Open-closed field theories, string topology, and Hochschild homology

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Abstract

In this expository paper we discuss a project regarding the string topology of a manifold, that was inspired by recent work of Moore-Segal [29], Costello [18], and Hopkins and Lurie [27] on “open-closed topological conformal field theories”. In particular, given a closed, oriented manifold $M$, we describe the “string topology category” $S_M$, which is enriched over chain complexes over a fixed field $k$. The objects of $S_M$ are connected, closed, oriented submanifolds $N$ of $M$, and the space of morphisms between $N_1$ and $N_2$ is a chain complex homotopy equivalent to the singular chains $C_*(\mathcal{P}_{N_1,N_2})$ where $\mathcal{P}_{N_1,N_2}$ is the space paths in $M$ that start in $N_1$ and end in $N_2$. The composition pairing in this category is a chain model for the open string topology operations of Sullivan [36], and expanded upon by Harrelson [24] and Ramirez [32].

We will describe a calculation yielding that the Hochschild homology of the category $S_M$ is the homology of the free loop space, $LM$. Another part of the project is to calculate the Hochschild cohomology of the open string topology chain algebras $C_*(\mathcal{P}_N)$ when $M$ is simply connected, and relate the resulting calculation to $H_*(LM)$. These calculations generalize known results for the extreme cases of $N = \text{point}$ and $N = M$, in which case the resulting Hochschild cohomologies are both isomorphic to $H_*(LM)$. We also discuss a spectrum level analogue of the above results and calculations, as well as their relations to various Fukaya categories of the cotangent bundle $T^*M$ with its canonical symplectic structure.

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Introduction

In an open-closed topological field theory, one studies cobordisms between compact one-dimensional manifolds, whose boundary components are labeled by an indexing set, \( \mathcal{D} \). The cobordisms are those of manifolds with boundary, that preserve the labeling sets in a specific way. The set of labels \( \mathcal{D} \) are referred to as “D-branes”, and in the string theory literature these are boundary values of “open strings”. An open-closed field theory is a monoidal functor from a category built out of such manifolds and cobordisms, that takes values in a linear category, such as vector spaces, chain complexes, or even the category of spectra. In this paper we will discuss two flavors of such open-closed field theories: “topological quantum field theories” (TQFT) as introduced by Moore and Segal [29], and “topological conformal field theories”, (TCFT), as studied by Getzler [22] and Costello [18].

The open part of such a theory \( \mathcal{F} \) is the restriction of \( \mathcal{F} \) to the “open subcategory”. This is the full subcategory generated by those compact one-manifolds, all of whose path components have nonempty boundary. As Moore and Segal originally pointed out, the data of an open field theory can be encoded in a category (or as Costello points out, an \( A_\infty \)-category when \( \mathcal{F} \) is an open-closed TCFT), \( \mathcal{C}_\mathcal{F} \). The objects of \( \mathcal{C}_\mathcal{F} \) are the set of D-branes, \( \mathcal{D} \). The space of morphisms between \( \lambda_0 \) and \( \lambda_1 \in \mathcal{D} \) is given by the value of the theory \( \mathcal{F} \) on the object \( I_{\lambda_0,\lambda_1} \), defined by the interval \([0,1]\) where the boundary component 0 is labeled by \( \lambda_0 \), and 1 is labeled by \( \lambda_1 \). We denote this vector space by \( \mathcal{F}(\lambda_0,\lambda_1) \). The composition rules in this \( (A_\infty) \) category are defined by the values of \( \mathcal{F} \) on certain “open-closed” cobordisms. Details of this construction will be given below.

In this paper we will report on a project whose goal is to understand how the “String Topology” theory of a manifold fits into this structure. This theory, as originally introduced by Chas and Sullivan [10] starts with a closed, oriented \( n \)-dimensional manifold \( M \). It was shown in [12] that there is a (positive boundary) TQFT \( \mathcal{S}_M \), which assigns to a circle the homology of the free loop space,

\[
\mathcal{S}_M(S^1) = H_*(LM;k)
\]

with field coefficients. This was recently extended by Godin [23] to show that string topology is actually an open-closed homological conformal field theory. In this theory the set of D-branes \( \mathcal{D}_M \) is the set of connected, closed, oriented, connected submanifolds of \( M \). The theory assigns to a
compact one-manifold $c$ with boundary levels, the homology of the mapping space,

$$S_M(c) = H_*(\text{Map}(c, \partial; M)).$$

Here $\text{Map}(c, \partial; M)$ refers to the space of maps $c \to M$ that take the labeled boundary components to the submanifolds determined by the labeling. In particular, we write $\mathcal{P}_{N_0,N_1} = \text{Map}(I_{N_0,N_1}, \partial; M)$ for the space of paths $\gamma : [0,1] \to M$ such that $\gamma(0) \in N_0$, and $\gamma(1) \in N_1$. In Godin’s theory, given any open-closed cobordism $\Sigma_{c_1,c_2}$ between one-manifolds $c_1$ and $c_2$, there are homological operations

$$\mu_{\Sigma_{c_1,c_2}} : H_*(\text{Diff}(\Sigma_{c_1,c_2}); k) \otimes H_*(\text{Map}(c_1, \partial; M); k) \to H_*(\text{Map}(c_2, \partial; M); k).$$

An open-closed topological conformal field theory in the sense of Costello is a chain complex valued theory, and it is conjectured that the string topology theory has the structure of such a theory. The following theorem, which we report on in this paper, gives evidence for this conjecture.

**Theorem 1.** Let $k$ be a field.

1. There exists a DG-category (over $k$) $S_M$, with the following properties:

   (a) The objects are the set of $D$-branes, $D_M = \{\text{connected, closed, oriented submanifolds of } M\}$

   (b) The morphism complex, $\text{Mor}_{S_M}(N_1,N_2)$, is chain homotopy equivalent to the singular chains on the path space $C_*(\mathcal{P}_{N_1,N_2})$.

   The compositions in $S_M$ realize the open-closed string topology operations on the level of homology.

2. The Hochschild homology of $S_M$ is equivalent to the homology of the free loop space,

   $$HH_*(S_M) \cong H_*(LM; k).$$

**Note.** In this theorem we construct a DG-category with strict compositions rather than an $A_\infty$ category. See section 2 below.

Given any fixed submanifold $N$, the space of self-morphisms, $\text{Mor}_{S_M}(N,N) \simeq C_*(\mathcal{P}_{N,N})$ is a differential graded algebra. Again, on the level of homology, this algebra structure is the string topology product introduced by Sullivan [36]. In this note we pose the following question and report on its answer in a variety of special cases. (See Theorem 12 below.) Details will appear in [7].

**Question:** Let $M$ be a simply connected, closed submanifold. For which connected, oriented, closed submanifolds $N \subset M$ is the Hochschild cohomology of $C_*(\mathcal{P}_{N,N})$ isomorphic to the homology of the free loop space,

$$HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N})) \cong H_*(LM).$$
as algebras? The algebra structure of the left hand side is given by cup product in Hochschild cohomology, and on the right hand side by the Chas-Sullivan product string topology product.

We observe that in the two extreme cases \((N = \text{a point}, \text{and} N = M)\), affirmative answers to this question are known. For example, when \(N\) is a point, \(P_{N,N}\) is the based loop space, \(\Omega M\), and the statement that \(HH_*(C_*(\Omega M), C_*(\Omega M)) \cong H_*(LM)\) was known in the 1980’s by work of Burghelea, Goodwillie, and others. The Hochschild cohomology statement then follows from Poincaré duality. Similarly, when \(N = M\), then \(P_{N,N} \simeq M\), and the string topology algebra on \(C_*(P_{N,N})\) corresponds, via Poincaré duality, to the cup product in \(C_*(M)\). The fact that the Hochschild cohomology of \(C^*(M)\) is isomorphic to \(H_*(LM)\) follows from work of J. Jones in the 1980’s, and the fact that the ring structure corresponds to the Chas-Sullivan product was proved in [13]. In this note we are able to report on a calculation of \(HH^*(C_*(P_{N,N}), C_*(P_{N,N}))\) which yields an affirmative answer to this question in many cases. (See Theorem 12 below.) These cases include when the inclusion map \(N \hookrightarrow M\) is null homotopic. Thus

\[ HH^*(C_*(P_{N,N}), C_*(P_{N,N})) \cong H_*(LS^n) \]

for every connected, oriented, closed submanifold of a sphere \(S^n\). Other cases when one gets an affirmative answer to the above question include when the inclusion \(N \hookrightarrow M\) is the inclusion of the fiber of a fibration \(p : M \rightarrow B\), or more generally, when \(N \hookrightarrow M\) can be factored as a sequence of embeddings, \(N = N_0 \hookrightarrow N_1 \hookrightarrow \cdots N_i \hookrightarrow N_{i+1} \cdots N_k = M\) where each \(N_i \subset N_{i+1}\) is the inclusion of a fiber of a fibration \(p_{i+1} : N_{i+1} \rightarrow B_{i+1}\).

We point out that an amusing aspect of this question is that for any \(N \hookrightarrow M\) for which the answer is affirmative, then one can use this submanifold as a single D-brane and recover \(H_*(LM)\) as a Hochschild cohomology ring (i.e “one brane is enough”), and that all such branes yield the same answer.

This paper is organized as follows. In section one below we discuss the relevant background of open-closed topological field theories, including a review of work of Moore and Segal [29], and of Costello [18]. In section 2 we describe the ingredients of the proof of Theorem 1 and discuss the Hochschild cohomology calculations of the chain algebras, \(C_*(P_{N,N})\) in Theorem 12 below. The methods involve generalized Morita theory, and so yield comparisons between certain module categories over the algebras \(C_*(P_{N,N})\). We present these in Theorem 13 below. In the extreme cases mentioned above, these comparisons reduce to the standard equivalences of certain module categories over the cochains \(C^*(M)\) and the chains of the based loop space, \(C_*(\Omega M)\) (originally obtained in [19]). In section 3 we discuss possible relationships between the categories described here and certain Fukaya categories of the cotangent bundle, \(T^*M\) with its canonical symplectic structure.

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1 Open-closed Topological Field Theories

As mentioned in the introduction, the objects of study in an open-closed field theory are parameterized, compact, oriented one-manifolds, $c$, together with a labeling of the components of the boundary, $\partial c$, by elements of a set, $D$. An “open-closed” cobordism $\Sigma_{c_1,c_2}$ between two objects $c_1$ and $c_2$ is an oriented surface $\Sigma$, whose boundary is partitioned into three parts: the incoming boundary, $\partial_{in} \Sigma$ which is identified with $c_1$, the outgoing boundary $\partial_{out} \Sigma$ which is identified with $c_2$, and the remaining part of the boundary, referred to as the “free part”, $\partial_{free} \Sigma$ whose path components are labeled by $D$, with the property that $\partial_{free} \Sigma$ is itself a cobordism between $\partial c_1$ and $\partial c_2$, preserving the labeling. This is the usual notion of a cobordism of manifolds with boundary, with the additional data of the labeling set $D$. Figure 1 below is a picture of a one-manifold whose boundary components are labeled by elements of $D$, and figure 2 is a picture of an open-closed cobordism. In this picture the free part of the boundary, $\partial_{free} \Sigma$ is highlighted in red. In figure 3 a smooth surface is shown that is homeomorphic to the open-closed cobordism given in figure 2. The free part of the boundary is again highlighted in red.

\begin{center}
\begin{tikzpicture}
    \node (circ) at (0,0) [circle, draw, minimum size=1cm] {};
    \node (lambda1) at (-0.5,0) {$\lambda_1$};
    \node (lambda2) at (0,-0.5) {$\lambda_2$};
    \node (lambda3) at (0,-1) {$\lambda_3$};
    \node (lambda4) at (0,-1.5) {$\lambda_4$};
    \draw (circ) -- (lambda1);
    \draw (circ) -- (lambda2);
    \draw (circ) -- (lambda3);
    \draw (circ) -- (lambda4);
\end{tikzpicture}
\end{center}

Figure 1: A one manifold with labels $\lambda_i \in D$

In [29], Moore and Segal describe basic properties of open-closed topological quantum field theories, and in a sense, Costello then gave a derived version of this theory when he gave a description of open-closed topological conformal field theories.
1.1 The work of Moore and Segal on open-closed TQFT’s

In [29] Moore and Segal describe how an open-closed topological quantum field theory \( \mathcal{F} \) assigns to each one manifold \( c \) with boundary components labeled by \( D \), a vector space over a field \( k \), \( \mathcal{F}(c) \). The theory \( \mathcal{F} \) also assigns to every diffeomorphism class of open closed cobordism \( \Sigma_{c_1,c_2} \) a linear map

\[
\mathcal{F}(\Sigma_{c_1,c_2}) : \mathcal{F}(c_1) \rightarrow \mathcal{F}(c_2).
\]

This assignment is required to satisfy two main properties:

1. **Gluing:** One can glue two open-closed cobordisms when the outgoing boundary of one is identified with the incoming boundary of the other:

\[
\Sigma_{c_1,c_2} \# \Sigma_{c_2,c_3} = \Sigma_{c_1,c_3}.
\]

In this case the operation \( \mathcal{F}(\Sigma_{c_1,c_2} \# \Sigma_{c_2,c_3}) \) is required to be the composition:

\[
\mathcal{F}(\Sigma_{c_1,c_2} \# \Sigma_{c_2,c_3}) : \mathcal{F}(c_1) \xrightarrow{\mathcal{F}(\Sigma_{c_1,c_2})} \mathcal{F}(c_2) \xrightarrow{\mathcal{F}(\Sigma_{c_2,c_3})} \mathcal{F}(c_3).
\]

This condition can be viewed as saying that \( \mathcal{F} \) is a functor \( \mathcal{F} : \mathcal{C}_D \rightarrow \text{Vect}_k \), where \( \mathcal{C}_D \) is the cobordism category whose objects are one manifolds with boundary labels in \( D \), and whose
morphisms are diffeomorphism classes of open-closed cobordisms. Here the diffeomorphisms are required to preserve the orientations, as well as the boundary structure ($\partial_{in}, \partial_{out}$, and the labeling). $\text{Vect}_k$ is the category of vector spaces over the field $k$, whose morphisms are linear transformations between them.

2. **Monoidal:** There are required to be natural isomorphisms,

$$F(c_1) \otimes F(c_2) \xrightarrow{\cong} F(c_1 \sqcup c_2)$$

that makes $F$ into a monoidal functor. (The monoid structure in $\mathcal{C}_D$ is given by disjoint union of both the object manifolds and the morphism cobordisms.)

Let $F$ be an open-closed TQFT. The value of $F$ on a single unit circle, $F(S^1)$ is known as the closed state space of the theory $F$. It is well known that $F$ is a commutative Frobenius algebra over $k$. That is, there is an associative multiplication $\mu_F : F(S^1) \otimes F(S^1) \to F(S^1)$ coming from the value of $F$ on the pair of pants cobordism from $S^1 \sqcup S^1$ to $S^1$. The unit disk, viewed as having one outgoing boundary component, $\partial_{out}D^2 = S^1$, is a cobordism from the emptyset $\emptyset$ to $S^1$, and therefore induces a map $\iota : k \to F(S^1)$, which is the unit in the algebra structure. Thinking of the disk as having one incoming boundary component, $\partial_{in}D^2 = S^1$, induces a map $\theta_F : F(S^1) \to k$ which is the “trace map” in the theory. That is, the bilinear form

$$\langle \cdot, \cdot \rangle : F(S^1) \times F(S^1) \xrightarrow{\mu_F} F(S^1) \xrightarrow{\theta_F} k$$

is nondegenerate.

![Figure 4: The pair of pants cobordism inducing the multiplication $\mu_F : F(S^1) \otimes F(S^1) \to F(S^1)$.](image)

There is more algebraic structure associated to an open closed field theory $F$. As described by Moore and Segal, there is a category, $\mathcal{C}_F$ associated to the open part of the field theory.

**Definition 2.** The category $\mathcal{C}_F$ associated to an open-closed TQFT $F$ has as its objects the set of $D$-branes, $\mathcal{D}$. The space of morphisms between objects $\lambda_1$ and $\lambda_2$ is given by the value of the field theory $F$ on the one-manifold $I_{\lambda_1, \lambda_2}$ which is given by the interval $[0, 1]$ with boundary components labeled by $\lambda_1$ and $\lambda_2$. We write this space as $F(\lambda_1, \lambda_2)$. 

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The composition law in the category $\mathcal{C}_F$ is defined by the open-closed cobordism shown in figure 6.

Notice that the endomorphism algebras in this category, $\mathcal{F}(\lambda, \lambda)$, are also Frobenius algebras. For simplicity we write these algebras as $\mathcal{F}(\lambda)$. The trace maps are induced by the open-closed cobordism between $I_{\lambda, \lambda}$ and the empty set given by the disk as in figure 7.

We observe that the closed state space $\mathcal{F}(S^1)$ is necessarily commutative as an algebra, because the cobordisms shown in figure 8 admit an orientation preserving diffeomorphism between them that fixes the boundary pointwise. However for a $D$-brane $\lambda \in D$, the fact that the open-closed cobordisms shown in figure 9 are not diffeomorphic via an orientation preserving diffeomorphism that fixes the incoming and outgoing boundaries, imply that the Frobenius algebra $\mathcal{F}(\lambda, \lambda)$ may not
be commutative.

These algebras are, of course, related to each other. For example, the “whistle” open-closed cobordism given in figure 10 defines a ring homomorphism \( \theta_\lambda : \mathcal{F}(S^1) \to \mathcal{F}(\lambda) \), which, is easy to see takes values in the center \( Z(\mathcal{F}(\lambda)) \) (see [29] for details.) So in particular one has the following result.

**Proposition 3.** Any open-closed TQFT \( \mathcal{F} \) comes equipped with map of algebras

\[
\theta_\lambda : \mathcal{F}(S^1) \to Z(\mathcal{F}(\lambda))
\]

where \( Z(\mathcal{F}(\lambda)) \) is the center of the endomorphism algebra \( \mathcal{F}(\lambda) \), for any \( \lambda \in \mathcal{D} \).

Turning the whistle cobordism around, so that its incoming boundary is \( I_{\lambda,\lambda} \), and its outgoing boundary is \( S^1 \), defines a homomorphism \( \theta_\lambda^* : \mathcal{F}(S^1) \to \mathcal{F}(\lambda) \), which is not difficult to see is adjoint to \( \theta_\lambda \), with respect to the inner products defined by the corresponding Frobenius algebras. Moreover, studying the relevant glued cobordisms, one can show that the composition, \( \theta_\lambda \circ \theta_\lambda^* \) satisfies the “Cardy formula”,

\[
\theta_\lambda \circ \theta_\lambda^*(\phi) = \sum_{i=1}^{n} \psi_i^i \phi \psi_i
\]  

(1)
Figure 9: These surfaces are not diffeomorphic as open-closed cobordisms, and thus the Frobenius algebras $\mathcal{F}(\lambda)$ may not be commutative.

Figure 10: The “whistle open-closed cobordism” inducing the map $\theta_\lambda : \mathcal{F}(S^1) \to Z(\mathcal{F}(\lambda))$.

where $\{\psi_1, \cdots, \psi_n\}$ is any basis of $\mathcal{F}(\lambda)$, and $\{\psi_1^*, \cdots, \psi_n^*\}$ is the dual basis (with respect to the inner product in the Frobenius algebra structure). Again, see [29] for the details of this claim.

1.2 The work of Costello on open-closed TCFT’s

In [18] Costello studied open-closed topological conformal field theories (TCFT). Such a theory can be viewed as a derived version of a topological quantum field theory, and in a sense, Costello’s work can, in part be viewed as a derived extension and generalization of the work of Moore and Segal.

More precisely, the TCFT’s Costello studied are functors,

$$\mathcal{F} : \mathcal{OC}_D \to Comp_k$$

where $\mathcal{OC}_D$ is an open-closed cobordism category, enriched over chain complexes, and $Comp_k$ is the symmetric monoidal category of chain-complexes over a ground field $k$. In Costello’s work, $Char(k) = 0$. By an “open-closed cobordism category, enriched over chain complexes”, Costello means the following. Let $\mathcal{D}$ be an indexing set of “D-branes” as above. Then the objects of $\mathcal{OC}_D$ are parameterized, compact, oriented one-manifolds, $c$, together with a labeling of the components of the boundary, $\partial c$, by elements of $\mathcal{D}$, as described in the previous section.

To describe the chain complex of morphisms between objects $c_1$ and $c_2$, one considers the moduli space of all Riemann surfaces that form open-closed cobordisms between $c_1$ and $c_2$. This moduli
space was originally described by Segal [34] when the $c_i$’s have no boundary. For the general situation we refer the reader to Costello’s paper [18]. These open-closed cobordisms are required to satisfy the additional “positive boundary” requirement, that every path component of an element $\Sigma \in \mathcal{M}_D(c_1, c_2)$ has a nonempty incoming boundary. It is standard to see that

$$\mathcal{M}_D(c_1, c_2) \simeq \bigsqcup \text{diff}^+(\Sigma, \partial \Sigma)$$

where the disjoint union is taken over all diffeomorphism classes of open-closed cobordisms from $c_1$ to $c_2$. These diffeomorphisms are diffeomorphisms of open-closed cobordisms, as defined in the previous section. They are orientation preserving, they preserve the incoming and outgoing boundaries pointwise, and they preserve the labelings in $D$. The morphisms in $\mathcal{O}_\mathcal{D}$ are then the singular chains with coefficients in $k$, $\text{Mor}_{\mathcal{O}_\mathcal{D}}(c_1, c_2) = C_*(\mathcal{M}_D(c_1, c_2); k)$.

A topological conformal field theory is then a functor $\mathcal{F} : \mathcal{O}_\mathcal{D} \to \text{Comp}_k$ which is “$h$-monoidal”, in the sense that there are natural transformations $\mathcal{F}(c_1) \otimes \mathcal{F}(c_2) \to \mathcal{F}(c_1 \sqcup c_2)$ which are quasi-isomorphisms of chain complexes. Costello calls $\mathcal{O}_\mathcal{D} - \text{mod}$ the functor category of topological conformal field theories.

Let $\mathcal{O}_\mathcal{D} \hookrightarrow \mathcal{O}_\mathcal{D}^\text{mod}$ be the full subcategory whose objects have no closed components. That is, every connected component of a one-manifold $c \in \text{Ob}(\mathcal{O}_\mathcal{D})$ has (labeled) boundary. Write $\mathcal{O}_\mathcal{D} - \text{mod}$ to be the functor category of $h$-monoidal functors $\phi : \mathcal{O}_\mathcal{D} \to \text{Comp}_k$. We refer to such a functor as an “open-field theory”.

Costello observed that an open topological conformal field theory $\phi : \mathcal{O}_\mathcal{D} \to \text{Comp}_k$ defines an $A_\infty$-category, enriched over chain complexes, in much the same way as an open topological quantum field theory defines a category (see Definition (2) above). This is most easily seen if the field theory is strictly monoidal, that is, the transformations $\phi(c_1) \otimes \phi(c_2) \to \phi(c_1 \sqcup c_2)$ are isomorphisms of chain complexes, rather than only quasi-isomorphisms. In this case the associated $A_\infty$-category, which we call $\mathcal{C}_\phi$, has objects given by the set of $D$-branes $\mathcal{D}$. The space of morphisms $\phi(\lambda_0, \lambda_1)$ is the chain complex given by the value of the functor $\phi$ on the object $I_{\lambda_0, \lambda_1}$. We call this space $\phi(\lambda_0, \lambda_1)$. The higher compositions

$$\phi(\lambda_1, \lambda_2) \otimes \phi(\lambda_2, \lambda_3) \otimes \cdots \otimes \phi(\lambda_{n-1}, \lambda_n) \to \phi(\lambda_1, \lambda_n)$$

are given by the the value of the functor $\phi$ on the open-closed cobordism between $\bigsqcup_{i=1}^{n-1} I_{\lambda_i, \lambda_{i+1}}$ and $I_{\lambda_1, \lambda_n}$ given by the connected, genus zero surface $D_{\lambda_1, \cdots, \lambda_n}$ pictured in figure 11 in the case $n = 4$.

The $A_\infty$-category defined by an open TCFT has additional properties that Costello referred to as a “Calabi-Yau” $A_\infty$ category. The following theorem of Costello describes the central nature of this category in open-field theory.

**Theorem 4.** (Costello [18]) a. The restriction functor $\rho : \mathcal{O}_\mathcal{D} - \text{mod} \to \mathcal{O}_\mathcal{D} - \text{mod}$ from open-closed TCFT’s to open TCFT’s has a derived left adjoint, $L_\rho : \mathcal{O}_\mathcal{D} - \text{mod} \to \mathcal{O}_\mathcal{D} - \text{mod}$.

b. If $\phi \in \mathcal{O}_\mathcal{D} - \text{mod}$ is an open TCFT, then the closed state space of the open-closed field theory $L_\rho(\phi)$ (i.e. the value of the functor on the object given by the circle, $L_\rho(\phi)(S^1)$) is a chain complex
Figure 11: The open-closed cobordism $D_{\lambda_1, \cdots, \lambda_4}$

whose homology is given by the Hochschild homology of the $A_\infty$-category, $\phi$. That is,

$$H_*(L_\rho(\phi)(S^1)) \cong HH_*(C_\phi).$$

Here the Hochschild homology of a category enriched over chain complexes is computed via the Hochschild complex, whose $n$-simplices are direct sums of terms of the form $\text{Mor}(\lambda_0, \lambda_1) \otimes \text{Mor}(\lambda_1, \lambda_2) \otimes \cdots \otimes \text{Mor}(\lambda_{n-1}, \lambda_n) \otimes \text{Mor}(\lambda_n, \lambda_0)$. This is a double complex whose boundary homomorphisms are the sum of the internal boundary maps in the chain complex of $n$-simplices, plus the Hochschild boundary homomorphism, which is defined as the alternating sum $\sum_{i=0}^{n} (-1)^i \partial_i$, where for $i = 0, \cdots, n-1$, $\partial_i$ is induced by the composition $\text{Mor}(\lambda_i, \lambda_{i+1}) \otimes \text{Mor}(\lambda_{i+1}, \lambda_{i+2}) \rightarrow \text{Mor}(\lambda_i, \lambda_{i+2})$. $\partial_n$ is induced by the composition $\text{Mor}(\lambda_n, \lambda_0) \otimes \text{Mor}(\lambda_0, \lambda_1) \rightarrow \text{Mor}(\lambda_n, \lambda_1)$. The Hochschild homology of an $A_\infty$-category enriched over chain complexes is defined similarly. See [18] for details.

Costello’s theorem can be interpreted as saying that there is a “universal” open-closed theory with a given value on the open cobordism category (i.e the value of the derived left adjoint $L_\rho$), and that its closed state space has homology equal to the Hochschild homology of the associated $A_\infty$-category. We note that in the interesting case when there is only one $D$-brane, that is, $D = \{\lambda\}$, then the $A_\infty$-category is an $A_\infty$-algebra, and so the closed state space of the associated universal open-closed theory would have homology given by the Hochschild homology of this algebra. In particular this says that for any open-closed field theory $\phi$ with one $D$-brane, which has the corresponding $A_\infty$-algebra $A$, then there is a well defined map from the Hochschild homology $HH_*(A) \rightarrow H_*(\phi(S^1))$. This can be viewed as a derived version of the Moore-Segal result (Proposition 3) that gives a map $\phi(S^1) \rightarrow Z(A)$. In the Moore-Segal setting, $\phi(S^1)$ is an ungraded Frobenius algebra, (or equivalently it has trivial grading) so we may identify it with $H_0(\phi(S^1))$. Furthermore the center $Z(A)$ may be identified with the zero dimensional Hochschild cohomology $HH^0(A)$, so that the Moore-Segal result
gives a map $H_0(\phi(S^1)) \to HH^0(A)$. By the self-duality of the Frobenius algebra structures of $\phi(S^1)$ and of $A$, this gives a dual map $HH_0(A) \to H_0(\phi(S^1))$. Costello’s map can be viewed as a derived version of this map.

We end this section by remarking that recently Hopkins and Lurie have described a generalization of Costello’s classification scheme that applies in all dimensions. The type of field theories they consider are called “extended topological quantum field theories”. We refer the reader to [27] for a description of their work.

2 The string topology category and its Hochschild homology

One of the goals of our project is to understand how string topology fits into Costello’s picture. The most basic operation in string topology is the loop product defined by Chas and Sullivan [10]:

$$\mu : H_p(LM) \otimes H_q(LM) \to H_{p+q-n}(M)$$

where $M$ is a closed, oriented, $n$-dimensional manifold. Now let $B : H_q(LM) \to H_{q+1}(LM)$ be the operation induced by the rotation $S^1$-action on $LM$, $r : S^1 \times LM \to LM$.

$$B : H_q(LM) \to H_{q+1}(S^1 \times LM) \xrightarrow{r_*} H_{q+1}(LM)$$

$$\alpha \to r_*([S^1] \times \alpha)$$

The following was one of the main theorems of [10]

Theorem 5. [10] Let $\mathbb{H}_\ast(LM) = H_{\ast+n}(LM)$ be the (regraded) homology of the free loop space. Then with respect to the loop product $\mu$ and the degree one operator $B$, $\mathbb{H}_\ast(LM)$ has the structure of a (graded) Batalin-Vilkovisky algebra. That is, it is a graded commutative algebra satisfying the following identities:

1. $B^2 = 0$, and

2. For $\alpha \in \mathbb{H}_p(LM)$, and $\beta \in \mathbb{H}_q(LM)$ the bracket operation

$$\{\alpha, \beta\} = (-1)^{|\alpha|}B(\alpha \cdot \beta) - (-1)^{|\alpha|}B(\alpha) \cdot \beta - \alpha \cdot B(\beta)$$

is a derivation in each variable.

Moreover, a formal argument given in [10] implies that the operation $\{ , \}$ satisfies the (graded) Jacobi identity, and hence gives $\mathbb{H}_\ast(LM)$ the structure of a graded Lie algebra.

The product is defined by considering the mapping space, $Map(P,M)$ where $P$ is the pair of pants cobordism (figure (4)) between two circles and one circle. By restricting maps to the incoming and outgoing boundaries, one has a correspondence diagram

$$LM \xleftarrow{\text{Pant}} Map(P,M) \xrightarrow{\text{Out}} LM \times LM.$$ (2)
By retracting the surface $P$ to the homotopy equivalent figure 8 graph, one sees that one has a homotopy cartesian square,
\[
\begin{array}{c}
\text{Map}(P, M) \\ \downarrow \rho_{\text{in}} \\
M \xrightarrow{\Delta} M \times M
\end{array}
\]
where $\Delta : M \hookrightarrow M \times M$ is the diagonal embedding. This then allows the construction of an “umkehr map” $\rho_{\text{in}}^! : H_*(LM \times LM) \to H_{*-n}(\text{Map}(P, M))$. This map was defined on the chain level in [10], and via a Pontryagin-Thom map $LM \times LM \to \text{Map}(P, M)^{TM}$ in [13]. Here $\text{Map}(P, M)^{TM}$ is the Thom space of the tangent bundle $TM$ pulled back over the mapping space via evaluation at a basepoint, $\text{Map}(P, M) \to M$. By twisting with the virtual bundle $-TM$, Cohen and Jones proved the following.

**Theorem 6.** [13] For any closed manifold $M$, the Thom spectrum $LM^{-TM}$ is a ring spectrum. When given an orientation of $M$, the ring structure of $LM^{-TM}$ induces, via the Thom isomorphism, the Chas-Sullivan algebra structure on $H^*(LM)$.

The Chas-Sullivan product was generalized to a TQFT by Cohen and Godin in [12]. Given a cobordism $\Sigma$ between $p$-circles and $q$-circles, they considered the following correspondence diagram analogous to (2).

\[
(LM)^q \xrightarrow{\rho_{\text{out}}} \text{Map}(\Sigma, M) \xrightarrow{\rho_{\text{in}}} (LM)^p.
\]

Using fat (ribbon) graphs to model surfaces, Cohen and Godin described an umkehr map
\[
\rho_{\text{in}}^1 : H_*(LM)^p \to H_{*+\chi(\Sigma)-n}(\text{Map}(\Sigma, M))
\]
which allowed the definition of an operation
\[
\mu_{\Sigma} = (\rho_{\text{out}})_* \circ \rho_{\text{in}}^1 : H_*(LM)^p \to H_{*+\chi(\Sigma)-n}(LM)^q
\]
which yielded the (closed) TQFT structure. In these formulae, $\chi(\Sigma)$ is the Euler characteristic of the cobordism $\Sigma$.

Open-closed operations were first defined by Sullivan in [36]. Somewhat later, Ramirez [32] and Harrelson [24] showed that these operations define a positive boundary, open-closed topological quantum field theory, in the Moore-Segal sense, except that the value of the theory lie in the category of graded vector spaces over a field $k$. In this theory, which we call $S_M$, the closed state space is given by
\[
S_M(S^1) = H_*(LM; k).
\]

The set of D-branes $D_M$ is defined to be the set of connected, closed submanifolds $N \subset M$. The value of this theory on the interval labeled by submanifolds $N_1$ and $N_2$ (see figure (5) is given by
\[
S_M(N_1, N_2) = S_M(I_{N_1, N_2}) = H_*(P_{N_1, N_2}),
\]

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where \( \mathcal{P}_{N_1,N_2} \) is the space of paths \( \alpha : [0,1] \to M \) with boundary conditions, \( \alpha(0), \in N_1, \alpha(1) \in N_2 \).

Finally, using families of ribbon graphs modeling both closed and open-closed cobordisms, in [23] Godin recently proved the following result.

**Theorem 7.** (Godin) [23] Let \( \mathcal{OC}^{\mathcal{H}_*}_{D_M} \) be the category with the same objects as \( \mathcal{OC}^D_{D_M} \), and whose morphisms are the homology of the morphisms in \( \mathcal{OC}^D_{D_M} \). That is, given objects \( c_1 \) and \( c_2 \), the morphisms from \( c_1 \) to \( c_2 \) are given by

\[
\text{Mor}_{\mathcal{OC}_{D_M}^{\mathcal{H}_*}}(c_1,c_2) = H_*(\mathcal{M}_D(c_1,c_2); k) \cong \bigoplus H_*(BDiff^+(\Sigma, \partial \Sigma); k)
\]

where the disjoint union is taken over all diffeomorphism classes of open-closed cobordisms from \( c_1 \) to \( c_2 \). Then the above string topology operations can be extended to a symmetric monoidal functor

\[
\mathcal{S}_M : \mathcal{OC}^{\mathcal{H}_*}_{D_M} \to \text{GrVect}
\]

where \( \text{GrVect} \) is the category of graded vector spaces over \( k \), whose monoidal structure is given by (graded) tensor product. In other words, the string topology of \( M \) is a positive boundary, open-closed "homological conformal field theory" (HCFT).

Notice that being a homological conformal field theory is a weaker property than being a topological conformal field theory, and so Costello’s results cannot be immediately applied to the string topology of a manifold \( M \). In order for the functor \( \mathcal{S}_M \) to actually induce a TCFT, the string topology operations must be defined on the chain level, and satisfy the appropriate compatibility and coherence properties. It is conjectured that in fact this can be done. In any case, Costello’s theorem (Theorem 4 above) suggests that there is an \( A_\infty \)-category associated to the string topology of \( M \), and that its Hochschild homology should be the value of the closed state space, \( \mathcal{S}_M(S^1) = H_*(LM; k) \).

Theorem 1 in the introduction asserted the existence of such a category; we will describe the construction in more detail below, although full proofs appear in [7].

Another interesting question arises when there is only a single \( D \)-brane \( D = \{ N \} \), where \( N \) is a fixed, connected submanifold of \( M \). In this case the corresponding \( A_\infty \)-category would be an \( A_\infty \)-algebra. Here it turns out that for Poincare duality reasons it is more appropriate to consider Hochschild cohomology. The question described in the introduction, regarding the relationship between these Hochschild cohomology algebras and the Chas-Sullivan algebra structure on \( H_*(LM) \), was based on the idea that string topology, even in this “one \( D \)-brane” setting should fit into Costello’s picture of a universal open-closed TCFT. In particular the calculations described below verify that for a large class of submanifolds \( N \subset M \), the full subcategory of \( \mathcal{S}_M \) consisting of the single object \( N \) still yields the full closed state space of string topology,

\[
HH^*(C_*(\mathcal{P}_{N,N}),C_*(\mathcal{P}_{N,N})) \cong H_*(LM) = \mathcal{S}_M(S^1).
\]
The proofs of these statements involve the following ideas. By an Eilenberg–Moore argument, there is a chain homotopy equivalence $C_*(P_{N_1,N_2}) \simeq C_*(F_{N_1}) \otimes_{L_{C_*(\Omega M)}} C_*(F_{N_1})$ where $F_{N_1}$ is an appropriate homotopy fiber of the inclusion $N_1 \hookrightarrow M$ which comes equipped with an action of a topological group of the homotopy type of $\Omega M$ (e.g., the Kan simplicial loop group). The homotopy orbit space of this action is equivalent to $N$. This tensor product is the derived tensor product in the category of complexes.

Next one uses an appropriate notion of Poincare duality with twisted coefficients. Here, instead of the classical setting where one has coefficients given by modules over the group ring $\mathbb{Z}[\pi_1(M)]$, one needs a derived version of Poincare duality, that applies for modules over $C_*(\Omega M)$. This was worked out by Malm in [28], using the work of Dwyer-Greenlees-Iyengar [19] and Klein [25]. Using this, one obtains the following equivalence of complexes (related to results of Klein in [26]).

$$C_*(P_{N_1,N_2}) \simeq C_*(F_{N_1}) \otimes_{C_*(\Omega M)} C_*(F_{N_1}) \simeq \text{Rhom}_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2}))$$

Here $\text{Rhom}$ denotes derived homomorphism complex. This chain equivalence allows the following definition of the category $S_M$. Fix a field $k$ (and assume that all chains are taken with respect to this ground field), and abusively let $C_*(\Omega M)$ denote a strictly associative model for the chains on $\Omega M$. We will also use the existence of a model structure on the category of DG-modules for $C_*(\Omega M)$.

**Definition 8.** The string topology category $S_M$ has as

1. **Objects** the pairs $(N, F_N)$, where $N$ is a closed, connected, oriented submanifolds $N \subset M$ and $F_N$ is a specific choice of model for the homotopy fiber of $N \to M$ with an action of $\Omega M$.

2. **Morphism** from $N_1$ to $N_2$ the derived homomorphism space

$$\text{Rhom}_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2})),$$

computed via functorial cofibrant-fibrant replacement of $C_*F_{N_1}$.

In other words, $S_M$ is the full subcategory of the DG-category of differential graded modules over $C_*(\Omega M)$ with objects cofibrant-fibrant replacements of $C_*(F_N)$ for $N \subset M$ a submanifold as above.

We remark also that this generalized notion of Poincare duality (i.e with coefficients being modules over $C_*(\Omega M)$ ) is at the heart of the argument that shows that composition in this category realizes the string topology compositions on the level of homology,

$$H_*(P_{N_1,N_2}) \otimes H_*(P_{N_2,N_3}) \to H_*(P_{N_1,N_3}).$$

When $M$ is simply-connected, we have a useful “dual” model of $S_M$ which follows from the following alternative description of the path spaces $C_*(P_{N_0,N_1})$, proved using the generalized form of Poincare duality. Here we regard $C^*(M)$ as an $E_\infty$-algebra and we are relying on the existence of model structures on modules over an $E_\infty$-algebra.
Lemma 9. Let $M$ be a closed, simply connected manifold, and $N_0, N_1 \subset M$ connected, oriented, closed submanifolds. Then there is a chain equivalence,

$$C_*(\mathcal{P}_{N_0,N_1}) \simeq \text{Rhom}_{C^*_{CM}}(C^*(N_1), C^*(N_0)).$$

Equivalently, there is a chain equivalence

$$\text{Rhom}_{C_*(\Omega M)}(C_*(F_{N_0}), C_*(F_{N_1})) \simeq \text{Rhom}_{C^*_{CM}}(C^*(N_1), C^*(N_0)).$$

As a consequence, we obtain the following comparison result.

Theorem 10. For $M$ a closed, simply connected manifold, there is a zigzag of Dwyer-Kan equivalences between the string topology category $\mathcal{S}_M$ and the full subcategory of the category of $C^*(M)$-submodules, with objects cofibrant-fibrant replacements (as $C^*(M)$-modules) of the cochains $C^*(N)$ for $N \subset M$ a connected, oriented submanifold.

The Hochschild homology statement in Theorem 1 follows from an identification $HH_*(\mathcal{S}_M) \cong HH_*(\Omega M)$. This in turn is obtained as a straightforward consequence of the general theory developed in [8]; the thick closure of $\mathcal{S}_M$ inside the category of $C_*(\Omega M)$-modules is the entire category of finite $C_*(\Omega M)$-modules. This is essentially a Morita equivalence result. When one restricts to a single object $N$, (the “one-brane” situation), the endomorphism algebra is equivalent to $C_*(\mathcal{P}_{N,N})$. In this case analysis of the Hochschild (co)homology requires a more involved Morita theory. This is because it is definitely not the case in general that there is a Morita equivalence between the category of $C_*(\Omega M)$-modules and the category of $C_*(\mathcal{P}_{N,N})$-modules.

Here the situation is a kind of Koszul duality, and so whereas the Hochschild homologies of the path algebras vary as $N$ varies (explicit descriptions will be given in [7]), it is reasonable to expect that the Hochschild cohomologies should coincide. To approach these calculations, the following basic principle is used in [7]:

Theorem 11. Let $R$ and $S$ be two differential graded algebras over a field $k$, and suppose there exist $R-S$ (differential graded) modules satisfying the following equivalences:

$$\text{Rhom}_R(P,Q) \simeq S \text{ and } \text{Rhom}_S(P,Q) \simeq R.$$  \hspace{1cm} (6)

Then their Hochschild cohomologies are isomorphic,

$$HH^*(R,R) \cong HH^*(S,S).$$

Using this result, given a submanifold $N \subset M$, one considers

$$R \simeq C_*(\Omega M), \quad S \simeq C_*(\mathcal{P}_{N,N}) \simeq \text{Rhom}_{C_*(\Omega M)}(C_*(F_N), C_*(F_N)).$$

The $R-S$ modules are both given by $P = Q \simeq C_*(F_N)$. We already know that $\text{Rhom}_R(P,Q) \simeq S$, for any $N \subset M$, when $M$ is simply connected. The Hochschild cohomology calculations are then
reduced to a question about “double centralizers”: \( HH^*(C_*(P_{N,N})) \) is equivalent to \( HH^*(C_*(\Omega M)) \) if there is an equivalence

\[
\text{Rhom}_{C_*(P_{N,N})}(C_*(F_N), C_*(F_N)) \simeq C_*(\Omega M).
\]

The question of the existence of such equivalences can be studied using the generalized Morita theory of Dwyer, Greenlees, and Iyengar [19], and leads to the following characterization:

**Theorem 12.** Assume \( M \) is simply connected. Then for any \( N \subset M \) in \( \mathcal{D} \),

\[
\text{Rhom}_{C_*(P_{N,N})}(C_*(F_N), C_*(F_N)) \simeq \hat{C}_*(\Omega M)
\]

Here \( \hat{C}_*(\Omega M) \) is the Bousfield localization of \( C_*(\Omega M) \) with respect to the homology theory \( h_\mathcal{N}^* \), defined on the category of \( C_*(\Omega M) \)-modules given by

\[
h_\mathcal{N}^*(P) = \text{Ext}_{C_*(\Omega M)}(C_*F_N, P) \cong \text{Ext}_{C_*(\Omega M)}(k, P).
\]

(Note that this is best regarded as a completion process, despite the terminology of localization; we will refer to local objects as \( C_*(F_N) \)-complete.)

An immediate corollary is that the double centralizer property holds if and only if \( C_*(\Omega M) \) is \( C_*(F_N) \)-complete. Therefore, we are immediately led to study the following question: For which submanifolds \( N \subset M \) is \( C_*(\Omega M) \) in fact \( C_*(F_N) \)-complete? Counterexamples (obtained in consultation with Bill Dwyer) exist that suggest that this does not always hold. However, we can show the result in certain useful special cases. In particular, we know that \( C_*(\Omega M) \) is \( C_*(F_N) \)-complete in the following cases:

1. The inclusion map \( N \hookrightarrow M \) is null homotopic. This implies that \( F_N \simeq \Omega M \times N \), and \( P_{N,N} \simeq \Omega M \times N \times N \).

2. The inclusion \( N \hookrightarrow M \) is the inclusion of the fiber of a fibration \( p : M \to B \). More generally there is a sequence of inclusions, \( N \subset N_1 \subset N_2 \subset \cdots \subset N_k = M \) where each \( N_i \subset N_{i+1} \) is the inclusion of the fiber of a fibration \( p_{i+1} : N_{i+1} \to B_{i+1} \).

These results also have consequences for certain module categories related to the rings we are considering. Denote by \( E_N \) the endomorphism ring \( \text{Rhom}_{C_*(\Omega M)}(C_*(F_N), C_*(F_N)) \), which as noted above provides a strictly multiplicative model of \( C_*(P_{N,N}) \).

**Theorem 13.** When the double centralizer condition holds, the following categories of modules are Dwyer-Kan equivalent:

1. The thick subcategory of \( C^*(M) \)-modules generated by \( C^*(M) \) (i.e., the perfect modules).

2. The thick subcategory of \( C_*(\Omega M) \)-modules generated by the trivial module \( k \).
3. The thick subcategory of $E\text{-}modules$ generated by $\text{Rhom}_{C,*(G)}(C_*(F_N), k)$. Notice that this latter module is equivalent to $C^*(N)$.

Note. The equivalence of the categories 1 and 2 was established in [19].

Finally, we point out that there are spectrum level analogues of the above theorems (with essentially similar proofs), in particular Theorems 1 as stated in the introduction.

Theorem 14. There is a string topology category, enriched over spectra, which by abuse of notation we still refer to as $S_M$, whose objects again are elements of $\mathcal{D}_M$. The morphism spectrum between $N_1$ and $N_2$ is the analogous mapping spectrum and has the homotopy type of $\mathcal{P}_{N_1,N_2}^{TN_1}$, the Thom spectrum of the virtual bundle $-TN_1$, where $TN_1$ is the tangent bundle of $N_1$, pulled back over $\mathcal{P}_{N_1,N_2}$ via the evaluation map that takes a path $\alpha \in \mathcal{P}_{N_1,N_2}$ to its initial point $\alpha(0) \in N_1$. Furthermore, the $\text{THH}$ of this category is the suspension spectrum of the free loop space (with a disjoint basepoint),

$$\text{THH}^\ast(S_M) \simeq \Sigma^\infty(LM_+).$$

(7)

Moreover, the analogue of the above Hochschild cohomology statement is the following:

Theorem 15. Assume that $M$ is a simply connected, closed manifold. Then given any connected, closed, oriented submanifold $N \subset M$ for which the double-centralizer condition holds, then the Thom spectrum $\mathcal{P}_{N,N}^{TN}$ is a ring spectrum, and its Topological Hochschild Cohomology is given by

$$\text{THH}^\ast(\mathcal{P}_{N,N}^{TN}) \simeq LM^{-TM}$$

(8)

and the equivalence is one of ring spectra.

3 Relations with the Fukaya category of the cotangent bundle

This section is speculative, regarding the possible relationships between the string topology category $S_M$, and the Fukaya category of the cotangent bundle, $T^*M$. The Fukaya category is an $A_\infty$-category associated to a symplectic manifold $(N^{2n}, \omega)$. Here $\omega \in \Omega^2(N)$ is a symplectic 2-form. Recall that for any smooth $n$-manifold $M^n$, $T^*M$ has the structure of an exact symplectic manifold. That is, it has a symplectic 2-form $\omega$ which is exact. In the case $T^*M$, $\omega = d\theta$, where $\theta$ is the Liouville one-form defined as follows. Let $p : T^*M \to M$ be the projection map. Let $x \in M$, and $t \in T^*_xM$. Then $\theta(x, t)$ is the given by the composition,

$$\theta(x, t) : T_{x,t}(T^*M) \xrightarrow{dp} T_xM \xrightarrow{t} \mathbb{R}.$$

There has been a considerable amount of work comparing the symplectic topology of $T^*M$ with the string topology of $M$. This relationship begins with a theorem of Viterbo [37], that the symplectic
Floer homology is isomorphic to the homology of the free loop space,

\[ SH_\ast(T^*M) \cong H_\ast(LM). \]

The symplectic Floer homology is computed via a Morse-type complex associated to the (possibly perturbed) “symplectic action functional”, \( A : L(T^*M) \to \mathbb{R} \). The perturbation is via a choice of Hamiltonian, and so long as the Hamiltonian grows at least quadratically near infinity, the symplectic Floer homology is described by the above isomorphism. The precise relationship between the Floer theory of the symplectic action functional \( A \) and Morse theory on \( LM \) was studied in great detail by Abbondandolo and Schwarz in [1]. In particular they were able to show that a “pair of pants” (or “quantum”) product construction in \( SH_\ast(T^*M) \) corresponds under this isomorphism to a Morse-theoretic analogue of the Chas-Sullivan product in \( H_\ast(LM) \). In [15] this product was shown to agree with the Chas-Sullivan construction.

The objects of the Fukaya category \( \text{Fuk}(T^*M) \) are exact, Lagrangian submanifolds \( L \subset T^*M \). The morphisms are the “Lagrangian intersection Floer cochains”, \( CF^\ast(L_0, L_1) \). These Floer cochain groups are also a Morse type cochain complex associated to a functional on the path space, \( \mathcal{A}_{L_0,L_1} : \mathcal{P}_{L_0,L_1}(T^*M) \to \mathbb{R} \).

If \( L_0 \) and \( L_1 \) intersect transversally, then the critical points are the intersection points (viewed as constant paths), and the coboundary homomorphisms are computed by counting holomorphic disks with prescribed boundary conditions. Of course if \( \mathcal{A}_{L_0,L_1} \) were actually a Morse function, satisfying the Palais-Smale convergence conditions, then these complexes would compute \( H_\ast(\mathcal{P}_{L_0,L_1}(T^*M)) \).

One knows that this Morse condition is not satisfied, but there are examples, when this homological consequence is nonetheless satisfied. Namely, let \( N \subset M \) be an oriented, closed submanifold. Let \( \nu_N \) be the conormal bundle. That is, for \( x \in N \), \( \nu_N(x) \subset T_x^*M \) consists of those cotangent vectors which vanish on the subspace \( T_xN \subset T_xM \). Notice that the conormal bundle is always an \( n \)-dimensional submanifold of the \( 2n \)-dimensional manifold \( T^*M \). It is a standard fact that the conormal bundle \( \nu_N \) is a (noncompact) Lagrangian submanifold of \( T^*M \). Notice that for any two closed, oriented submanifolds \( N_0, N_1 \subset M \) the following path spaces in the cotangent bundle and in the base manifold \( M \) are homotopy equivalent:

\[ \mathcal{P}_{\nu_{N_0},\nu_{N_1}}(T^*M) \simeq \mathcal{P}_{N_0,N_1}(M). \]

The following was recently proven by Abbondandalo, Portaluri, and Schwarz [3]:

**Theorem 16.** Given any closed, oriented submanifolds \( N_0, N_1 \subset M \), then the intersection Floer cohomology of the \( HF^\ast(\nu_{N_0},\nu_{N_1}) \) is isomorphic to the homology of the path space,

\[ HF^\ast(\nu_{N_0},\nu_{N_1}) \cong H_\ast(\mathcal{P}_{N_0,N_1}(M)). \]

If one could realize these isomorphisms on the level of chain complexes, in such a way that the compositions correspond, then one would have a proof of the following conjecture:
Conjecture 17. Let $\text{Fuk}_{\text{conor}}(T^*M)$ be the full subcategory of the Fukaya category generated by conormal bundles of closed, connected, submanifolds of $M$. Then there is a Dwyer-Kan equivalence with the string topology category,

$$\text{Fuk}_{\text{conor}}(T^*M) \simeq \mathcal{S}_M.$$ 

Remarks.

1. In this conjecture one probably wants to study the “wrapped” Fukaya category as defined by Fukaya, Seidel, and Smith in [21].

2. When $N = \text{point}$, its conormal bundle is a cotangent fiber, $T_xM$. Abouzaid [4] has recently described the $A_\infty$ relationship between the Floer cochains $CF^*(T_xM,T_xM)$ and the chains of the based loop space, $C_*(\Omega M)$.

There are other potential relationships between the Fukaya category and the string topology category as well. For example, Fukaya, Seidel, and Smith [21] as well as Nadler [30] building on work of Nadler and Zaslow [31] showed that when $M$ is simply connected, the Fukaya category $\text{Fuk}_{\text{cpt}}(T^*M)$ generated by compact, exact Lagrangians with Maslov index zero, has a fully faithful embedding into the derived category of modules over the Floer cochains, $CF^*(M,M)$ where $M$ is viewed as a Lagrangian submanifold of $T^*M$ as the zero section. Furthermore, one knows that the Floer cohomology, $HF^*(M,M)$ is isomorphic to $H^*(M)$, and recently Abouzaid [5] proved that $CF^*(M,M) \simeq C^*(M)$ as $A_\infty$-differential graded algebras. So $\text{Fuk}_{\text{cpt}}(T^*M)$ can be viewed as a sub-$A_\infty$-category of the derived category of $C^*(M)$-modules.

When $M$ is simply connected, recall that the string topology category $\mathcal{S}_M$ can also be viewed as a subcategory of the category of $C^*(M)$-modules. From Nadler’s work we see that the relationship between the compact Fukaya category $\text{Fuk}_{\text{cpt}}(T^*M)$ should be equivalent to the “one-brane” string topology category, $\mathcal{S}_M^{1}$, which is the full subcategory of of $\mathcal{S}_M$ where the only D-brane is the entire manifold itself. This category, in turn, generates the derived category of perfect $C^*M$-modules.

The significance of this potential relationship is amplified when one considers recent work of Hopkins and Lurie [27] classifying “extended” topological conformal field theories. This can be viewed as a direct generalization of the work of Moore-Segal, and of Costello discussed above. In their classification scheme, such a field theory is determined by an appropriately defined “Calabi-Yau” category enriched over chain complexes. The category of perfect $C^*(M)$ is such a category. Moreover generalized Morita theory implies that this category is Dwyer-Kan equivalent to the category of $k$-finite $C_*(\Omega M)$-modules. Thus these categories should determine an extended field theory, which should correspond to string topology. On the other hand, by the above remarks, the Fukaya category $\text{Fuk}_{\text{can}}(T^*M)$ should also determine a field theory, presumably the “Symplectic Field Theory” of Eliashberg, Givental, and Hofer [20] applied to $T^*M$. One can therefore speculate that this line of reasoning may produce an equivalence of the symplectic field theory of $T^*M$, and of the string topology of $M$. There is evidence that such an equivalence may exist, for example the work of
Cieliebak and Latchev [9]. Pursuing this relationship using the Hopkins-Lurie classification scheme could lead to a very satisfying understanding of the deep connections between these two important theories.

References


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