussion of manifolds with corners and a proof of this theorem will be given in the appendices.

A

Appendix: Manifolds with Corners

In the first section of this appendix we give a brief description of a categorical notion of manifold with corners. In the second section we discuss applications to Morse theory, including a generalization of the fact that the compactified moduli space of flow lines is a framed manifold with corners. Our discussion follows the description in [89], and also [32].

A.1 A categorical notion of manifolds with corners and $\langle k \rangle$-manifolds

An n-dimensional manifold with corners, M, has charts that are local diffeomorphisms with $\mathbb{R}^n_+$. Here $\mathbb{R}_+$ denotes the nonnegative real numbers and $\mathbb{R}^n_+$ is the n-fold cartesian product of $\mathbb{R}_+$. Let $\psi : U \to \mathbb{R}^n_+$ be a chart of a manifold with corners M. For $x \in U$, the number of zeros of this chart, $c(x)$ is independent of the chart. One defines a *face* of M to be a connected component of the space $\{x{\in}M \text{ such that } c(x) = 1\}$. Given an integer k, there is a notion of a manifold with corners having "codimension k", or a $\langle k \rangle$-manifold. We recall the definition from [89].

Definition A.1. *A $\langle k \rangle$-manifold is a manifold with corners, M, together with an ordered k-tuple $(\partial_1 M, ..., \partial_k M)$ of unions of faces of M satisfying the following properties.*

1. *Each $x{\in}M$ belongs to $c(x)$ faces*
2. $\partial_1 M \cup \cdots \cup \partial_k M = \partial M$
3. *For all $1{\leq}i{\neq}j{\leq}k$, $\partial_i M \cap \partial_j M$ is a face of both $\partial_i M$ and $\partial_j M$.*

The archetypical example of a $\langle k \rangle$-manifold is $\mathbb{R}^k_+$. In this case the face $F_j \subset \mathbb{R}^k_+$ consists of those k-tuples with the j^{th}-coordinate equal to zero.

As described in [89], the data of a $\langle k \rangle$-manifold can be encoded in a categorical way as follows. Let $\underline{2}$ be the partially ordered set with two objects, $\{0, 1\}$, having a single nonidentity morphism $0 \to 1$. Let $\underline{2}^k$ be the product of k-copies of the category $\underline{2}$. A $\langle k \rangle$-manfold M then defines a functor from $\underline{2}^k$ to the category of topological spaces, where for an object $a = (a_1, \cdots, a_k) \in \underline{2}^k$, $M(a)$ is the intersection of the faces $\partial_i(M)$ with $a_i = 0$. Such a functor is a k-dimensional cubical diagram of spaces, which, following Laures's terminology, we refer to as a $\langle k \rangle$-diagram, or a $\langle k \rangle$-space. Notice that $\mathbb{R}^k_+(a) \subset \mathbb{R}^k_+$ consists of those k-tuples of nonnegative real numbers so that the i^{th}-coordinate is zero for every i with $a_i = 0$. More generally, consider the $\langle k \rangle$-Euclidean space, $\mathbb{R}^k_+ \times \mathbb{R}^n$, where the value on $a \in \underline{2}^k$ is $\mathbb{R}^k_+(a) \times \mathbb{R}^n$. In general we refer to a functor $\phi : \underline{2}^k \to \mathcal{C}$ as a $\langle k \rangle$-object in the category $\mathcal{C}$.

In this section we will consider embeddings of manifolds with corners into Euclidean spaces $M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^n$ of the form given by the following definition.

Definition A.2. *A "neat embedding" of a $\langle k \rangle$-manifold M into $\mathbb{R}^k_+ \times \mathbb{R}^m$ is a natural transformation of $\langle k \rangle$-diagrams*

$$e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$$

that satisfies the following properties:

1. *For each $a \in \underline{2}^k$, $e(a)$ is an embedding.*
2. *For all $b < a$, the intersection $M(a) \cap \left(\mathbb{R}^k_+(b) \times \mathbb{R}^m\right) = M(b)$, and this intersection is perpendicular. That is, there is some $\epsilon > 0$ such that*

$$M(a) \cap \left(\mathbb{R}^k_+(b) \times [0, \epsilon)^k(a - b) \times \mathbb{R}^m\right) = M(b) \times [0, \epsilon)^k(a - b).$$

Here $a - b$ denotes the object of $\underline{2}^k$ obtained by subtracting the k-vector b from the k-vector a.

In [89] it was proved that every $\langle k \rangle$-manifold neatly embeds in $\mathbb{R}^k_+ \times \mathbb{R}^N$ for N sufficiently large. In fact it was proved there that a manifold with corners, M, admits a neat embedding into $\mathbb{R}^k_+ \times \mathbb{R}^N$ *if and only if* M has the structure of a $\langle k \rangle$-manifold. Also, in analogy to the situation with respect to closed manifolds, it was shown in [59] that the connectivity of the space of neat embeddings, $Emb_{\langle k \rangle}(M; \mathbb{R}^k_+ \times \mathbb{R}^N)$ increases with the dimension N.

Notice that an embedding of manifolds with corners, $e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$, has a well defined normal bundle. In particular, for any pair of objects in $\underline{2}^k$, $a > b$, the normal bundle of $e(a) : M(a) \hookrightarrow \mathbb{R}^k_+(a) \times \mathbb{R}^m$, when restricted to $M(b)$, is the normal bundle of $e(b) : M(b) \hookrightarrow \mathbb{R}^k_+(b) \times \mathbb{R}^m$.

These embedding properties of $\langle k \rangle$ - manifolds make it clear that these are the appropriate manifolds to study for cobordism - theoretic information. In particular, given an embedding $e : M \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^m$ the Thom space of the normal bundle, $T(\nu_e)$, has the structure of an $\langle k \rangle$-space, where for $a \in \underline{2}^k$, $T(\nu_e)(a)$ is the Thom space of the normal bundle of the associated embedding, $M(a) \hookrightarrow \mathbb{R}^k_+(a) \times \mathbb{R}^N$. We can then desuspend and define the Thom spectrum,

$M_e^\nu = \Sigma^{-N}\Sigma^\infty T(\nu_e)$, to be the associated $\langle k\rangle$-spectrum. The Pontrjagin-Thom construction defines a map of $\langle k\rangle$-spaces,

$$\tau_e : \left(\mathbb{R}_+^k \times \mathbb{R}^N\right) \cup \infty = ((\mathbb{R}_+^k) \cup \infty) \wedge S^N \to T(\nu_e).$$

Desuspending we get a map of $\langle k\rangle$-spectra, $\Sigma^\infty((\mathbb{R}_+^k)\cup\infty) \to M_e^\nu$. Notice that the homotopy type (as $\langle k\rangle$-spectra) of M_e^ν is independent of the embedding e. We denote the homotopy type of this normal Thom spectrum as M^ν, and the Pontrjagin-Thom map, $\tau : \Sigma^\infty((\mathbb{R}_+^k) \cup \infty) \to M^\nu$.

We now begin to describe the relevance of this notion of manifold with corners to our study of Morse theory, as developed in Chapters 12 and 13. We continue to use the notation and terminology used there.

A.2 The moduli space of piecewise flow lines as a compact framed manifold with corners

Consider a smooth, closed, Riemannian n-manifold M^n, and a smooth Morse function $f : M^n \to \mathbb{R}$ that satisfies the Morse-Smale condition. Notice that there is a partial order on the finite set of critical points given by $a \geq b$ if the moduli space of flow lines $\mathcal{M}(a,b)$ is nonempty. (Notice that one needs the Morse-Smale property to know that this is indeed a partial ordering on the critical points.)

Recall that

$$\bar{\mathcal{M}}(a,b) = \bigcup_{a=a_1>a_2>\cdots>a_k=b} \mathcal{M}(a_1,a_2) \times \cdots \times \mathcal{M}(a_{k-1},a_k), \tag{A.1}$$

Let $\mathcal{C}_f$ denote the flow category, whose objects are the critical points, and whose space of morphisms from a to b is given by $\bar{\mathcal{M}}(a,b)$.

It turns out that the moduli spaces $\mathcal{M}(a,b)$ have natural framings on their stable normal bundles that play an important role in this theory. These framings are defined in the following manner. Let $a > b$ be critical points. Let $\epsilon > 0$ be chosen so that there are no critical values in the half open interval $[f(a)-\epsilon, f(a))$. Define the *unstable sphere* to be the level set of the unstable manifold,

$$S^u(a) = W^u(a) \cap f^{-1}(f(a)-\epsilon).$$

The sphere $S^u(a)$ has dimension $\mu(a)-1$. Notice there is a natural diffeomorphism,

$$\mathcal{M}(a,b) \cong S^u(a) \cap W^s(b).$$

This leads to the following diagram,

$$\begin{array}{ccc} W^s(b) & \xrightarrow{\hookrightarrow} & M \\ \cup\uparrow & & \uparrow\cup \\ \mathcal{M}(a,b) & \xrightarrow[\hookrightarrow]{} & S^u(a). \end{array} \tag{A.2}$$

From this diagram one sees that the normal bundle ν of the embedding $\mathcal{M}(a,b) \hookrightarrow S^u(a)$ is the restriction of the normal bundle of $W^s(b) \hookrightarrow M$. Since $W^s(b)$ is a disk, and therefore contractible, this bundle is trivial. Indeed an orientation of $W^s(b)$ determines a homotopy class of trivialization, or a framing. In fact this framing determines a diffeomorphism of the bundle to the product, $W^s(b) \times W^u(b)$. Thus these orientations give the moduli spaces $\mathcal{M}(a,b)$ canonical normal framings, $\nu \cong \mathcal{M}(a,b) \times W^u(b)$.

As was pointed out in [34], these framings extend to the boundary of the compactifications, $\bar{\mathcal{M}}(a,b)$. In order to describe what it means for these framings to be "coherent" in an appropriate sense, the following categorical approach was used in [31]. The first step is to abstract the basic properties of a flow category of a Morse functon.

Definition A.3. *A smooth, compact category is a topological category $\mathcal{C}$ whose objects form a discrete set, and whose whose morphism spaces, $Mor(a,b)$ are compact, smooth $\langle k\rangle$ -manifolds, where $k = dim\, Mor(a,b)$. The composition maps, $\nu : Mor(a,b) \times Mor(b,c) \to Mor(a,c)$, are smooth codimension one embeddings (of manifolds with corners) whose images lie in the boundary. Moreover every point in the boundary of $Mor(a,c)$ is in the image under ν of a unique maximal sequence in $Mor(a,b_1) \times Mor(b_1,b_2) \times \cdots \times Mor(b_{k-1},b_k) \times Mor(b_k,c)$ for some objects $\{b_1, \cdots, b_k\}$.*

A smooth, compact category $\mathcal{C}$ is said to be a "Morse-Smale" category if the following additional properties are satisfied.

1. *The objects of $\mathcal{C}$ are partially ordered by the condition*

$$a \geq b \quad if \quad Mor(a,b) \neq \emptyset.$$

2. *$Mor(a,a) = \{identity\}$.*

3. *There is a set map, $\mu : Ob(\mathcal{C}) \to \mathbb{Z}$, which preserves the partial ordering, such that if $a > b$,*

$$dim\, Mor(a,b) = \mu(a) - \mu(b) - 1.$$

The map μ is known as an "index" map. A Morse-Smale category such as this is said to have finite type, if there are only finitely many objects of any given index, and for each pair of objects $a > b$, there are only finitely many objects c with $a > c > b$. For ease of notation we write $k(a,b) = \mu(a) - \mu(b) - 1$.

The following is a folk theorem that goes back to the work of Smale and Franks [48] although a proof of this fact did not appear in the literature until much later [130].

Proposition A.1. *Let $f : M \to \mathbb{R}$ be smooth Morse function on a closed Riemannian manifold with a Morse-Smale metric. Then the compactified moduli space of piecewise flow-lines, $\bar{\mathcal{M}}(a,b)$ is a smooth $\langle k(a,b) \rangle$ - manifold.*

Using this result, as well as an associativity result for the gluing maps $\bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) \to \bar{\mathcal{M}}(a,c)$ which was eventually proved in [130], it was proven in [34] that the flow category $\mathcal{C}_f$ of such a Morse-Smale function is indeed a Morse-Smale smooth, compact category according to Definition A.3.

Remark. The fact that [34] was never submitted for publication was due to the fact that the "folk theorem" mentioned above (Proposition A.1), as well as the associativity of gluing, both of which the authors of [34] assumed were "well known to the experts", were indeed not in the literature, and their proofs, which was eventually provided in [130], were analytically more complicated than the authors imagined.

In order to define the notion of "coherent framings" of the moduli spaces $\bar{\mathcal{M}}(a,b)$, so that we may apply the Pontrjagin-Thom construction coherently, we need to study an associated category, enriched in spectra, defined using the stable normal bundles of the moduli spaces of flows.

We first describe a category defined in [35] that the authors used to classify finite spectra that realize a given finite complex. The category is called $\mathcal{J}$. Its objects are the nonnegative integers $\mathbb{Z}^+$, and its non-identity morphisms from i to j is empty for $i \leq j$, for $i > j+1$ it is defined to be the one point compactification,

$$Mor_{\mathcal{J}}(i,j) \cong (\mathbb{R}_+)^{i-j-1} \cup \infty$$

and $Mor_{\mathcal{J}}(j+1,j)$ is defined to be the two point space, S^0. Here $\mathbb{R}_+$ is the space of nonnegative real numbers. Composition in this category can be viewed in the following way. Notice that for $i > j+1$ $Mor_{\mathcal{J}}(i,j)$ can be viewed as the one point compactification of the space $J(i,j)$ consisting of sequences of real numbers $\{\lambda_k\}_{k\in\mathbb{Z}}$ such that

$$\begin{aligned} \lambda_k &\geq 0 \quad \text{for all } k \\ \lambda_k &= 0 \quad \text{unless } i > k > j. \end{aligned}$$

For consistency of notation we write $Mor_{\mathcal{J}}(i,j) = J(i,j)^+$. Composition of morphisms $J(i,j)^+ \wedge J(j,k)^+ \to J(i,k)^+$ is then induced by addition of sequences. In this smash product the basepoint is taken to be ∞. Notice that this map is basepoint preserving. Given integers $p > q$, then there are subcategories $\mathcal{J}_q^p$ defined to be the full subcategory generated by integers $q \geq m \geq p$. The category $\mathcal{J}_q$ is the full subcategory of $\mathcal{J}$ generated by all integers $m \geq q$.

We now define the "normal Thom spectrum" of a smooth, compact category.

Definition A.4. *Let $\mathcal{C}$ be a smooth, compact category of finite type satisfying the Morse-Smale condition. Then a "normal Thom spectrum" of the category $\mathcal{C}$ is a category, $\mathcal{C}^\nu$, enriched over spectra, that satisfies the following properties.*

1. *The objects of $\mathcal{C}^\nu$ are the same as the objects of $\mathcal{C}$.*
2. *The morphism spectra $Mor_{\mathcal{C}^\nu}(a,b)$ are $\langle k(a,b)\rangle$-spectra, having the homotopy type of the normal Thom spectra $Mor_{\mathcal{C}}(a,b)^\nu$, as $\langle k(a,b)\rangle$-spectra. The composition maps,*

$$\circ : Mor_{\mathcal{C}^\nu}(a,b) \wedge Mor_{\mathcal{C}^\nu}(b,c) \to Mor_{\mathcal{C}^\nu}(a,c)$$

   have the homotopy type of the maps,

$$Mor_{\mathcal{C}}(a,b)^\nu \wedge Mor_{\mathcal{C}}(b,c)^\nu \to Mor_{\mathcal{C}}(a,c)^\nu$$

   of the Thom spectra of the stable normal bundles corresponding to the composition maps in $\mathcal{C}$, $Mor_{\mathcal{C}}(a,b) \times Mor_{\mathcal{C}}(b,c) \to Mor_{\mathcal{C}}(a,c)$. Recall that these are maps of $\langle k(a,c)\rangle$-spaces induced by the inclusion of a component of the boundary.

3. *The morphism spectra are equipped with "Pontrjagin-Thom maps" $\tau_{a,b} : \Sigma^\infty(J(\mu(a),\mu(b))^+) = \Sigma^\infty((\mathbb{R}_+^{k(a,b)}) \cup \infty)) \to Mor_{\mathcal{C}^\nu}(a,b)$ such that the following diagram commutes:*

$$\begin{array}{ccc}
\Sigma^\infty(J(\mu(a),\mu(b))^+) \wedge \Sigma^\infty(J(\mu(b),\mu(c))^+) & \longrightarrow & \Sigma^\infty(J(\mu(a),\mu(c))^+) \\
{\scriptstyle \tau_{a.b}\wedge\tau_{b,c}}\downarrow & & \downarrow{\scriptstyle \tau_{a,c}} \\
Mor_{\mathcal{C}^\nu}(a,b) \wedge Mor_{\mathcal{C}^\nu}(b,c) & \longrightarrow & Mor_{\mathcal{C}^\nu}(a,c).
\end{array}$$

   Here the top horizontal map is defined via the composition maps in the category $\mathcal{J}$, and the bottom horizontal map is defined via the composition maps in $\mathcal{C}^\nu$.

With the notion of a "normal Thom spectrum" of a flow category $\mathcal{C}$, the notion of a coherent E^*-orientation was defined in [31]. Here E^* is a generalized cohomology theory represented by a commutative (E_∞) ring spectrum E. We recall that definition now.

First observe that a commutative ring spectrum E induces a $\langle k\rangle$-diagram in the category of spectra ("$\langle k\rangle$-spectrum"), $E\langle k\rangle$, defined in the following manner.

For $k = 1$, we let $E\langle 1\rangle : \underline{2} \to \mathit{Spectra}$ be defined by $E\langle 1\rangle(0) = S^0$, the sphere spectrum, and $E\langle 1\rangle(1) = E$. The image of the morphism $0 \to 1$ is the unit of the ring spectrum $S^0 \to E$.

To define $E\langle k\rangle$ for general k, let a be an object of $\underline{2}^k$. We view a as a vector of length k, whose coordinates are either zero or one. Define $E\langle k\rangle(a)$ to be the multiple smash product of spectra, with a copy of S^0 for every every zero coordinate, and a copy of E for every string of successive ones. For example, if $k = 6$, and $a = (1, 0, 1, 1, 0, 1)$, then $E\langle k\rangle(a) = E \wedge S^0 \wedge E \wedge S^0 \wedge E$.

Given a morphism $a \to a'$ in $\underline{2}^k$, one has a map $E\langle k\rangle(a) \to E\langle k\rangle(a')$ defined by combining the unit $S^0 \to E$ with the ring multiplication $E \wedge E \to E$.

Said another way, the functor $E\langle k\rangle : \underline{2}^k \to \mathit{Spectra}$ is defined by taking the k-fold product functor $E\langle 1\rangle : \underline{2} \to \mathit{Spectra}$ which sends $(0 \to 1)$ to $S^0 \to E$, and then using the ring multiplication in E to "collapse" successive strings of E's.

This structure allows us to define one more construction. Suppose $\mathcal{C}$ is a smooth, compact, Morse-Smale category of finite type as in Definition A.3. We can then define an associated category, $E_{\mathcal{C}}$, whose objects are the same as the objects of $\mathcal{C}$ and whose morphisms are given by the spectra,

$$Mor_{E_{\mathcal{C}}}(a, b) = E\langle k(a, b)\rangle$$

where $k(a, b) = \mu(a) - \mu(b) - 1$. Here $\mu(a)$ is the index of the object a as in Definition A.3. The composition law is the pairing,

$$\begin{aligned} E\langle k(a,b)\rangle \wedge E\langle k(b,c)\rangle &= E\langle k(a,b)\rangle \wedge S^0 \wedge E\langle k(b,c)\rangle \\ &\xrightarrow{1\wedge u\wedge 1} E\langle k(a,b)\rangle \wedge E\langle 1\rangle \wedge E\langle k(b,c)\rangle \\ &\xrightarrow{\mu} E\langle k(a,c)\rangle. \end{aligned}$$

Here $u : S^0 \to E = E\langle 1\rangle$ is the unit. This category encodes the multiplication in the ring spectrum E.

Definition A.5. *An E^*-orientation of a smooth, compact category of finite type satisfying the Morse-Smale condition, $\mathcal{C}$, is a functor, $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$, where $\mathcal{C}^\nu$ is a normal Thom spectrum of $\mathcal{C}$, such that on morphism spaces, the induced map*

$$Mor_{\mathcal{C}^\nu}(a, b) \to E\langle k(a, b)\rangle$$

is a map of $\langle k(a, b)\rangle$-spectra that defines an E^* orientation of $Mor_{\mathcal{C}^\nu}(a, b) \simeq \bar{\mathcal{M}}(a, b)^\nu$.

The functor $u : \mathcal{C}^\nu \to E_{\mathcal{C}}$ should be thought of as a coherent family of E^*- Thom classes for the normal bundles of the morphism spaces of $\mathcal{C}$. When $E = \mathbb{S}$, the sphere spectrum, then an E^*-orientation, as defined here, defines a coherent family of framings of the morphism spaces, and is equivalent to the notion of a *framing* of the category $\mathcal{C}$, as defined in [35].

In [35] the following was proved modulo the results of [130] which appeared much later.

Theorem A.2. *Let $f : M \to \mathbb{R}$ be a Morse function on a closed Riemannian manifold satisfying the Morse-Smale condition. Then the flow category $\mathcal{C}_f$ has a canonical structure as a "$\mathbb{S}$-oriented, smooth, compact Morse-Smale category of finite type". That is, it is a "*framed, smooth compact Morse-Smale category*". The induced framings of the morphism manifolds $\bar{\mathcal{M}}(a,b)$ are canonical extensions of the framings of the open moduli spaces $\mathcal{M}(a,b)$ descrbed above (A.2).*

B

Appendix: Classifying Spaces and Morse Theory

In this appendix we give a proof of Theorem 13.17. We will then give another proof of a weaker version of this theorem that is much simpler and less anaytic in nature. As we mentioned earlier our proof of Theorem 13.17 is a slight modification of the argument originally given in the unpublished preprint by the author, Jones, and Segal [34] in the 1990's. Here we take advantage of various more recent results that allow us to fill in the missing pieces of the original argument. We continue to use all the notation and definitions in chapters 12 and 13.

Let $f : M \to \mathbb{R}$ be a Morse function on a closed Riemannian manifold. As was done in Chapter 13, if a and b are critical points, we let $\mathcal{M}(a, b)$ be the space of flow lines connecting a to b. Recall that when f satisfies the Morse-Smale condition, $\mathcal{M}(a, b)$ is a manifold of dimension $index(a) - index(b) - 1$. As in Chapter 13, we let $\bar{\mathcal{M}}(a, b)$ be the compactification of $\mathcal{M}(a, b)$ given by *"piecewise flows"*.

B.0.1 The ends of the moduli space of flow lines

In this section we describe the ends of the moduli space of flow-lines $\mathcal{M}(a, b)$. We then use this analysis to construct a combinatorial model $\mathcal{R}_f$ for the manifold M. We show that $\mathcal{R}_f$ is homeomorphic to the classifying space $B\mathcal{C}_f$ in the next section.

In [139] Smale described a partial ordering of the set of critical points of f. Namely, one writes $a > b$ if $a \neq b$ and $\mathcal{M}(a, b) \neq \emptyset$. Notice that the Morse-Smale condition is necessary for the critical points be partially ordered.

We say that a sequence $\mathbf{a} = (a_0, \cdots, a_{l+1})$ of critical points is **ordered** if $a_i > a_{i+1}$ for all i. Given such a sequence we define

$$\begin{aligned} s(\mathbf{a}) &= a_0, \\ e(\mathbf{a}) &= a_{l+1}, \\ l(\mathbf{a}) &= l, \\ \mathcal{M}(\mathbf{a}) &= \mathcal{M}(a_0, a_1) \times \cdots \times \mathcal{M}(a_l, a_{l+1}). \end{aligned} \tag{B.1}$$

In [34] the authors anticipated the following result, however it was not written down carefully in a proof until the recent work of Qin [130] and Hutchings [80]. In particular they showed that the moduli spaces $\bar{\mathcal{M}}(a,b)$ have the structure of smooth, compact manifolds with corners. In examining the corner structure, the following was proved in [130]. It is very much related to Theorem A.2.

Theorem B.1. *There exists an $\epsilon > 0$ and maps*

$$\mu : (0, \epsilon] \times \mathcal{M}(a, a_1) \times \mathcal{M}(a_1, b) \longrightarrow \mathcal{M}(a, b),$$

which we write as

$$(t, \gamma_1, \gamma_2) \longrightarrow \gamma_1 \circ_t \gamma_2,$$

such that:

1. *The map μ satisfies the following associativity law*

$$(\gamma_1 \circ_s \gamma_2) \circ_t \gamma_3 = \gamma_1 \circ_s (\gamma_2 \circ_t \gamma_3)$$

for all s, $t \leq \epsilon$.

2. *Let $\mathbf{a}$ be an ordered sequence with $s(\mathbf{a}) = a$, $e(\mathbf{a}) = b$, and $l(\mathbf{a}) = l$. Then the map*

$$\mu : (0, \epsilon]^l \times \mathcal{M}(\mathbf{a}) \longrightarrow \mathcal{M}(a, b)$$

defined by

$$(s_1, \dots, s_l; \gamma_0, \dots, \gamma_l) \mapsto \gamma_0 \circ_{s_1} \gamma_1 \circ_{s_2} \cdots \circ_{s_l} \gamma_l$$

is a diffeomorphism onto its image.

3. *Define $\mathcal{K}(a, b) \subset \mathcal{M}(a, b)$ to be*

$$\mathcal{K}(a, b) = \mathcal{M}(a, b) - \bigcup \mu\left((0, \epsilon)^l \times \mathcal{M}(\mathbf{a})\right)$$

where the union is taken over all ordered sequences $\mathbf{a}$ with $s(\mathbf{a}) = a$, $e(\mathbf{a}) = b$, and $l(\mathbf{a}) \geq 1$. Then $\mathcal{K}(a, b)$ is compact.

4. *There are homeomorphisms*

$$\bar{\mathcal{M}}(a, b) \cong \mathcal{M}(a, b) \cup_\mu \bigcup [0, \epsilon]^l \times \mathcal{M}(\mathbf{a}) \tag{B.2}$$

$$\bar{\mathcal{M}}(a, b) \cong \mathcal{K}(a, b). \tag{B.3}$$

This theorem shows that the ends of the moduli space $\mathcal{M}(a, b)$ consist of unions of half-open cubes parameterized by composable sequences of flow lines. The compact space $\mathcal{K}(a, b)$ is formed by removing the associated *open* cubes.

The compactification $\bar{\mathcal{M}}(a,b)$ is formed by formally *closing* the cubes, or equivalently, but formally adjoining the piecewise flows. It follows tha $\mathcal{K}(a,b)$ and $\bar{\mathcal{M}}(a,b)$ are homeomorphic.

It also follows from this theorem that

$$\lim_{t\to 0} \gamma_1 \circ_t \gamma_2 = \gamma_1 \circ \gamma_2$$

where $\circ$ is the composition of piecewise flow-lines described in chapter 13. In view of this fact we sometimes use the notation $\circ_0$ for $\circ$.

The homeomorphism between $\mathcal{K}(a,b)$ and $\bar{\mathcal{M}}(a,b)$ allows us to define the category $\mathcal{C}_f$ in two equivalent (isomorphic) ways. The first way, described in Chapter 13, is to define the spaces of morphisms between critical points a and b to be the space $\bar{\mathcal{M}}(a,b)$, and the composition law is given by

$$\begin{aligned} \bar{\mathcal{M}}(a,b) \times \bar{\mathcal{M}}(b,c) &\to \bar{\mathcal{M}}(a,c) \\ (\gamma_1, \gamma_2) &\to \gamma_1 \circ_0 \gamma_2. \end{aligned}$$

The second way is to define the space of morphisms between critical points a and b to be $\mathcal{K}(a,b)$, and this time the composition law is given by

$$\begin{aligned} \mathcal{K}(a,b) \times \mathcal{K}(b,c) &\to \mathcal{K}(a,c) \\ (\gamma_1, \gamma_2) &\to \gamma_1 \circ_\epsilon \gamma_2. \end{aligned}$$

We now use Theorem B.1 to produce a combinatorial model we call $\mathcal{R}_f$ of the manifold M. We begin by describing a filtration of the spaces $\bar{\mathcal{M}}(a,b)$. By scaling if necessary, we can assume that the constant ϵ in the statement of Theorem B.1 is 1. If $\gamma_1 \in \mathcal{M}(a,a_1)$ and $\gamma_2 \in \mathcal{M}(a_1,b)$, then the parameter $t \in (0,1]$ in the flow $\gamma_1 \circ_t \gamma_2 \in \mathcal{M}(a,b)$ can be viewed as a measure of how close this flow comes to the critical point a_1. This interpretation will become clearer in the proof. Thus the fact that the pairing μ is a diffeomorphism onto its image allows us to view the space $\mathcal{K}(a,b)$ as the space of flows that stay at least 1 away from all critical points other than a and b (in this undefined measure).

Next we look at the curves in $\bar{\mathcal{M}}(a,b)$ which get within a distance 1 of at most one intermediate critical point. More generally we can filter the space $\bar{\mathcal{M}}(a,b)$ by saying that a curve in $\bar{\mathcal{M}}(a,b)$ has filtration k if it gets within a distance of less than 1 of at most k intermediate critical points. We now make this description precise.

For any ordered sequence $\mathbf{a} = (a_0, \ldots, a_{l+1})$ of critical points define

$$\mathcal{K}(\mathbf{a}) = \mathcal{K}(a_0,a_1) \times \cdots \times \mathcal{K}(a_l, a_{l+1}).$$

Now define

$$\begin{aligned} \mathcal{K}^0(a,b) &= \mathcal{K}(a,b) \\ \mathcal{K}^{(k)}(a,b) &= \bigcup_{s(\mathbf{a})=a, e(\mathbf{a})=b, l(\mathbf{a})\le k} \mu\left([0,1]^{l(\mathbf{a})} \times \mathcal{K}(\mathbf{a})\right). \end{aligned} \tag{B.4}$$

In this definition of $\mathcal{K}^{((k)}(a,b)$ we interpret $\mu\left([0,1]^{l(\mathbf{a})}\times\mathcal{K}(\mathbf{a})\right)$ in the case where $l(\mathbf{a})=0$, in other words $\mathbf{a}=(a,b)$, to be $\mathcal{K}(a,b)$.

We therefore have that

$$\mathcal{K}^{(k-1)}(a,b)\subset\mathcal{K}^{(k)}(a,b),$$

and γ is in $\mathcal{K}^{(k)}(a,b)$ if and only if γ can be decomposed as

$$\gamma=\gamma_0\circ_{s_1}\cdots\circ_{s_l}\gamma_l$$

where $\gamma_i\in\mathcal{K}(a_i,a_{i-1})$, $0\leq s_i\leq 1$, and $l\leq k$.

Notice that

1. $$\bigcup\mathcal{K}^{(k)}(a,b)=\mathcal{M}(a,b).$$

2. $$\mathcal{K}^{(k)}(a,b)-\mathcal{K}^{(k-1)}(a,b)\cong\coprod_{s(\mathbf{a})=a,e(\mathbf{a})=b,l(\mathbf{a})=k}[0,1)^k\times\mathcal{K}(\mathbf{a}),$$

It then follows that the map

$$\coprod_{s(\mathbf{a})=a,e(\mathbf{a})=b}[0,1]^{l(\mathbf{a})}\times\mathcal{K}(\mathbf{a})\longrightarrow\bar{\mathcal{M}}(a,b)$$

defined by

$$(s_1,\cdots,s_l;\gamma_0,\cdots,\gamma_l)\longrightarrow\gamma_0\circ_{s_1}\cdots\circ_{s_l}\gamma_l$$

is surjective. Therefore $\bar{\mathcal{M}}(a,b)$ can be recovered by imposing an equivalence relation on the above disjoint union. From (2) it follows that this equivalence relation is generated by

$$(s_1,\ldots,s_{i-1},1,s_{i+1},\ldots,s_l;\gamma_1,\ldots,\gamma_l)\simeq (s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_l;\gamma_1,\ldots,\gamma_{i-1}\circ_1\gamma_i,\ldots,\gamma_l). \tag{B.5}$$

Note that the relations only involve the faces of the cubes $[0,1]^{l(\mathbf{a})}$ which do not contain the point $(0,\ldots,0)$. From this argument we draw the following conclusion.

Theorem B.2.

$$\bar{\mathcal{M}}(a,b)\cong\coprod_{\mathbf{a}}[0,1]^{l(\mathbf{a})}\times\mathcal{K}(\mathbf{a})/\simeq$$

The next step is to go from this description of the spaces $\bar{\mathcal{M}}(a,b)$ to one of the manifold M. Recall that, by definition, $\bar{\mathcal{M}}(a, b)$ consists of continuous curves that are differentiable away from finitely many points,

$$\gamma : [f(b), f(a)] \to M$$

which satisfy

$$\frac{d\gamma}{dt} = -\frac{\nabla(f)}{\|\nabla(f)|^2} \tag{B.6}$$

with boundary conditions

$$\gamma(f(b)) = b, \quad \gamma(f(a)) = a. \tag{B.7}$$

Thus we get a map

$$\phi : [f(b), f(a)] \times \bar{\mathcal{M}}(a, b) \to M$$

whose image is the closure of the space $W(a, b) \subset M$.

Let us simplify the notation slightly by writing

$$J_{\mathbf{a}} = [f(e(\mathbf{a})), f(s(\mathbf{a}))], \quad I^{\mathbf{a}} = [0, 1]^{l(\mathbf{a})}.$$

Then the previous observation shows that the map

$$\bigsqcup_{\mathbf{a}} J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a}) \to M$$

defined by

$$(t; s_1, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \longrightarrow (\gamma_0 \circ_{s_1} \cdots \circ_{s_l} \gamma_l)(t)$$

is surjective. Once more it is not difficult to extract the appropriate equivalence relation on the disjoint union.

Define

$$\mathcal{R}_f = \bigsqcup_{\mathbf{a}} J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a}) / \sim \tag{B.8}$$

where the relations $\sim$ are given by

$$\begin{aligned}(t;\, s_1, \cdots, s_{i-1}, 1, s_{i+1}, \cdots, s_l;\, \gamma_0, \cdots, \gamma_l) \sim \\ (t;\, s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_l;\, \gamma_0, \cdots, \gamma_0, \cdots, \gamma_{i-1} \circ_1 \gamma_i, \cdots, \gamma_l)\end{aligned} \tag{B.9}$$

and

$$(t;\, s_1, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim \begin{cases} (t;\, s_1, \cdots, s_{i-1}; \gamma_0, \cdots, \gamma_{i-1}), & \text{if}\, t \in [f(a_i), f(a_0)] \\ (t;\, s_{i+1}, \cdots, s_l; \gamma_{i+1}, \cdots, \gamma_l), & \text{if}\, t \in [f(a_{l+1}), f(a_i)]. \end{cases} \tag{B.10}$$

The map ϕ respects the equivalence relation $\sim$ so gives a well defined map

$$\mathcal{R}_f \longrightarrow M.$$

An elementary analysis now shows that all the identifications which can take place are consequences of B.9 and B.10. This leads to the following theorem.

Theorem B.3. *The map*

$$\phi : \mathcal{R}_f \to M$$

is a homeomorphism.

Proof. The first step is to check that the relations B.9 are the only relations which can occur if all the s_i are non-zero. In this case we are dealing with genuine flow lines so the result follows from the definition of the relation $\simeq$ used to construct the model for $\bar{\mathcal{M}}(a,b)$, Theorem B.2, and the fact that two flow lines with a common point are equal.

If one of the s_i's is zero then we are dealing with a piecewise flow line. If two piecewise flow lines have a point in common then this point must be one of the joining points or else they have a common segment. The first set of relations in B.10 account for the identifications which occur between points on piecewise flow-lines of the form $\gamma \circ \delta_1$ and $\gamma \circ \delta_2$. The second set of relations in B.10 account for the identifications between points on two piecewise flow lines of the form $\delta_1 \circ \gamma$ and $\delta_2 \circ \gamma$.

An elementary analysis now shows that all the identifications which can take place are consequences of B.9 and B.10. □

B.0.2 The classifying space of a Morse function

In this section we continue to assume that $f : M \to \mathbb{R}$ is a Morse-Smale function. The goal of this section is to prove the following theorem.

Theorem B.4. *There is a natural homeomorphism*

$$\psi : \mathcal{R}_f \to B\mathcal{C}_f.$$

In view of Theorem B.3 this shows that $B\mathcal{C}_f$ is homeomorphic to M, and completes the proof of Theorem 13.17.

Recall from Section 5.4.3 that the classifying space of $\mathcal{C}_f$ is given by

$$B\mathcal{C}_f = \coprod_{\mathbf{a}} \Delta^{l(\mathbf{a})+1} \times \bar{\mathcal{M}}(\mathbf{a})/\sim$$

where Δ^n is the standard n-simplex. The identifications $\sim$ are given by the following rules. If $t \in \Delta^{l(\mathbf{a})}$ and $x \in \bar{\mathcal{M}}(\mathbf{a})$, then

$$(t, d_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{l(\mathbf{a})+2}$ and $x \in \bar{\mathcal{M}}(\mathbf{a})$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

Here

1. $\delta_i : \Delta^n \to \Delta^{n+1}$ is the inclusion of the i-th face;
2. $\sigma_j : \Delta^{n+1} \to \Delta^n$ is the j-th degeneracy, given by projecting linearly onto the j-th face;
3. $d_i : \bar{\mathcal{M}}(a_0, \ldots, a_{l+1}) \to \bar{\mathcal{M}}(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{l+1})$ is given by

$$d_i(\gamma_0, \ldots, \gamma_l) = \begin{cases} (\gamma_1, \cdots, \gamma_n) & \text{for } i = 0 \\ (\gamma_0, \cdots, \gamma_i \circ \gamma_{i+1}, \cdots \gamma_n) & \text{for } 1 \leq i \leq l \\ (\gamma_0, \cdots, \gamma_{l-1}) & \text{for } i = l. \end{cases}$$

4. $s_j : \bar{\mathcal{M}}(a_0, \ldots, a_{l+1}) \to \bar{\mathcal{M}}(a_0, \ldots, a_j, a_j, \ldots, a_{l+1})$ is given by

$$s_j(\gamma_0, \cdots, \gamma_l) = (\gamma_0, \cdots \gamma_j, 1, \gamma_{j+1}, \cdots, \gamma_l)$$

Recall that $\mathcal{R}_f$ is the union of spaces of the form

$$J_{\mathbf{a}} \times I^{\mathbf{a}} \times \mathcal{K}(\mathbf{a})$$

where $\mathbf{a} = (a_0, \cdots, a_{l+1})$ is an ordered sequence of critical points. Recall also that the spaces $\mathcal{K}(a_{i-1}, a_i)$ are homeomorphic to the compactified spaces $\bar{\mathcal{M}}(a_{i-1}, a_i)$ in such a way that the composition in the category corresponds to $\circ_1$.

Thus the construction of $\mathcal{R}_f$ is very similar to that of $B\mathcal{C}_f$. the main difference is that $\mathcal{R}_f$ is constructed from the cubes $J_{\mathbf{a}} \times I^{\mathbf{a}}$ whereas $B\mathcal{C}_f$ is constructed from simplices. The main point in the argument to prove Theorem 13.17 is to show that the equivalence relations used to define $\mathcal{R}_f$ can be imposed in two steps; the first step turns the cubes into simplices and the second step imposes the identifications among the simplices that make up $B\mathcal{C}_f$.

Proof of Theorem 13.17

First we look at the image of a single cube

$$J_{\mathbf{a}} \times I^{\mathbf{a}} = J_{\mathbf{a}} \times I^{\mathbf{a}} \times (\gamma_0, \cdots, \gamma_l),$$

where $\mathbf{a}$ is the ordered sequence $(a_0, \ldots, a_{l+1})$, in the quotient space $\mathcal{R}_f$. For each i with $0 \leq i \leq l+1$ define

$$\mathbf{a}_i = (a_0, \cdots a_{i-1}, a_{i+1}, \cdots a_{l+1})$$

and a map

$$\partial_i : J_{\mathbf{a}_i} \times I^{l-1} \longrightarrow J_{\mathbf{a}} \times I^l$$

by the formula

$$\partial_i(t; s_1, \cdots, s_{l-1}) = \begin{cases} (t; 0, s_1, \cdots, s_{l-1}), & \text{if}\, i = 0 \\ (t; s_1, \cdots, s_{i-1}, 1, s_i, \cdots, s_{l-1}), & \text{for}\, 1 \leq i \leq l \\ (t; s_1, \cdots, s_{l-1}, 0), & \text{if}\, i = l+1. \end{cases}$$

Now consider the spaces

$$J_{\mathbf{a}} \times I^{l-1} / \sim$$

where we make the following list of identifications: If $1 \leq i \leq l$, so that $J_{\mathbf{a}} = J_{\mathbf{a}_i}$, then

$$(t; s_1, \cdots s_{i-1}, 0, s_{i+1}, \cdots, s_l) \sim \begin{cases} (t; s_1, \cdots, s_{i-1}, 0, s'_{i+1}, \cdots s'_l) & \text{if}\, f \in [f(a_i), f(a_0)] \\ (t; s'_1, \cdots, s'_{i-1}, 0, s_{i+1}, \cdots, s_l), & \text{if}\, t \in [f(a_{l+1}, f(a_i)]; \end{cases} \quad \text{(B.11)}$$

if $i = 0, l+1$ then

$$\begin{aligned} (t; 0, s_2, \cdots, s_l) &\sim (t; 0, s'_2, \cdots s'_l) && \text{if}\, t \in [f(a_1), f(a_0)] \\ (t; s_1, \cdots, s_{l-1}, 0) &\sim (t; s'_1, \cdots, s'_{l-1}, 0) && \text{if}\, t \in [f(a_{l+1}, a_l)]; \end{aligned} \quad \text{(B.12)}$$

finally

$$\begin{aligned} (f(a_{l+1}); s_1, \cdots, s_l) &\sim (f(a_{l+1}); s'_1, \cdots, s'_l) \\ (f(a_0; s_1, \cdots, s_l) &\sim (f(a_0); s'_1, \cdots, s'_l). \end{aligned} \quad \text{(B.13)}$$

It is straightforward to check that if two points in $J_{\mathbf{a}} \times I^{\mathbf{a}}$ are identified then they have the same image in $\mathcal{R}_f$. So we can construct $\mathcal{R}_f$ from the spaces $J_{\mathbb{A}} \times I^{\mathbb{A}} / \sim$. However the space $J_{\mathbf{a}} \times I^l / \sim$ is naturally homeomorphic to an $(l+1)$-simplex, and using these homeomorphisms the map ∂_i corresponds to the map δ_i, that is the inclusion of the i-th face. More precisely, we have the following combinatorial result, whose verification is straightforward.

Lemma B.5. *There are homeomorphisms*

$$h_{\mathbf{a}} : J_{\mathbf{a}} \times I^{\mathbf{a}} / \sim \to \Delta^{l+1}$$

which make the following diagrams commute

$$\begin{array}{ccc} J_{\mathbf{a}} \times I^{\mathbf{a}} / \sim & \xrightarrow{h_{\mathbf{a}}} & \Delta^{l+1} \\ \partial_i \uparrow & & \uparrow \delta_i \\ J_{\mathbf{a}_i} \times I^{\mathbf{a}_i} / \sim & \xrightarrow[h_{\mathbf{a}_i}]{} & \Delta^l. \end{array}$$

At this stage we have used up the first relations B.10 in the definition of $\mathcal{R}_f$. Now we impose the relations B.9 to get the following result. Once more the proof is straightforward.

Lemma B.6. *There is a homeomorphism*

$$\mathcal{R}_f \cong \bigsqcup_{\mathbf{a}} \Delta^{l(\mathbf{a})+1} \times \mathcal{K}(\mathbf{a})/\sim$$

where, in the coordinates for Δ^n given by

$$\Delta^n = \left\{ (s_1, \ldots, s_n) \in \mathbb{R}^n : 0 \leq s_i \leq 1, \ \textit{and} \ \sum_{i=1}^{n} s_i \leq 1 \right\},$$

$$(s_0, \cdots, s_{i-1}, 0, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_l) \sim \tag{B.14}$$

$$\begin{cases} (s_1, \cdots s_l; \gamma_1, \cdots, \gamma_l) & \textit{if } i = 0 \\ (s_0, \cdots, s_{i-1}, s_{i+1}, \cdots, s_l; \gamma_0, \cdots, \gamma_{i-1} \circ_1 \gamma_i, \cdots \gamma_l) & \textit{if } 1 \leq i \leq l-1 \\ (s_0, \cdots, s_{l-1}; \gamma_0, \cdots, \gamma_{l-1}) & \textit{if } i = l \end{cases} \tag{B.15}$$

We can now complete the proof of Theorem 13.17. First we must regard the category $\mathcal{C}_f$ as the category with spaces of morphisms $\mathcal{K}(a, b)$ and composition law defined by $\circ_1$. Now recall that we are assuming that the sequence $\mathbf{a}$ is strictly ordered, that is $\mathbf{a} = (a_0, \ldots, a_{l+1})$ with $a_i > a_{i+1}$ using Smale's partial ordering. If we now compare the model for $\mathcal{R}_f$ given by Lemma B.6 with the definition of $B\mathcal{C}_f$ we see that the difference is that in Lemma B.6 we have used the space of *non-degenerate* simplices rather than the space of all simplices. Thus using Lemma B.6 we have constructed a map

$$\mathcal{R}_f \to B\mathcal{C}_f.$$

Now using the following two properties of $\mathcal{C}_f$

1. the only morphism in $\mathcal{C}_f$ from the object a to itself is the identity,
2. if $\alpha_1, \beta_1 : a \to b$, $\alpha_2, \beta_2 : b \to c$ are morphisms in $\mathcal{C}_f$ such that

   $$\alpha_2 \circ \alpha_1 = \beta_2 \circ \beta_1$$

   then it follows that

   $$\alpha_1 = \beta_1, \qquad \alpha_2 = \beta_2,$$

one simply verifies that the map

$$\mathcal{R}_f \to B\mathcal{C}_f.$$

is a bijection with an obviously continuous inverse.

□

We end this appendix by pointing out that a somewhat weaker version of Theorem 13.17 can be proved in a much simpler fashion. Indeed one only needs basic facts about the homotopy theory of topological categories, such as is contained in [129] or [135]. More specifically we give a quick proof of the following:

Theorem B.7. *Let $f : M \to \mathbb{R}$ be a Morse function on a closed Riemannian manifold satisfying the Morse-Smale property. Then there is a map*

$$\phi : B\mathcal{C}_f \to M$$

that induces an isomorphism in homology.

Proof. Let $\mathcal{C}$ be a small category. Following [110] define a subdivision of $\mathcal{C}$, $sd\mathcal{C}$, as follows:

Objects. The objects of $sd\mathcal{C}$ are the morphisms in $\mathcal{C}$.

Morphisms. Let $\gamma_1 : a_1 \to b_1$ and $\gamma_2 : a_2 \to b_2$ be objects in $sd\mathcal{C}$. A morphism $\gamma_1 \to \gamma_2$ consists of a pair of morphisms in $\mathcal{C}$

$$\alpha : a_1 \to a_2, \qquad \beta : b_2 \to b_1$$

such that the following diagram commutes

$$\begin{array}{ccc} a_1 & \xrightarrow{\gamma_1} & b_1 \\ {\scriptstyle\alpha}\downarrow & & \uparrow{\scriptstyle\beta} \\ a_2 & \xrightarrow[\gamma_2]{} & b_2. \end{array}$$

The composition law is the natural one.

For technical reasons we need to enlarge the category $sd\mathcal{C}_f$. Define $\tilde{sd}(\mathcal{C}_f)$ as follows:

Objects. The objects of $\tilde{sd}(\mathcal{C}_f)$ are pairs (γ, x) where γ is a piecewise flow line and x is a point on γ.

Morphisms There are no morphisms from (γ_1, x_1) to (γ_2, x_2) unless $x_1 = x_2$. If $x_1 = x_2$ then the morphisms from (γ_1, x_1) to (γ_2, x_2) are the same as the morphisms from γ_1 to γ_2 in the category $sd(\mathcal{C}_f)$.

There is an obvious functor

$$\Psi : \tilde{sd}(\mathcal{C}_f) \to sd(\mathcal{C}_f)$$

given by forgetting the preferred point. This functor induces a map of classifying spaces

$$B\Psi : B\tilde{sd}(\mathcal{C}_f) \to Bsd(\mathcal{C}_f).$$

Lemma B.8. *The map $B\Psi : B\tilde{sd}(\mathcal{C}_f) \to Bsd(\mathcal{C}_f)$ is a homotopy equivalence.*

Proof. The map induced by the forgetful functor Ψ on the space of chains of composable morphisms of length n is a fibration with contractible fibers and therefore a homotopy equivalence. It is now standard, from the theory of simplicial spaces (see [110]) that the map on the geometric realizations of the nerves of the categories, $B\Psi$ is a homotopy equivalence. □

From the manifold M construct a topological category $\underline{M}$ whose objects are the points of M and whose morphisms consist only of the identity maps. Thus there are no morphisms in $\underline{M}$ from x to y if $x \neq y$. It is clear from the construction of the classifying space that

$$B\underline{M} = M.$$

There is a functor $\Theta : \tilde{sd}(\mathcal{C}_f) \to \underline{M}$ defined by sending (γ, x) to x and this induces a map of classifying spaces

$$B\Theta : B\tilde{sd}(\mathcal{C}_f) \to B\underline{M} = M.$$

To complete the proof of Theorem B.7 it now suffices to prove the following:

Lemma B.9. *The map in homology $(B\Theta)_* : H_*(B\tilde{sd}(\mathcal{C}_f)) \to H_*(M)$ is an isomorphism.*

Proof. For a point $x \in M$, consider the fiber, $B\Theta^{-1}(x) \subset B\tilde{sd}(\mathcal{C}_f)$. This is the classifying space of the subcategory of $\Theta^{-1}(x)$ of $\tilde{sd}(\mathcal{C}_f)$ whose objects are pairs (γ, x) where γ is a piecewise flow lines in M that goes through x. Notice that this category has a terminal object. It is the pair (γ_x, x) where γ_x is the unique flow line (as opposed to piecewise flow line) that goes through x. The reader can check that this is a terminal object by reviewing what the morphisms are in the category $\Theta^{-1}(x) \subset \tilde{sd}(\mathcal{C}_f)$. Thus $B\Theta^{-1}(x)$ is contractible for every $x \in M$.

Now if we knew that $B\Theta$ was a fibration, we would conclude that it is a homotopy equivalence, since all of its fibers are contractible. Failing this we rely on a classical result in algebraic topology called the *"Vietoris - Begle Theorem"*. (see [141] chapter 6, theorem 14).

Theorem B.10. *. Let $f : X' \to X$ be a closed continuous, surjective map between paracompact Hausdorff spaces. Assume that $\tilde{\bar{H}}^*(f^{-1}(x)) = 0$ for all $x \in X$. Then*

$$f^* : \bar{H}^*(X) \to \bar{H}^*(X')$$

is an isomorphism. Here $\bar{H}^*$ denotes Cech cohomology.

We may therefore conclude that $(B\Theta)^*$ is an isomorphism in Cech cohomology. But since M is a manifold, it is a CW complex, and so its Cech cohomology is equal to its ordinary (singular) cohomology. Now this is true for $B\tilde{sd}(\mathcal{C}_f)$ as well, for the following reason. Since we assumed that $f : M \to \mathbb{R}$ satisfied the Morse-Smale property, the moduli spaces of piecewise flows $\bar{\mathcal{M}}(a,b)$ are all manifolds, and hence CW-complexes. But the morphism spaces of $\tilde{sd}(\mathcal{C}_f)$ are built out of these spaces and are hence CW-complexes themselves. One then checks quickly that this implies the classifying space $B\tilde{sd}(\mathcal{C}_f)$ is therefore a CW-complex. Hence we can conclude that its Cech cohomology is isomorphic to its singular cohomology. Thus $B\Theta$ induces an isomorphism on singular cohomology, and therefore on singular homology. □

□

We end by noting that the only place in the proof of Theorem B.7 that the Morse-Smale assumption about the Morse function $f : M \to \mathbb{R}$ was used, was in using the fact that the moduli spaces $\bar{\mathcal{M}}(a,b)$ were manifolds (actually them being CW-complexes was sufficient). Theorem B.7 would hold if one simply knew this fact.

Bibliography

[1] M. Ando, A. Blumberg, D. Gepner, M.J Hopkins, and C. Rezk, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology*. J. Topol. 7 (2014), no. 3, 869–893.

[2] J.F. Adams, *On the structure and applications of the Steenrod algebra*, Comm. Math. Helv. **32** (1958), 180-214

[3] J.F. Adams, *On the non-existence of elements of Hopf invariant one*, Annals of Math. **2** 72, (1960), 20 - 104.

[4] J.F. Adams, *Vector fields on spheres*, Annals of Math. **2**, (1963), 181-195. 75, (1962), 603-632.

[5] J. F. Adams, *On the groups $J(X)$ I*, Topology **2**(3),

[6] J. F. Adams, *A variant of E. H. Brown's representability theorem*, Topolopgy **10,** (1971), 185-198

[7] J. F. Adams, *Stable Homotopy and Generalised Homology*, The University of Chicago Press (1974).

[8] M.F. Atiyah, *Thom complexes* , Proc. London Math. Soc. **(3)** , no. 11 (1961), 291–310.

[9] M. Atiyah and R. Bott, *The Yang - Mills equations over Riemann surfaces*, Phil. Trans. Royal Soc. Lond. **308**, (1982), 523-615.

[10] M. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Symp. Pure Math. **v. III,** AMS, Providence, R.I., (1961), 7-38.

[11] M. Atiyah, and G. Segal, *Equivariant K-theory and completion*, J. Diff. Geom. **3** (1969), 1 - 18.

[12] M.G. Barratt, J.D.S Jones, M.E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*, J. Lond. Math. Soc. (2) **30,** (1984), 533-550.

[13] A.J Blumberg, R.L Cohen, and C. Schlichtkrull, *Topological Hochschild homology of Thom spectra and the free loop space*, Geometry and Topology **14,** (2010), 1165-1242.

[14] R. Bott, *The stable homotopy of the classical groups* Ann. of Math. **70** (1959), 313–337.

[15] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer Grad. Texts **82**, 1982.

[16] Glen E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics **vol. 139,**Springer Verlag, 1993.

[17] W. Browder, *The Kervaire invariant of framed manifolds and its generalization*, Annals of Math. (2) **90,** (1969), 157-186.

[18] W. Browder, *Surgery on Simply Connected Manifolds* Springer (1972),

[19] E.H. Brown Jr. *Cohomology theories*, Annals of Math. **Vol. 75** no. 3 (1962), 467-484.

[20] E. H. Brown Jr. *Abstract homotopy theory*, Trans. of AMS, **119,** (1965), 79-85.

[21] E. H. Brown Jr. *Generalizations of the Kervalre invariant*, Annals of Mathematics **95,**368-383 (1972)

[22] E. H. Brown Jr. and F. P. Peterson, *Relations among characteristic classes I*, Topology **3** (1964), 39-52.

[23] E. H. Brown Jr. and F. P. Peterson, *Relations among characteristic classes II*, Ann. of Math. **81** (1965), 356-363.

[24] E.H. Brown Jr. and F.P. Peterson, *The Kervaire invariant of 8k+2 -manifolds*, American J. of Math. **(88),** no. 4 (1966), 815-826.

[25] E. H. Brown Jr. and F. P. Peterson, *A universal space for normal bundles of n-manifolds*, Comment. Math. Helv. **54** (1979), 405-430.

[26] S.S. Chern, *On the multiplication in the characteristic ring of a sphere bundle*, Annals of Math. **49** (1948), 362 - 372.

[27] R.L. Cohen, *Odd primary infinite families in stable homotopy theory*, Memoirs of A.M.S, **30** no. 242 (1981)

[28] R. L. Cohen, *The immersion conjecture for differentiable manifolds*, Ann. of Math. **122** no. 2 (1985), 237-328.

[29] R. L. Cohen, *Multiplicative properties of Atiyah duality*, Homology, Homotopy, and its Applications, **vol. 6 no. 1,** (2004), 269-281.

[30] R.L. Cohen, *The Topology of Fiber Bundles*, Lecture Notes, originally written in 2002, AMS Open Math Notes Series, https://www.ams.org/open-math-notes/omn-view-listing?listingId=110706, (2017)

[31] R. L. Cohen, *Floer homotopy theory, realizing chain complexes by module spectra, and manifolds with corners,* in Proc. of Fourth Abel Symposium, Oslo, 2007, ed. N. Baas, E.M. Friedlander, B. Jahren, P. Ostvaer, Springer Verlag (2009), 39-59. preprint: arXiv:0802.2752

[32] R. L. Cohen, *Floer homotopy theory, revisited*, Handbook of Homotopy Theory, edited by H. Miller, CRC Press (2020), 369-404.

[33] R.L. Cohen, J.D.S. Jones, and M.E. Mahowald, *The Kervaire invariant of immersions*, Inventionnes math. **vol. 79,** (1985), 95-123

[34] R.L. Cohen, J.D.S Jones, and G.B. Segal, *Morse theory and classifying spaces*, preprint: http://math.stanford.edu/ ralph/morse.ps (1995)

[35] R. L. Cohen, J. D. S. Jones, and G. B. Segal, *Floer's infinite-dimensional Morse theory and homotopy theory*, The Floer memorial volume, Progr. Math., vol. 133, Birkhauser, Basel, 1995, pp. 297 -325.

[36] R.L. Cohen and M.E. Mahowald

[37] P. Conner and E. Floyd, *Differentiable periodic maps*, Berlin-Gottingen-Heidelberg-New York, Springer Verlag (1964).

[38] E.B. Curtis, *Simplicial homotopy theory*, Adv. in Math **6** (1971), 107-209.

[39] S.K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geometry **18** (2), (1983) ,279-315.

[40] S.K. Donaldson, *Irrationality and the h-cobordism conjecture*, J. Differential Geometry **26** (1) (1987), 141-168.

[41] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Springer Verlag, (1956).

[42] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four - Manifolds*, Oxford Math. Monographs, Oxford Univ. Press, 1990.

[43] C.J. Earle and J. Eells, *A fibre bundle description of Teichmuller theory*, J. Diff. Geom. **3,** (1969), 19-43.

[44] S. Eilenberg, and S. MacLane, *On the groups* $H(\pi, n)$ *I, II, III*, Annals of Math. **58**, (1953), 55-106, **60** (1954), 49 - 139, 513 - 557.

[45] C. Ehresmann, *Sur la topologie de certains espaces homogenes*, Annals of Math, **35**, (1934), 396 - 443.

[46] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Grad. Studies in Math. **(48)** , Amer. Math. Soc., (2002).

[47] A.D. Elmendorf, I. Kriz, M.A Mandell, and J. P May *Rings, modules and algebras in stable homotopy theory* , with an appendix by M. Colem Mathematical Surveys and Monographs, **47,** (1997)

[48] J. M. Franks, *Morse-Smale flows and homotopy theory*, Topology **18** (1979), 119-215.

[49] D.S. Freed and K. Uhlenbeck, *Instantons and Four - manifolds*, MSRI publications 1, Springer Verlag 1984.

[50] M.H. Freedman, *The topology of four-dimensional manifolds*, Jour. Differential Geometry, **17** (3), (1982), 357-453.

[51] M. H. Freedman and F. Quinn, *Topology of* 4*-manifolds*, Princeton Math Series, **vol. 39,** Princeton Univ. Press (1990).

[52] , H. Freudenthal, *(*1938), Über die Klassen der Sphärenabbildungen. I. Große Dimensionen, Compositio Mathematica, **(5)** (1938): 299–314.

[53] S. Galatius, *Mod p homology of the stable mapping class group*, Topology **43,** (2004), 1105-1132.

[54] S. Galatius, I. Madsen, U. Tillmann, and M. Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2009), 195-239.

[55] Søren Galatius and Oscar Randal-Williams, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377,

[56] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds I*, J. Amer. Math. Soc., **31 (1),** (2018), 215-264

[57] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds II*, Annals of Math. **(2), 186 (1)m**](2017), 127-204.

[58] J. Genauer, *Cobordism categories, corners, and surgery*, Ph.D thesis, Stanford University, (2009).

[59] J. Genauer, *Cobordism categories of manifolds with corners*, preprint arXiv:0810.0581 (2008).

[60] A. Gleason, *Spaces with a compact Lie group of transformations*, Proc. of A.M.S **1**, (1950), 35 - 43.

[61] M. Gromov, *Partial differential relations*, Springer, ISBN 3-540-12177-3 (1986).

[62] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory II* Geometry and Topology **3** (1999), 103 -118.

[63] T. Goodwillie, J.R Klein, and M. S. Weiss, *Spaces of smooth embeddings, disjunction and surgery*, Surveys on surgery theory, Vol. 2, 221284, Ann. of Math. Stud., 149, Princeton Univ. Press, Princeton, NJ, 2001.

[64] R. Hamilton, *Three manifolds with positive Ricci Curvature*, J. Diff. Geometry **17,** (2), (1982), 255-306.

[65] J.L. Harer, *Stability of the homology of mapping class groups of oriented surfaces*, Ann, if Math., **121,** 215-249.

[66] Hartman, *Ordinary Differential Equations* Wiley, (1964).

[67] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[68] A. Haefliger, *Plongements différentiables dans le domain stable*, Comment. Math. Helv. **37** (1963), 155-176.

[69] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Acad. Press, 1978.

[70] M. A. Hill, M. J. Hopkins, and D.C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Annals of Math. **184,** (2016), 1-262

[71] M. A. Hill, M. J. Hopkins, and D.C. Ravenel, *Equivariant Stable Homotopy Theory and the Kervaire Invariant Problem*, New Mathematical Monographs **40,** Cambridge University press, 2021.

[72] M. Hirsch, *Differential Topology*, Graduate Texts in Mathematics, **vol. 33**, Springer Verlag, 1976.

[73] M. Hirsch, *Immersions of manifolds*, Trans. A.M.S **93** (1959), 242-276

[74] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure and Applied Algebra, **165** issue 1 (2001), 63-127.

[75] M. Hovey, *Model Categories*, American Math Society, 1999.

[76] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149-208.

[77] W. Hurewicz, *Beitrage zur Topologie der Deformationen I - IV*, Nederl. Akad. Wetensch. Proc., **Ser. A 38** (1935), 112 - 119, 521 - 528. **39** (1936), 117 - 126, 215 - 224.

[78] W. Hurewicz, *On the concept of a fiber space*, Proc. Nat. Acad. Sci. U.S.A **41**, (1955), 956 - 961.

[79] D. Husemoller, *Fibre Bundles*, MacGraw - Hill, 1966, Springer-verlag, 1975.

[80] M. Hutchings, *Morse theory notes* U.C. Berkeley (1998)

[81] N. V. Ivanov, *Stabilization of the homology of the Teichmuller modular groups*, Algebra i Analiz **1,** (1989), 120-126; translation in Leningrad Math. J. **1,** (1990), 675-691.

[82] J.D.S. Jones, *The Kervaire invariant of extended power manifolds*, Topology, **17 (3), (1978)** 249-266

[83] M. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv., **34** (1960), 257-270.

[84] M.A Kervaire and J.W. Milnor, *A procedure for killing the homotory groups of diferentiable manifolds*, Symposia in Pure Math. A.M.S., **vol. III** (1961), 39-55.

[85] M.A. Kervaire and J.W. Milnor, *Groups of homotopy spheres: I*, Annals of Math., **vol. 77,** no. 3 (1963), 504-537

[86] A. Kriegl and P.W. Michor, *The Convenient Setting of Global Analysis*, Math. Surveys and Monographs, **53,** AMS, Providence (1997).

[87] P.B. Kronheimer and T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters, **1** (1994), 797-808.

[88] S. Lang, *Algebra*, Addison Wesley, 1971.

[89] G. Laures, *On cobordism of manifolds with corners*, Trans. of AMS **352** no. 12, (2000), 5667-5688.

[90] H.B. Lawson Jr., *The Theory of Gauge Fields in Four Dimensions* CBMS Regional Conf. Series **58**, AMS 1986.

[91] L.G. Lewis, J.P. May, and M. Steinberger, *Equivariant Stable Homotopy Theory*, Lecture Notes in Math. **1213** Springer Verlag, 1986.

[92] E. L. Lima, *The Spanier-Whitehead duality in new homotopy categories*, Summa Brasil. Math. **4** (1958-1959), 91-148.

[93] A. Liulevicius, *The factorization of cyclic reduced powers by secondary cohomology operations*, Memoirs of A.M.S **42** (1962).

[94] S. MacLane, *Homology*, Springer Verlag, (1963).

[95] Ib Madsen and Ulrike Tillmann, *The stable mapping class group and* $Q\mathbb{CP}^\infty$, Inventionnes of Math. **145,** (2001), 509-544.

[96] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) 165 (2007), no. **3,** 843–941.

[97] M.E Mahowald, *A new infinite family in* ${}_2\pi_*^s$, Topology **vol. 16** (1977), 249-256.

[98] M.E. Mahowald and M.C. Tangora, *On secondary operations which detect homotopy classes*, Bd. Soc. Mat. Mer. (1967), 71-75.

[99] M.E. Mahowald and M.C. Tangora, *Some differentials in the Adams spectral sequence*, Topology **6,** (1967), 349-369.

[100] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, American Mathematical Society (2002)

[101] W. S. Massey, *On the Stiefel-Whitney classes of a manifold*, Amer. J. Math. **82** (1960), 92-102

[102] J.P. May, *The cohomology of restricted Lie algebras and of Hopf algebras*, dissertation, Princeton University, 1964.

[103] J.P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, 1967.

[104] J.P. May, *An algebraic approach to the Steenrod operations*, Steenrod's 60th birthday conference, Springer Lecture Notes

[105] J.P. May, *The geometry of infinite loop spaces*, Springer Lecture Notes **vol. 271** (1972)

[106] M.A. Mandell, J.P. May, S. Schwede, B. Shipley, *Model categories of diagram spectra*, Proc. Lond. Math. Soc. 82 (2001), 441-512.

[107] A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May, *Rings, modules, and algebras in stable homotopy theory*, Math. Surveys and Monographs, **47** A.M.S. (1997).

[108] J. P. May and K. Ponto, *More concise algebraic topology. Localization, completion, and model categories*, Univ. of Chicago Press, 2012.

[109] J. McCleary, *User's Guide to Spectral Sequences*, Math. Lect. Series **12** Publish or Perish 1985.

[110] D. McDuff and G. Segal, *Homology fibrations and the "Group Competion" theorem*, Inventionnes Math., **31** (1976), 279-284.

[111] R.J. Milgram, *The bar construction and abelian H-spaces*, Illinois J. Math. **11** (1967), 242–250.

[112] J. Milnor, *Morse Theory*, Annals of Math. Studies, vol **51**, Princeton Univ. Press, 1963, J. Milnor, *On spaces having the homotopy type of a CW - complex*, Trans. Amer. Math. Soc. **90**, (1959), 272-280.

[113] J. Milnor, *Lectures on the h-Cobordism Theorem*, Princeton Legacy Library, (1965)

[114] J. Milnor, *On spaces having the homotopy type of a CW - complex*, Trans. Amer. Math. Soc. **90**, (1959), 272-280.

[115] J. Milnor, *Construction of universal bundles. I. and II.* Ann. of Math. **2** 63, (1956), 272–284, and Ann. of Math. **2** 63, (1956), 430–436.

[116] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Annals of Mathematics, **64** (2), (1956), 399 - 405,

[117] J. Milnor, *The Steenrod algebra and its dual*, Annals of Mathematics **67,** (1958), 150-171.

[118] J. Milnor, *On the cobordism ring* Ω^* *and a complex analogue, part I*, American J. of Math. **82 no. 3,**(1960), 505-521.

[119] J. Milnor, *A procedure for killing the homotopy groups of differentiable manifolds*, Symposia in Pure Math. A.M.S., **vol. III** (1961), 39-55.

[120] J. Milnor, *The realization of a semi-simplicial complex*, Ann. of Math **65,** (1957), 357-362.

[121] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Annals of Math. Studies, **76**, Princeton Univ. Press, 1974.

[122] S. Mitchell, *Notes on principal bundles and classifying spaces*, Univ. of Washington preprint, https://sites.math.washington.edu/ mitchell/Notes/prin.pdf

[123] J. W. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Math. Monographs **Vol. 3**, American Math. Society (2007).

[124] R. E. Mosher and M.C. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Harper and Row, 1968.

[125] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry, Vol. II, Prog. Math. **36,** Birkhauser, Boston (1983), 271 - 328.

[126] S. P. Novikov, *Some problems in the topology of manifolds connected with the theory of Thom spaces*, Doklady Akadami Nauk SSR, **vol. 132,** (1960), 1031-1034.

[127] L.S. Pontrjagin, *Characteristic cycles on differentiable manifolds* Mat. Sbornik N.S. **21** 63 (1947), 233-284. AMS translation **32** (1950).

[128] D. Quillen, *Homotopical Algebra*, Springer Lecture Notes **(vol. 43),**Berlin, New York, 1967.

[129] D. Quillen, *Higher Algebraic K-theory I*, Springer Lecture Notes **(vol. 341),** (1972) 77 - 139.

[130] L. Qin, *On the Associativity of Gluing*, Journal of Topology and Analysis, **vol. 10, no. 03** (2018), 585-604 preprint: https://arxiv.org/abs/1107.5527

[131] D.C. Ravenel, *The non-existence of odd primary Arf invariant elements in stable homotopy*, Math. Proc. Camb. Phil. Soc. **83** (1978), 429-443.

[132] D.C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press (1986), reprinted as AMS Chelsea Publishing, **vol. 347)** (2004).

[133] H. Samelson, *Beitrage zur Topologie der Gruppenmannigfaltigkeiten*, Ann. of Math. **2**, 42, (1941), 1091 - 1137.

[134] H. Samelson, *Notes on Lie Algebras*, Springer Verlag, 1990.

[135] G.B. Segal, *Classifying spaces and spectral sequences*, Publ. Math. I.H.E.S. **34,** (1968), 105-112.

[136] J.-P Serre, *Homologie singuliere des espaces fibres*, Ann. Math. **2**, 54, (1951), 425-505.

[137] B. Shipley, *Symmetric spectra and topological Hochschild homology, K*-theory, **19** (2) (2000), 155-183. ArXiv: math.AT/9801079

[138] S. Smale, *A classification of immersions of the two-sphere*, Trans. AMS **90** no. 2 (1959), 281-290.

[139] S. Smale, *On Gradient Dynamical Systems*, Annals of Math., **74**, (1961), 199–206

[140] S. Smale, *Generalized Poincaré conjecture in dimensions greater than four*, Annals of Math., **74** (1961), 391-406.

[141] E.H. Spanier, *Algebraic Topology*, McGraw Hill, 1966 (reprinted by Springer Verlag)

[142] M. Spivak, *Differential Geometry II*, Publish or Perish, 1970.

[143] N.E. Steenrod *The topology of fibre bundles*, Princeton Mathematical Series **14**, Princeton Univ. Press, 1951.

[144] N.E. Steenrod and D.B.A Epstein, *Cohomology Operations*, Annals of Math. Studies **no. 50,** Princeton University Press, 1962.

[145] E. Stiefel, *Richtungsfelder und Ferparallismus in Mannigfaltigkeiten*, Comment. Math. Helv. **8** (1936), 3 - 51.

[146] R. E. Stong *Notes on cobordism theory* Princeton Univ. press, 1968.

[147] M. C. Tangora, *The cohomology of the Steenrod algebra*, dissertation, Northwestern Univ., 1966.

[148] T. Tao, *Perelman's proof of the Poincaré Conjecture: a nonlinear PDE perspective*, preprint arXiv:math/0610903v1 (2006).

[149] C. H. Taubes *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geometry **25** (3) (1987), 363-430.

[150] R. Thom, *Quelques propriétés globales des variétés differentiables*, Comment. Math. Helv., **28** (1954), 17-86.

[151] U. Tillmann, *On the homotopy type of the stable mapping class group*, Inventionnes Math., **130,** (1997), 257-275.

[152] F. Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, pp. 35–60, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

[153] F. Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Mathematics, vol. 1126 (1985), 318–419.

[154] C.T.C Wall, *Surgery on Compact Manifolds* Math. Surveys and Monographs **vol. 69** , AMS, (1970).

[155] C.T.C Wall, *Determination of the cobordism ring*, Annals of Math., **vol. 72,** (1960), 292-311.

[156] K. Wehrheim, *Smooth structures on Morse trajectory spaces, featuring finite ends and associative gluing* preprint: https://arxiv.org/abs/1205.0713

[157] T. Weston, *An introduction to cobordism theory*, http://www.math.umass.edu/ weston/ oldpapers/cobord.pdf.

[158] G. W. Whitehead, *Generalized homology theories*, Transactions of the American Mathematical Society **Vol. 102, No. 2** (1962), pp. 227-283

[159] G.W. Whitehead, *Elements of Homotopy Theory*, Grad. Texts in Mathematics, **61**, Springer Verlag, 1978,

[160] G.W. Whitehead, *Fifty years of homotopy theory*, Bull. Am. Math. Soc. (N.S) **8**, (1983), 1 - 29.

[161] J.H.C. Whitehead, *Combinatorial homotopy I and II* Bull. Am. Math. Soc. **55**, (1949), 213 - 245.

[162] M Weiss, *Calculus of embeddings*, Bull. Amer. Math. Soc. **33** (1996), 177 - 187

[163] M Weiss, *Embeddings from the point of view of immersion theory, Part I*, Geometry and Topology, **3** (1999), 67 - 101.

[164] H. Whitney, *The singularities of a smooth n-manifold in $(2n-1)$-space*, Annals of Mathematics **45** (1944), 247-293.

[165] H. Whitney, *On the theory of sphere bundles*, Proc. Nat. Acad. Sci. USA **26** (1940), 148 - 153.

[166] Z. Xu, *The strong Kervaire invariant problem in dimension 62*, Geom. Topol., **20 (3),** (2016), 1611-1624.