GLOBAL WELL-POSEDNESS OF QUASILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY DE SITTER SPACES

PETER HINTZ

ABSTRACT. We establish the small data solvability of suitable quasilinear wave and Klein-Gordon equations in high regularity spaces on a geometric class of spacetimes including asymptotically de Sitter spaces. We obtain our results by proving the global invertibility of linear operators with coefficients in high regularity $L^2$-based function spaces and using iterative arguments for the nonlinear problems. The linear analysis is accomplished in two parts: Firstly, a regularity theory is developed by means of a calculus for pseudodifferential operators with non-smooth coefficients, similar to the one developed by Beals and Reed, on manifolds with boundary. Secondly, the asymptotic behavior of solutions to linear equations is studied using standard $b$-analysis, introduced in this context by Vasy; in particular, resonances play an important role.

1. INTRODUCTION

Consider the $n$-dimensional de Sitter space $\{|x|^2 - t^2 = 1\} \subset \mathbb{R}^{1+n}$, compactified to a cylinder $\tilde{M}$ by adding boundaries $\tau := \langle t \rangle^{-1} = 0$ at future and past infinity; taking a point $p$ at future infinity, consider a neighborhood of the lift of the backward light cone from $p$ to the blow-up of $\tilde{M}$ at $p$ near the front face; see Figure 1. Notice that $\tau$ lifts to a boundary defining function for the front face within $\Omega$. Let $g_0$ be the Lorentzian metric on $\Omega$ induced by the standard metric on $\mathbb{R}^{1+n}$. Denote by $V_b(\Omega)$ the space of vector fields on $\Omega$ which are tangent to the front face; then $V_b(\Omega)$ consists of smooth sections of a natural vector bundle $^bT\Omega$, and $g_0$ is in fact a $b$-metric, i.e. a nondegenerate section of the second symmetric tensor power of $^bT\Omega$. For $k \in \mathbb{N}_0$, define the $b$-Sobolev space

$$H^k_b(\Omega) = \{ u \in L^2(\Omega, d\text{vol}_{g_0}) : X_1 \cdots X_k u \in L^2(\Omega, d\text{vol}_{g_0}), X_1, \ldots, X_k \in V_b(\Omega) \}.$$  

Theorem 1.1. For $u \in C(M)$, let $g(u)$ be a $b$-metric s.t. $g(0) = g_0$, and in local coordinates, $g(u) = (g_{ij}(u))$ with $g_{ij} \in C^\infty(\mathbb{R})$. Moreover, let

$$q(u, du) = \sum_j u^{e_j} \prod_{l=1}^{N_j} X_{jl} u, \quad e_j + N_j \geq 2, N_j \geq 1, X_{jl} \in V_b(\Omega).$$

Fix $k > n/2 + 7$ and $\delta \in (0, 1)$. Then there exist $R, C > 0$ such that for all $f \in C^\infty_c(\Omega^2; \mathbb{R})$ with $\|\tau^{-1+\delta} f\|_{H^{k-1}_b(\Omega)} \leq C$, the equation

$$\Box_{g(u)} u = f + q(u, du)$$

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has a unique forward solution $u = c + u'$, $c \in \mathbb{R}$, $u' \in \tau^{1-\delta}H^k_b(\Omega;\mathbb{R})$, with $|c| + \|\tau^{-1+\delta}u'\|_{H^k_b(\Omega)} \leq R$; that is, $\text{supp } u \subset \{t \geq t_0\}$ for all $t_0$ such that $\text{supp } f \subset \{t \geq t_0\}$.

See Theorem 8.8 for the full result, in particular for statements regarding stability and higher regularity. Theorem 1.1 follows from Theorem 1.2 below which is in a more general geometric setting and allows for a vastly larger class of nonlinearities. Even more generally, one can consider equations on natural vector bundles; see the discussion later in the introduction. In a different direction, we can also solve backward problems in spaces with high decay at $\tau = 0$, see Theorem 8.16, where we can in fact replace $\square_g(u)$ by $\square_g(u) + L$ for first order operators $L$.

![Figure 1. Geometric setup of the static (asymptotically) de Sitter problem. Indicated are the blow-up of $\tilde{M}$ at $p$ and the front face of the blow-up, further the lift of the backward light cone to $[\tilde{M}; p]$ (solid), and lifts of backward light cones from points near $p$ (dotted); moreover, $\Omega$ is bounded by the front face and the dashed spacelike boundaries.](image)

The novelty of our analysis of quasilinear wave and Klein-Gordon equations lies in combining the methods used by Vasy and the author [18] to treat semilinear equations on static asymptotically de Sitter (and more general) spaces with the technology of pseudodifferential operators with non-smooth coefficients in the spirit of Beals and Reed [7] which is used to understand the regularity properties of operators like $\square_g(u)$ in the above theorem. We expect that our approach, appropriately adapted, also works in a variety of other settings, for instance on asymptotically Minkowski spaces in the sense of Baskin, Vasy and Wunsch [6], or even on asymptotically Kerr-de Sitter spaces, where however a much more delicate analysis is necessary in view of issues coming from trapping.

Compactifying our spacetime at infinity puts equation (1.2) into a $b$-framework, where it reveals a rich microlocal structure (in particular, the operator $\square_g(u)$ is a perturbation of one that has radial points at the boundary). Then, as in [18], rather than solving an evolution equation for a short amount of time, controlling the solution using (almost) conservation laws and iterating, we use a different iterative procedure, where at each step we solve a linear equation, with non-smooth coefficients, of the form

$$P_{u_k}u_{k+1} \equiv (\square_g(u_k) - \lambda)u_{k+1} = f + q(u_k, du_k)$$

1Here ‘b’ refers to analysis based on vector fields tangent to the boundary of the (compactified) space. The b-analysis originates in Melrose’s work on the propagation of singularities on manifolds with smooth boundary; Melrose described a systematic framework for elliptic $b$-equations in [24]. We will give more details later in the introduction.
globally on $L^2$-based b-Sobolev spaces or analogous spaces that encode partial expansions. Since the non-linearity $q$ (as well as $g$) must be well-behaved relative to these, we work on high regularity spaces; recall here that $H^s(\mathbb{R}^n)$ is an algebra for $s > n/2$. Moreover, we need to prove decay (or at least non-growth) for solutions of (1.3) so that $q$ can be considered a perturbation. The allowed asymptotics of solutions to the linear equation (1.3) are captured by the normal operator family of $P_{u_k}$ at infinity, encoded as a compactification of the space. By virtue of the asymptotics of linear waves on (approximately) static (asymptotically) de Sitter spaces, this family will for all $k$ be a family of operators with smooth coefficients, thus one can use results of Vasy [34, 32] to understand its behavior, in particular resonances, i.e. the location of the poles of the inverse Mellin transformed family and their structure, as well as stability results. Just as in the semilinear setting, we need to require the resonances to lie in the ‘unphysical half-plane’ $\text{Im} \sigma < 0$ (a simple resonance at 0 is fine as well), since resonances in the ‘physical half-plane’ $\text{Im} \sigma > 0$ would allow growing solutions to the equation, making the non-linearity non-perturbative and thus causing our method to fail. The linear analysis of equations like (1.3) is carried out in Section 7 in two steps: the invertibility on high regularity spaces which however contain functions that are growing at $\infty$ (see Theorem 7.8) and the proof of decay corresponding to the location of resonances (see Theorem 7.9).

In the iteration scheme (1.3), notice that if $u_k \in H^s$ (more precisely, an $H^s_b$-based space), then the right hand side is in $H^{s-1}$. Now $P_{u_k}$ has leading order coefficients in $H^s$ and subprincipal terms with regularity $H^{s-1}$, and to keep the iteration running, we need that the solution operator for $P_{u_k}$ maps $H^{s-1}$ to $H^s$ (the loss of one derivative being standard for hyperbolic problems). In other words, there is a delicate balance of the regularities involved; at the heart of this paper thus lies a robust regularity theory for operators like $P_{u_k}$ on manifolds with boundary.

The main ingredient of the framework in which will analyze b-operators with non-smooth coefficients on manifolds with boundary\(^2\) is a partial calculus for what we call $b$-Sobolev $b$-pseudodifferential operators; for brevity, we will refer to these as ‘non-smooth operators’ to distinguish them from ‘smooth operators’, by which we mean standard b-pseudodifferential operators, recalled below. b-Sobolev b-ps.d.o’s are (generalizations of) b-ps.d.o’s with coefficients in b-Sobolev spaces, which partly extends a corresponding partial calculus on manifolds without boundary in the form developed by Beals and Reed [7].\(^3\) This calculus allows us to prove microlocal regularity results – that are standard in the smooth setting – for b-Sobolev b-ps.d.o’s, namely elliptic regularity, real principal type propagation of singularities, including with (microlocal) complex absorbing potentials, and propagation near radial points; see Section 6. We only develop a local theory since all regularity results we need are local in character due to the compactness of the underlying (compactified!) manifold. The exposition of the calculus and its consequences in Sections 2–6 comprises the bulk of the paper.

In order to emphasize the generality of the method, let us point out that given an appropriate structure of the null-geodesic flow at $\infty$, for example radial points as

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\(^2\)The framework readily generalizes to manifolds with corners.

\(^3\)Beals and Reed consider coefficients in microlocal Sobolev spaces; this generality is not needed for our purposes, even though including it would only require more care in bookkeeping.
above, the only obstruction to the solvability of quasilinear equations are resonances in the upper half plane.

To set up the main theorem, recall from [32] that an asymptotically de Sitter space \( \tilde{M} \) is an appropriate generalization of the Riemannian conformally compact spaces of Mazzeo and Melrose [23] to the Lorentzian setting, namely a smooth manifold with boundary, with the interior of \( \tilde{M} \) equipped with a Lorentzian signature (taken to be \((1, n-1)\)) metric \( \tilde{g} \), and with a boundary defining function \( \tau \) such that \( \tilde{g} = \tau^2 \hat{g} \) is a smooth symmetric 2-tensor of signature \((1, n-1)\) up to the boundary of \( \tilde{M} \), and \( \hat{g}(d\tau, d\tau) = 1 \) so that the boundary defining function is timelike and the boundary itself is spacelike. In addition, \( \partial \tilde{M} \) has two components \( X_{\pm} \), each of which may be a union of connected components, with all null-geodesics \( \gamma(s) \) tending to \( X_{\pm} \) as \(|s| \to \infty \).

We now blow up a point \( p \in X_+ \), which amounts to introducing polar coordinates around \( p \), and obtain a manifold with corners \([\tilde{M}; p] \to \tilde{M} \). The backward light cone from \( p \) lifts to a smooth manifold transversal to the front face of \([\tilde{M}; p] \) and intersects the front face in a sphere. The interior of this backward light cone, at least near the front face, is a generalization of the static model of de Sitter space; we will refer to a neighborhood \( \Omega \subset \tilde{M} \), bounded by (a part of) \( Y \) and two ‘artificial’ spacelike hypersurfaces \( H_1 \) and \( H_2 \), see Figure 1. For definiteness, let us assume \( H_1 = \{ \tau = 1 \} \).

On \( \tilde{M} \), we naturally have the b-tangent bundle \( bTM \), whose sections are the b-vector fields \( \mathcal{V}_b(M) \), i.e. vector fields tangent to the boundary; in local coordinates \( \tau, y \) near the boundary, \( bTM \) is spanned by \( \tau \partial_\tau \) and \( \partial_y \). The enveloping algebra of \( \mathcal{V}_b \) of b-differential operators is denoted \( \text{Diff}_b^*(M) \). The b-cotangent bundle, the dual of \( bTM \), is denoted \( bT^*M \) and spanned by \( \frac{dx}{\tau} \) and \( dy \), and we have the b-differential \( b\te : C^\infty(M; \mathbb{C}) \xrightarrow{d} C^\infty(M; T^*M) \to C^\infty(M; bT^*M) \), where the last map comes from the natural map \( bTM \to TM \). Now, the metric \( g \) on \( M \) is a smooth, symmetric, Lorentzian signature (taken to be \((1, n-1)\)) section of the second tensor power of \( bTM \). The associated d’Alembertian (or wave operator) \( \Box_g \) is an element of \( \text{Diff}_b^2(M) \) and therefore naturally acts on weighted b-Sobolev spaces \( H^{s,\alpha}_b(M) = \tau^\alpha H^s_b(M) \), where we define \( H^s_b(M) \) for \( k \in \mathbb{N}_0 \) as in (1.1) and for general \( s \in \mathbb{R} \) using duality and interpolation. Denote by \( H^{s,\alpha}_b(\Omega)^* \) the space of restrictions of \( H^{s,\alpha}_b(M) \)-functions with support in \( \{ \tau \leq 1 \} \) to \( \Omega \); that is, elements of \( H^{s,\alpha}_b(\Omega)^* \) are supported at \( H_1 \) and extendible at \( H_2 \) in the sense of Hörmander [20, Appendix B]. Finally, let \( X^{s,\alpha} \) be the space of all \( u \) which near \( \tau = 0 \) asymptotically look like a constant plus an \( H^{s,\alpha}_b(\Omega)^* \)-function, i.e. for some \( c \in \mathbb{C} \), \( u^* = u - c \chi(\tau) \in H^{s,\alpha}_b(\Omega)^* \), where \( \chi \in C^\infty(\mathbb{R}) \), \( \chi \equiv 1 \) near 0, is a cutoff near \( Y \); for such a function \( u \), define its squared norm by

\[
\|u\|_{X^{s,\alpha}}^2 = |c|^2 + \|u^*\|_{H^{s,\alpha}_b(\Omega)^*}^2.
\]

\[\text{See [32, 34] for relating the ‘global’ and ‘static’ problems.}\]
Our main theorem then is:

**Theorem 1.2.** Let \( s > n/2 + 7 \) and \( 0 < \alpha < 1 \). Assume that for \( j = 0, 1 \),

\[
\begin{align*}
g & : \mathcal{X}^{s-j,\alpha} \to (C^\infty + H_0^{s-j,\alpha})(M; \text{Sym}^2 b TM), \\
q & : \mathcal{X}^{s-j,\alpha} \times H_0^{s-1-j,\alpha}(\Omega; b T^*_\Omega M) \to H_b^{s-1-j,\alpha}(\Omega)\cdot
\end{align*}
\]

are continuous, \( g \) is locally Lipschitz, and

\[
\|q(u, b^* du) - q(v, b^* dv)\|_{H_b^{s-1-j,\alpha}(\Omega)\cdot} \leq L_q(R)\|u - v\|_{\mathcal{X}^{s-j,\alpha}}
\]

for \( u, v \in \mathcal{X}^{s-j,\alpha} \) with norm \( \leq R \), where \( L_q : \mathbb{R}_{\geq 0} \to \mathbb{R} \) is continuous and non-decreasing. Then there is a constant \( C_L > 0 \) so that the following holds: If \( L_q(0) < C_L \), then for small \( R > 0 \), there is \( C_f > 0 \) such that for all \( f \in H_0^{s-1,\alpha}(\Omega)\cdot \) with norm \( \leq C_f \), there exists a unique solution \( u \in \mathcal{X}^{s,\alpha} \) of the equation

\[
\Box g(u)u = f + q(u, b^* du)
\]

with norm \( \leq R \), and in the topology of \( \mathcal{X}^{s-1,\alpha} \), \( u \) depends continuously of \( f \).

See Theorem 8.5 for a slightly more general statement; in particular, we can still guarantee the existence of solutions if we merely make a continuity assumption on \( g \) for \( j = 0 \) and a weak Lipschitz assumption on \( q \). Another case we study is \( g(u) = \mu(u)g \), i.e. we only allow conformal changes of the metric; here, one can partly improve the above theorem, in particular allow non-linearities of the form \( q(u, b^* du, \Box g(u)u) \); see Section 8.3. The point of the Lipschitz assumptions on \( q \) in all these cases is to ensure that \( q(u, b^* du) \) has a sufficient order of vanishing at \( u = 0 \) so that \( q(u, b^* du) \) can be considered a perturbation of \( \Box g(u) \); quadratic vanishing is enough, but slightly less (simple vanishing will small Lipschitz constant near or at 0) also suffices.

Similar results hold for quasilinear Klein-Gordon equations with positive mass, where the asymptotics of solutions, hence the function spaces used, are different, namely the leading order term is now decaying; see Section 8.4 for details.

In Section 8.5 finally, we will discuss backward problems; it is expected that the results there extend to the setting of Einstein’s equations (after fixing a gauge) on static de Sitter and even on Kerr-de Sitter spacetimes, thus enabling scattering constructions of dynamical black hole spacetimes in the spirit of recent work by Dafermos, Holzegel and Rodnianski [9].

While all results were stated for scalar equations, corresponding results hold for operators acting on natural vector bundles, provided that all resonances lie in the unphysical half-plane \( \text{Im} \sigma < 0 \) (with a simple resonance at 0 being fine as well): Indeed, the linear arguments go through in general for operators with scalar principal symbols; only the numerology of the needed regularities depends on estimates of the subprincipal symbol at (approximate) radial points.

Lastly, let us mention that paradifferential methods would give sharper results with respect to the regularity of the spaces in which we solve equation (1.2), and correspondingly we have not made any efforts here to push the regularity down. However, our entirely \( L^2 \)-based method is both conceptually and technically relatively straightforward, powerful enough for our purposes, and lends itself very easily to generalizations in other contexts.
Non-linear wave and Klein-Gordon equations on asymptotically de Sitter space-times (or static patches thereof) have been studied in various contexts:\(^5\) Friedrich [15, 14] proved the global non-linear stability of 4-dimensional asymptotically de Sitter spaces using a conformal method, see also [13] for a discussion of more recent developments; also in four dimensions, Rodnianski and Speck [28] proved the stability of the Euler-Einstein system. Anderson [2] proved the nonlinear stability of all even-dimensional asymptotically de Sitter spaces by generalizing Friedrich’s argument. On the semilinear level, Baskin [4, 5] established Strichartz estimates for the linear Klein-Gordon equation using his parametrix construction [3] and used them to prove global well-posedness results for classes of semilinear equations with no derivatives; Yagdjian and Galstian [38] derived explicit formulas for the fundamental solution of the Klein-Gordon equation on exact de Sitter spaces, which were subsequently used by Yagdjian [36, 37] to solve semilinear equations with no derivatives. Vasy and the author [18] proved global well-posedness results for a large class of semilinear wave and Klein-Gordon equations on (static) asymptotically de Sitter spaces, where the non-linearity can also involve derivatives; however, just as in the present paper, the (b-)microlocal, high regularity approach used does not apply to low-regularity non-linearities covered by the results of Baskin and Yagdjian.

The study of ps.d.o’s with non-smooth coefficients has a longer history: Beals and Reed [7] developed a partial calculus with coefficients in \(L^2\)-based Sobolev spaces on Euclidean space, which is the basis for our extension to manifolds with boundary. Marschall [22] gave an extension of the calculus to \(L^p\)-based Sobolev spaces (and even more general spaces) and in addition proved the invariance of certain classes of non-smooth operators under changes of coordinates. Witt [35] extended the \(L^2\)-based calculus to contain elliptic parametrices. Pseudodifferential calculi for coefficients in \(C^k\) spaces have been studied by Kumano-go and Nagase [21]. In a slightly different direction, paradifferential operators, pioneered by Bony [8] and Meyer [27], are a widely used tool in non-linear PDE; see e.g. Hörmander [19] and Taylor [31, 30] and the references therein.

1.1. b-preliminaries and outline of the paper. We will now give some background on b-pseudodifferential operators and microlocal regularity results along with indications as to how to generalize them to the non-smooth setting, thereby giving a brief, mostly chronological, outline of some of the technical aspects of the paper.

We recall from Melrose [24] that the small calculus of b-ps.d.o’s on a compact manifold \(M\) with boundary is the microlocalization of the algebra of b-differential operators on \(M\), and the kernels of b-ps.d.o’s are conceptually best described as conormal distributions on a certain blow-up \(M^b\) of \(M \times M\), smooth up to the front face, and vanishing to infinite order at the left/right boundary faces. More prosaically, using local coordinates \((x, y) \in \mathbb{R}^n_+ := [0, \infty)_x \times \mathbb{R}^{n-1}_y\) near the boundary of \(M\), i.e. \(x\) is a local boundary defining function, and using the corresponding coordinates \(\lambda, \eta\) in the fibers of \(b\)\(^*\)\(M\), i.e. writing b-covectors as

\[
\lambda \frac{dx}{x} + \eta \, dy,
\]

\(^5\)There is more work on the linear problem in de Sitter spaces; see e.g. the bibliography of [32].
the action of a b-ps.d.o $A \in \Psi_b^m$ of order $m$ on $u \in \dot{C}^\infty_c(\mathbb{R}^n_+)$, the dot referring to
infinite order of vanishing at the boundary, is computed by
$$Au(x, y) = \int_{\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1}} e^{i(x-y') \eta} s^\lambda a(x, y, \lambda, \eta) u(x/s, y') d\lambda d\eta \frac{ds}{s} dy', \quad (1.4)$$
where $a(x, y, \lambda, \eta)$, the full symbol of $A$ in the local coordinate chart, lies in the
symbol class $S^m(\mathbb{b}T^*\mathbb{R}^n_+)$, i.e. satisfies the symbolic estimates
$$|\partial^\alpha_x \partial^\beta_y a(x, y, \lambda, \eta)| \leq C_{\alpha \beta}(\lambda, \eta)^{m-|\beta|} \text{ for all multiindices } \alpha, \beta.$$  

We say that $A$ is a left quantization of $a$. Using the formula for the behavior of the
full symbol under a coordinate change, one finds that one can invariantly define a
principal symbol
$$\sigma^m_b(A) \in S^m(\mathbb{b}T^*\mathbb{M})/S^{m-1}(\mathbb{b}T^*\mathbb{M})$$
of $A$, which is locally just given by (the equivalence class of) $a$. If the principal
symbol admits a homogeneous representative $a_m$, meaning $a_m(z, \lambda \zeta) = \lambda^m a_m(z, \zeta)$
for $\lambda \geq 1$, then we say that $A$ has a homogeneous principal symbol and, by a slight
abuse of notation, set $\sigma^m_b(A) = a_m$. We will sometimes identify homogeneous
functions on $\mathbb{b}T^*\mathbb{M}\setminus o$ with functions on the unit cosphere bundle $\mathbb{b}S^*\mathbb{M}$, viewed
as the boundary of the fiber-radial compactification $\mathbb{b}^*T^*\mathbb{M}$ of $\mathbb{b}T^*\mathbb{M}$.\footnote{Strictly speaking, this identification is only well-defined for functions which are homogeneous
of order 0; in the general case, one should identify homogeneous functions with sections of a
natural line bundle on $\mathbb{b}S^*\mathbb{M}$ which encodes the differential of a boundary defining function
of fiber infinity.}

The first key point now is that there is a symbolic calculus for b-ps.d.o’s, with the most
important features being that for $A \in \Psi^m_b(M), B \in \Psi^m_b(M),$
$$\sigma^m_b(I) = 1, \quad \sigma^m_b(A^*) = \overline{\sigma^m_b(A)}, \quad \sigma^m_b(A \circ B) = \sigma^m_b(A)\sigma^m_b(B),$$
where we fixed a b-density on $M$, which in local coordinates is of the form $a \frac{dx}{x}$
with $a > 0$ smooth down to $x = 0$, to define the adjoint. For local computations, it
is very useful to have the asymptotic expansion
$$\sigma^m_b(A \circ B)(z, \zeta) \sim \sum_{\beta \geq 0} \frac{1}{|\beta|!} (\partial^\beta_x a \partial^\beta_y B)(z, \zeta) \quad (1.5)$$
for the full symbol of a composition of b-ps.d.o’s,\footnote{Here, ‘$\sim$’ is to be understood in the sense that the difference of the left hand side and the
sum on the right hand side, restricted to $|\beta| < N$, lies in $S^{m+m'}-N(\mathbb{b}T^*\mathbb{R}^n_+)$, for all $N$.} where $a$ and $b$ are the full symbols
of $A$ and $B$, and $bD_z = (xD_x, D_y)$, where $D = -i\partial$. In particular, this gives that
for $A \in \Psi^m_b, B \in \Psi^m_{b'}$ with principal symbols $a, b$, the principal symbol of the
commutator is $\sigma^m_{b+m'}([A, B]) = \frac{1}{i} H_a b$, where
$$H_a = (\partial_\lambda a)(x\partial_x + (\partial_\rho a) \partial_y - (x\partial_x a) \partial_\lambda - (\partial_\rho a) \partial_\eta).$$
This follows from the expansion (1.5) if we keep track of terms up to first order.

The vector field $H_a$ is in fact the smooth extension to the boundary of the standard
Hamilton vector field $H_a \in \dot{C}^\infty(T^*\mathbb{M}, TT^*\mathbb{M})$ of $a \in \dot{C}^\infty(T^*\mathbb{M})$.

The second key point for us is that b-ps.d.o’s naturally act on weighted b-Sobolev
spaces $H^{s,\alpha}_b(M)$, defined above:
$$\Psi_b^m(M) \ni A; H^{s,\alpha}_b(M) \rightarrow H^{s-m,\alpha}_b(M), \quad s, \alpha \in \mathbb{R}.$$
We will collect some more information on \(b\)-Sobolev spaces and \(b\)-ps.d.o’s in Section 2.

The analogous ‘non-smooth’ operators that play the starring role in this paper, \(b\)-Sobolev \(b\)-ps.d.o’s, are locally defined by (1.4), but we now allow the symbol \(a\) to be less regular. As an example, for many remainder terms in our computations, it will suffice to merely have

\[
\left\| a(z, \zeta) \right\|_{H^s_{\zeta}(\mathbb{R^n}_\zeta)} \leq C, \text{ uniformly in } \zeta \in \mathbb{R^n},
\]

which already implies that \(A = a(z, bD_z)\) defines a continuous map

\[
A: H^{s'}_b \to H^{s'-m}_b, \quad s \geq s' - m, s > n/2 + \max(0, m - s');
\]

see Proposition 3.9. Assuming more regularity of the symbols in \(\zeta\), we can study compositions of such non-smooth operators; the main tool here is the asymptotic expansion (1.5), which must be cut off after finitely many terms in view of the limited regularity of the symbols, and the remainder term will be estimated carefully. In Section 3, we will develop the (partial) calculus of \(b\)-Sobolev \(b\)-ps.d.o’s as far as needed for the remainder of the paper, in particular for the proofs of microlocal regularity results, which will be essential for the linear analysis of equation (1.3).

Let us briefly recall a few such regularity results in the smooth setting.\(^8\) First, we define the \(b\)-wavefront set \(WF_b^b(u) \subset bT^*M \setminus o\) of \(u \in H^\infty_b(M)\) as the complement of the set of all \(\omega \in bT^*M \setminus o\) such that \(Au \in L^2_b(M) = H^0_b(M)\) for some \(A \in \Psi^1_b(M)\) elliptic at \(\omega\); recall that a \(b\)-ps.d.o \(A \in \Psi^1_b(M)\) with homogeneous principal symbol \(a\) is elliptic at \(\omega \in bT^*M \setminus o\) iff \(|a(\lambda\omega)| \geq c|\lambda|^m\) for \(\lambda \geq 1\), where we let \(\mathbb{R}_+\) act on \(bT^*M \setminus o\) by dilations in the fiber. We informally say that \(u\) is in \(H^s_b\) microlocally at \(\omega\) iff \(\omega \not\in WF_b^b(u)\). By definition, the wavefront set is closed and conic, thus we can view it as a subset of \(bS^*M\); moreover, it can capture global \(H^\infty_b\)-regularity in the sense that \(WF_b^b(u) = \emptyset\) implies \(u \in H^s_b(M)\) (and vice versa). **Elliptic regularity** then states that if \(u \in H^{-\infty}_b\) satisfies \(Pu \in H^{s-m}_b\) for \(P \in \Psi^m_b\) which is elliptic at \(\omega\), then \(u\) is in \(H^s_b\) microlocally at \(\omega\). The proof is an easy application of the symbolic calculus – one essentially takes the reciprocal of the symbol of \(P\) near \(\omega\) to obtain an approximate inverse of \(P\) there – and readily generalizes to the non-smooth setting as shown in Section 5; the main technical task is to understand reciprocals of non-smooth symbols, which we will deal with in Section 4.

Next, given an operator \(P \in \Psi^m_b\) with real homogeneous principal symbol \(p\), we need to study the singularities for solutions \(u \in H^{-\infty}_b\) of \(Pu = f \in H^s_b\) within the characteristic set \(\Sigma = p^{-1}(0)\) of \(u\),\(^9\) where we assume \(dp \neq 0\) at \(\Sigma\) (so that \(\Sigma\) is a smooth conic codimension 1 submanifold of \(bT^*M \setminus o\)). The **real principal type propagation of singularities**, in the setting of closed manifolds originally due to Duistermaat and Hörmander [12], then states that \(WF^b_b(u)\) is invariant under the flow of the Hamilton vector field \(H_p\) of \(p\). In other words, \(WF^b_b(u)\) is the union of maximally extended null-bicharacteristics of \(P\), which are by definition flow lines of \(H_p\). One proof of this statement uses a positive commutator estimate in the form given by de Hoop, Uhlmann and Vasy [10] (see Section 6 for further

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\(^8\)We use unweighted \(b\)-Sobolev spaces here for brevity.

\(^9\)Note that elliptic regularity gives \(u \in H^{s+1}_b\) microlocally off \(\Sigma\).
references), which roughly goes as follows: \(^{10}\) Suppose \(\omega \notin \WF^a_b(u)\); we want to propagate microlocal \(H^\sigma_b\)-regularity of \(u\) along a null-bicharacteristic strip \(\gamma\) of \(P\) from \(\omega\) to a nearby point \(\omega' \in \gamma\). To do so, we choose a symbol \(a \in S^{2\sigma-m+1}\) with support localized near \(\gamma\), which is decreasing along the Hamilton flow of \(p\) except near \(\omega\) (where we have a priori information on \(u\)), i.e. \(H_p a = -b^2 + e\), where \(e \in S^{2\sigma}\) is supported near \(\omega\), and \(b \in S^\sigma\) is elliptic near \(\omega'\). Then, denoting by \(A, B, E\) quantizations of \(a, b, e\), respectively, we obtain
\[
\|Bu\|_{L^2_b(M)}^2 = \langle B^* Bu, u \rangle = \langle Eu, u \rangle - \langle i[P, A]u, u \rangle + \langle Gu, u \rangle, \tag{1.8}
\]
where \(\langle \cdot, \cdot \rangle\) denotes the (sesquilinear) dual pairing on \(L^2_b(M)\), and \(G = B^* B - E + i[P, A] \in \Psi^{2\sigma-1}_b\). For simplicity, let us assume \(u \in H^\sigma_{b-1/2} (M)\); then, expanding the commutator and using that \(Pu \in H^\sigma_{b-m+1}\), moreover using that \(\langle Eu, u \rangle\) is bounded by the regularity assumption on \(u\) at \(\omega\) and \(\langle Gu, u \rangle\) is bounded since \(u\) is in \(H^\sigma_{b-1/2}\), we obtain \(Bu \in L^2_b\). Hence by elliptic regularity, \(u \in H^\sigma_{b}\) microlocally near \(\omega'\), finishing the argument. Notice the loss of one derivative compared to the elliptic setting, which naturally comes about by use of a commutator: We can only propagate \(H^\sigma_{b}\)-regularity of \(u\), not \(H^\sigma_{b+1}\)-regularity, even though \(Pu \in H^\sigma_{b+1-m}\).

We will generalize this statement to the case of non-smooth \(P\) in Section 6.3 by a similar proof. Since \(P\) now only acts on a certain range of \(b\)-Sobolev spaces, the allowed degrees \(\sigma\) of regularity that we can propagate have bounds both from above and from below in terms of the regularity \(s\) of the coefficients of \(P\); also, since non-smooth operators like the ones given by symbols as in (1.6) have very restricted mapping properties on low or negative order spaces, see (1.7), we need to assume higher regularity \(H^\sigma_{b}\) of the coefficients of \(P\) when we want to propagate low regularity \(H^\sigma_{b}\) of solutions \(u\). The main bookkeeping overhead of the proof of the propagation of singularities thus comes from the need to make sense of all compositions, dual pairings, adjoints and actions of non-smooth operators that appear in the course of the positive commutator argument. On a more technical side, we will be choosing most operators in the argument \((A, B, E, G\) in the above notation\) to be smooth ones and thus have to absorb certain non-smooth terms into an additional error term \(F\) of symbolic order \(2\sigma\), which would render the above argument invalid; by judiciously choosing \(B\) and \(E\), we can however ensure that the symbol of \(F\) in fact has a sign, thus the additional term \(\langle Fu, u \rangle\) appearing in (1.8) can be bounded by a version of the sharp Gårding inequality which we will prove in Section 6.1.

In order to complete the microlocal picture, we also need to consider the propagation of singularities near radial points, which are points in the \(b\)-cotangent bundle where the Hamilton vector field \(H_p\) is radial, i.e. a multiple of the generator of dilations in the fiber. The above propagation of singularities statement does not give any information at radial points. Now, in many geometrically interesting cases, the Hamilton flow near the set of radial points has a lot of structure, e.g. if the radial set is a set of sources/sinks/saddle points for the flow. The proof of a microlocal estimate near a class of radial points in Section 6.4 (see the introduction to Section 6 for references in the smooth setting) again proceeds via positive commutators, thus similar comments about the interplay of regularities as in the real principal type setting apply.

\(^{10}\)We omit a number of terms and gloss over the fact that the argument needs to be regularized in order to make sense of the appearing dual pairings.
In Section 7.2, we will combine the microlocal regularity results with standard energy estimates for second order hyperbolic equations from Section 7.1, see e.g. Hörmander [20, Chapter XXIII] or [18, §2], and prove the existence and higher regularity of global forward solutions to linear wave equations with non-smooth coefficients under certain geometric and dynamical assumptions, in particular non-trapping.\textsuperscript{11} The idea is to start off with forward solutions in a space $H^0_{\mathbf{b}, r}$, $r \ll 0$, obtain higher regularity at elliptic points, propagate higher regularity (from the ‘past,’ where the solution vanishes) using the real principal type propagation of singularities, propagate this regularity into radial points, which lie over the boundary, and propagate from there within the boundary; the non-trapping assumption guarantees that by piecing together all such microlocal regularity statements, we get a global membership in a high regularity b-Sobolev space, however still with weight $r \ll 0$. To improve the decay of the solution, we use a contour deformation argument using the normal operator family as in [34, §3].

Finally, to apply the machinery developed thus far to quasilinear wave and Klein-Gordon equations on ‘static’ asymptotically de Sitter spaces, we check in Section 8 that they fit into the framework of Section 7.2, thereby proving Theorems 1.1 and 1.2. To keep the discussion in Section 8 simple, we will in fact only consider quasi-linear equations on static patches of de Sitter space explicitly, but the reader should keep in mind that the arguments apply in more general settings; see Section 8, in particular Remark 8.3, for further details.

2. Function and symbol spaces for local b-analysis

We work on an $n$-dimensional manifold $M$ with boundary $\partial M$. Since almost all results we will describe are local, we consider a product decomposition $\mathbb{R}^n_T = (\mathbb{R}^+_x) \times \mathbb{R}^{n-1}_y$ near a point on $\partial M$. Whenever convenient, we will assume that all distributions and kernels of all operators we consider have compact support. Whenever the distinction between $x$ and $y$ (or their dual variables, $\lambda$ and $\eta$) is unimportant, we also write $z = (x, y)$ (or $\zeta = (\lambda, \eta)$).

On $\mathcal{S}(\mathbb{R}^{n-1}_y)$, we have the Fourier transform $(\mathcal{F} v)(\eta) = \int e^{-iy \cdot \eta} v(y) dy$ with inverse $(\mathcal{F}^{-1} v)(y) = \int e^{iy \cdot \eta} v(\eta) d\eta$, where we normalize the measure $d\eta$ to absorb the factor $(2\pi)^{-(n-1)}$. Likewise, on $\mathcal{C}^\infty_c(\mathbb{R}^+_x)$, i.e. functions vanishing to infinite order at 0 with compact support, we have the Mellin transform $(\mathcal{M} u)(\lambda) = \int_0^\infty x^{-i\lambda} u(x) \frac{dx}{x}$ with inverse $(\mathcal{M}_\alpha^{-1} u)(x) = \int_{\text{Im} \lambda = -\alpha} x^{i\lambda} u(\lambda) d\lambda$, where $\alpha \in \mathbb{R}$ is arbitrary; here, we also normalize $d\lambda$ to absorb the factor $(2\pi)^{-1}$. For any function $u = u(x, y) \in \mathcal{C}^\infty_c(\mathbb{R}^n_T)$, we shall write

$$\hat{u}(\lambda, \eta) = (\mathcal{M}_x \rightarrow \lambda, \mathcal{F}_y \rightarrow \eta) u(\lambda, \eta).$$

Weighted b-Sobolev spaces on $\mathbb{R}^n_T$ can then be defined by

$$u \in H^\alpha_0(\mathbb{R}^n_T) \iff (\zeta)^\alpha (\mathcal{M} |_{\text{Im} \lambda = -\alpha} \mathcal{F} u)(\zeta) \in L^2(\mathbb{R}^n_T),$$

where the restriction to $\text{Im} \lambda = -\alpha$ effectively removes the weight $x^\alpha$. We will also write $L^2_0(\mathbb{R}^n_T) = H^0_0(\mathbb{R}^n_T)$, which agrees with the usual definition of $L^2_0(\mathbb{R}^n_T)$.

\textsuperscript{11}The use of energy estimates, although they are microlocally inconvenient since they restrict the allowed range of Sobolev spaces one can work in, is necessary to guarantee the forward character of solutions and is moreover unproblematic for present purposes, since we only work in high regularity spaces and do not need estimates for adjoints on low regularity spaces. See also the discussion in [18, Section 2.1].
distribution a symbol \( \text{lb}(a) \) are conormal distributions, namely the partial Fourier transform of a symbol \( k \) kernels of \( b \)-ps.d.o’s explicitly, as done by Melrose [24]: On the one hand, they are appropriately chosen \( \chi \) thus \( C \) holds (with \( \phi \)). Let \( \chi \equiv \chi_k \) be any characteristic function of \( K \).

Proof. It suffices to prove the lemma when \( \chi_k \) is replaced by \( \chi_k \in C^\infty(\mathbb{R}^n) \), where \( \chi_k = 1 \) on the half line \( \mathbb{R}_{\geq 1} \). Given such a \( \chi_k \) and \( \phi \in C^\infty(\mathbb{R}^n) \) so that (2.1) holds (with \( \chi_k \) replaced by \( \chi_k \)), the map \( A: v \mapsto (\chi_k(bD)(bD)^* + r(bD))(\phi v) \) is an element of \( \Psi_{b,c}(\mathbb{R}^n) \) for an appropriate choice of \( r(\xi) \in S^{-\infty} \) (see Lemma 2.5). Since \( r(bD): H_b^{-\infty}(\mathbb{R}^n) \to H_b^{-\infty}(\mathbb{R}^n) \), we conclude that \( \widehat{Au} \in L^2(\mathbb{R}^n) \), which by Plancherel’s theorem gives \( (z_0, \zeta_0) \notin \WF^b_\alpha(u) \).

For the converse direction, given \( A \in \Psi_{b,c}(\mathbb{R}^n) \), \( \sigma^*_b(A)(z_0, \zeta_0) \neq 0 \), take \( \phi \in C^\infty(\mathbb{R}^n) \) and \( \chi_k \in C^\infty(\mathbb{R}^n) \) with \( \phi(z_0) \neq 0, \chi_k(\zeta_0) \neq 0 \) such that \( A \) is elliptic on \( \WF^b_\alpha(B) \), where \( B = (\chi_k(bD)(bD)^* + r(bD))(\phi) \in \Psi_{b,c}(\mathbb{R}^n) \), again with an appropriately chosen \( r \in S^{-\infty} \). A straightforward application of the symbol calculus gives the existence of \( C \in \Psi_{b,c}(\mathbb{R}^n) \), \( R' \in \Psi_{b,c}(\mathbb{R}^n) \) such that \( B = CA - R' \); thus \( Bu = C(Au) - R'u \in L^2(\mathbb{R}^n) \). Since \( r(bD): H_b^{-\infty} \to H_b^{-\infty} \), we conclude that \( \chi_k(\zeta_0)\zeta_0^* \phi u \in L^2(\mathbb{R}^n) \), and the proof is complete. \( \square \)

It is convenient to build up the calculus of smooth \( b \)-ps.d.o’s on \( M \) using the kernels of \( b \)-ps.d.o’s explicitly, as done by Melrose [24]: On the one hand, they are conormal distributions, namely the partial Fourier transform of a symbol\(^{12} \) \( a(x, y; \lambda, \eta) \) near the diagonal of the \( b \)-stretched product \( M_b^2 \), smoothly up to the front face, and on the other hand, they vanish to infinite order at the two boundaries \( \text{lb}(M_b^2) \) and \( \text{rb}(M_b^2) \), which in particular ensures that \( b \)-ps.d.o’s act on weighted spaces. However, we will refrain from describing the kernels of the non-smooth \( b \)-operators to be considered later and rather keep track of more information on the symbol \( a \), wherever this is necessary. The idea is the following: Given a conormal distribution

\[
\tilde{I}_a(s) := \int e^{i\lambda \log s} a(\lambda) \, d\lambda, \quad a \in S^m(\mathbb{R}),
\]

\(^{12}\)For clarity, the semicolon ‘;’ will often be used to separate base and fiber variables (resp. differential operators) in symbols (resp. operators).
The function $I_a(t) := \hat{I}_a(t)$ is rapidly decaying as $|t| \to \infty$. If we require however that $I_a(s)$ be rapidly decaying as $s \to 0$ and $s \to \infty$, i.e. $I_a$ is super-exponentially decaying as $|t| \to \infty$, it turns out that the symbol $a(\lambda)$ can be extended to an entire function of $\lambda$ with symbol bounds in $\text{Re} \lambda$ which are locally uniform in $\text{Im} \lambda$; see Lemma 2.3 below.

**Definition 2.2.** Let $m \in \mathbb{R}$. Then $S^m_b((\mathbb{R}^+)_x \times \mathbb{R}^{n-1}; \mathbb{R}_x \times \mathbb{R}^n_\sigma)$ is the space of all symbols $a \in S^m((\mathbb{R}^+)_x \times \mathbb{R}^{n-1}; \mathbb{R}_x \times \mathbb{R}^n_\sigma)$ such that the partial inverse Fourier transform $\mathcal{F}^{-1}_x a$ is super-exponentially decaying as $|t| \to \infty$, i.e. for all $\mu \in \mathbb{R}$, there is $C_\mu < \infty$ such that $|\mathcal{F}^{-1}_{x,y} a(x,y,t,\eta)| \leq C_\mu e^{-\mu|t|}$ for $|t| > 1$.

**Lemma 2.3.** Let $m \in \mathbb{R}$. Then $a(\lambda) \in S^m_b(\mathbb{R})$ if and only if $a$ extends to an entire function, also denoted $a(\lambda)$, which for all $N, K \in \mathbb{N}$ satisfies an estimate

$$|D^N_{\lambda} a(\lambda)| \leq C_{N,K} \langle \lambda \rangle^{m-k}, \quad |\text{Im} \lambda| \leq N, k \leq K. \quad (2.2)$$

for a constant $C_{N,K} < \infty$.

We will need the following simple estimate.

**Lemma 2.4.** For $\alpha \geq 0$, $(x+y)^\alpha \lesssim (x)^\alpha + (y)^\alpha$.

**Proof.** The statement is obvious if $\alpha = 0$, so let us assume $\alpha > 0$. Put $\beta = \alpha/2$. Using $|x+y|^2 \leq 2(|x|^2 + |y|^2)$, we get

$$(x+y)^\alpha = (1 + |x+y|^2)^\beta \leq (1 + 2|x|^2 + 2|y|^2)^\beta \leq (4 + 2|x|^2 + 2|y|^2)^\beta = 2^\beta(1 + |x|^2) + (1 + |y|^2)^\beta.$$ 

Thus, putting $u = 1 + |x|^2$, $v = 1 + |y|^2$, it suffices to show $(u+v)^\beta \lesssim u^\beta + v^\beta$. Notice that $u, v \geq 1$, hence, introducing $w = v/u \in (0, \infty)$, this reduces to $(1 + w)^\beta \lesssim 1 + w^\beta$. Define the continuous function $f: (0, \infty) \to \mathbb{R}$, $f(w) = (1 + w)^\beta/(1 + w^\beta)$. Since $\beta > 0$, $\lim_{w \to 0} f(w) = 1$, and we also have $\lim_{w \to \infty} f(w) = 1$. Thus, $f$ is bounded, which was to be shown. \hfill $\square$

**Proof of Lemma 2.3.** Given $a \in S^m_b$, we write $a = a_0 + a_1$, where for $\phi \in C^\infty_c(\mathbb{R})$, $\phi \equiv 1$ near 0,

$$a_0 = \mathcal{F} \phi \mathcal{F}^{-1} a, \quad a_1 = \mathcal{F} ((1 - \phi) \mathcal{F}^{-1} a).$$

Since $\mathcal{F}^{-1} a_1 \in C^\infty(\mathbb{R})$ is super-exponentially decaying, we easily get the estimate (2.2) for $a_1$ (in fact, the estimate holds for arbitrary $m$); see e.g. [24, Theorem 5.1]. Next, $\phi \mathcal{F}^{-1} a \in \mathcal{E}'(\mathbb{R})$, thus $a_0$ is entire, and we write for $\lambda, \mu \in \mathbb{R}$:

$$a_0(\lambda + i\mu) = \int_{\mathbb{R}^2} e^{i(\sigma - \lambda - i\mu)x} \phi(x) a(\sigma) \, d\sigma \, dx$$

$$= \int_{\mathbb{R}} a(\sigma + \lambda) \left( \int_{\mathbb{R}} e^{i|\sigma|} e^{i\mu x} \phi(x) \, dx \right) \, d\sigma.$$ 

Since $e^{i\mu x} \phi(x)$ is a locally bounded (in $\mu$) family of Schwartz functions, we have for $|\mu| \leq N, N \in \mathbb{N}$ arbitrary,

$$|a_0(\lambda + i\mu)| \leq C_N \int \langle \sigma + \lambda \rangle^m \langle \sigma \rangle^{-N} \, d\sigma \quad (2.3)$$

$$= C_N \left( \int_{|\sigma| \leq |\lambda|} \langle \sigma + \lambda \rangle^m \langle \sigma \rangle^{-N} \, d\sigma + \int_{|\sigma| > |\lambda|} \langle \sigma + \lambda \rangle^m \langle \sigma \rangle^{-N} \, d\sigma \right). \quad (2.4)$$
First, we consider the case $m \geq 0$. Then the first integral in (2.4) is bounded by
\[
\int_{\mathbb{R}} \langle \lambda \rangle^m \langle \sigma \rangle^{-N} d\sigma \leq C_N \langle \lambda \rangle^m
\]
for $N > 1$, and the second integral is bounded by
\[
\int_{\mathbb{R}} \langle \sigma \rangle^{-N+m} d\sigma \leq C_{N,m} \leq C_N \langle \lambda \rangle^m
\]
for $-N + m < -1$ in view of $m \geq 0$; thus we obtain (2.2) for $k = 0$.

Next, we consider the case $m < 0$. The integral in (2.3) is dominated by
\[
\langle \lambda \rangle^m \int_{\mathbb{R}} \frac{\langle \lambda \rangle^{-m}}{(\sigma + \lambda)^{-m}} \langle \sigma \rangle^{-N-m} d\sigma \leq \langle \lambda \rangle^m \int_{\mathbb{R}} C_m \langle \sigma \rangle^{-N-m} d\sigma \leq C_{N,m} \langle \lambda \rangle^m;
\]
we use Lemma 2.4 to see that the fraction in the integral is uniformly bounded by a constant $C_m$. This proves (2.2) for $k = 0$. To get the estimate for the derivatives of $a_0$, we compute
\[
D^k_\sigma a_0(\lambda) = \iint (-x)^k e^{i(\sigma - \lambda - i\mu)x} \phi(x) a(\sigma) d\sigma dx
\]
and the above estimates yield (2.2) for arbitrary $K$, since $|D^k a(\sigma)| \leq C_k \langle \sigma \rangle^{m-k}$.

For the converse direction, it suffices to prove the super-exponential decay of $\mathcal{F}^{-1} a$. Fix $\mu \in \mathbb{R}$. Then for $|x| > 1$, $k \in \mathbb{N}$, we compute
\[
e^{x\mu} \mathcal{F}^{-1} a(x) = e^{x\mu} \int_{\mathbb{R}} e^{ix\lambda} a(\lambda) d\lambda = \frac{e^{x\mu}}{x^k} \int_{\mathbb{R}} e^{ix\lambda} (-D_\lambda)^k a(\lambda) d\lambda.
\]
Choose $k$ such that $m - k < -1$, then we can shift the contour of integration to $\text{Im} \lambda = \mu$, thus
\[
|e^{x\mu} \mathcal{F}^{-1} a(x)| \leq |x|^{-k} \int_{\mathbb{R}} |D^k_\lambda a(\lambda + i\mu)| d\lambda \in L^\infty_x.
\]
Since this holds for any $\mu \in \mathbb{R}$, this gives the super-exponential decay of $\mathcal{F}^{-1} a$ for $|x| \to \infty$, and the proof is complete. \qed

In particular, the operator with full symbol $\langle \zeta \rangle^s$ is not a b-\text{p.s.d.o.} unless $s \in 2\mathbb{N}$. However, we can fix this by changing $\langle \zeta \rangle^s$ by a symbol of order $-\infty$; more generally:

**Lemma 2.5.** For any symbol $a \in S^m((\mathbb{R}^n_+) \times \mathbb{R}^{n-1}_y \times \mathbb{R}^\lambda \times \mathbb{R}^\eta)$, there is a symbol $\tilde{a} \in S^m_0$ with $a - \tilde{a} \in S^{-\infty}$.

**Proof.** Fix $\phi \in C^\infty_c(\mathbb{R})$ identically 1 near 0 and put
\[
\tilde{a}(x,y;\lambda,\eta) = \mathcal{F}_{\lambda \rightarrow t}((\mathcal{F}^{-1}_{\lambda \rightarrow t} a)(x,y;\lambda,\eta)\phi(t)).
\]
Then $\tilde{a} \in S^m_0$ by the proof of Lemma 2.3. Moreover, $\mathcal{F}^{-1}_{\lambda \rightarrow t} (a - \tilde{a})$ is smooth and rapidly decaying, thus the lemma follows. \qed

**Corollary 2.6.** For each $s \in \mathbb{R}$, there is $\Lambda_s \in \Psi^s_0((\mathbb{R}^n_+) \times \mathbb{R}^{n-1}_y \times \mathbb{R}^\lambda \times \mathbb{R}^\eta)$ with full symbol $\lambda_s \in S^m_0$, $\lambda_s(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^n$, such that $\lambda_s - \langle \zeta \rangle^s \in S^{-\infty}$. 
Proof. The only statement left to be proved is that \( \lambda_s \) can be arranged to be non-vanishing. Let \( \lambda_s \in S^*_b \) be the symbol constructed in Lemma 2.5. Since \( \hat{\lambda}_s \) differs from the positive function \( \langle \zeta \rangle^s \in S^s \setminus S^{s-1} \) by a symbol of order \( S^{-\infty} \), it is automatically positive for large \( |\zeta| \); thus we can choose \( C = C(s) \) large such that \( \lambda_s(\zeta) = \hat{\lambda}_s(\zeta) + C(s)e^{-\zeta^2} \) is positive for all \( \zeta \in \mathbb{R}^n \). Since \( e^{-\zeta^2} \in S^*_b \), the proof is complete. \( \square \)

3. A Calculus for Operators with b-Sobolev Coefficients

We continue to work in local coordinates on \( M \). To analyze the action of operators with non-smooth coefficients on b-Sobolev functions, we need a convenient formula. Given \( A \in \Psi^m_0(M) \) with full symbol \( a(x, y; \lambda, \eta) \in S^m(b^*TM) \), compactly supported in \( x, y \), we have for \( u \in C^\infty(M) \)

\[
Au(x, y) = \iiint e^{i\lambda \log(x/x')} e^{i\eta(y-y')} a(x, y; \lambda, \eta) u(x', y') \frac{dx'}{x'} \frac{dy'}{y'} d\lambda d\eta
\]

Writing \( \hat{a} \) for the Mellin transform in \( x \) and the Fourier transform in \( y \), we obtain

\[
\hat{A}u(\sigma, \gamma) = \iiint x^{-i(\sigma-\lambda)} e^{-i(\gamma-\eta)} a(x, y; \lambda, \eta) \hat{u}(\lambda, \eta) d\lambda d\eta \frac{dx}{x} dy
\]

(3.1)

Even though this makes sense as a distributional pairing, it is technically inconvenient to use directly: The problem is that if \( a \) does not vanish at \( x = 0 \), then \( \hat{a}(\sigma, \gamma; \lambda, \eta) \) has a pole at \( \sigma = 0 \) (cf. [24, Proposition 5.27]). This is easily dealt with by decomposing

\[
a = a^{(0)}(y; \lambda, \eta) + a^{(1)}(x, y; \lambda, \eta),
\]

(3.2)

where \( a^{(0)}(y; \lambda, \eta) = a(0, y; \lambda, \eta) \) and \( a^{(1)}(x, y; \lambda, \eta) = x \hat{a}^{(1)}(x, y; \lambda, \eta) \) with \( \hat{a}^{(1)} \in S^m \).

Then \( a^{(1)}(\sigma, \gamma; \lambda, \eta) \) is smooth and rapidly decaying in \( (\sigma, \gamma) \), and we write

\[
(A^{(1)}u)^\sim(\zeta) = \int a^{(1)}(\zeta - \xi; \zeta) \hat{u}(\xi) d\xi.
\]

(3.3)

For \( A^{(0)} = a^{(0)}(y, bD) \), we obtain

\[
(A^{(0)}u)^\sim(\sigma, \gamma) = \int \mathcal{F} a^{(0)}(\gamma - \eta; \sigma, \eta) \hat{u}(\sigma, \eta) d\eta,
\]

(3.4)

and \( \mathcal{F} a^{(0)}(\gamma; \sigma, \eta) \) is rapidly decaying in \( \gamma \).

Remark 3.1. Either we read off equation (3.4) directly from equation (3.1), where we observe that the symbol \( a^{(0)} \) is independent of \( x \), thus the integrals over \( x \) and \( \lambda \) are Mellin transform and inverse Mellin transform, respectively, and therefore cancel; or we observe that, with \( a^{(0)}(x, y; \lambda, \eta) := a^{(0)}(y; \lambda, \eta) \), we have \( \hat{a}^{(0)}(\sigma - \lambda, \gamma - \eta; \lambda, \eta) = 2\pi \delta_x = \mathcal{F} a^{(0)}(0, \gamma - \eta; \lambda, \eta) \). The second argument also shows that many manipulations on integrals that compute \( A^{(1)}u \) (or compositions of b-operators) also

\[\text{\textsuperscript{13}}\text{Of course, } a^{(0)} \text{ in general no longer has compact support; however, this will be completely irrelevant for the analysis, due to the fact that } a^{(0)} \text{ has ‘nice’ behavior in } y, \text{ independently in } x.\]
apply to the computation of $A^{(0)}u$ if one reads integrals as appropriate distributional pairings.

Notice that (3.3) is, with the change in meaning of $\hat{a}^{(1)}$ and $\hat{u}$ and keeping in mind that $a^{(1)} = x\hat{a}^{(1)}$ is a rather special symbol, the same formula as for pseudo-differential operators on a manifold without boundary used by Beals and Reed [7]. Since also the characterization of $H^s_\mathbb{F}$ functions in terms of their mixed Mellin and Fourier transform (Lemma 2.1) is completely analogous to the characterization of $H^s$ functions in terms of their Fourier transform, the arguments presented in [7] carry over to this restricted b-setting. In order to introduce necessary notation and construct a (partial) calculus in the full b-setting, containing weights, we will go through most arguments of [7], extending and adapting them to the b-setting; and of course we will have to treat the term $A_0$ separately.

The class of operators we are interested in are b-differential operators whose coefficients lie in (weighted) b-Sobolev spaces of high order. Let us remark that we do not attempt to develop an invariant calculus that can be transferred to a manifold; in particular, all definitions are on $\mathbb{R}^n_+$, see also the beginning of Section 2.

We thus define the following classes of non-smooth symbols:

**Definition 3.2.** For $m, s \in \mathbb{R}$, define the spaces of symbols

$$H^s_\mathbb{F}^m = \left\{ \sum_{\text{finite}} a_j(z)p_j(z, \zeta) : a_j \in H^s_\mathbb{F}, p_j \in S^m \right\},$$

and denote by $H^s_\mathbb{F}^m$ the corresponding spaces of operators, i.e.

$$H^s_\mathbb{F}^m = \left\{ a(z, bD) : a(z, \zeta) \in H^s_\mathbb{F}^m \right\}.$$

Moreover, let $\Psi^m = \left\{ a(z, bD) : a(z, \zeta) \in S^m \right\}.$

**Remark 3.3.** In this paper, we will only deal with operators that are quantizations of symbols on the b-cotangent bundle, and thus with $\Psi^m$ we will always mean the space defined above.

**Remark 3.4.** In a large part of the development of the calculus for non-smooth b-ps.d.o’s in this section, we will keep track of additional information on the symbols of most ps.d.o’s, encoded in the space of symbols $S^m_\mathbb{F}$, in order to ensure that they act on weighted b-Sobolev spaces. Although this requires a small conceptual overhead, it simplifies some computations later on.

The spaces $H^s_\mathbb{F}^m$ are not closed under compositions, in fact they are not even left $\Psi^m$-modules. To get around this, which will be necessary in order to develop a sufficiently powerful calculus, we will consider less regular spaces, which however are still small enough to allow for good analytic (i.e. mapping and composition) properties.

**Definition 3.5.** For $s, m \in \mathbb{R}, k \in \mathbb{N}_0$, define the space

$$S^{m;0}_\mathbb{F} = \left\{ p(z, \zeta) : p \in \left( \zeta \right)^m L^\infty_\mathbb{F} \left( \left( H^s_\mathbb{F} \right) z \right) \right\} = \left\{ p(z, \zeta) : \langle \eta \rangle^m \hat{p}(\eta, \zeta) \in L^\infty_\mathbb{F}^2 \right\}.$$ 

Let $S^{m;0}_\mathbb{F}^s$ be the space of all symbols $p(x, y; \lambda, \eta) \in S^{m;0}_\mathbb{F}^s$ which are entire in $\lambda$ with values in $\langle \eta \rangle^m L^\infty_\mathbb{F} \left( \left( H^s_\mathbb{F} \right) z \right)$ s.t. for all $N$ the following estimate holds:

$$\|p(z; \lambda + i\mu, \eta)|_{H^s_\mathbb{F}} \leq C_N \langle \lambda, \eta \rangle^m, \quad |\mu| \leq N.$$ (3.5)
Finally, define the spaces 
\[ S^{m,k}_{(b)} H^s_b = \left\{ p(z, \zeta) : \partial^\beta \partial^s p \in S^{m-|\beta|,0}_{(b)} H^s_b, |\beta| \leq k \right\}. \]
The spaces of operators which are left quantizations of these symbols are denoted by \( \Psi^{m,0} H^s_b, \Psi^{m,0}_{(b)} H^s_b \) and \( \Psi^{m,k}_{(b)} H^s_b \), respectively.

Weighted versions of these spaces, involving \( H^s_b, \alpha \) for \( \alpha \in \mathbb{R} \), are defined analogously.

We can also define similar symbol and operator classes for operators acting on bundles: Let \( E, F, G \) be the trivial (complex or real) vector bundles over \( \mathbb{R}^n \) of ranks \( d_E, d_F, d_G \), respectively, equipped with a smooth metric (Hermitian for complex bundles) on the fibers which is the standard metric on the fibers over the complement of a compact subset of \( \mathbb{R}^n \); then we can define
\[ H^s_b S^{m} (\mathbb{R}^n; G) := \{ (a_i)_{1 \leq i \leq d_G} : a_i \in H^s_b S^{m} \}. \]
We then define the space \( H^s_b \Psi^{m} (\mathbb{R}^n; E, F) \) to consist of left quantizations of symbols in \( H^s_b S^{m} (\mathbb{R}^n; \text{Hom}(E, F)) \); likewise for all other symbol and operator classes.\(^\text{14}\) We shall also write \( H^s_b \Psi^{m} (\mathbb{R}^n; E, E) := H^s_b \Psi^{m} (\mathbb{R}^n; E, E) \).

Remark 3.6. If we considered, as an example, the wave operator corresponding to a non-smooth metric acting on differential forms, the natural metric on the fibers of the form bundle would be non-smooth. Even though this could be dealt with directly in this setting, we simplify our arguments by choosing an ‘artificial’ smooth metric to avoid regularity considerations when taking adjoints, etc.

The first step is to prove mapping properties of operators in the classes just defined; compositions will be discussed in Section 3.2.

3.1. Mapping properties. The mapping properties of operators in \( \Psi^{m,0} H^s_b \) are easily proved using the following simple integral operator estimate.

Lemma 3.7. (Cf. [7, Lemma 1.4].) Let \( g(\eta, \xi) \in L^\infty_\xi L^2_\eta \) and \( G(\eta, \xi) \in L^\infty_\eta L^2_\xi \). Then the operator
\[ Tu(\eta) = \int G(\eta, \xi) g(\eta - \xi, \xi) u(\xi) \, d\xi \]
is bounded on \( L^2 \) with operator norm \( \leq ||G||_{L^\infty_\eta L^2_\xi} ||g||_{L^\infty_\xi L^2_\eta} ||u||_{L^2} \).

Proof. Cauchy-Schwartz gives
\[
||Tu||^2_{L^2} \leq \left( \int |G(\eta, \xi)|^2 \, d\xi \right) \left( \int |g(\eta - \xi, \xi) u(\xi)|^2 \, d\xi \right) \, d\eta \\
\leq ||G||^2_{L^\infty_\eta L^2_\xi} \int \left( \int |g(\eta - \xi, \xi)|^2 \, d\eta \right) |u(\xi)|^2 \, d\xi \\
\leq ||G||^2_{L^\infty_\eta L^2_\xi} ||g||^2_{L^\infty_\xi L^2_\eta} ||u||^2_{L^2}. \]

\(^\text{14}\)Since we are only concerned with local constructions, we use the sloppy notation just introduced; the proper class that the symbol of a \( b \)-pseudodifferential operator (with smooth coefficients), mapping sections of \( E \) to sections of \( F \), lies in, is \( S^m (bT^* M; \pi^* \text{Hom}(E, F)) \), where \( \pi : bT^* M \to M \) is the projection; see [24].
The most common form of $G$ in this paper is given by and estimated in the following lemma. We use the notation

$$a_+ := \max(a, 0), \quad a \in \mathbb{R}.$$ 

**Lemma 3.8.** Suppose $s, r \in \mathbb{R}$ are such that $s \geq r, s > n/2 + (-r)_+$, then

$$G(\eta, \xi) = \frac{(\eta)^r}{(\eta - \xi)^s} \in L^\infty_\eta (\mathbb{R}^n; L^2_\xi(\mathbb{R}^n)).$$

**Proof.** First, suppose $r \geq 0$. Then we use Lemma 2.4 to obtain

$$G(\eta, \xi)^2 \leq \frac{1}{(\eta - \xi)^{2(s-r)}} \frac{r}{(\eta - \xi)^{2s}}.$$

Since $s > n/2$, the $\xi$-integral of the second fraction is finite and $\eta$-independent. For the $\xi$-integral of the first fraction, we split the domain of integration into two parts and obtain

$$\int_{|\xi| \leq |\eta - \xi|} \frac{1}{(\eta - \xi)^{2(s-r)}} |\xi|^r d\xi + \int_{|\eta - \xi| \leq |\xi|} \frac{1}{(\eta - \xi)^{2(s-r)}} |\xi|^r d\xi$$

$$\leq \int \frac{1}{|\xi|^{2s}} d\xi + \int \frac{1}{(\eta - \xi)^{2s}} d\xi \in L^\infty_\eta.$$

Next, if $r < 0$, then we again use Lemma 2.4 to estimate

$$G(\eta, \xi)^2 = \frac{(\xi)^{-2r}}{(\eta - \xi)^{2s}} \leq \frac{1}{(\eta - \xi)^{2s}} \frac{1}{(\eta - \xi)^{-2r}},$$

where in the first fraction, we discarded the term $(\eta)^{-2r} \geq 1$. Since $s - (-r) > n/2$, the integrals of both fractions are finite, and the proof is complete. □

**Proposition 3.9.** Let $m \in \mathbb{R}$. Suppose $s \geq s' - m$ and $s > n/2 + (m - s')_+$. Then every $A = a(z, bD) \in \Psi^{m,0}_b H^s_b(\mathbb{R}^n_+; E, F)$ is a bounded operator $H^{s'}_b(\mathbb{R}^n_+; E) \to H^m_b(\mathbb{R}^n_+; F)$. If $A \in \Psi^{m,0}_b H^s_b(\mathbb{R}^n_+; E, F)$, then $A$ is also a bounded operator $H^{s'}_b(\mathbb{R}^n_+; E) \to H^m_b(\mathbb{R}^n_+; F)$ for all $\alpha \in \mathbb{R}$.

Note that this proposition also deals with ‘low’ regularity in the sense that negative $b$-Sobolev orders are permitted in the target space. We shall have occasion to use this in arguments involving dual pairings in Section 6.

**Proof of Proposition 3.9.** Let us first prove the statement without bundles, i.e. for complex-valued symbols and functions. Let $u \in H^s_b$ be given. Then

$$\langle \zeta \rangle^{s'-m} \widehat{Au}(\zeta) = \int \frac{\langle \zeta \rangle^{s'-m} \langle \xi \rangle^m}{(\zeta - \xi)^s} a_0(\zeta - \xi) u_0(\xi) d\xi$$

for $a_0(\zeta; \xi) \in L^\infty_\zeta L^2_\xi, u_0 \in L^2$. Lemma 3.8 ensures that the fraction in the integrand is an element of $L^\infty_\zeta L^2_\xi$, and then Lemma 3.7 implies $\langle \zeta \rangle^{s'-m} \widehat{Au}(\zeta) \in L^2_\zeta$.

In order to prove the second statement, we write for $u \in \mathcal{C}_c^\infty$

$$a(x, y, xD_x, D_y) u(x, y) = \int_{\text{Im} \lambda = 0} e^{i\lambda \log x} e^{i\eta} a(x, y; \lambda, \eta) \hat{u}(\lambda, \eta) d\lambda d\eta$$

$$= \int_{\text{Im} \lambda = 0} \hat{a}(\lambda)(\eta; x, y) \hat{u}(\lambda, \eta) d\lambda d\eta,$$

where

$$\hat{a}(\lambda)(\eta; x, y) = x^{i\lambda} e^{i\eta} a(x, y; \lambda, \eta);$$
we want to shift the contour of integration to \( \text{Im} \lambda = -\alpha \). Assuming that \( \text{supp}_{x,y} a \) is compact, we have that for any \( N \),
\[
\| \hat{a}(\lambda) (\eta, \cdot, \cdot) \|_{H^{s,-N}_b} \leq C_N (\lambda, \eta)^{m+\delta}, \quad |\text{Im} \lambda| < N,
\]
and \( \hat{a}(\lambda) \) is holomorphic in \( \lambda \) with values in \( H^{s,-N}_b \) for fixed \( \eta \). Since \( \hat{a}(\lambda, \eta) \) is rapidly decaying, we infer for all sufficiently large \( M > 0 \)
\[
\int \| \hat{a}(\lambda) (\eta, \cdot, \cdot) \|_{H^{s,-N}_b} |\hat{u}(\lambda, \eta)| \, d\eta \leq C_N \int (\lambda, \eta)^{m+\delta-M} \, d\eta = C_N M (\lambda)^{m+\delta-M+n-1},
\]
thus
\[
\hat{a}'(\lambda)(x, y) := \int \hat{a}(\lambda)(\eta; x, y) \hat{u}(\lambda, \eta) \, d\eta \in \langle \lambda \rangle^{-M} L^\infty(N(H^{s,-N}_b))
\]
for all \( M > 0 \), and \( \hat{a}' : \mathbb{C} \to H^{s,-N}_b \) is holomorphic. Therefore, if we choose \( N > |\alpha| \), we can shift the contour of integration to the horizontal line \( \mathbb{R} - i\alpha \):
\[
a(x, y, xD_x, yD_y)u(x, y) = \int_{\text{Im} \lambda = -\alpha} \hat{a}'(\lambda)(x, y) \, d\lambda
= e^{i\alpha \log x} e^{i\eta y} a(x, y; \lambda - i\alpha, \eta) (x^{-\alpha} u)^{\langle \lambda, \eta \rangle} \, d\lambda \, d\eta.
\] (3.7)
By definition, \( a \big| \text{Im} \lambda = -\alpha \) satisfies symbolic bounds just like \( a \big| \text{Im} \lambda = 0 \), thus we are done by the first half of the proof.

Adding bundles is straightforward: Write \( A \in \Psi^{m; 0} H^s_b(\mathbb{R}^n_+; E, F) \) as \( A = (A_{ij}) \), \( A_{ij} \in \Psi^{m; 0} H^s_b(\mathbb{R}^n_+) \) and \( u \in H^s_b(\mathbb{R}^n_+; E) \) as \( u = (u_j) \), \( u_j \in H^s_b(\mathbb{R}^n_+) \). Then \( Au = \left( \sum_{j=1}^n A_{ij} u_j \right) \), thus \( Au \in H^{s-m}_b(\mathbb{R}^n_+; F) \) follows by component-wise application of what we just proved. \( \square \)

**Corollary 3.10.** Let \( s > n/2 \). Then \( H^s_b(\mathbb{R}^n_+; \text{End}(E)) \) is an algebra. Moreover, \( H^s_b(\mathbb{R}^n_+; \text{Hom}(E, F)) \) is a left \( H^s_b(\mathbb{R}^n_+; \text{End}(E)) \)- and a right \( H^s_b(\mathbb{R}^n_+; \text{End}(F)) \)-module for \( |s'| \leq s \).

**Proof.** As in the proof of Proposition 3.9, we can reduce the proof to the case of complex-valued functions. For \( s' \geq 0 \), the claim follows from \( H^s_b \subset \Psi^{s; 0}_b H^s_b \) and the previous Proposition. For \( s' \leq 0 \), use duality. \( \square \)

### 3.2. Compositions.

The basic idea is to mimic the formula for the asymptotic expansion of the full symbol of an operator which is the composition of \( P = p(z, bD) \in \Psi^m_b \) and \( Q = q(z, bD) \in \Psi^{m'}_b \), namely
\[
\sigma_{\text{full}}(P \circ Q)(z, \zeta) \sim \sum_{\beta \geq 0} \frac{1}{\beta!} (\partial^{\beta}_\zeta (bD^\beta q))(z, \zeta).
\]
If \( p \) or \( q \) only have limited regularity in \( \zeta \) or \( z \), we only keep finitely many terms of this expansion and estimate the resulting remainder term carefully. More precisely, we compute\(^{15}\) for \( u \in \mathcal{C}_c^\infty \)
\[
(PQ u)^{\langle \cdot, \cdot \rangle}(\eta) = \int \int \hat{p}(\eta - \xi; \xi) \hat{q}(\xi - \zeta; \zeta) \hat{u}(\zeta) \, d\xi \, d\zeta
= \int \left( \int \hat{p}(\eta - \zeta; \zeta + \xi) \hat{q}(\xi; \zeta) \, d\xi \right) \hat{u}(\zeta) \, d\zeta, \quad \text{(3.8)}
\]
\(^{15}\)Keep Remark 3.1 in mind.
and
\[(\partial_\xi^k \hat{p} D_\xi^k q)(z, bD)\hat{u}(\eta) = \int (\partial_\xi^k \hat{p} D_\xi^k q)^\sim (\eta - \zeta; \zeta) \hat{u}(\zeta) d\zeta \]
\[= \int \left( \int \hat{p}(\eta - \zeta - \xi; \zeta) \xi^k \hat{q}(\xi; \zeta) d\xi \right) \hat{u}(\zeta) d\zeta. \]

We now apply Taylor’s theorem to the second argument of \(\hat{p} \) at \(\xi = 0\) in the inner integral in (3.8), keeping track of terms up to order \(k - 1 \in \mathbb{N}_0\),\(^{16}\) and obtain a remainder
\[\hat{r}(\eta - \zeta; \zeta) = \sum_{|\beta| = k} \frac{k}{|\beta|!} \int_0^1 \left( \int_0^1 (1-t)^{k-1} \partial_\xi^\beta \hat{p}(\eta - \zeta - \xi + t\xi) dt \right) \xi^k \hat{q}(\xi; \zeta) d\xi, \]

corresponding to the operator
\[r(z, bD) = P \circ Q - \sum_{|\beta| < k} \frac{1}{|\beta|!} (\partial_\xi^\beta \hat{p} D_\xi^\beta q)(z, bD). \quad (3.9)\]

We rewrite the remainder as
\[\hat{r}(\eta; \zeta) = \sum_{|\beta| = k} \frac{k}{|\beta|!} \int_0^1 \left( \int_0^1 (1-t)^{k-1} \partial_\xi^\beta \hat{p}(\eta - \zeta - \xi + t\xi) dt \right) (bD_\xi^\beta q)^\sim (\xi; \zeta) d\xi. \quad (3.10)\]

We will start by analyzing the terms in an expansion like (3.9) when the symbols involved are not smooth. When we deal with smooth b-operators by using the decomposition (3.2) of their symbols, we will need multiple sets of dual variables of \(x\) and \(y\). For clarity, we will stick to the following names for them:

- (Mellin-)dual variables of \(x\): \(\sigma, \lambda, \rho\)
- (Fourier-)dual variables of \(y\): \(\gamma, \eta, \theta\).

**Lemma 3.11.** Let \(s, s', m, m' \in \mathbb{R}\) be such that \(s > n/2, |s'| \leq s\). Then
\[S^{m, 0}H_0^{s'}, S^{m', 0}H_0^{s'} \subset S^{m+m', 0}H_0^{s'}, \]
\[S^m, S^{m', 0}H_0^{s'} \subset S^{m+m', 0}H_0^{s'}. \]

The same statements are true if all symbol classes are replaced by the corresponding b-symbol classes.

**Proof.** In light of the definitions of the symbol classes, we can assume \(m = m' = 0\). The first statement then is an immediate consequence of Corollary 3.10. In order to prove the second statement, we simply observe that, given \(p \in S^0\), \(p(\cdot, \xi)\) is a uniformly bounded family of multipliers on \(H_0^{s'}\). A direct proof of the sort that we will use in the sequel goes as follows: Decompose the symbol \(p\) as in (3.2). The part \(p^{(1)} \in S^{0, 0}H_0^\infty\) can then be dealt with using the first statement. Thus, we may assume \(p = p^{(0)}\), i.e. \(p = p(y; \lambda, \eta)\) is \(x\)-independent. Let \(q \in S^{0, 0}H_0^{s'}\) be given. Choose \(N\) large and put
\[p_0(\gamma; \lambda, \eta) = (\gamma)^N |\mathcal{F}p(\gamma; \lambda, \eta)|, \quad q_0(\sigma, \gamma; \lambda, \eta) = (\sigma, \gamma)^{s'} |\hat{q}(\sigma, \gamma; \lambda, \eta)|,\]
\[r_0(\sigma, \gamma; \lambda, \eta) = (\sigma, \gamma)^{s'} \hat{p}(\sigma, \gamma; \lambda, \eta)\].

\(^{16}\)The case \(k = 0\) is handled easily.
Then
\[
\int \int r_0(\sigma, \gamma; \lambda, \eta)^2 \, d\sigma \, d\gamma
\leq \int \int \left( \int \frac{(|\sigma, \gamma|)_{s'}}{(|\gamma - \theta|N(\sigma, \theta))_{s'}} p_0(\gamma - \theta; \lambda, \eta) q_0(\sigma, \theta; \lambda, \eta) \, d\theta \right)^2 \, d\sigma \, d\gamma
\leq \|p_0(\gamma; \lambda, \eta)\|_{L_0^\infty L_0^2}^2 \|q_0(\sigma, \theta; \lambda, \eta)\|_{L_0^\infty L_0^2}^2
\]
by Cauchy-Schwartz. \(\square\)

Recall Remark 3.3 for the notation used in the following theorem on the composition properties of non-smooth operators:

**Theorem 3.12.** Let \(m, m', s, s' \in \mathbb{R}, k, k' \in \mathbb{N}_0\). For two operators \(P = p(z, bD)\) and \(Q = q(z, bD)\) of orders \(m\) and \(m'\), respectively, let
\[
R = P \circ Q - \sum_{|\beta| < k} \frac{1}{|\beta|!} (\partial_z^\beta p (bD z) q)(z, bD).
\]
Denote the sum of the terms in the expansion for which \(|\beta| = j\) by \(E_j\).

(1) Composition of non-smooth operators, \(k \geq m + k', k \geq k'\).
(a) Suppose \(s > n/2\) and \(s \leq s' - k\). If \(P \in \Psi^{m,k} H^s_b\), \(Q \in \Psi^{m',0} H^{s'}_{b'}\), then \(E_j \in \Psi^{m+m'-j,0} H^s_{b'}\) and \(R \in \Psi^{m-m'-k,0} H^{s'}_{b'}\).
(b) If \(P \in \Psi^{m,k} H^s_b\), \(Q \in \Psi^{m',0} H^{s'}_{b'}\), then \(E_j \in \Psi^{m+m'-j,0} H^s_{b'}\) and \(R \in \Psi^{m-m'-k,0} H^{s'}_{b'}\).

(2) Composition of smooth with non-smooth operators.
(a) Suppose \(k \geq m + k'\). If \(P \in \Psi^{m} Q \in \Psi^{m',0} H^{s'}_{b'}\), then \(E_j \in \Psi^{m+m'-j,0} H^s_{b'}\) and \(R \in \Psi^{m-m'-k,0} H^{s'}_{b'}\).
(b) Suppose \(s \leq m + k'\). If \(P \in \Psi^{m,k} H^s_b\), \(Q \in \Psi^{m'} H^s_{b'}\), then \(E_j \in \Psi^{m+m'-j,0} H^s_{b'}\) and \(R \in \Psi^{m-m'-k,0} H^s_{b'}\).

(3) Composition of smooth with non-smooth operators.
\[k \leq m + k'\). If \(P \in \Psi^{m} Q \in \Psi^{m',0} H^{s'}_{b'}\), then \(E_j \in \Psi^{m+m'-j,0} H^s_{b'}\) and \(R = R_1 \Lambda_{m+k'-k} + \Lambda_{m+k'-k} R_2\), where \(R_1, R_2 \in \Psi^{m'-k,0} H^{s'-k}_{b'}\).

Moreover, (1)-(2) hold as well if all operator spaces are replaced by the corresponding b-spaces. Also, all results hold, mutatis mutandis, if \(P\) maps sections of \(F\) to sections of \(G\), and \(Q\) maps sections of \(E\) to sections of \(F\).

**Proof.** The statements about the expansion terms \(E_j\) follow from Lemma 3.11. It remains to analyze the remainder operators. We will only treat the case \(k > 0\); the case \(k = 0\) is handled in a similar way. We prove parts (1), (2a) and (3) of the theorem for \(k' = 0\) first.

1. (a) Consider the case \(s \leq s' - k\). We use formula (3.10) and define
\[
p_0(\eta, \xi; \zeta) = \sum_{|\beta| = k} \frac{1}{|\beta|!} (\eta)^s \int_0^1 |\partial^\beta \hat{p}(\eta; \xi + t\zeta)| \, dt, \quad q_0(\xi; \zeta) = \frac{(|\xi|^{s' - k})^m}{(|\zeta|^{m'})^{s' - k}},
\]
where \(bD^k\) denotes the vector \((bD^\beta)_{|\beta| = k}\). Since \(p_0 \in L_0^\infty L_0^2\) in view of \(k \geq m\), i.e. \(\partial^\beta \hat{p}\) is a symbol of order \(m - k \leq 0\), and \(q_0 \in L_0^\infty L_0^2\), we obtain
\[
\frac{(\eta)^s |\partial^\beta \hat{p}(\eta; \zeta)\|_{L_0^\infty L_0^2}}{|\zeta|^{m'}} \leq \int \frac{(|\eta|)^s}{(|\eta - \xi|^{s' - k})} \, p_0(\eta - \xi; \zeta) q_0(\xi; \zeta) \, d\xi \in L_0^\infty L_0^2,
\]
by Lemma 3.7, as claimed. Next, if \( s \leq s' - 2k + m \), we instead define

\[
p_0(\eta, \xi; \zeta) = \sum_{|\beta|=k} \frac{k}{\beta!} \langle \eta \rangle^s \int_0^1 \langle \zeta + t \xi \rangle^{k-m} |\partial_\zeta^\beta \hat{p}(\eta; \zeta + t \xi) | \, dt \in L^2_{\zeta} L^2_{\eta},
\]

(3.11)

thus

\[
\frac{\langle \eta \rangle^s |\hat{p}(\eta; \zeta) |}{\langle \zeta \rangle^m + m' - k} \leq \int \frac{\langle \eta \rangle^s}{\langle \eta - \xi \rangle^{s' - k}} \frac{\langle \zeta \rangle^{k-m}}{\inf_{0 \leq t \leq 1} \langle \zeta + t \xi \rangle^{k-m} p_0(\eta - \xi; \zeta) q_0(\eta; \zeta) \, d\xi}
\]

with \( q_0 \in L^\infty_{\xi} L^2_{\xi} \) as above. Now

\[
\frac{\langle \zeta \rangle^{k-m}}{\inf_{0 \leq t \leq 1} \langle \zeta + t \xi \rangle^{k-m}} \lesssim \langle \xi \rangle^{k-m},
\]

(3.12)

since for \(|\xi| \leq |\zeta|/2\), the left hand side is uniformly bounded, and for \(|\zeta| \leq 2|\xi|\), we estimate the infimum from below by 1 and the numerator from above by \( \langle \xi \rangle^{k-m} \). Therefore, we get \( r_0 \in L^\infty_{\xi} L^2_{\xi} \) in this case as well.

(b) is proved similarly: Define \( q_0(\eta; \zeta) \) as above, and choose \( N \) large and put

\[
p_0(\eta, \xi; \zeta) = \sum_{|\beta|=k} \frac{k}{\beta!} \langle \eta \rangle^N \int_0^1 |\partial_\zeta^\beta \hat{p}(\eta; \zeta + t \xi) | \, dt.
\]

Then

\[
\frac{\langle \eta \rangle^{s' - k} |\hat{p}(\eta; \zeta) |}{\langle \zeta \rangle^{m'}} \leq \int \frac{\langle \eta \rangle^{s' - k}}{\langle \eta - \xi \rangle^{N} \langle \zeta \rangle^{s' - k}} p_0(\eta - \xi; \zeta) q_0(\eta; \zeta) \, d\xi,
\]

and the fraction in the integrand is an element of \( L^\infty_{\eta} L^2_{\eta} \) by Lemma 3.8, thus an application of Lemma 3.7 yields \( R \in \Psi^{m+\theta} H_{b}^{s' - k} \). \( R \in \Psi^{m + m' - k + \theta} H_{b}^{s' - 2k + m} \) is proved in the same way using (3.12).

(2) Decomposing the smooth operator as in (3.2), the \( x \)-dependent part has coefficients in \( H^\infty_{\gamma} \), thus we can apply part (1). Therefore, we may assume that the smooth operator is \( x \)-independent in both cases.

(a) The remainder is

\[
\hat{r}(\sigma, \gamma; \lambda, \eta) = \sum_{|\beta|=k} \frac{k}{\beta!} \int \left( \int_0^1 (1 - t)\langle \gamma \rangle^{N} |\partial_\lambda^\beta \mathcal{F} p(\gamma - \theta; \lambda + t \sigma, \eta + t \theta) | \, dt \right)
\]

\[
\times \langle b D^\beta_\gamma \rangle (\sigma, \theta; \lambda, \eta) \, d\theta;
\]

therefore, choosing \( N \) large and defining

\[
p_0(\sigma, \theta; \lambda, \eta) = \sum_{|\beta|=k} \frac{k}{\beta!} \langle \gamma \rangle^{N} \int_0^1 |\partial_\lambda^\beta \mathcal{F} p(\gamma; \lambda + t \sigma, \eta + t \theta) | \, dt \in L^\infty_{\sigma, \theta, \lambda, \eta} L^2_{\gamma},
\]

\[
q_0(\sigma, \theta; \lambda, \eta) = \langle \sigma, \theta \rangle^{s' - k} |\langle b D^\beta_\gamma \rangle (\sigma, \theta; \lambda, \eta) \rangle \in L^\infty_{\sigma, \theta, \lambda, \eta} L^2_{\sigma, \theta},
\]

we get

\[
\frac{\langle \sigma, \gamma \rangle^{s' - k} |\hat{r}(\sigma, \gamma; \lambda, \eta) |}{\langle \lambda, \eta \rangle^{m'}} \leq \int \frac{\langle \sigma, \gamma \rangle^{s' - k}}{\langle \gamma - \theta \rangle^{N} \langle \sigma, \theta \rangle^{s' - k}} p_0(\gamma - \theta; \sigma, \theta; \lambda, \eta) q_0(\sigma, \theta; \lambda, \eta) \, d\theta,
\]

which is an element of \( L^\infty_{\lambda, \eta} L^2_{\gamma, \sigma} \) by Lemmas 3.8 and 3.7. Hence \( R \in \Psi^{m + \theta} H_{b}^{s' - k} \), and in a similar way we obtain \( R \in \Psi^{m + m' - k + \theta} H_{b}^{s' - 2k + m} \).
(b) Here, the remainder is
\[ \hat{r}(\sigma, \gamma; \lambda, \eta) = \sum_{|\beta|=k} \frac{k!}{\beta!} \int_0^1 \left( \int_0^1 (1 - t)^{k-1} \partial_{\lambda, \eta}^\beta \hat{\rho}(\sigma, \gamma - \theta; \lambda, \eta + t\theta) \, dt \right) \times \mathcal{F}(bD^2 \eta)(\theta; \lambda, \eta) \, d\theta, \]
and arguments similar to those used in (a) give the desired conclusion if \( k = k' \). If \( k < k' \), we just truncate the expansion after \( E_{k-1} \) and note that the resulting remainder term, which is the sum of the remainder term after expanding to order \( k' \) and the expansion terms \( E_k, \ldots, E_{k'-1} \), indeed lies in \( \Psi^{m+m'-k';0} H_b \).

(3) We again use formula (3.10) for the remainder term and put
\[ \hat{r}_1(\eta; \zeta) = \frac{\hat{r}(\eta; \zeta) \chi(|\zeta| \geq |\eta + \zeta|)}{\lambda_{m-k}(\zeta)}, \quad \hat{r}_2(\eta; \zeta) = \frac{\hat{r}(\eta; \zeta) \chi(|\zeta| < |\eta + \zeta|)}{\lambda_{m-k}(\eta + \zeta)}, \]
the point being that, by equation (3.3), for any \( u \in \mathcal{C}_c^\infty \),
\[ (r(z, bD)u)^m(\eta) = \int \hat{r}(\eta - \zeta, \zeta)\hat{u}(\zeta) \, d\zeta \]
\[ = \int \hat{r}_1(\eta - \zeta, \zeta)(\Lambda_{m-k}u)^m(\zeta) \, d\zeta + \lambda_{m-k}(\eta) \int \hat{r}_2(\eta - \zeta, \zeta)\hat{u}(\zeta) \, d\zeta \]
\[ = (r_1(z, bD)\Lambda_{m-k}u)^m(\eta) + (\Lambda_{m-k}r_2(z, bD)u)^m(\eta). \]
It remains to prove that \( r_1(z, bD), r_2(z, bD) \in \Psi^{m';0} H_b^{s'-k} \). First, we treat the case \( P \in x^0 \psi^m \). Then for any \( N \in \mathbb{N} \), we obtain, using
\[ \sup_{0 \leq t \leq 1} \langle \zeta + t \xi \rangle^{m-k} \lesssim \langle \zeta \rangle^{m-k} + \langle \xi \rangle^{m-k}, \]
that
\[ \langle \eta \rangle^{s'-k}|\hat{r}_1(\eta, \zeta)| \lesssim \int \frac{\langle \eta \rangle^{s'-k}(1 + \langle \xi \rangle^{m-k}/\langle \zeta \rangle^{m-k})}{\langle \eta - \xi \rangle N \langle \xi \rangle^{s'-k}} \chi(|\zeta| \geq |\eta + \zeta|) \times p_0(\eta - \xi, \zeta) q_0(\xi, \zeta) \, d\xi \]
\[ \equiv \int G(\eta, \xi; \zeta) p_0(\eta - \xi, \zeta) q_0(\xi, \zeta) \, d\xi, \]
where \( p_0(\eta, \xi; \zeta) \in L^\infty_{\xi, \zeta} L^2_{\eta} \) is defined as in (3.11) and \( q_0 \in L^\infty_{\xi, \zeta} L^2_{\eta} \) as before. We have to show \( G(\eta, \xi; \zeta) \in L^\infty_{\eta, \xi} L^2_{\zeta} \) in order to be able to apply Lemma 3.7. For \( |\xi| \geq 2|\eta| \), we immediately get, for \( N \) large enough,
\[ G(\eta, \xi; \zeta) \lesssim \frac{1}{\langle \xi \rangle^N} \left( 1 + \frac{\langle \eta \rangle^{m-k}}{\langle \zeta \rangle^{m-k}} \right) \in L^\infty_{\eta, \xi} L^2_{\zeta}(|\zeta| \geq 2|\eta|), \]
where \( N' = N - (k - s')_+ \). On the other hand, if \( |\xi| < 2|\eta| \), we estimate
\[ G(\eta, \xi; \zeta) \lesssim \frac{\langle \eta \rangle^{s'-k}}{(\eta - \xi)^N \langle \xi \rangle^{s'-k}} \left( 1 + \frac{\langle \eta \rangle^{m-k}}{\langle \zeta \rangle^{m-k}} \right) \chi(|\zeta| \geq |\eta + \zeta|) \]
and use that \( |\eta| \geq |\eta + \zeta| + | - \zeta | \leq 2|\zeta| \), hence the product of the last two factors is uniformly bounded, giving \( G(\eta, \xi; \zeta) \in L^\infty_{\eta, \xi} L^2_{\zeta}(|\zeta| < 2|\eta|) \) by Lemma 3.8.
In the case \( P = p(0, y; xD_x, D_y) \), we get the estimate

\[
\frac{\langle \sigma, \gamma \rangle^{s' - k} |\hat{f}_1(\sigma, \gamma; \lambda, \eta)|}{(\lambda, \eta)^{m'}} \leq \int G(\sigma, \gamma, \theta; \lambda, \eta)p_0(\gamma - \theta, \sigma, \theta; \lambda, \eta)q_0(\sigma, \theta; \lambda, \eta) \, d\theta,
\]

where \( p_0(\gamma, \sigma, \theta; \lambda, \eta) \in L_{\sigma,\theta,\lambda,\eta}^\infty L_\gamma^2 \), \( q_0(\sigma, \theta; \lambda, \eta) \in L_{\lambda,\eta}^\infty L_\sigma^2 \), and

\[
G(\sigma, \gamma, \theta; \lambda, \eta) = \frac{\langle \sigma, \gamma \rangle^{s' - k} - \langle \gamma - \theta \rangle^N \langle \sigma, \theta \rangle^{s' - k}}{(\lambda, \eta)^{m-k}} \chi(|(\lambda, \eta)| \geq |(\sigma, \gamma) + (\lambda, \eta)|).
\]

As above, separating the cases \(|(\sigma, \theta)| \geq 2|\langle \sigma, \gamma \rangle|\) and \(|(\sigma, \theta)| < 2|\langle \sigma, \gamma \rangle|\), one obtains \( G \in L_{\sigma,\lambda,\eta}^\infty L_\gamma^2 \), and we can again apply Lemma 3.7.

The second remainder term \( r_2 \) is handled in the same way.

Next, we prove that (1)-(2) also hold for the corresponding \( b \)-operator spaces. Using exactly the same estimates as above, one obtains the respective symbolic bounds for the remainders on each line \( \text{Im} \lambda = \alpha_0 \). What remains to be shown is the holomorphicity of the remainder operator in \( \lambda \). This is a consequence of the fact that the derivatives \( \partial_\lambda \partial_\nu^k p, |\nu| = k \), and \( \partial_\lambda q \), satisfy the same (in the case of symbols of smooth \( b \)-ps.d.o.'s, even better by one order) symbol estimates as \( \partial_\nu^k p \) and \( q \), respectively. Indeed, for (a), i.e. for non-smooth \( b \)-symbols, this follows from the Cauchy integral formula, which for \( \partial_\lambda q \) gives

\[
\partial_\lambda q(z; \lambda, \eta) = \frac{1}{2\pi i} \oint_{\gamma(\lambda)} \frac{q(z; \sigma, \eta)}{(\sigma - \lambda)^2} \, d\sigma
\]

where \( \gamma(\lambda) \) is the circle around \( \lambda \) with radius 1. Namely, since \(|\sigma - \lambda| = 1 \) for \( \sigma \in \gamma(\lambda) \), we get the desired estimate for \( \partial_\lambda q \) from the corresponding estimate for \( q \) itself. We handle \( \partial_\lambda \partial_\nu^k p \) similarly. (1b) and (2) for \( b \)-operators follow in the same way.

Finally, let us prove (1), (2a) and (3) for \( k' > 0 \) following the argument of Beals and Reed in [7, Corollary 1.6], starting with (1a): Choose a partition of unity on \( \mathbb{R}^n \) consisting of smooth non-negative functions \( \chi_0, \ldots, \chi_n \) with \( \text{supp} \chi_0 \subset \{ |\zeta| \leq 2 \} \), and \( |\zeta_i| \geq 1 \) on \( \text{supp} \chi_i \). Then

\[
P \circ Q\chi_0(bD) \in \Psi^{m,k} H^{k'}_s \circ \Psi^{-\infty,0} H^0_b
\]

can be treated using (1a) with \( k' = 0 \), taking an expansion up to order \( k \geq m + k' \geq m \); all terms in the expansion as well as the remainder term are elements of \( \Psi^{-\infty,0} H^0_s \), hence \( P \circ Q\chi_0(bD) \in \Psi^{-\infty,0} H^0_s \) can be put into the remainder term of the claimed expansion.

Let us now consider \( P \circ Q\chi_1(bD) \). For brevity, let us replace \( Q \) by \( Q\chi_1(bD) \) and thus assume \( |\zeta_i| \geq 1 \) on \( \text{supp} q(z, \zeta) \). Then by the Leibniz rule,

\[
P \circ Q^{bD}_{z_i} = P^{bD}_{z_i} \circ Q - \sum_{j=1}^{k'} c_{jk'} P^{bD}_{z_i} D^{k'-j} \circ (bD_{z_i} q)(z, bD)
\]
for some constants $c_{jk'} \in \mathbb{R}$. Composing on the right with $D_{z_i}^{-k'}$ thus shows that $P \circ Q$ is an element of the space

$$
\sum_{j=0}^{k'} \Psi^{m+k'-j,0} H^s_b \circ \Psi^{m'-k',j} H^{s'}_b.
$$

In view of the part of (1a) already proved, the $j$-th summand has an expansion to order $k-j \geq m+k'-j$ with error term in $\Psi^{m'+k'-j,0} H^s_b \circ \Psi^{m+m'-k,0} H^{s'}_b$, where we use $k-j \geq k-k' \geq 0$ and $s \leq (s'-j)-(k-j) \leq (s'-j)-2(k-j)+(m+k'-j)$. Using the same idea, one can prove (1b), (2a) and (3).

Notice that we do not claim in (3) that $q$ does. The issue is that $1/\lambda_m(\zeta)$ in general has singularities for non-real $\zeta$. In applications later in this paper, we will only need the proposition as stated, with the additional assumption that $p$ is a b-symbol, since instead of letting the operators in the expansion and the remainder operator act on weighted spaces, we will conjugate $P$ and $Q$ by the weight before applying the theorem.

### 4. Reciprocals of and compositions with $H^s_b$ functions

In this section, we recall some basic results about $1/u$ and, more generally, $F(u)$, for $u$ in appropriate b-Sobolev spaces on an $n$-dimensional compact manifold with boundary $M$, and smooth/analytic functions $F$.

**Remark 4.1.** We will give direct proofs which in particular do not give Moser-type bounds; see [31, §§13.3, 13.10] for examples of the latter. However, at least special cases of the results below (e.g. when $C^\infty(M)$ is replaced by $\mathbb{C}$ or $\mathbb{R}$) can easily be proved in a way as to obtain such bounds: The point is that the analysis can be localized and thus reduced to the case $M = \mathbb{R}^n_+$; a logarithmic change of coordinates then gives an isometric isomorphism of $H^s_b(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$, and on the latter space, Moser-type reciprocal/composition results are standard, see [31].

**4.1. Reciprocals.** Let $M$ be a compact $n$-dimensional manifold with boundary.

**Lemma 4.2.** Let $s > n/2 + 1$. Suppose $u, w \in H^s_b(M)$ and $a \in C^\infty(M)$ are such that $|a+u| \geq c_0 > 0$ on $\text{supp } w$. Then $w/(a+u) \in H^s_b(M)$, and one has an estimate

$$
\left\| \frac{w}{a+u} \right\|_{H^s_b} \leq C_K \|w\|_{H^s_b} \left(1 + \|u\|_{H^s_b}\right)^{[s]} \left(1 + \left\| \frac{1}{a+u} \right\|_{L^\infty(K)} \right)^{[s]+1}
$$

for any neighborhood $K$ of $\text{supp } w$.

**Proof of Lemma 4.2.** We can assume that $\text{supp } w$ and $\text{supp } u$ are small enough to lie in a coordinate patch of $M$. Note that clearly $w/(a+u) \in L^2_a$. We will give an iterative argument that improves on the regularity of $w/(a+u)$ by (at most) 1 at each step, until we can eventually prove $H^s_b$-regularity.

To set this up, let us assume $w/(a+u) \in H^{s'-1}_b$ for some $1 \leq s' \leq s$. Recall the operator $\Lambda_{s'} = \lambda_{s'}(bD)$ from Corollary 2.6, and choose $\psi_0, \psi \in C^\infty(M)$ such that

$$
\Lambda \psi_0 = \psi_0, \quad \Lambda \psi = \psi,
$$

where $\chi_0 \equiv 1$ on $\text{supp } \chi_0$ and $|\zeta| \geq 1/2$ on $\text{supp } \chi_0$.

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17To be precise, one should take $bD_{z_i}^{-k'} \chi_l(bD)$, where $\chi_l \equiv 1$ on $\text{supp } \chi_l$ and $|\zeta| \geq 1/2$ on $\text{supp } \chi_l$. 
Then, assuming that for integer \(1 \leq u\), it suffices to show that \([\Lambda_s^\psi]'(\psi_0w/a + u)\) estimate uniformly in \(t\), and \([\Lambda_s^\psi]'\) is \([\Lambda_s^\psi]'\) \(|\psi/w/a + u|\) \(L^\infty(K)\)

\[
\left\|\Lambda_s^\psi \frac{w}{a + u}\right\|_{L^2_b} \leq \left\|\left(1 - \psi\right)\Lambda_s^\psi \frac{\psi_0w}{a + u}\right\|_{L^2_b} + \left\|\psi\Lambda_s^\psi \frac{\psi_0w}{a + u}\right\|_{L^2_b}
\]

\[
\lesssim \left\|\frac{w}{a + u}\right\|_{L^2_b} + \left\|\frac{1}{a + u}\right\|_{L^\infty(K)} \left\|\psi(a + u)\Lambda_s^\psi \frac{w}{a + u}\right\|_{L^2_b}
\]

\[
\lesssim \left\|\frac{w}{a + u}\right\|_{L^2_b} + \left\|\frac{1}{a + u}\right\|_{L^\infty(K)} \left(\left\|\psi\Lambda_s^\psi w\right\|_{L^2_b} + \left\|\psi[\Lambda_s^\psi, a + u] \frac{w}{a + u}\right\|_{L^2_b}\right)
\]

\[
\lesssim \left\|\frac{w}{a + u}\right\|_{L^2_b} + \left\|\frac{1}{a + u}\right\|_{L^\infty(K)} \times \left(\|w\|_{H^0_b} + \left\|\frac{w}{a + u}\right\|_{H^0_b'_{s-1}} + \left\|\psi[\Lambda_s^\psi, w] - \frac{w}{a + u}\right\|_{L^2_b}\right),
\tag{4.2}
\]

where we used that the support assumptions on \(\psi_0\) and \(\psi\) imply \((1 - \psi)\Lambda_s^\psi \psi_0w \in \Psi^{-\infty}\), and \([\Lambda_s^\psi, a] \in \Psi^{s'-1}\). Hence, in order to prove that \(w/(a + u) \in H^s_b\), it is enough to observe that

\[
\|\psi' \frac{w}{a + u}\|_{L^2_b'} \lesssim \|\psi\|_{L^\infty(K)} \|w\|_{L^2_b} \left(1 + \left\|\frac{1}{a + u}\right\|_{L^\infty(K)}\right).
\]

We will to prove that this is an element of \(L^2_\zeta\) using Lemma 3.7. Since for \(|\beta| = 1\),

\[
|\partial_\zeta^\beta \Lambda_s^\psi(\eta + t(\zeta - \eta))| \lesssim \langle \eta + t(\zeta - \eta)\rangle^{s'-1},
\]

\[
|\langle bD^0_\zeta u \rangle(\zeta - \eta)| = \frac{u_0(\zeta - \eta)}{\langle \zeta - \eta\rangle^{s-1}}, \quad \|\hat{v}(\eta)\| = \frac{v_0(\eta)}{\langle \eta\rangle^{s'-1}}
\]

for \(u_0, v_0 \in L^2\), it is enough to observe that

\[
\langle \eta + t(\zeta - \eta)\rangle^{s'-1} \lesssim \frac{1}{\langle \zeta - \eta\rangle^{s-1}} + \frac{1}{\langle \zeta - \eta\rangle^{s'-1}} \in L^\infty_\zeta L^2_\eta,
\]

uniformly in \(t \in [0, 1]\), since \(s - 1 > n/2\).

To obtain the estimate (4.1), we proceed inductively, starting with the obvious estimate

\[
\|w/(a + u)\|_{L^2_b} \leq \|w\|_{L^2_b} \langle 1/(a + u)\rangle_{L^\infty(K)} \leq \|w\|_{H^0_b}(1 + \left\|\frac{1}{a + u}\right\|_{L^\infty(K)}).
\]

Then, assuming that for integer \(1 \leq m \leq s\), one has

\[
\|w/(a + u)\|_{H^m_b} \lesssim \|w\|_{H^m_b}(1 + \left\|\frac{1}{a + u}\right\|_{L^\infty(K)})^m (1 + \|w\|_{H^0_b})^{m-1}
\]
we conclude, using the estimate (4.2),
\[ \| \frac{w}{a + u} \|_{H^m_b} \leq \| \frac{w}{a + u} \|_{L^1_b} + \| \frac{1}{a + u} \|_{L^\infty(K)} \left( \| w \|_{H^m_b} + (1 + \| u \|_{H^1_b}) \right) \| \frac{w}{a + u} \|_{H^m_b} \]
\[ \leq \| w \|_{H^m_b} \left( 1 + \| \frac{1}{a + u} \|_{L^\infty(K)} \right)^{m+1} \left( 1 + \| u \|_{H^1} \right)^m. \]
Thus, one gets such an estimate for \( m = |s| \); then the same type of estimate gives (4.1), since one has control over the \( H^{s-1}_b \)-norm of \( w/(a+u) \) in view of \( s-1 < |s| \) and the bound on \( \| w/(a+u) \|_{H^{|s|}_b} \).

\[ \square \]

In particular:

**Corollary 4.3.** Let \( s > n/2 + 1 \).

1. If \( u \in H^s_b(M) \) does not vanish on \( \text{supp} \phi \), where \( \phi \in \mathcal{C}^\infty_c(M) \), then \( \phi/u \in H^s_b(M) \).
2. Let \( \alpha \geq 0 \). If \( u \in H^{s,\alpha}_b(M) \) is bounded away from \(-1\), then \( 1/(1 + u) \in 1 + H^{s,\alpha}_b(M) \).

**Proof.** The second statement follows from
\[ 1 - \frac{1}{1 + u} = \frac{u}{1 + u} \in H^{s,\alpha}_b(M). \]

We also obtain the following result on the inversion of non-smooth elliptic symbols:

**Proposition 4.4.** Let \( s > n/2 + 1 \), \( m \in \mathbb{R} \), \( k \in \mathbb{N}_0 \).

1. Suppose \( p(z,\zeta) \in S^{m:k}_b(F^0,H^0_b;\text{Hom}(E,F)) \) and \( a(z,\zeta) \in S^0 \) are such that \( \| p(z,\zeta)^{-1} \|_{\text{Hom}(F,E)} \leq c_0 \| \zeta \|^{-m} \), \( c_0 < \infty \), on \( \text{supp} \ a \). Then \( ap^{-1} \in S^{-m:k}_b(F^0,H^0_b;\text{Hom}(F,E)) \).
2. Let \( \alpha \geq 0 \). Suppose that \( p'(z,\zeta) \in S^{m:k}_b(F^0,H^0_b;\text{Hom}(E,F)) \), \( p''(z,\zeta) \in S^m_b(F^0,H^0_b;\text{Hom}(E,F)) \) and \( a(z,\zeta) \in S^0 \) are such that \( \| (p')^{-1} \|_{\text{Hom}(F,E)}, \| (p' + p'')^{-1} \|_{\text{Hom}(F,E)} \leq c_0 \| \zeta \|^{-m} \) on \( \text{supp} \ a \). Then \( a(p' + p'')^{-1} \in a(p'^{-1}) + S^{-m:k}_b(F^0,H^0_b;\text{Hom}(F,E)) \).

**Proof.** By multiplying the symbols \( p \) and \( p' \) by \( \langle \zeta \rangle^{-m} \), we may assume that \( m = 0 \).

1. Let us first treat the case of complex-valued symbols. By Corollary 4.3, \( a(\cdot,\zeta)/p(\cdot,\zeta) \in H^s_b \), uniformly in \( \zeta \); thus \( a/p \in S^{0,0}_b \). Moreover, for \( |\alpha| \leq k \),
\[ \partial^\alpha_{\zeta} \left( \frac{a}{p} \right) = \sum c_{\beta_1,\ldots,\gamma_{\nu}} \prod_{j=1}^u \partial_{\zeta}^\beta \prod_{i=1}^\nu \partial_\zeta^\gamma p, \]
where the sum is over all \( \beta_1 + \cdots + \beta_\mu + \gamma_1 + \cdots + \gamma_{\nu} = \alpha \) with \( |\gamma_j| \geq 1, 1 \leq j \leq \nu \).

Hence, using that \( H^s_b \) is an algebra and that the growth order of the numerator is \(-|\alpha|\), we conclude, again by Corollary 4.3, that \( \partial^\alpha_{\zeta}(a/p) \in S^{-|\alpha|,0} H^s_b \); thus \( a/p \in S^{0,k}_b H^s_b \).

Now, if \( p \) is bundle-valued, we obtain \( ap^{-1} \in S^{0,0}_b(F^0,H^0_b;\text{Hom}(F,E)) \) using the explicit formula for the inverse of a matrix and Corollaries 3.10 and 4.3; then, by virtue of
\[ \partial^\alpha_{\zeta}(ap^{-1}) = (\partial_\zeta a - p^{-1}(\partial_\zeta p))p^{-1}, \]
similarly for higher derivatives, we obtain \( a p^{-1} \in S^{0:k}H^s_b(\mathbb{R}^n_+; \text{Hom}(F, E)) \).

(2) Since \( a(p' + p'')^{-1} = (a(p'')^{-1})(I + p'(p'')^{-1})^{-1} \), we may assume \( p'' = I \), \( a \in S^0(\mathbb{R}^n_+; \text{Hom}(F, E)) \) and \( p' \in S^{0:k}H^s_b(\mathbb{R}^n_+; \text{End}(F)) \), and we need to show
\[
(I + p')^{-1} - I \in S^{0:k}H^s_b(\mathbb{R}^n_+; \text{End}(F)).
\]
But we can write
\[
(I + p')^{-1} - I = -p'(I + p')^{-1},
\]
which is an element of \( S^{0:k}H^s_b(\mathbb{R}^n_+; \text{End}(F)) \) by assumption. Then, by an argument similar to the one employed in the first part, we obtain the higher symbol estimates. \( \square \)

4.2. Compositions. Using the results of the previous subsection and the Cauchy integral formula, we can prove several results on the regularity of \( F(u) \) for \( F \) smooth or holomorphic and \( u \) in a weighted b-Sobolev space. The main use of such results for us will be that they allow us to understand the regularity of the coefficients of wave operators associated to non-smooth metrics.

In all results in this section, we shall assume that \( M \) is a compact \( n \)-dimensional manifold with boundary, \( s > n/2 + 1 \), and \( \alpha \geq 0 \).

**Proposition 4.5.** Let \( u \in H^s_b(M) \). If \( F : \Omega \to \mathbb{C} \) is holomorphic in a simply connected neighborhood \( \Omega \) of \( u(M) \), then \( F(u) - F(0) \in H^s_b(M) \). Moreover, there exists \( \epsilon > 0 \) such that \( F(v) - F(0) \in H^s_b(M) \) depends continuously on \( v \in H^s_b(M) \), \( ||u - v||_{H^s_b(M)} < \epsilon \).

**Proof.** Observe that \( u(M) \) is compact. Let \( \gamma \subset \mathbb{C} \) denote a smooth contour which is disjoint from \( u(M) \), has winding number 1 around every point in \( u(M) \), and lies within the region of holomorphicity of \( F \). Then, writing \( F(z) - F(0) = z F_1(z) \) with \( F_1 \) holomorphic in \( \Omega \), we have
\[
F(u) - F(0) = \frac{u}{2\pi i} \oint_{\gamma} F_1(\zeta) \frac{1}{\zeta - u} d\zeta,
\]
Since \( \gamma \ni \zeta \mapsto u/(\zeta - u) \in H^s_b(M) \) is continuous by Lemma 4.2, we obtain the desired conclusion \( F(u) - F(0) \in H^s_b(M) \).

The continuous dependence of \( F(v) - F(0) \) on \( v \) near \( u \) is a consequence of Proposition 4.2 and Corollary 3.10. \( \square \)

**Proposition 4.6.** Let \( u' \in C^\infty(M) \), \( u'' \in H^s_b(M) \); put \( u = u' + u'' \). If \( F : \Omega \to \mathbb{C} \) is holomorphic in a simply connected neighborhood \( \Omega \) of \( u(M) \), then \( F(u) \in C^\infty(M) + H^s_b(M) \); in fact, \( F(v) \) depends continuously on \( v \) in a neighborhood of \( u \) in the topology of \( C^\infty(M) + H^s_b(M) \).

**Proof.** Let \( \gamma \subset \mathbb{C} \) denote a smooth contour which is disjoint from \( u(M) \), has winding number 1 around every point in \( u(M) \), and lies within the region of holomorphicity of \( F \). Since \( u'' = 0 \) at \( \partial M \) and \( u'' \) is continuous by the Riemann-Lebesgue lemma, we can pick \( \phi \in C^\infty(M) \), \( \phi \equiv 1 \) near \( \partial M \), such that \( \gamma \) is disjoint from \( \omega'(\text{supp} \phi) \). Then
\[
\phi F(u) = \frac{1}{2\pi i} \oint_{\gamma} \phi \frac{F(\zeta)}{\zeta - u'} d\zeta + \frac{1}{2\pi i} \oint_{\gamma} \phi \frac{F(\zeta)}{\zeta - u''} d\zeta;
\]
the first term equals \( \phi F(u') \), and the second term is an element of \( H_b^{s,\alpha} \) by Corollary 4.3. Next, let \( \hat{\phi} \in C^\infty(M) \) be identically equal to 1 on \( \text{supp}(1 - \phi) \), and \( \hat{\phi} \equiv 0 \) near \( \partial M \). Then \( \hat{\phi}u \in H_b^{s,\alpha} \); in fact, it lies in any weighted such space. Thus,

\[
(1 - \phi)F(u) = \frac{1}{2\pi i} \oint_\gamma \frac{(1 - \phi)F(\zeta)}{\zeta - \hat{\phi}u} \, d\zeta \in H_b^{s,\alpha},
\]

and the proof is complete. \( \square \)

If we only consider \( F(u) \) for real-valued \( u \), it is in fact sufficient to assume \( F \in C^\infty(\mathbb{R}; \mathbb{C}) \) using almost analytic extensions, see e.g. Dimassi and Sjöstrand [11, Chapter 8]: For any such function \( F \) and an integer \( N \in \mathbb{N} \), let us define

\[
\tilde{F}_N(x + iy) = \sum_{k=0}^N \frac{(iy)^k}{k!} (\partial_x^k F)(x)\chi(y), \quad x, y \in \mathbb{R},
\]

(4.3)

where \( \chi \in C^\infty(\mathbb{R}) \) is identically 1 near 0. Then, writing \( z = x + iy \), we have for \( y \) close to 0:

\[
\partial_z \tilde{F}_N(z) = \frac{1}{2} (\partial_x + i\partial_y)\tilde{F}_N(z) = \frac{(iy)^N}{2N!} (\partial_x^{N+1} F)(x)\chi(y) = O(|\text{Im } z|^N).
\]

(4.4)

Observe that all \( u \in C^\infty(M) + H_b^{s,\alpha}(M) \) are bounded, hence in analyzing \( F(u) \), we may assume without restriction that \( F \in C^\infty(\mathbb{R}; \mathbb{C}) \).

**Proposition 4.7.** Let \( F \in C^\infty(\mathbb{R}; \mathbb{C}) \). Then for \( u \in H_b^{s,\alpha}(M; \mathbb{R}) \), we have \( F(u) - F(0) \in H_b^{s,\alpha}(M) \); in fact, \( F(u) - F(0) \) depends continuously on \( u \).

**Proof.** Write \( F(x) - F(0) = x F_1(x) \). Then, with \( (F_1)_N \) defined as in (4.3), the Cauchy-Pompeiu formula gives the pointwise identity

\[
F(u) - F(0) = -\frac{u}{\pi} \int_{\mathbb{C}} \frac{\partial_x (\tilde{F}_1)_N(\zeta)}{\zeta - u} \, dx \, dy, \quad \zeta = x + iy.
\]

Here, note that the integrand is compactly supported, and \( 1/(\zeta - u(z)) \) is locally integrable for all \( z \). In particular, we can rewrite

\[
F(u) - F(0) = -\frac{1}{\pi} \lim_{\delta \searrow 0} \int_{|\text{Im } \zeta| > \delta} \frac{\partial_x (\tilde{F}_1)_N(\zeta)}{\zeta - u} \, dx \, dy.
\]

(4.5)

Now Lemma 4.2 gives

\[
\left\| \frac{u}{\zeta - u} \right\|_{H_b^{s,\alpha}} \leq C(\|u\|_{H_b^{s,\alpha}}) |\text{Im } \zeta|^{-s-2},
\]

since \( u \) is real-valued. Thus, if we choose \( N \geq s + 2 \), then

\[
\mathbb{C} \setminus \mathbb{R} \ni \zeta \mapsto \partial_x (\tilde{F}_1)_N(\zeta) \frac{u}{\zeta - u} \in H_b^{s,\alpha}
\]

is bounded by (4.4), hence integrable, and therefore the limit in (4.5) exists in \( H_b^{s,\alpha} \), proving the proposition. \( \square \)

We also have an analogue of Proposition 4.6.

**Proposition 4.8.** Let \( F \in C^\infty(\mathbb{R}; \mathbb{C}) \), and \( u' \in C^\infty(M; \mathbb{R}) \), \( u'' \in H_b^{s,\alpha}(M; \mathbb{R}) \); put \( u = u' + u'' \). Then \( F(u) \in C^\infty(M) + H_b^{s,\alpha}(M) \); in fact, \( F(u) \) depends continuously on \( u \).
Proof. As in the proof of the previous proposition, we have the pointwise identity
\[ F(u' + u'') - F(u') = \frac{1}{\pi} \lim_{\delta \to 0} \int_{|\zeta| > \delta} \partial_\zeta (f_{\zeta}) (\frac{1}{\zeta - w' - u'' - \zeta} - \frac{1}{\zeta - w'}) \, dx \, dy \]
\[ = \frac{1}{\pi} \lim_{\delta \to 0} \int_{|\zeta| > \delta} \partial_\zeta (f_{\zeta}) \frac{u''}{\zeta - w'} \frac{1}{(\zeta - u') - u''} \, dx \, dy \]
Writing \( f_N := \partial_\zeta (f_{\zeta}) \), we estimate the \( H^{s,\alpha}_b \)-norm of the integrand for \( \zeta \in \mathbb{C} \setminus \mathbb{R} \) using Lemma 4.2 by
\[ \left\| \frac{f_N(\zeta)}{\zeta - w'} \right\|_{L(H^{s,\alpha}_b)} \left\| \frac{u''}{(\zeta - u') - u''} \right\|_{H^{s,\alpha}_b} \lesssim \left\| \frac{f_N(\zeta)}{\zeta - w'} \right\|_{L(H^{s,\alpha}_b)} |\text{Im} \, \zeta|^{-s-2}; \]
here, we denote by \( \|h\|_{L(H^{s,\alpha}_b)} \), for a function \( h \), the operator norm of multiplication by \( h \) on \( H^{s,\alpha}_b \). We claim that the operator norm
\[ b_s := \left\| \frac{f_N(\zeta)}{\zeta - w'} \right\|_{L(H^{s,\alpha}_b)} \]
is bounded by \( |\text{Im} \, \zeta|^{N+s-1} \); then choosing \( N \geq 2s + 3 \) finishes the proof as before.
To prove this bound, we use interpolation: First, since \( u' \) is real-valued, we have \( b_0 = O(|\text{Im} \, \zeta|^{-1} f_N(\zeta)) = O(|\text{Im} \, \zeta|^{-N}) \) by (4.4). Next, for integer \( k \geq 1 \), the Leibniz rule gives
\[ b_k \lesssim \sum_{j=0}^k |\text{Im} \, \zeta|^{-1-j} |\partial_x^{k-j} f_N(\zeta)| \lesssim |\text{Im} \, \zeta|^{N-k-1}, \]
where we use that \( |\partial_x^\ell f_N(\zeta)| = O(|\text{Im} \, \zeta|^N) \) for all \( \ell \), as follows directly from the definition of \( f_N \). By interpolation, we thus obtain \( b_s \lesssim |\text{Im} \, \zeta|^{N-s-1} \), as claimed. □

5. Elliptic regularity

With the partial calculus developed in Section 3, it is straightforward to prove elliptic regularity for \( b \)-Sobolev \( b \)-pseudodiﬀerential operators. Notice that operators with coefficients in \( H^s_b \) for \( s > n/2 \) must vanish at the boundary by the Riemann-Lebesgue lemma, thus they cannot be elliptic there. A natural class of operators which can be elliptic at the boundary is obtained by adding smooth \( b \)-p.d.o’s to \( b \)-Sobolev \( b \)-p.d.o’s, and we will deal with such operators in the second part of the following theorem.

**Theorem 5.1.** Let \( m, s, r \in \mathbb{R} \) and \( \zeta_0 \in \{ T^* \mathbb{R}^n_+ \setminus \partial \}. \) Suppose \( P' = p'(z, bD) \in H^m_b \Psi^m_n((\mathbb{R}^n_+) ; E, F) \) has a homogeneous principal symbol \( p'_m \). Moreover, let \( R \in \Psi^{m-1,0}_b H^{s-1}_b((\mathbb{R}^n_+) ; E, F) \).

1. Let \( P = P' + R \), and suppose \( p_m = p'_m \) is elliptic at \( \zeta_0 \), or
2. Let \( P = P' + P'' + R \), where \( P'' \in \Psi^m_b (\mathbb{R}^n_+) ; E, F \) has a homogeneous principal symbol \( p''_m \), and suppose \( p_m = p'_m + p''_m \) is elliptic at \( \zeta_0 \).

Let \( \delta \in \mathbb{R} \) be such that \( \delta \leq s - 1 \) and \( s > n/2 + 1 + (-s)_+ \). Then in both cases, if \( u \in H^{s+m-1, r}_b((\mathbb{R}^n_+) ; E) \) satisfies
\[ Pu = f \in H^{s+r}_b((\mathbb{R}^n_+) ; F), \]
it follows that \( \zeta_0 \notin WF^{s+m,r}_b (u) \).
We will only prove the theorem without bundles; adding bundles only requires simple notational changes. In both cases, we can assume that \( r = 0 \) by conjugating \( P \) by \( x^{-r} \); moreover, \( Ru \in H^s_b \) by Proposition 3.9 by the assumptions on \( s \) and \( \tilde{s} \), thus we can absorb \( Ru \) into the right hand side and hence assume \( R = 0 \). Choose \( a_0 \in S^0 \) elliptic at \( \zeta_0 \) such that \( p_m \) is elliptic on \( \text{supp} \ a_0 \).

(1) Let \( \lambda_m \) be as in Corollary 2.6. By Proposition 4.4,
\[
q(z, \zeta) := a_0(z, \zeta)\lambda_m(\zeta)/p_m(z, \zeta) \in S^{0,\infty}_b H^s_b.
\]
Put \( Q = q(z, bD) \). Then by Theorem 3.12 (1a), using \( P = P' \in \Psi^{m,0} H^s_b \),
\[
Q \circ P = a_0(z, bD)\Lambda_m + R'
\]
with \( R' \in \Psi^{-1:0} H^{s-1}_b \), hence by Proposition 3.9
\[
a_0(z, bD)\Lambda_m u = Qf - R'u \in H^s_b.
\]
Then standard microlocal ellipticity implies \( \zeta_0 \notin \text{WF}^{\tilde{s}+m}_b(u) \).

(2) If \( \zeta_0 \notin bT^*_\partial R^\infty_{\tilde{+}} \), then the proof of part (1) applies, since away from \( \partial R^\infty_{\tilde{+}} \), one has \( \Psi^m_b \subset H^s_b \). Thus, assuming \( \zeta_0 \in bT^*_\partial R^\infty_{\tilde{+}} \), we note that the ellipticity of \( p_m \) at \( \zeta_0 \) implies \( p''_m \neq 0 \) near \( \zeta_0 \), since the function \( p''_m \) vanishes at \( \partial R^\infty_{\tilde{+}} \). Therefore, Proposition 4.4 applies if one chooses \( a_0 \in S^0 \) as in the proof of part (1), yielding
\[
q(z, \zeta) := a_0(z, \zeta)\lambda_m(\zeta)/p_m(z, \zeta) = q'_0(z, \zeta) + q''_0(z, \zeta),
\]
where \( q'_0 \in S^{0,\infty} H^s_b, q''_0 \in S^0 \). Put \( Q_0 = q'_0(z, bD), Q''_0 = q''_0(z, bD) \), then
\[
(Q'_0 + Q''_0) \circ (P' + P'') = a_0(z, bD)\Lambda_m + R'
\]
with
\[
R' \in \Psi^{-1:0} H^s_{b-1} + \Psi^{-1:0} H^s_b + \Psi^{-1:0} H^{s-1}_b + \Psi^{-m-1}_b \subset \Psi^{-1:0} H^s_{b-1} + \Psi^{-m-1}_b,
\]
where the terms are the remainders of the first order expansions of \( Q'_0 \circ P' \), \( Q''_0 \circ P'' \), \( Q'_0 \circ P'' \) and \( Q''_0 \circ P'_0 \), in this order; to see this, we use Theorem 3.12 (1a), (2b), (2a) and composition properties of \( b\)-ps.d.o’s, respectively. Hence
\[
a_0(z, bD)\Lambda_m u = Q'_0 f + Q''_0 f - R'u \in H^s_b,
\]
which implies \( \zeta_0 \notin \text{WF}^{\tilde{s}+m}_b(u) \). \( \square \)

Remark 5.2. Notice that it suffices to have only local \( H^s_b \)-membership of \( f \) near the base point of \( \zeta_0 \). Under additional assumptions, even microlocal assumptions are enough, see in particular [7, Theorem 3.1]; we will not need this generality though.

\(^{18}\)And non-vanishing, which only matters near the zero section.

\(^{19}\)For \( Qf \in H^s_b \), we need \( s \geq \tilde{s} \) and \( s > n/2 + (-\tilde{s})_+ \). For \( R'u \in H^s_b \), we need \( s - 1 \geq \tilde{s} \) and \( s - 1 > n/2 + (-\tilde{s})_+ \).
6. PROPAGATION OF SINGULARITIES

We next study the propagation of singularities (equivalently the propagation of regularity) for certain classes of non-smooth operators. The results cover operators that are of real principal type (Section 6.3) or have a specific radial point structure (Section 6.4). For a microlocally more complete picture, we also include a brief discussion of complex absorption (Section 6.3.3).

The statements of the theorems and the ideas of their proofs are (mostly) standard in the context of smooth pseudodifferential operators; see for example Hörmander [20] and Vasy [34] for statements on manifolds without boundary and Hassell, Melrose and Vasy [16], Baskin, Vasy and Wunsch [6] as well as [18] for the propagation of b-regularity near radial points in various settings. Beals and Reed [7] discuss the propagation of singularities on manifolds without boundary for non-smooth ps.d.o’s, and parts of Sections 6.1 and 6.3 follow their exposition closely.

6.1. Sharp Gårding inequalities. We will need various versions of the sharp Gårding inequality, which will be used to obtain one-sided bounds for certain terms in positive commutator arguments later. For the first result, we follow the proof of [7, Lemma 3.1].

Proposition 6.1. Let \(s,m \in \mathbb{R}\) be such that \(2s \geq 2 - m\) and \(s > n/2 + 2 + m_+\). Let \(p(z,\zeta) \in S^{2m+1,2}_{b0}(\mathbb{R}^n; \text{End}(E))\) be a symbol with non-negative real part, i.e. \(\text{Re}\langle p(z,\zeta)e,e\rangle \geq 0\) \(z \in \mathbb{R}^n_+, \zeta \in \mathbb{R}^n, e \in E\), where \(\langle \cdot, \cdot \rangle\) is the inner product on the fibers of \(E\). Then there is \(C > 0\) such that \(P = p(z, bD)\) satisfies the estimate

\[\text{Re}\langle Pu,u\rangle \geq -C\|u\|_{H^m}^2, \quad u \in \mathcal{C}_c^\infty(\mathbb{R}^n_+; E).\]

Proof. Let \(q \in \mathcal{C}_c^\infty(\mathbb{R}^n)\) be a non-negative even function, supported in \(|\zeta| \leq 1\), s.t. \(\int q^2(\zeta) d\zeta = 1\), and put

\[F(\zeta, \xi) = \frac{1}{|\xi|^{n/4}}q(\frac{\xi - \zeta}{|\zeta|^{1/2}}).\]

Define the symmetrization of \(p\) to be

\[p_{\text{sym}}(\eta, z, \zeta) = \int F(\eta, \xi)p(z, \xi)F(\zeta, \xi) d\xi.\]

Observe that the integrand has compact support in \(\xi\) for all \(\eta, z, \zeta\), therefore \(p_{\text{sym}}\) is well-defined. Moreover,

\[(p_{\text{sym}}(bD, z, bD)u)(\eta) = \int \hat{p}_{\text{sym}}(\eta, \eta - \zeta, \zeta)\hat{u}(\zeta) d\zeta,\]

hence, writing \(u = (u_j)\), \(p = (p_{ij})\), \(p_{\text{sym}} = ((p_{\text{sym}})_{ij})\), and summing over repeated indices,

\[\text{Re}\langle p_{\text{sym}}(bD, z, bD)u, u\rangle = \text{Re} \int \hat{p}_{\text{sym}}(\eta, \eta - \zeta, \zeta)_{ij}\hat{u}(\zeta)_{j} \overline{\hat{u}(\eta)} (\eta) d\eta d\zeta = \text{Re} \int \int e^{iz\zeta}F(\zeta, \xi)\hat{u}(\zeta) d\zeta \overline{\int e^{iz\eta}F(\eta, \xi)\hat{u}(\eta) d\eta}_{j} p_{ij}(z, \xi) d\xi dz.\]

\(^{20}\)Recall the notation \(a_+ = \max(a, 0)\) for \(a \in \mathbb{R}\).
Indeed, this follows from Cauchy-Schwartz:

\[ \langle \eta \rangle^{s-2} \langle \hat{r}(\eta; \zeta) \rangle_{\text{End}(E)} \leq r_{0}(\eta; \zeta), \quad r_{0}(\eta; \zeta) \in L_{\zeta}^{\infty} L_{\eta}^{2}. \]  

(6.1)

in order to conclude the proof, since Proposition 3.9 then implies the continuity of

\[ r(z, bD) : H^{m}_{b}(\mathbb{R}_{+}^{n}; E) \rightarrow H^{m}_{b}(\mathbb{R}_{+}^{n}; E). \]

From now on, we will suppress the bundle \( E \) in our notation and simply write \(| \cdot | \) for \( \| \cdot \|_{\text{End}(E)} \). Now, \( r(z, bD) \) acts on \( \mathcal{C}^{\infty} \) by

\[ (r(z, bD)u)(\eta) = \int \hat{r}(\eta - \zeta, \zeta) \hat{u}(\zeta) \, d\zeta; \]

hence

\[ \hat{r}(\eta; \zeta) = \hat{p}_{\text{sym}}(\eta + \zeta, \eta, \zeta) - \hat{p}(\eta; \zeta) \]

\[ = \int F(\eta + \zeta, \xi) \hat{p}(\eta; \xi) F(\zeta, \xi) \, d\xi - \hat{p}(\eta; \zeta) \quad (6.2) \]

\[ = \int F(\eta + \zeta, \xi) (\hat{p}(\eta; \xi) - \hat{p}(\eta; \zeta)) F(\zeta, \xi) \, d\xi \]

\[ + \int (F(\eta + \zeta, \xi) - F(\zeta, \xi)) \hat{p}(\eta; \xi) F(\zeta, \xi) \, d\xi, \quad (6.3) \]

where we use \( \int F(\xi, \eta) \, d\xi = 1 \). To estimate \( \hat{r}(\eta; \zeta) \), we use that

\[ |\hat{p}(\eta; \zeta)| = \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle^{s}} p_{0}(\eta; \zeta), \quad p_{0}(\eta; \zeta) \in L_{\zeta}^{\infty} L_{\eta}^{2}. \]

We get a first estimate from (6.2):

\[ |\hat{r}(\eta; \zeta)| \lesssim \int_{S} \frac{1}{\langle \eta + \zeta \rangle^{n/4} \langle \xi \rangle^{n/4} \langle \eta \rangle^{s}} \langle \zeta \rangle^{2m+1} p_{0}(\eta; \xi) \, d\xi + \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle^{s}} p_{0}(\eta; \zeta), \]

where \( S \) is the set

\[ S = \{ |\xi - \zeta| \leq \langle \xi \rangle^{1/2}, |\xi - (\eta + \zeta)| \leq \langle \eta + \zeta \rangle^{1/2} \}. \]

In particular, we have \( \zeta \sim \langle \xi \rangle \sim \langle \eta + \zeta \rangle \) on \( S \), which yields

\[ |\hat{r}(\eta; \zeta)| \lesssim \frac{\langle \zeta \rangle^{2m+1-n/2}}{\langle \eta \rangle^{s}} \int_{|\xi - \zeta| \leq \langle \zeta \rangle^{1/2}} p_{0}(\eta; \xi) \, d\xi + \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle^{s}} p_{0}(\eta; \zeta). \]

We contend that

\[ p_{0}'(\eta; \zeta) := \langle \zeta \rangle^{-n/2} \int_{|\xi - \zeta| \leq \langle \zeta \rangle^{1/2}} p_{0}(\eta; \xi) \, d\xi \in L_{\zeta}^{\infty} L_{\eta}^{2}. \]

Indeed, this follows from Cauchy-Schwartz:

\[ \int \left| \int_{|\xi - \zeta| \leq \langle \zeta \rangle^{1/2}} p_{0}(\eta; \xi) \, d\xi \right|^{2} \, d\eta \lesssim \int \zeta^{n} \int_{|\xi - \zeta| \leq \langle \zeta \rangle^{1/2}} |p_{0}(\eta; \xi)|^{2} \, d\xi \, d\eta \lesssim \langle \zeta \rangle^{n} \|p_{0}(\eta; \xi)\|_{L_{\eta}^{\infty} L_{\zeta}^{2}}^{2}. \]

We deduce

\[ |\hat{r}(\eta; \zeta)| \lesssim \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle^{s}} p_{0}'(\eta; \zeta), \quad p_{0}'(\eta; \zeta) \in L_{\zeta}^{\infty} L_{\eta}^{2}. \]
If $|\eta| \geq |\zeta|/2$, this implies
\[
\frac{(\eta)^{s-1} |\hat{r}(\eta; \zeta)|}{(\zeta)^{2m}} \leq \frac{\langle \zeta \rangle}{\langle \eta \rangle} p_0''(\eta; \zeta) \lesssim p_0''(\eta; \zeta),
\] (6.4)
thus we obtain a fortiori the desired estimate (6.1) in the region $|\eta| \geq |\zeta|/2$.

From now on, let us thus assume $|\eta| \leq |\zeta|/2$. We estimate the first integral in (6.3). By Taylor’s theorem,
\[
\hat{p}(\eta; \xi) - \hat{p}(\eta; \zeta) = \partial_\zeta \hat{p}(\eta; \zeta) \cdot (\xi - \zeta) + \int_0^1 (1-t) (\xi - \zeta, \partial_\zeta^2 \hat{p}(\eta; \zeta + t(\xi - \zeta)) \cdot (\xi - \zeta)) \, dt,
\]
and since $\langle \xi \rangle \sim \langle \zeta \rangle$ on supp $F(\zeta, \xi)$, this gives
\[
\hat{p}(\eta; \xi) - \hat{p}(\eta; \zeta) = \partial_\zeta \hat{p}(\eta; \zeta) \cdot (\xi - \zeta) + \langle \xi - \zeta \rangle^2 O((\zeta)^{2m-1}) \text{ on supp } F(\zeta, \xi),
\]
where we say $f \in O(g)$ if $|f| \leq |g|h$ for some $h \in L^\infty L^2$. The first integral in (6.3) can then be rewritten as
\[
\partial_\zeta \hat{p}(\eta; \zeta) \cdot \int (\xi - \zeta) (F(\eta + \zeta, \xi) - F(\zeta, \xi)) F(\zeta, \xi) \, d\xi
\]
\[
+ O(\langle \zeta \rangle^{2m-1}) \int |\xi - \zeta|^2 F(\eta + \zeta, \xi) F(\zeta, \xi) \, d\xi,
\]
where we use $\int (\xi - \zeta) F(\zeta, \xi)^2 \, d\xi = 0$, which is a consequence of $q$ being even.

Taking the second integral in (6.3) into account, we obtain
\[
|\hat{r}(\eta; \zeta)| \lesssim (M_1 + M_2 + M_3)p_0'''(\eta; \zeta), \quad p_0'''(\eta; \zeta) \in L^\infty L^2, \tag{6.5}
\]

where
\[
M_1(\eta, \zeta) = \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle} \int |\xi - \zeta| \langle \zeta \rangle F(\eta + \zeta, \xi) - F(\zeta, \xi) \, d\xi
\]
\[
M_2(\eta, \zeta) = \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle} \int |\xi - \zeta|^2 \langle \zeta \rangle \langle \zeta \rangle F(\eta + \zeta, \xi) F(\zeta, \xi) \, d\xi
\]
\[
M_3(\eta, \zeta) = \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle} \int \left| \int (F(\eta + \zeta, \xi) - F(\zeta, \xi)) F(\zeta, \xi) \, d\xi \right|.
\]

$M_2$ is estimated easily: On the support of the integrand, one has $|\xi - \zeta|^2 \leq \langle \zeta \rangle$, thus
\[
M_2(\eta, \zeta) \lesssim \frac{\langle \zeta \rangle^{2m}}{\langle \eta \rangle} \cdot \frac{\langle \zeta \rangle^{n/2}}{\langle \eta + \zeta \rangle^{n/4} \langle \zeta \rangle^{n/4}};
\]
here, the term $\langle \zeta \rangle^{n/2}$ in the numerator is (up to a constant) an upper bound for the volume of the domain of integration. Since we are assuming $|\eta| \leq |\zeta|/2$, we have $\langle \eta + \zeta \rangle \gtrsim \langle \zeta \rangle$, which gives $M_2(\eta, \zeta) \lesssim \langle \zeta \rangle^{2m}/\langle \eta \rangle^s$.

In order to estimate $M_1$ and $M_3$, we will use
\[
\partial_\zeta F(\zeta, \xi) = \frac{a_0(\zeta)}{\langle \zeta \rangle^{n/4+1}} q_1 \left( \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \right) + \frac{a_1(\zeta)}{\langle \zeta \rangle^{n/4+1/2}} \partial_\zeta q \left( \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \right),
\]
\[
\partial_\zeta^2 F(\zeta, \xi) = \frac{a_2(\zeta)}{\langle \zeta \rangle^{n/4+1}} q_2 \left( \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \right),
\]
where the $a_j$ are scalar-, vector- or matrix-valued symbols of order 0, and $q_j \in C^\infty_c(\mathbb{R}^n)$.
Hence, writing $F(\eta + \zeta, \xi) - F(\zeta, \xi) = \eta \cdot \partial_\xi F(\zeta + t\eta, \xi)$ for some $0 \leq t \leq 1$, we get

$$M_1(\eta, \zeta) \lesssim \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle^s} \cdot \frac{\langle \zeta \rangle^{n/2} |\eta|}{\langle \zeta \rangle^{1/2} (\zeta + t\eta)^{n/4+1/2} (\zeta)^{n/4}} \lesssim \frac{\langle \zeta \rangle^{2m}}{\langle \eta \rangle^{s-1}},$$

where we again use $|\eta| < |\zeta|/2$ and $\langle \zeta + t\eta \rangle \gtrsim \langle \zeta \rangle$.

Finally, to bound $M_3$, we write

$$F(\eta + \zeta, \xi) - F(\zeta, \xi) = \eta \cdot \partial_\xi F(\zeta + t\eta, \xi) + \int_0^1 (1-t) \langle \eta, \partial_\xi^2 F(\zeta + t\eta, \xi) \cdot \eta \rangle \, dt$$

and deduce

$$M_3(\eta, \zeta) \lesssim \frac{\langle \zeta \rangle^{2m+1}}{\langle \eta \rangle^s} \left( \frac{\langle \zeta \rangle^{n/2} |\eta|}{\langle \zeta \rangle^{n/4+1} (\zeta)^{n/4}} \right) + \frac{|\eta|}{\langle \zeta \rangle^{n/4+1/2}} \left| \int (\partial_\xi q) \left( \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \right) \, \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \, d\xi \right|$$

$$+ \frac{(\langle \zeta \rangle^{n/2} |\eta|)^2}{\langle \zeta \rangle^{n/4+1} (\zeta)^{n/4}} \lesssim \frac{\langle \zeta \rangle^{2m}}{\langle \eta \rangle^{s-2}},$$

where we use

$$\int (\partial_\xi q) \left( \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \right) \, \frac{\xi - \zeta}{\langle \zeta \rangle^{1/2}} \, d\xi = 0,$$

which holds since $q$ has compact support. Plugging the estimates for $M_j$, $j = 1, 2, 3$, into (6.5) proves that (6.1) holds. The proof is complete. 

The idea of the proof can also be used to prove the sharp Gårding inequality for smooth b-ps.d.o’s:

**Proposition 6.2.** Let $m \in \mathbb{R}$, and let $p(z, \zeta) \in S^{2m+1}(\mathbb{R}^n; \text{End}(E))$ be a symbol with non-negative real part. Then there is $C > 0$ such that $P = p(z, bD)$ satisfies the estimate

$$\text{Re}(Pu, u) \geq -C||u||_{H^m_b}, \quad u \in C^\infty_c(\mathbb{R}^n; E).$$

**Proof.** Write $p(x, y; \zeta) = p^{(0)}(y; \zeta) + p^{(1)}(x, y; \zeta)$, where $p^{(0)}(y; \zeta) = p(0, y; \zeta)$ and $p^{(1)} = xp \in H^m_b S^{2m+1}$. The symmetrized operator $p^{(bD, z, bD)}$, defined as in the proof of Proposition 6.1, is again non-negative, and the symbol of the remainder operator $r(z, bD) = p_{\text{sym}}(bD, z, bD) - p(z, bD)$ is the sum of two terms $p_{\text{sym}}^{(0)} - p^{(0)}$ and $p_{\text{sym}}^{(1)} - p^{(1)}$. The proof of Proposition 6.1 shows that $p_{\text{sym}}^{(1)} - p^{(1)} \in S^{2m+1} H^m_b$. It thus suffices to assume that $p = p^{(0)}$ is independent of $x$, which implies that $p_{\text{sym}}$ is independent of $x$ as well, and to prove $r(y, bD) = (p_{\text{sym}} - p)(y, bD) : H^m_b \to H^{-m}_b$.

Similarly to the proof of Proposition 6.1, we put

$$F(\lambda, \eta; \sigma, \gamma) = \frac{1}{\langle \lambda, \eta \rangle^{n/4}} q \left( \frac{(\sigma - \lambda, \gamma - \eta)}{\langle \lambda, \eta \rangle^{1/2}} \right)$$

$$p_{\text{sym}}(\rho, \theta; y; \lambda, \eta) = \iint F(\rho, \theta; \sigma, \gamma) p(y; \sigma, \gamma) F(\lambda, \eta; \sigma, \gamma) \, d\sigma d\gamma$$

and obtain

$$(p_{\text{sym}}(bD; y; bD) u)(\rho, \theta) = \int \mathcal{F} p_{\text{sym}}(\rho, \theta; \eta - \rho; \eta) \tilde{u}(\rho, \eta) \, d\eta$$
\[\mathcal{F} r(\theta; \lambda, \eta) = \mathcal{F} p_{\text{sym}}(\lambda, \theta + \eta; \lambda, \eta) - \mathcal{F} p(\theta; \lambda, \eta),\]

thus

\[\mathcal{F} r(\theta; \lambda, \eta) = \iint F(\lambda, \theta + \eta; \sigma, \gamma)(\mathcal{F} p(\theta; \sigma, \gamma) - \mathcal{F} p(\theta; \lambda, \eta)) F(\lambda, \eta; \sigma, \gamma) \, d\sigma \, d\gamma + \iint (F(\lambda, \theta + \eta; \sigma, \gamma) - F(\lambda, \eta; \sigma, \gamma)) \mathcal{F} p(\theta; \lambda, \eta) F(\lambda, \eta; \sigma, \gamma) \, d\sigma \, d\gamma.\]

Then, following the argument in the previous proof, we obtain

\[|\mathcal{F} r(\theta; \lambda, \eta)| \leq \frac{(\lambda, \eta)^{2m}}{(\theta)^{N}} r_{0}(\theta; \lambda, \eta), \quad r_{0}(\theta; \lambda, \eta) \in L^{\infty}_{\lambda, \eta} L^{2}_{\theta}, \quad (6.6)\]

where we use

\[|\mathcal{F} p(\theta; \lambda, \eta)| = \frac{(\lambda, \eta)^{2m+1}}{(\theta)^{N+2}} p_{0}(\theta; \lambda, \eta), \quad p_{0}(\theta; \lambda, \eta) \in L^{\infty}_{\lambda, \eta} L^{2}_{\theta},\]

which holds for every integer \(N\) (with \(p_{0}\) depending on the choice of \(N\)). An estimate similar to the one used in the proof of Proposition 3.9 shows that (6.6) implies \(r(y, bD) : H^{s}_{b} \to H^{s-2m}_{b}\) for all \(s \in \mathbb{R}\). \(\square\)

Finally, we merge Propositions 6.1 and 6.2.

**Corollary 6.3.** Let \(s, m \in \mathbb{R}\) be such that \(s \geq 2 - m, s > n/2 + 2 + m_{+}\). Let \(p'(z, \xi) \in S^{2m+1, 2}_{L^{\infty}(\mathbb{R}^{n}_{+}; \text{End}(E))}, p''(z, \xi) \in S^{2m+1}_{L^{\infty}(\mathbb{R}^{n}_{+}; \text{End}(E))}\) be symbols such that \(p = p' + p''\) is non-negative real part. Then there is \(C > 0\) such that \(P = p(z, bD)\) satisfies the estimate

\[\text{Re}(Pu, u) \geq -C\|u\|^{2}_{L^{m}_{b}}, \quad u \in C_{c}^{\infty}(\mathbb{R}^{n}_{+}; E).\]

**Proof.** The symmetrized operator \(p_{\text{sym}}(bD, z, bD)\) is again non-negative, and the symbol of the remainder operator \(r(z, bD) = p_{\text{sym}}(bD, z, bD) - p(z, bD)\) is the sum of two terms \(p'_{\text{sym}} - p'\) and \(p''_{\text{sym}} - p''\). The proofs of Propositions 6.1 and 6.2 show that \((p'_{\text{sym}} - p')(z, bD)\) and \((p''_{\text{sym}} - p'')(z, bD)\) map \(H^{m}_{b}\) to \(H^{-m}_{b}\), hence \(r(z, bD)\) maps \(H^{m}_{b}\) to \(H^{-m}_{b}\), and the proof is complete. \(\square\)

**6.2. Mollifiers.** In order to deal with certain kinds of non-smooth terms in Sections 6.3 and 6.4, we will need smoothing operators in order to smooth out and approximate non-smooth functions in a precise way. We only state the results for unweighted spaces, but the corresponding statements for weighted spaces hold true by the same proofs.

**Lemma 6.4.** Let \(s \in \mathbb{R}\), \(\chi \in C_{c}^{\infty}(\mathbb{R}^{+})\). Then \(\chi(x/\epsilon) \to 0\) strongly as a multiplication operator on \(H^{m}_{b}(\mathbb{R}^{n}_{+})\) as \(\epsilon \to 0\), and in norm as a multiplication operator from \(H^{m+\alpha}_{b}(\mathbb{R}^{n}_{+}) \to H^{m}_{b}(\mathbb{R}^{n}_{+})\) for \(\alpha > 0\).

**Proof.** We start with the first half of the lemma: For \(s = 0\), the statement follows from the dominated convergence theorem. For \(s\) a positive integer, we use that

\[(x \partial x)^{s} \left(\chi\left(\frac{x}{\epsilon}\right)\right) = \sum_{j=1}^{s} c_{sj} \left(\frac{x}{\epsilon}\right)^{j} \chi^{(j)}\left(\frac{x}{\epsilon}\right), \quad c_{sj} \in \mathbb{R},\]

is bounded and converges to 0 pointwise in \(x > 0\) as \(\epsilon \to 0\), thus by virtue of the Leibniz rule and the dominated convergence theorem, we obtain \(\chi(x/\epsilon)u(x, y) \to 0\) in \(H^{0}_{b}(\mathbb{R}^{n}_{+})\) for \(u \in H^{m}_{b}(\mathbb{R}^{n}_{+})\). For \(s \in -\mathbb{N}\), the statement follows by duality.
Finally, to treat the case of general \( s \), we first show that \( \chi(\cdot/\epsilon) \) is a uniformly bounded family (in \( \epsilon > 0 \)) of multiplication operators on \( H^s_b(\mathbb{R}^n_+ \cap \mathbb{R}^n) \) for all \( s \in \mathbb{R} \). For \( s \in \mathbb{N}_0 \), this follows from the above estimates, for \( s \in \mathbb{Z} \) again by duality, and then for general \( s \in \mathbb{R} \) by interpolation. Now, put \( M = \sup_{0 < \epsilon \leq 1} \| \chi(\cdot/\epsilon) \|_{H^s_b(M^*)} < \infty \).

Let \( w \in H^s_b \) and \( \delta > 0 \) be given, and choose \( w' \in H^s_b \) such that \( \|w - w'\|_{H^s_b} < \delta/2M \). By what we have already proved, we can choose \( \epsilon_0 > 0 \) so small that

\[
\|\chi(\cdot/\epsilon)w\|_{H^s_b} \leq \|\chi(\cdot/\epsilon)w'\|_{H^s_b} < \delta/2, \quad \epsilon < \epsilon_0;
\]

then

\[
\|\chi(\cdot/\epsilon)w\|_{H^s_b} \leq \|\chi(\cdot/\epsilon)(w - w')\|_{H^s_b} + \|\chi(\cdot/\epsilon)w'\|_{H^s_b} < M \frac{\delta}{2M} + \frac{\delta}{2} = \delta.
\]

Concerning the second half of the lemma, the case \( s = 0 \) is clear since \( x^{\alpha} \chi(x/\epsilon) \to 0 \) in \( L^\infty(\mathbb{R}^n_+) \) as \( \epsilon \to 0 \); as above, this implies the statement for \( s \) a positive integer, and the case of real \( s \) again follows by duality and interpolation. \( \square \)

**Lemma 6.5.** Let \( M \) be a compact manifold with boundary. Then there exists a family of operators \( J_\epsilon : H^s_b(M^*) \to C^\infty_c(M^*) \), \( \epsilon > 0 \), such that \( J_\epsilon \in \Psi^{-\infty}_b(M) \), and for all \( s \in \mathbb{R} \), \( J_\epsilon \) is a uniformly bounded family of operators on \( H^s_b(M) \) that converges strongly to the identity map \( I \) as \( \epsilon \to 0 \).

**Proof.** Choosing a product decomposition \( \partial M \times [0, \epsilon_0) \times [0, \epsilon_0] \) near the boundary of \( M \) and \( \chi \in C^\infty_c(\mathbb{R}^n_+) \), \( \chi \equiv 1 \) near 0, \( \text{supp} \chi \subset [0, 1/2] \), we can define the multiplication operators \( \chi(\cdot/\epsilon) \) globally on \( H^s_b(\mathbb{R}^n_+) \). By the previous lemma, \( I - \chi(\cdot/\epsilon) \) converges strongly to \( I \) on \( H^s_b(M) \); moreover, \( \text{supp}(u - \chi(\cdot/\epsilon)u) \subset \{ x \geq \epsilon \} \). Thus, if we let \( \tilde{J}_\epsilon \) be a family of mollifiers, \( J_\epsilon \in \Psi^{-\infty}_b(M) \), \( \tilde{J}_\epsilon \to I \) in \( \Psi^{-\infty}_b(M) \) for \( \delta' > 0 \), such that on the support of the Schwartz kernel of \( \tilde{J}_\epsilon \), we have \( |x_1 - x_2| < \epsilon/2 \) near \( \partial M \times \partial M \) where \( x_1, x_2 \) are the lifts of \( x \) to the left and right factor of \( M \times M \), then we have that \( J_\epsilon(u - \chi(\cdot/\epsilon)u) \) is an element of \( H^s_b(M) \) with support in \( \{ x \geq \epsilon/2 \} \), thus is smooth. Therefore, the family \( J_\epsilon := \tilde{J}_\epsilon \circ (I - \chi(\cdot/\epsilon)) \) satisfies all requirements. \( \square \)

### 6.3. Real principal type propagation, complex absorption

We will prove real principal type propagation estimates of b-regularity for operators with non-smooth coefficients by following the arguments outlined in the introduction in the smooth coefficient case as closely as possible.

**Theorem 6.6.** Let \( m, r, s, \tilde{s} \in \mathbb{R}, \alpha > 0 \). Suppose \( P'_m \in H^{s+\alpha}_b(\mathbb{R}^n_+; E) \) has a real, scalar, homogeneous principal symbol \( p'_m \), and let \( P'_{m-1} \in H^{s-1+\alpha}_b(\mathbb{R}^n_+; E), R \in \Psi^{-2}_b(\mathbb{R}^n_+; E) + \Psi^{-2}_b H^{s-1+\alpha}_b(\mathbb{R}^n_+; E) \). Suppose \( s \) and \( \tilde{s} \) are such that

\[
\tilde{s} \leq s - 1, \quad s > n/2 + 7/2 + (2 - \tilde{s})_+.
\]

(1) Let \( P = P'_m + P'_{m-1} + R \), and let \( p_m = p'_m \), or

(2) let \( P = P'_m + P''_m + P'_{m-1} + R \), where \( P''_m \in \Psi^{-\alpha}_b(\mathbb{R}^n_+; E) \) has a real, scalar, homogeneous principal symbol \( p''_m \), and let \( P''_{m-1} \in \Psi^{-\alpha}_b(\mathbb{R}^n_+; E) \).

Denote \( p_m = p'_m + p''_m \).

In both cases, if \( u \in H^{\tilde{s}+\alpha}_{b} \mathbb{R}^n_+ (\mathbb{R}^n_+; E) \) is such that \( Pu \in H^{r+\alpha}_{b} \mathbb{R}^n_+ (\mathbb{R}^n_+; E) \), then WF\(_b^{\tilde{s}+\alpha}_{b} \mathbb{R}^n_+ (u) \) is a union of maximally extended null-bicharacteristics of \( p_m \), i.e. of integral curves of the Hamilton vector field \( H_{p_m} \) within the characteristic set \( p_m^{-1}(0) \subset bT^* \mathbb{R}^n_+ \setminus \partial \).
The proof, given in Sections 6.3.1 and 6.3.2, in fact gives an estimate (which can also be recovered from the above statement by the closed graph theorem as in Hörmander [20, Proof of Theorem 26.1.7]) for the $H^s_{\phi} [20, \text{Proof of Theorem 26.1.7}]$ for the $H^s$ can also be recovered from the above statement by the closed graph theorem as in $G$ $WF$ $acteristics from $Q$ $\psi$, we assume $m$ away from the boundary, as explained below. Thus, let $\alpha$ first half of Theorem 6.6, where we can in fact assume $\alpha = 0$ since we are working away from the boundary, as explained below. Thus, let $P = P_m + P_{m-1} + R$, where we assume $m \geq 1$ for now,

$$P_m \in H^s_{\phi} \Psi^m_b$$

with real homogeneous principal symbol,

$$P_{m-1} \in H^s_{\phi} \Psi^{m-1}_b$$

$$R \in \Psi^{m-2,0}_b H^s_{\phi}$$

in the sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds. In particular, it suffices to have only microlocal $H^r$-membership of $Pu$ near the parts of null-bicharacteristics along which we want to propagate $H^s_{\phi}$-regularity of $u$. The term involving $\psi Pu$ comes from the local requirements for elliptic regularity, see Remark 5.2.

In Section 6.3.3, we will add complex absorption and obtain the following statement.

**Theorem 6.7.** Under the assumptions of Theorem 6.6, let $Q \in \Psi^m_b (\mathbb{R}^n_+; E)$, $Q = Q^*$. Suppose $A, B, G \in \Psi^0_b$ are such that all forward, resp. backward, bicharacteristics from $WF^b_0 (B)$ reach the elliptic set of $A$ while remaining in the elliptic set of $G$, and suppose moreover that $q \leq 0$, resp. $q \geq 0$, on $WF^b_0 (G)$, further let $\psi \in C^\infty (\mathbb{R}^n_+)$ be identically 1 on $\pi (WF^b_0 (B))$, then

$$\| Bu \|_{H^{s+1-\epsilon}_b, -r} \leq C (\| GPu \|_{H^{s-\epsilon}_b, r} + \| Au \|_{H^{s+1-\epsilon}_b, -r} + \| \psi Pu \|_{H^{s-\epsilon}_b, r} + \| u \|_{H^{s+1-\epsilon}_b, -r})$$

(6.8)

in the sense that if all quantities on the right hand side are finite, then so is the left hand side, and the inequality holds.

in other words, we can propagate estimates from the elliptic set of $A$ forward along the Hamilton flow to $WF^b_0 (B)$ if $q \geq 0$, and backward if $q \leq 0$.

Conjugating by $x^s$ (where $x$ is the standard boundary defining function), it suffices to prove Theorems 6.6 and 6.7 for $r = 0$. Moreover, as in the smooth setting, we can apply Theorem 5.1 on the elliptic set of $P$ in both cases and deduce microlocal $H^s_{\phi}$-regularity of $u$ there, which implies that $WF^s_{\phi} (u)$ is a subset of the characteristic set of $P$, and thus we only need to prove the propagation result within the characteristic set. We will begin by proving the first part of Theorem 6.6 in Section 6.3.1; the proof is then easily modified in Section 6.3.2 to yield the second part of Theorem 6.6. To keep the notation simple, we will only consider the case of complex-valued symbols (hence, operators acting on functions); in the general, bundle-valued case, all arguments go through with purely notational changes.

6.3.1. Propagation in the interior. For brevity, denote $M = \mathbb{R}^n_+$. We start with the first half of Theorem 6.6, where we can in fact assume $\alpha = 0$ since we are working away from the boundary, as explained below. Thus, let $P = P_m + P_{m-1} + R$, where we assume $m \geq 1$ for now,
and let us assume that we are given a solution
\[ u \in H^{\sigma - 1/2}_b, \]
(6.10)
to the equation
\[ Pu = f \in H^{\sigma - m + 1}_b, \]
where \( \sigma = \tilde{s} + m - 1 \) with \( \tilde{s} \) as in the statement of Theorem 6.6. In fact, since
\[ R: H^{\sigma - 1/2}_b \subset H^{\sigma - 1}_b \to H^{\sigma - m + 1}_b \]
by Proposition 3.9, we may absorb the term \( Ru \) into the right hand side; thus, we can assume \( R = 0 \), hence \( P = P_m + P_{m-1} \).

Moreover, let \( \gamma \) be a null-bicharacteristic of \( p_m \), and assume that \( H_{p_m} \) is never radial on \( \gamma \). Note that this in particular means that \( \gamma \cap b^{T \partial M} M = \emptyset \) since \( p_m \) vanishes identically at the boundary, and in fact this setup is the correct one for the discussion of real principal type propagation in the interior of \( M \). All functions we construct in this section are implicitly assumed to have support away from \( \partial M \). Even though we are working away from the boundary, we will still employ the \( b \)-notation throughout this section, since the proof of the real principal type propagation result (near and) within the boundary will only require minor changes compared to the proof of the interior result given here.

The objective is to propagate microlocal \( H^{\sigma}_b \)-regularity along \( \gamma \) to a point \( \zeta_0 \in b^{T^* M \setminus \partial} \), assuming a priori knowledge of microlocal \( H^{\sigma}_b \)-regularity of \( u \) near a point \( \zeta_* \) on the backward bicharacteristic from \( \zeta_0 \); the location and size of this region will be specified later, see Proposition 6.8. We will use a positive commutator argument.

The idea, following [10, §2], is to arrange for \( H_{p_m} = \rho^{1-m} H_{p_m}, \rho = \langle \zeta \rangle \),
\[ H_{p_m} a = -b^2 + e - f, \]
(6.11)
where \( a, b, e \) are smooth symbols and \( f \) is a non-smooth symbol, absorbing non-smooth terms of \( H_{p_m} a \) in an appropriate way, which however has a definite sign; by virtue of the sharp Gårding inequality, we will be able to bound terms involving \( f \) using the a priori regularity assumptions on \( u \). As in the smooth case, terms involving \( e \) will be controlled by the a priori assumptions of \( u \) near \( \zeta_* \). If \( b \) is elliptic at \( \zeta_0 \), we are thus able to prove the desired \( H^{\sigma}_b \)-regularity at \( \zeta_0 \). The actual commutant to be used, which has the correct symbolic order and is regularized, will be constructed later; see Proposition 6.8 for its relevant properties.

The general strategy for choosing the non-smooth symbol \( f \) is as follows: Non-smooth terms \( T \), which arise in the computation and are positive, say \( T \geq c > 0 \), are smoothed out using a mollifier \( J \), giving a smooth function \( JT \), but only as much as to still preserve some positivity \( JT - c/4 \geq c/4 > 0 \), and in such a way that the error \( T - JT + c/4 \) is non-negative; then \( b^2 = JT - c/4 \) is a smooth, positive term, and \( f = T - JT + c/4 \) is non-smooth, but has a sign, and \( T = b^2 + f \). The mollifiers we shall use were constructed in Lemma 6.5.

To start, choose \( \tilde{\eta} \in C^\infty(b^{S^* M}) \) with \( \tilde{\eta}^{(\zeta_0)} = 0 \), \( H_{p_m} \tilde{\eta}^{(\zeta_0)} > 0 \), i.e. \( \tilde{\eta} \) measures, at least locally, propagation along the Hamilton flow. Choose \( \sigma_j \in C^\infty(b^{S^* M}) \), \( j = 1, \ldots, 2n - 2 \), with \( \sigma_j^{(\zeta_0)} = 0 \) and \( H_{p_m} \sigma_j^{(\zeta_0)} = 0 \), and such that \( d\tilde{\eta}, d\sigma_j \) span \( T^{(\zeta_0)}_{\zeta_0}(b^{S^* M}) \). Put \( \omega = \sum_{j=1}^{2n-2} \sigma_j^2 \), so that \( \omega^{1/2} \) approximately measures how far away one is from the bicharacteristic through \( \zeta_0 \). Thus, \( |\tilde{\eta}| + \omega^{1/2} \) is, near \( \zeta_0 \), equivalent to the distance from \( \zeta_0 \) with respect to any distance function given by a Riemannian
metric on $bS^*M$. Then for $\delta \in (0, 1), \epsilon \in (0, 1], \beta \in (0, 1]$ and $F > 0$ (large) to be chosen later, let

$$\phi = \tilde{\eta} + \frac{1}{\epsilon^2 \delta} \omega,$$

and, taking $\chi_0(t) = e^{-1/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, and $\chi_1 \in C^\infty(\mathbb{R})$, $\chi_1 \geq 0$, $\sqrt{\chi_1} \in C^\infty(\mathbb{R})$, supp $\chi_1 \subset (0, \infty)$, supp $\chi_1' \subset (0, 1)$, and $\chi_1 \equiv 1$ in $[1, \infty)$, consider

$$a = \chi_0 \left( F^{-1} \left( 2\beta - \frac{\phi}{\delta} \right) \right) \chi_1 \left( \frac{\tilde{\eta} + \delta}{\epsilon \delta} + 1 \right).$$

First, we observe that $H_{p\mu} \phi(\zeta_0) = H_{p\mu} \tilde{\eta}(\zeta_0) > 0$; thus, since $\chi_1 \left( \frac{\tilde{\eta} + \delta}{\epsilon \delta} + 1 \right) \equiv 1$ near $\zeta_0$,

$$H_{p\mu} a(\zeta_0) = -F^{-1} \delta^{-1} H_{p\mu} \phi(\zeta_0) \chi_0'(2F^{-1} \beta) < 0$$

has the right sign at $\zeta_0$.

Next, we analyze the support of $a$: First of all, If $\zeta \in \text{supp } a$, then

$$\phi(\zeta) \leq 2\beta \delta, \quad \tilde{\eta}(\zeta) \geq -\delta - \epsilon \delta \geq -2\delta.$$

Since $\omega \geq 0$, we get $\tilde{\eta} = \phi - \omega^2 \delta \leq \phi \leq 2\beta \delta \leq 2\delta$, thus $\omega = \epsilon^2 \delta (\phi - \tilde{\eta}) \leq 4\epsilon^2 \delta^2$, i.e.

$$-\delta - \epsilon \delta \leq \tilde{\eta} \leq 2\beta \delta, \quad |\omega| \leq 2\epsilon \delta$$

on $\text{supp } a$.\hspace{1cm} (6.12)

In particular, we can make $\text{supp } a$ to be arbitrarily close to $\zeta_0$ by choosing $\delta > 0$ small, hence there is $\delta_0 > 0$ such that $H_{p\mu} \tilde{\eta} \geq c_0 > 0$ whenever $|\tilde{\eta}| \leq 2\delta_0$ and $\omega^{1/2} \leq 2\delta_0$. The support of $a$ becomes localized near $\omega = 0$ by choosing $\epsilon > 0$ small. The parameter $\beta$ then allows one to localize $\text{supp } a$ near the segment $\tilde{\eta} \in [-\delta; 0]$. Moreover, we have

$$-\delta - \epsilon \delta \leq \tilde{\eta} \leq -\delta, \quad |\omega| \leq 2\epsilon \delta \quad \text{on } \text{supp } a \cap \text{supp } \chi_1', \hspace{1cm} (6.13)$$

which is the region where we will assume a priori microlocal control on $u$. Observe that by taking $\epsilon > 0$ small, we can make this region arbitrarily closely localized at $\tilde{\eta} = -\delta, \omega = 0$.

Choose $\tilde{\chi}_1 \in C^\infty(\mathbb{R})$, $\tilde{\chi}_1 \geq 0$, such that $\tilde{\chi}_1 \equiv 1$ on $\text{supp } \chi_1'$, and supp $\tilde{\chi}_1 \subset [0, 1]$. Since the coefficients of $H_{p\mu}$ are continuous because of $s > n/2 + 1$, we can choose a mollifier $J$ as in Lemma 6.5 such that\hspace{1cm} (6.14)

$$e = \chi_0 \left( F^{-1} \left( 2\beta - \frac{\phi}{\delta} \right) \right) (H_{p\mu} J) \chi_1 \left( \frac{\tilde{\eta} + \delta}{\epsilon \delta} + 1 \right) + \tilde{\chi}_1 \left( \frac{\tilde{\eta} + \delta}{\epsilon \delta} + 1 \right),$$

$$f' = \chi_0 \left( F^{-1} \left( 2\beta - \frac{\phi}{\delta} \right) \right) \left[ \tilde{\chi}_1 \left( \frac{\tilde{\eta} + \delta}{\epsilon \delta} + 1 \right) + (H_{p\mu} - H_{p\mu}) \chi_1 \left( \frac{\tilde{\eta} + \delta}{\epsilon \delta} + 1 \right) \right],$$

hence $e - f' = \chi_0 H_{p\mu} \chi_1, \text{ we have } f' \geq 0$. Note that $e \in C^\infty$ has support as indicated in (6.13), and $f' \in H^{s-1}_\omega$ in the base variables.

In order to have (6.11), it remains to prove that the remaining term of $H_{p\mu} a$, namely $\chi_1 H_{p\mu} \chi_0$, is non-positive; for this, it is sufficient to require $H_{p\mu} \phi \geq c_0/2$ on $\text{supp } a$ if $\delta < \delta_0$. From the definition of $\phi$, this would follow provided

$$|H_{p\mu} \omega| \leq c_0 \epsilon^2 \delta^2 / 2$$

\hspace{1cm} (6.15)

\hspace{1cm} 22\text{We let } J \text{ act on a function } f \text{ defined on } bT^*\mathbb{R}^n_+ \text{ by } (Jf)(z, \zeta) = J(f(\cdot, \zeta))(z).\hspace{1cm}
on \text{supp } a. Now, since for \( s > n/2 + 2 \), \( H_{p_m} \sigma_j \) is Lipschitz continuous and vanishes at \( \zeta_0 \), we have
\[
|H_{p_m} \omega| \leq 2 \sum_{j=1}^{2n-2} |\sigma_j||H_{p_m} \sigma_j| \leq C \omega^{1/2} \left( |\tilde{\eta}| + \omega^{1/2} \right),
\]
(6.16)
hence (6.15) holds if \( 2C \epsilon \delta (2\delta + 2\epsilon \delta) \leq c_0 \epsilon^2 \delta / 2 \), which is satisfied provided \( 16C \delta / c_0 \leq \epsilon \). Let us choose \( \epsilon = 16C \delta / c_0 \), with \( \delta \) small enough such that \( \epsilon \leq 1 \). For later use, let us note that then near \( \tilde{\eta} = -\delta \), the ‘width’ of the support of \( a \) is
\[
\omega^{1/2} \leq \frac{c_0 \epsilon^2 \delta / 2}{C(\omega^{1/2} + |\tilde{\eta}|)} \lesssim \delta^2,
\]
(6.17)
hence by (6.13), the region where we will assume a priori microlocal control on \( u \) (i.e. \( \text{supp } e \)) has size \( \sim \delta^2 \).

Next, we have to make the commutant, \( a \), a symbol of order \( 2\sigma - (m-1) \), so that the ‘principal symbol’ of \( i[P,A] \), i.e. \( H_{p_m} a \), is of order \( 2\sigma \), hence \( b \) has order \( \sigma \), which is what we need, since we want to prove \( H_b^\sigma \)-regularity of \( u \) at \( \zeta_0 \). Thus, define
\[
\tilde{a} = \rho^{\sigma -(m-1)/2} a^{1/2},
\]
and let
\[
\varphi_t = (1 + t \rho)^{-1}
\]
(6.18)
be a regularizer, \( \varphi_t \in S^{-1} \) for \( t > 0 \), which is uniformly bounded in \( S^0 \) for \( t \in [0,1] \) and satisfies \( \varphi_t \rightarrow 1 \) in \( S^\ell \) for \( \ell > 0 \) as \( t \rightarrow 0 \). We define the regularized symbols to be \( \tilde{a}_t = \varphi_t \tilde{a} \) and \( a_t = \varphi_t^2 \rho^{2\sigma -(m-1)} a = \tilde{a}_t^2 \).

We compute \( H_{p_m} \varphi_t = -t \varphi_t^{2} H_{p_m} \rho \). Amending (6.11) by another term which will be used to absorb certain terms later on, we aim to show that we can choose \( b_t, e_t \) and \( f_t \) such that, in analogy to (6.11), for \( M > 0 \) fixed, to be specified later,
\[
H_{p_m} a_t = \varphi_t^2 \rho^{2\sigma} (H_{p_m} a + ((2\sigma - m + 1) - 2t \varphi_t \rho)(\rho^{-1} H_{p_m} \rho)a)
= -b_t^2 - M^2 \rho^{m-1} a_t + e_t - f_t,
\]
that is to say,
\[
\varphi_t^2 \rho^{2\sigma} (H_{p_m} a + [(2\sigma - m + 1) - 2t \varphi_t \rho)(\rho^{-1} H_{p_m} \rho) + M^2] a) = -b_t^2 + e_t - f_t.
\]
(6.19)
Here, note that, using the definition of \( \varphi_t \), \( t \rho \varphi_t \) is a uniformly bounded family of symbols of order 0. To achieve (6.19), let us take
\[
e_t = \varphi_t^2 \rho^{2\sigma} e,
\]
\[
f_t = f_t' + f_t'', \quad f_t' = \varphi_t^2 \rho^{2\alpha} f',
\]
(6.20)
where \( e, f' \) are given by (6.14); we will define \( f'' \) momentarily. Using \( \chi_0(t) = t^2 \chi_0'(t) \), we obtain
\[
H_{p_m} a + \left[ (2 \sigma - m + 1) - 2 t \phi (\rho^{-1} H_{p_m} \rho) + M^2 \right] a
\[
= e - f' - (F \delta)^{-1} \left( H_{p_m} \phi \right)
\[
- \left[ (2 \sigma - m + 1) - 2 t \phi (\rho^{-1} H_{p_m} \rho) + M^2 \right] F^{-1} \delta \left( 2 \beta - \frac{\phi}{\delta} \right)^2
\[
\times \chi_0 \left( F^{-1} \left( 2 \beta - \frac{\phi}{\delta} \right) \right) \chi_1 \left( \frac{\tilde{\eta} + \delta}{c \delta} + 1 \right).
\]

Thus, if \( F \) is large enough, the term in the large parentheses is bounded from below by \( 3c_0/8 \) on \( \text{supp} \, a \), since \( |2 \beta - \phi| / |\delta| \leq 4 \) there. (The last statement follows from \(-2 \delta \leq \tilde{\eta} \leq \phi \leq 2 \beta \delta \leq 2 \delta \) and \( \beta \leq 1 \).) Therefore, we can put
\[
b_t = (F \delta)^{-1/2} \phi (J H_{p_m}) \phi
\]
\[
- \left[ (2 \sigma - m + 1) - 2 t \phi (\rho^{-1} (J H_{p_m}) \rho) + M^2 \right] F^{-1} \delta \left( 2 \beta - \frac{\phi}{\delta} \right)^2 + \frac{c_0}{8}
\]
\[
\times \chi_0 \left( F^{-1} \left( 2 \beta - \frac{\phi}{\delta} \right) \right) \chi_1 \left( \frac{\tilde{\eta} + \delta}{c \delta} + 1 \right),
\]
\[
f''_t = (F \delta)^{-1/2} \phi^2 (J H_{p_m} - J H_{p_m}) \phi
\]
\[
- \left[ (2 \sigma - m + 1) - 2 t \phi (\rho^{-1} (J H_{p_m} - J H_{p_m}) \rho) \right] F^{-1} \delta \left( 2 \beta - \frac{\phi}{\delta} \right)^2 + \frac{c_0}{8}
\]
\[
\times \chi_0 \left( F^{-1} \left( 2 \beta - \frac{\phi}{\delta} \right) \right) \chi_1 \left( \frac{\tilde{\eta} + \delta}{c \delta} + 1 \right),
\]
with \( f''_t \geq 0 \) if the mollifier \( J \) is close enough to \( I \), and thus obtain (6.19).

We now summarize this construction, slightly rephrased, retaining only the important properties of the contracted symbols. Let us fix any Riemannian metric on \( S^\ast M \) near \( \zeta_0 \) and denote the metric ball around a point \( p \) with radius \( r \) in this metric by \( B(p, r) \).

**Proposition 6.8.** There exist \( \delta_0 > 0 \) and \( C_0 > 0 \) such that for \( 0 < \delta \leq \delta_0 \), the following holds: For any \( M > 0 \), there exist a symbol \( \check{a} \in S^{\sigma-(m-1)/2} \) and uniformly bounded families of symbols \( \check{a}_t = \varphi_t \check{a} \in S^{\sigma-(m-1)/2} \) (with \( \varphi_t \) defined by (6.18)), \( b_t \in S^{\sigma} \), \( e_t \in S^{2\sigma} \) and \( f_t \in S^{2\sigma} \); \( f_t \geq 0 \), supported in a coordinate neighborhood (independent of \( \delta \)) of \( \zeta_0 \) and supported away from \( \partial M \), that satisfy the following properties:

1. \( \check{a}_t H_{p_m} \check{a}_t = -b_t^2 - M^2 \rho^{-1} a_t^2 + e_t - f_t \).
2. \( b_t \to b_0 \) in \( S^{\sigma+\ell} \) for \( \ell > 0 \), and \( b_0 \) is elliptic at \( \zeta_0 \).
3. The support of \( e_t \) is contained in \( B(\zeta_0 - \delta H_{p_m} (\zeta_0), C_0 \delta^2) \).
4. For \( t > 0 \), the symbols have lower order: \( \check{a}_t \in S^{\sigma-(m-1)/2-1} \), \( b_t \in S^{\sigma-1} \), \( e_t \in S^{2\sigma-2} \) and \( f_t \in S^{2\sigma-2} H_b^{-1} \).

The commutant given by this proposition will now be used to deduce the propagation of regularity in a direction which agrees with the Hamilton flow to first order.
Let $\tilde{A} \in \Psi_b^{-(m-1)/2}$ be a quantization of $\hat{a}$ with $\text{WF}_b(\tilde{A}) \subset \text{supp} \hat{a}$, let $\Phi_t$ be a quantization of $\varphi_t$, i.e. $\Phi_t \in \Psi^0_b$ is a uniformly bounded family, $\Phi_t \in \Psi^{-1}_b$ for $t > 0$, and let $\tilde{A}_t = \tilde{A}\Phi_t$. Moreover, let $B_t \in \Psi^n_b$ be a quantization of $b_t$, with uniform b-microsupport contained in a conic neighborhood of $\gamma$, such that $B_t \in \Psi^{-1}_b$ is uniformly bounded, and $B_t \in \Psi^{-1}_b$ for $t > 0$. Similarly, let $E_t \in \Psi^{2\ast}_b$ be a quantization of $e_t$ with uniform b-microsupport disjoint from $\text{WF}_b(u)$ in the sense that

$$\|E_t u\|_{H^s_b}$$

is uniformly bounded for $t > 0$. (6.22)

This is the requirement that $u$ is in $H^s_b$ on a part of the backwards bicharacteristic from $\zeta_0$, more precisely in the ball specified in Proposition 6.8.

In a sense that we will make precise below, the principal symbol of the commutator $i\tilde{A}^*_t [P_m, \tilde{A}^*_t]$, is given by $\tilde{a}_t H_{P_m,a_t}$, which is what we described in Proposition 6.8. We compute for $t > 0$, following the proof of [7, Theorem 3.2]:

$$\text{Re} \langle i\tilde{A}^*_t [P_m, \tilde{A}^*_t] u, u \rangle = \text{Re} \langle i(P_m - P_m^*) \tilde{A}^*_t u, \tilde{A}^*_t u \rangle - \text{Re} \langle i\tilde{A}^*_t f, \tilde{A}^*_t u \rangle + \text{Re} \langle i\tilde{A}^*_t P_m u, \tilde{A}^*_t u \rangle,$$

(6.23)

where $\langle \cdot, \cdot \rangle$ denotes the sesquilinear pairing between spaces which are dual to each other relative to $L^2_b$. The adjoints here are taken with respect to the b-density $\frac{d^{\frac{1}{2}}}{dy}$, and in the case where $P$ acts on a vector bundle, we use the smooth metric in the fibers of $E$ for the adjoint. This computation needs to be justified, namely we must check that all pairings are well-defined by the a priori assumptions on $u$ so that we can perform the integrations by parts.

First, we observe that

$$\tilde{A}^*_t \tilde{A}^*_t P_m u \in \tilde{A}^*_t \tilde{A}_t H^s_b \cdot H^{\sigma-m-1/2}_b \subset H^{\sigma+1/2}_b,$$

because of $s \geq \sigma - m - 1/2$ and $\tilde{A}^*_t \tilde{A}^*_t \in \Psi^{2\sigma-m-1}_b$. Since $(\sigma-1/2) + (-\sigma+1/2) = 0$ is non-negative, the pairing $\langle \tilde{A}^*_t \tilde{A}_t P_m u, u \rangle$ is well-defined. By the same token, the pairing $\langle \tilde{A}_t P_m u, \tilde{A}_t u \rangle$ is well-defined, hence we can integrate by parts, justifying half of the first equality in (6.23). For the second half of the first equality, we use $P_m \in H^s_b \Psi^m_b$ and Corollary 3.10 to obtain

$$P_m \tilde{A}^*_t u \in P_m H^{m/2}_b \subset H^{-m/2}_b,$$

$$\tilde{A}^*_t P_m \tilde{A}_t u \in H^{-\sigma+1/2}_b,$$

which by the same reasoning as above proves the first equality in (6.23). For the second equality, we write $P_m$ as a sum of terms of the form $w Q_m$ with $w \in H^s_b$, $Q_m \in \Psi^m_b$, for which we have

$$\langle \tilde{A}_t u, w Q_m \tilde{A}_t u \rangle = \langle \tilde{w} \tilde{A}_t u, Q_m \tilde{A}_t u \rangle = \langle Q_m \tilde{w} \tilde{A}_t u, \tilde{A}_t u \rangle,$$

(6.24)

where the first equality follows from $\tilde{A}_t u \in H^{m/2}_b$ and $Q_m \tilde{A}_t u \in H^{-m/2}_b$ and for the second equality, one observes that the two pairings on the right hand side in (6.24) are well-defined, and we can integrate by parts, i.e. move $Q_m$ to the other side, taking its adjoint.

Now, since the principal symbol of $P_m$ is real, we can apply Theorem 3.12 (3) with $k = 1, k' = 0$ to obtain $P_m - P_m^* \in \Psi^{m-1} \circ \Psi^{0,0} H^{s-1}_b + \Psi^{m-1} H^{s-1}_b$. Therefore, 23

23 This requires $s \geq m/2$; recall that we are assuming $m \geq 1$.

24 We need $s \geq m/2$ and can then use Corollary 3.10.
Proposition 3.9 implies that $P_m - P_m^*$ defines a continuous map from $H_{b}^{(m-1)/2}$ to $H_{b}^{-(m-1)/2}$, thus

$$\langle (P_m - P_m^*) \hat{A}_t u, \hat{A}_t u \rangle \leq C_1 \| \hat{A}_t u \|^2_{H_{b}^{(m-1)/2}}$$

with a constant $C_1$ only depending on $P_m$.

Looking at the next term in (6.23), we estimate

$$\langle \hat{A}_t f, \hat{A}_t u \rangle \leq \frac{1}{4} \| \hat{A}_t f \|^2_{H_{b}^{-(m-1)/2}} + \| \hat{A}_t u \|^2_{H_{b}^{(m-1)/2}} \leq C_2 + \| \hat{A}_t u \|^2_{H_{b}^{(m-1)/2}},$$

where we use that

$$\hat{A}_t f \in H_{b}^{\sigma-m+1-\sigma+(m-1)/2} = H_{b}^{-(m-1)/2}$$

uniformly.

For the last term on the right hand side of (6.23), the well-definedness is easily checked. To bound it, we rewrite it as

$$\langle \hat{A}_t P_{m-1} u, \hat{A}_t u \rangle = \langle P_{m-1} \hat{A}_t u, \hat{A}_t u \rangle + \langle (\hat{A}_t, P_{m-1}) u, \hat{A}_t u \rangle.$$

The first term on the right hand side is bounded by $C_3 \| \hat{A}_t u \|^2_{H_{b}^{(m-1)/2}}$ for some constant $C_3$ only depending on $P_{m-1}$; indeed, $P_{m-1} : H_{b}^{(m-1)/2} \rightarrow H_{b}^{-(m-1)/2}$ is continuous. For the second term, note that $P_{m-1} \hat{A}_t \in H_{b}^{\sigma-m} \Psi_{b}^{\sigma+(m-1)/2}$ can be expanded to zeroth order, the first (and only) term being $p_{m-1} \hat{a}_t$ and the remainder being $R_1' \in H_{b}^{\sigma-m} \Psi_{b}^{\sigma+(m-1)/2-1}$. Next, we can expand $\hat{A}_t P_{m-1}$ to zeroth order by Theorem 3.12 (3) with $k' = 0$ again obtaining $p_{m-1} \hat{a}_t$ as the first term – which yields a remainder term $R_1'' + R_2$, where

$$R_1'' \in \Psi_{b}^{\sigma+(m-1)/2-1} \circ H_{b}^{s-2}$$

$$R_2 \in \Psi_{b}^{\sigma-(m-1)/2-1} \circ H_{b}^{s-2}. $$

We can then use Proposition 3.9 to conclude that

$$R_1 := R_1'' - R_1' \in \Psi_{b}^{\sigma+(m-1)/2-1} \circ H_{b}^{s-2}$$

is a uniformly bounded family of maps

$$R_1 : H_{b}^{\sigma-1/2} \rightarrow H_{b}^{-(m-1)/2}$$

which shows that $\langle R_1 u, \hat{A}_t u \rangle$ is uniformly bounded. Moreover, we can apply Proposition 3.9 and use the mapping properties of smooth b-ps.d.o’s to prove that $R_2 u \in H_{b}^{-(m-1)/2}$ is uniformly bounded. We thus conclude that

$$|\langle \hat{A}_t, P_{m-1} u, \hat{A}_t u \rangle| \leq C_4(M) + \| \hat{A}_t u \|^2_{H_{b}^{(m-1)/2}},$$

25 Provided $s - 1 \geq (m - 1)/2$ and $s - 1 > n/2 + (m - 1)/2$.

26 We need $s - 1 \geq |\sigma-m+1/2|$ and can then use Corollary 3.10 to obtain $P_{m-1} u \in H_{b}^{\sigma-m+1/2}$.

27 This requires $s - 1 \geq (m - 1)/2$ and $s - 1 > n/2$.

28 For notational convenience, we drop the explicit $t$-dependence here; inclusions are understood to be statements about a $t$-dependent family of operators being uniformly bounded in the respective spaces.

29 Assuming $\sigma - (m - 1)/2 \geq 1$.

30 The requirements are $s - 2 \geq -m/2 + 1$, $s - 2 > n/2 + (m/2 - 1)$.

31 Indeed, we have $u \in H_{b}^{\sigma-1/2} \subset H_{b}^{\sigma-1}$, and $\Psi_{b}^{m-1} H_{b}^{s-2} : H_{b}^{\sigma-1} \rightarrow H_{b}^{\sigma-m}$ is continuous if $s - 2 \geq \sigma - m$, $s - 2 > n/2 + (m - \sigma)$. 


where $C_4$, while it depends on $M$ in the sense that it depends on a seminorm of the $M$-dependent operator $\hat{A}$ constructed in Proposition 6.8, is independent of $t$.

Plugging all these estimates into (6.23), we thus obtain
\[
\text{Re} \left( i \hat{A}^*_t [P_m, \hat{A}_t] u, u \right) \geq -(C_2 + C_4(M)) - (C_1 + 1 + C_3 + 1) \| \hat{A}_t u \|_{H^s_{b(-1/2)}}^2,
\]
where all constants are independent of $t > 0$, and $C_1, C_2, C_3$ are in addition independent of the real number $M$. Choosing $M^2 > C_1 + C_3 + 2$, this implies that there is a constant $C < \infty$ such that for all $t > 0$, we have
\[
\text{Re} \left( (i \hat{A}^*_t [P_m, \hat{A}_t] + M^2 (\Lambda \hat{A}_t)^*(\Lambda \hat{A}_t)) u, u \right) \geq -C,
\]
(6.25)

where $\Lambda := \Lambda_{(m-1)/2}$. Therefore,
\[
\text{Re} \left( (i \hat{A}^*_t [P_m, \hat{A}_t] + B_t^* B_t + M^2 (\Lambda \hat{A}_t)^*(\Lambda \hat{A}_t) - E_t) u, u \right) \geq -C + \| B_t u \|_{L^2_b}^2.
\]
(6.26)

Here, we use that $\langle E_t u, u \rangle$ is uniformly bounded by (6.22).

The next step is to exploit the commutator relation in Proposition 6.8 in order to find a $t$-independent upper bound for the left hand side of (6.26). Theorem 3.12 (3) \footnote{Applicable with $k = 2, k' = 0$ if $\sigma - (m-1)/2 \geq 2$.}, gives
\[
i [P_m, \hat{A}_t] = (H_{p_m} \tilde{a}_t)(z, bD) + \tilde{R}_1 + \tilde{R}_2
\]
with uniformly bounded families of operators
\[
\tilde{R}_1 \in \Psi^{\sigma + (m-1)/2 - 1, 0} H^s_b
\]
\[
\tilde{R}_2 \in \Psi^{\sigma - (m-1)/2 - 2} \circ \Psi^{m, 0} H^s_b.
\]

Notice that $H_{p_m} \tilde{a}_t \in H^s_b S^{\sigma + (m-1)/2}$ uniformly. If we applied Theorem 3.12 (3) directly to the composition $\hat{A}_t^* (H_{p_m} \tilde{a}_t)(z, bD)$, the regularity of the remainder operator, say $R$, obtained by applying Theorem 3.12 (3), would be too weak in the sense that we could not bound $\langle R u, u \rangle$. To get around this difficulty, choose
\[
J^+ \in \Psi^\sigma_{b^-} \circ \Psi^\sigma_{b^+} \circ \Psi^\sigma_{b^+} \circ \Psi^\sigma_{b^-} \circ \Psi^\sigma_{b^+} \circ \Psi^\sigma_{b^-}
\]
with real principal symbols $j^+, j^-$ such that
\[
J^+ J^- = I + \tilde{R}, \quad \tilde{R} \in \Psi^{-\infty}.
\]
(6.27)

Observe that $J^- \hat{A}_t^*$ is uniformly bounded in $\Psi^1_{b^+}$. Then by Theorem 3.12 (3),
\[
i J^- \hat{A}_t^* [P_m, \hat{A}_t] = (\tilde{J}^- \tilde{a}_t H_{p_m} \tilde{a}_t)(z, bD) + R_1 + R_2 + R_3 + R_4,
\]
where
\[
R_1 = J^- \hat{A}_t^* \tilde{R}_1 \in \Psi^\sigma_{b^+} \circ \Psi^\sigma_{b^-} \circ \Psi^\sigma_{b^-} \circ \Psi^{m, 0} H^s_b
\]
\[
R_2 = J^- \hat{A}_t^* \tilde{R}_2 \in \Psi^\sigma_{b^-} \circ \Psi^\sigma_{b^+} \circ \Psi^\sigma_{b^+} \circ \Psi^{m, 0} H^s_b
\]
\[
R_3 \in \Psi^{\sigma + (m-1)/2 - 1, 0} H^s_b
\]
\[
R_4 \in \Psi^{\sigma + (m-1)/2} \circ \Psi^{\sigma + (m-1)/2} H^s_b.
\]
(6.28)

Applying Proposition 3.9, \footnote{The conditions $s - 2 \geq -m/2 + 1$ and $s - 2 > n/2 + m/2$ are sufficient to treat $R_1, R_3$ and $R_4$. For $R_2$, we need $s - 2 \geq \sigma - m - 1/2$ and $s - 2 > n/2 + (m + 1/2 - \sigma)_+$.} we conclude that $R_j$ $(1 \leq j \leq 4)$ is a uniformly bounded family of operators
\[
H^s_b \rightarrow H^{-m/2}_b.
\]
therefore the pairings \( \langle R_j u, (J^+)^* u \rangle \) are uniformly bounded in light of \( (J^+)^* u \in H_b^{m/2} \).

Hence, Proposition 6.8 implies
\[
J^+ \left( iJ^- \hat{A}_t^* [P_m, \hat{A}_t] + J^- B_t^* B_t + J^- M^2 (\Lambda \hat{A}_t)^* (\Lambda \hat{A}_t) - J^- E_t \right) \\
= J^+ \left( \left( j^- (\hat{a}_t H_{p_m} \hat{a}_t + \hat{b}_t^2 + M^2 \rho^{m-1} \hat{a}_t - \epsilon_t) \right)(z, bD) + R + G \right) \\
= J^+ \left( (-j^- f_1)(z, bD) + R + G \right),
\]
where \( R = R_1 + R_2 + R_3 + R_4 \) and \( G \in \Psi_b^{\sigma+(m-1)/2} \); \( G \) appears because the principal symbols of the smooth operators on both sides are equal. We already proved that \( \langle J^+ R u, u \rangle \) is uniformly bounded; also, \( \langle J^+ G u, u \rangle \) is uniformly bounded, since \( J^+ G \in \Psi_b^{2-1} \) and \( u \in H_b^{\sigma-1/2} \).

It remains to prove a uniform lower bound on \( \Re \langle (J^+ f_1)(z, bD) u, u \rangle = \Re \langle (j^- f_1)(z, bD) u, (J^+)^* u \rangle \).

In order to be able to apply the sharp Gårding inequality, Proposition 6.1, we need to rewrite this. Since \( j^+ \) is bounded away from 0, we can write
\[
(j^- f_1)(z, bD) = \left[ \frac{j^- f_1}{j^+} \right] (z, bD) \circ (J^+)^* + R, \quad R \in \Psi^{\sigma+(m-1)/2;0} H_b^{-1}
\]
by Theorem 3.12 (2b), since \( j^- f_1/j^+ \in S^{m+1;\infty} H_b^{-1} \). Now \( \langle R u, (J^+)^* u \rangle \) is uniformly bounded, since \( (J^+)^* u \in H_b^{m/2} \) and \( R u \in H_b^{-m/2} \) are uniformly bounded.\(^{35}\)

We can now apply the sharp Gårding inequality to deduce that
\[
\Re \left\langle \left[ \frac{j^- f_1}{j^+} \right] (z, bD) (J^+)^* u, (J^+)^* u \right\rangle \geq -C \| (J^+)^* u \|_{H_b^{m/2}}^2 \geq -C,
\]
where the constant \( C \) only depends on the uniform \( S^{2;\infty} H_b^{-1} \)-bounds on \( f_1 \) and the \( H_b^{\sigma^{-1/2}} \)-norm of \( u \).\(^{36}\)

Putting (6.26), (6.29) and (6.30) together by inserting \( I = J^+ J^- - \tilde{R} \) in front of the large parenthesis in (6.26) and observing that the error term
\[
\Re \left\langle \tilde{R} \left( i\hat{A}_t^* [P_m, \hat{A}_t] + B_t^* B_t + M^2 (\Lambda \hat{A}_t)^* (\Lambda \hat{A}_t) - E_t \right) u, u \right\rangle
\]
is uniformly bounded,\(^{37}\) we deduce that \( \| B_t u \|_{L^2_b} \) is uniformly bounded for \( t > 0 \). Therefore, a subsequence \( B_{t_k} u \), \( t_k \to 0 \), converges weakly to \( v \in L^2_b \) as \( k \to \infty \). On the other hand, \( B_{t_k} u \to B u \) in \( H_{b}^{-\infty} \); hence \( B u = v \in L^2_b \), which implies that \( u \in H^\sigma_b \) microlocally on the elliptic set of \( B \).

---

\(^{34}\)To justify the integration by parts here, note that, for \( t > 0 \), \( j^- f_1 \in S^{\sigma+(m-1)/2-1;\infty} H_b^{-1} \), thus \( (j^- f_1)(z, bD) u \in H_b^{-m/2+1} \) provided \( s - 1 \geq -m/2 + 1 \), \( s - 1 > n/2 + (m/2 - 1)_+ \), which follows from the conditions in Footnote 33.

\(^{35}\)For \( R u \), we need \( s - 1 > n/2 + m/2 \), which follows from the conditions in Footnote 33.

\(^{36}\)This requires \( s - 1 \geq 2 - m/2 \) and \( s - 1 > n/2 + m/2 \).

\(^{37}\)Indeed, \( \hat{A}_t^* \hat{A}_t P_m u \in H_b^{\sigma-3/2} \) is uniformly bounded because of \( s \geq |\sigma - m - 1/2| \); and \( \hat{A}_t u \in H_b^{\sigma-m/2-1} \) is uniformly bounded, hence so is \( P_m \hat{A}_t u \in H_b^{-m/2-1} \) in view of \( s \geq m/2 + 1 \), which follows from the condition in Footnote 27, and therefore \( \hat{A}_t^* P_m \hat{A}_t u \in H_b^{-\sigma-3/2} \) is uniformly bounded.
To eliminate the assumption that \( m \geq 1 \), notice that the above propagation estimate for a general \( m \)-th order operator can be deduced from the \( m_0 \)-th order result for any \( m_0 \geq 1 \), simply by considering

\[
P \Lambda^+ (\Lambda^- u) = f + P R u,
\]

where \( \Lambda^+ \in \Psi_b^{-(m-m_0)} \) is elliptic with parametrix \( \Lambda^- \in \Psi_b^{m-m_0} \), and \( \Lambda^+ \Lambda^- = I + R \), \( R \in \Psi_b^{-\infty} \). If we pass from \( P \) to \( P \Lambda^+ \), which means passing from \( m \) to \( m_0 \), we correspondingly have to pass from \( \sigma \) to \( \sigma_0 = \sigma - m + m_0 \) in equation (6.10); in other words, the difference \( \sigma - m = \sigma_0 - m_0 \) remains the same. Thus, let us collect the conditions on \( s \) and \( \tilde{s} = \sigma - m + 1 \) as given in Footnotes 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 36:\(^{38}\) All conditions are satisfied provided

\[
\begin{align*}
3/2 - s &\leq \tilde{s} \leq s - 1, \\
s &> n/2 + 2 + (3/2 - \tilde{s})_+,
\end{align*}
\]

for some \( m_0 \geq 1 \). The optimal choice for \( m_0 \) is thus \( m_0 = \max(1, 5 - 2\tilde{s}) = 1 + 2(2 - \tilde{s})_+ \); plugging this in, we obtain the conditions in the statement of Theorem 6.6:

\[
s > n/2 + 7/2 + (2 - \tilde{s})_+,
\]

Thus, we have proved a propagation result which propagates estimates in a direction which is ‘correct to first order’. To obtain the final form of the propagation result, we use an argument by Melrose and Sjöstrand \([25, 26]\), in the form given in \([10, \text{Lemma 8.1}]\). This finishes the proof of the first part of Theorem 6.6.

**Remark 6.9.** For second order real principal type operators of the form considered above, with the highest order derivative having \( H^s_b \)-coefficients, the maximal regularity one can prove for a solution \( u \) with right hand side \( f \in H^s_{-1} \) is \( H^{s+1} \) with \( \tilde{s} \) being at most \( s - 1 \), i.e. one can prove \( u \in H^s_b \), which is exactly what we will need in our quest to solve quasilinear wave equations.

### 6.3.2. Propagation near the boundary.

We now aim to prove the corresponding propagation result (near and) within the boundary \( \partial M \): Let \( P = P_m + P_{m-1} + R \), where

\[
\begin{align*}
P_m &= P'_m + P''_m, \\
P'_m &\in H^{s,\alpha}_b \Psi^m_b, P''_m \in \Psi^m_b
\end{align*}
\]

with real homogeneous principal symbols,

\[
\begin{align*}
P_{m-1} &= P'_{m-1} + P''_{m-1}, \\
P'_{m-1} &\in H^{s-1,\alpha}_b \Psi^{m-1}_b, P''_{m-1} \in \Psi^{m-1}_b,
\end{align*}
\]

\[
R = R' + R'', \\
R' \in \Psi^{m-2,0}_b H^{s-1,\alpha}_b, R'' \in \Psi^{m-2}_b,
\]

and let us assume that we are given a solution

\[
u \in H^{\sigma-1/2}_b
\]

to the equation

\[
P u = f \in H^{\sigma-m+1}_b,
\]

where \( \sigma = \tilde{s} + m - 1 \). In fact, since

\[
R: H^{\sigma-1/2}_b \subseteq H^{\sigma-1}_b \rightarrow H^{\sigma-m+1}_b,
\]

\(^{38}\)The definition of \( \tilde{s} \) reflects the loss of 1 derivative in this real principal type setting as compared to the elliptic one.
we may absorb the term $Ru$ into the right hand side; thus, we can assume $R = 0$, hence $P = P_m + P_{m-1}$.

Moreover, let $\gamma$ be a null-bicharacteristic of $p_m = p_m' + p_m''$; we assume $H_{p_m}$ is never radial on $\gamma$. Since $H_{p_m} = 0$ at $bT_{\partial M}^* M$, this in particular implies that $H_{p_m}$ is not radial on $\gamma \cap bT_{\partial M}^* M$, and the positivity of the principal symbol $\hat{a}_t H_{p_m} \hat{a}_t$ of the commutator there comes from the positivity of $\hat{a}_t H_{p_m} \hat{a}_t$.

The proof of the interior propagation, with small adaptations, carries over to the new setting. We indicate the changes: First, in the notation of Section 6.3.1, $H_{p_m} \sigma_j$ now only is Hölder continuous with exponent $\alpha$, thus (6.16) becomes

$$|H_{p_m} \omega| \leq C \omega^{1/2} (|\bar{\eta}| + \omega^{1/2})^\alpha.$$  

Hence, for (6.15) to hold, we need

$$C \omega^{1/2} (|\bar{\eta}| + \omega^{1/2})^\alpha \leq c_0 \epsilon^2 \delta / 2,$$

which holds if $2^{1+2 \alpha} C \delta^{1+\alpha} \leq c_0 \epsilon \delta / 2$, which suggests the choice $\epsilon = 4^{1+\alpha} C \delta^\alpha / c_0$; in particular $\epsilon \leq 1$ for $\delta$ small enough. Thus, the size of the a priori control region near $\bar{\eta} = -\delta$, cf. (6.17), becomes

$$\omega^{1/2} \leq \frac{c_0 \epsilon^2 \delta}{2C(|\bar{\eta}| + \omega^{1/2})^\alpha} = C \delta^{1+\alpha},$$

which is small enough for the argument in [10, Lemma 8.1] to work. Further, defining the commutant $a$ as before, we replace the a priori control terms $e, f'$ in (6.14) by

$$e = \chi_0 (JH_{p_m} + H_{p_m}') \chi_1 + \bar{\chi}_1,$$

$$f' = \chi_0 (JH_{p_m} - H_{p_m}') \chi_1 + \bar{\chi}_1,$$  

(6.33)

where we choose the mollifier $J$ to be so close to $I$ that $f' \geq 0$; here, we use that the first summand in the definition of $f'$ is an element of $H^{s-1}_b$ in the base variables, hence for $s > n/2 + 1$ in particular continuous and vanishing at the boundary $\partial M$, and can therefore be dominated by $\bar{\chi}_1$. We then let $e_t$ and $f''_t$ be defined as in (6.20) with the above $e$ and $f'$. We change the terms $b_t$ and $f''_t$ in (6.21) in a similar way: We take

$$b_t = (F \delta)^{-1/2} \varphi_t \rho (JH_{p_m} + H_{p_m}') \phi$$

$$- \left[ ((2\sigma - m + 1) - 2 t \varphi_t \rho) (\rho^{-1} (JH_{p_m}' + H_{p_m}'') \rho) + M^2 \right]$$

$$\times F^{-1} \delta (2 \beta - \frac{\phi}{\delta})^2 - \frac{c_0}{8} \left( F^{-1} \left( 2 \beta - \frac{\phi}{\delta} \right) \right)^{1/2} \chi_0 \left( F^{-1} (\bar{\eta} + \delta - \frac{\phi}{\delta}) \right) \chi_1 \left( \frac{\eta + \delta}{\epsilon \delta} + 1 \right).$$

$$f''_t = (F \delta)^{-1} \varphi_t^2 \rho^2 \left( H_{p_m}' - JH_{p_m} \right) \phi$$

$$- \left[ ((2\sigma - m + 1) - 2 t \varphi_t \rho) (\rho^{-1} (H_{p_m}' - JH_{p_m}) \rho) \right] F^{-1} \delta \left( 2 \beta - \frac{\phi}{\delta} \right)^2 + \frac{c_0}{8}$$

$$\times \chi_0 \left( F^{-1} \left( 2 \beta - \frac{\phi}{\delta} \right) \right) \chi_1 \left( \frac{\eta + \delta}{\epsilon \delta} + 1 \right).$$

As before, we can control the term $|E_t u, u|$ in (6.26) by the a priori assumptions on $u$. The new feature here is that $f''_t, f'''_t \geq 0$ are not just symbols with coefficients having regularity $H^{s-1}_b$, but there are additional smooth terms involving $\bar{\chi}_1$ and
therefore, we need to appeal to the version of the sharp Gårding inequality given in Corollary 6.3 to obtain a uniform lower bound on the term $\langle J^+ (j^{-f_1}) (z, bD) u, u \rangle$ in (6.29).

Since the computation of compositions and commutators in the proof of the previous section for $P_m''$ is standard as $P_m''$ is a smooth b-ps.d.o, and since $P_m'$ lies in the same space as the operator called $P_m$ there, all arguments now go through after straightforward changes that take care of the smooth b-ps.d.o. $P_m''$.

This finishes the proof of Theorem 6.6.

6.3.3. Complex absorption. We next aim to prove Theorem 6.7, namely we add a complex absorbing potential $Q = q(z, bD) \in \Psi^m_b$ with $Q = Q^*$ and prove the propagation of $H^\sigma_b$-regularity of solutions $u \in H^{\sigma-m+1}_b$ to the equation

$$(P - iQ)u = f \in H^{\sigma-m+1}_b,$$

where $\gamma$ is a null-bicharacteristic of $P$, in a direction which depends on the sign of $q$ near $\gamma$. Namely, we can propagate $H^\sigma_b$-regularity forward along the flow of the Hamilton vector field $H_{p_m}$ if $q \geq 0$ near $\gamma$, and backward along the flow if $q \leq 0$ near $\gamma$.

Let $\Gamma$ be an open neighborhood of $\gamma$. It suffices to consider the case that $q \geq 0$ in $\Gamma$. The only step that we have to change in the proofs of the previous propagation results\textsuperscript{39} is the right hand side equation (6.23), where we have an additional term in view of $P_m u = f - P_{m-1} u + iQu$, namely

$$- \text{Re}(i \hat{A}_t iQu, \hat{A}_t u) = \text{Re}(\hat{A}_t Q u, \hat{A}_t u) = \text{Re}(\hat{A}_t [\hat{A}_t, Q] u, u),$$

The first term on the right is bounded from below by $-C_5 \| \hat{A}_t u \|_{H^{\sigma-m+1}_b}$ and will be absorbed as in (6.25), and the second term is bounded by the a priori microlocal $H^{\sigma-m+1/2}_b$-regularity of $u$ in $\Gamma$, since

$$\text{Re}(\hat{A}_t [\hat{A}_t, Q] u, u) = \frac{1}{2} \langle \hat{Q}_t u, u \rangle$$

with

$$\hat{Q}_t = \hat{A}_t^* [\hat{A}_t, Q] + [Q, \hat{A}_t^*] \hat{A}_t = (\hat{A}_t^* - \hat{A}_t) [\hat{A}_t, Q] + [\hat{A}_t, [\hat{A}_t, Q]] + [Q, \hat{A}_t^* - \hat{A}_t] \hat{A}_t$$

uniformly bounded in $\Psi^{2\sigma-1}_b$ in view of the principal symbol of $\hat{A}_t$ being real and the presence of double commutators.

This finishes the proof of Theorem 6.7.

6.4. Propagation near radial points. We will only consider the class of radial points which will be relevant in our applications; see Section 8, where an example of an operator with this radial point structure is presented. The setting is very similar to the one in [18, §2]: There, the authors consider an operator $P \in \Psi^m(M; E)$ with real, scalar, homogeneous principal symbol $p$ on a compact manifold $M$ with boundary $Y = \partial M$ and boundary defining function $x$, where the assumptions on $p$ are as follows:

(1) At $p = 0$, $dp \neq 0$, and at $bS^*_x M \cap p^{-1}(0)$, $dp$ and $dx$ are linearly independent; hence $\Sigma = p^{-1}(0) \subset bS^*_x M$ is a smooth codimension 1 submanifold transversal to $bS^*_x M$.

\textsuperscript{39}Recall that the proofs given there only show the propagation forward along the flow; the backward propagation is proved by a completely analogous argument.
(2) \( L = L_+ \cup L_- \), where \( L_\pm \) are smooth disjoint submanifolds of \( bS^r \Sigma \), given
by \( L_+ = L_+ \cap bS^r \Sigma \), where \( L_\pm \) are smooth disjoint submanifolds of \( \Sigma \)
transversal to \( bS^r \Sigma \), defined locally near \( bS^r \Sigma \). Moreover, \( H_p = \rho^{1-m} H_p \)
is tangent to \( L_\pm \), where, as before, \( \rho = \langle \zeta \rangle \), and \( H_p \) is the Hamilton vector
field of \( p \).

(3) There are functions \( \beta_0, \bar{\beta} \in C^\infty(L_\pm) \), \( \beta_0, \bar{\beta} > 0 \), such that
\[
\rho H_p \rho^{-1}|_{L_\pm} = \mp \beta_0, \quad -x^{-1} H_p x|_{L_\pm} = \mp \bar{\beta} \beta_0.
\] (6.34)

(4) For a homogeneous degree 0 quadratic defining function \( \rho_0 \) of \( \mathcal{L} = L_+ \cup L_- \)
within \( \Sigma \),
\[
\mp H_p \rho_0 - \beta_1 \rho_0 \geq 0 \text{ modulo terms that vanish cubically at } L_\pm,
\] (6.35)
where \( \beta_1 \in C^\infty(\Sigma) \), \( \beta_1 > 0 \) at \( L_\pm \).

(5) The imaginary part of the subprincipal symbol is homogeneous, and equals
\[
\sigma_b^{m-1} \left( \frac{1}{2i} (P - P^*) \right) = \pm \bar{\beta} \beta_0 \rho^{m-1} \text{ at } L_\pm,
\] (6.36)
where \( \bar{\beta} \in C^\infty(L_\pm; \pi^* \text{ End}(E)) \), \( \pi : L_\pm \to M \) being the projection to the
base; note that \( \bar{\beta} \) is self-adjoint at every point.

These conditions imply that \( L_\pm \) is a sink, resp. source, for the bicharacteristic
flow within \( bS^r \Sigma \), in the sense that nearby null-bicharacteristics tend to \( L_\pm \) in
the forward, resp. backward, direction; but at \( L_\pm \) there is also an unstable, resp.
stable, manifold, namely \( L_\pm \).

In the non-smooth setting, we will make the exact same assumptions on the
’smooth part’ of the operator; the guiding principle is that non-smooth operators
with coefficients in \( H_b^{\alpha} \), \( \alpha > 0 \), \( s > n/2 + 1 \), have symbols and associated Hamilton
vector fields that vanish at the boundary in view of the Riemann-Lebesgue lemma,
thus would not affect the above conditions anyway, with the exception of condition (4),
which however is only used close to, but away from \( L_\pm \), and the positivity of \( \mp H_p \rho_0 \)
there is preserved when one adds small non-smooth terms in \( H_b^{\alpha} \) to \( p \).

In order to be able to give a concise expression for the threshold regularity (determining
whether one can propagate into or out of the boundary), let us define for a
function \( b \in C^\infty(L_\pm; \pi^* \text{ End}(E)) \) with values in self-adjoint endomorphisms of the
fiber,
\[
\inf_{L_\pm} b := \inf \{ \lambda \in \mathbb{R} : b \geq \lambda I \text{ everywhere on } L_\pm \},
\]
\[
\sup_{L_\pm} b := \sup \{ \lambda \in \mathbb{R} : b \leq \lambda I \text{ everywhere on } L_\pm \}.
\]

We then have the following theorem:

**Theorem 6.10.** Let \( m, r, s, \hat{s} \in \mathbb{R} \), \( \alpha > 0 \). Let \( P = P_m + P_{m-1} + R \), where
\( P_j = P_j^r + P_j^s \), \( j = m, m-1 \), with \( P_m \in H_b^{\alpha} \Psi^m_b(\mathbb{R}^n_+; E) \) having a real, scalar, homoge-
nous principal symbol \( \rho_m^r \), \( \rho_m^s \in \Psi^m_b(\mathbb{R}^n_+; E) \) a real, scalar, homogeneous principal
symbol \( \rho_m^s \); moreover \( P_m^r \in H_b^{\alpha-1} \Psi^m_b(\mathbb{R}^n_+; E) \), \( P_m^s \in \Psi^{m-1} - H_b^{\alpha-1} \Psi^{m-1} \Psi^m_b(\mathbb{R}^n_+; E) \)
and \( R \in \Psi^{m-2} \Psi^m_b(\mathbb{R}^n_+; E) \). Suppose that the above conditions (1)-(4) hold for \( p = \rho_m^r \), and
\[
\sigma_b^{m-1} \left( \frac{1}{2i} \left( P_m^r + P_m^s \right) - (P_m^r + P_m^s)^* \right) = \pm \bar{\beta} \beta_0 \rho^{m-1} \text{ at } L_\pm.
\] (6.37)
where $\hat{\beta} \in \mathcal{C}^\infty(L_\pm; \text{End}(E))$ is self-adjoint at every point. Finally, assume that $s$ and $\hat{s}$ satisfy
\begin{equation}
\hat{s} \leq s - 1, \quad s \geq n/2 + 7/2 + (2 - \hat{s}). \tag{6.38}
\end{equation}
Suppose $u \in H^\hat{s} + m - 3/2/r (\mathbb{R}^n_+; E)$ is such that $Pu \in H^s_0 (\mathbb{R}^n_+; E)$.

1. If $\hat{s} + (m - 1)/2 - 1 + \inf_{L_\pm} (\hat{\beta} - r\hat{\beta}) > 0$, let us assume that in a neighborhood of $L_\pm$, $L_\pm \cap \{x > 0\}$ is disjoint from $\text{WF}^\hat{s} + m - 1/r(u)$.

2. If $\hat{s} + (m - 1)/2 + \sup_{L_\pm} (\hat{\beta} - r\hat{\beta}) < 0$, let us assume that a punctured neighborhood of $L_\pm$, with $L_\pm$ removed, in $\Sigma \cap b^* S_{b^* \Sigma_\pm} \mathbb{R}^n_+$ is disjoint from $\text{WF}^\hat{s} + m - 1/r(u)$.

Then in both cases, $L_\pm$ is disjoint from $\text{WF}^\hat{s} + m - 1/r(u)$.

Adjoint are again taken with respect to the b-density $d\xi dy$ and the smooth metric on the vector bundle $E$.\textsuperscript{40}

**Remark 6.11.** Since $\text{WF}^\hat{s} + m - 1/r(u)$ is closed, we in fact have the conclusion that a neighborhood of $L_\pm$ is disjoint from $\text{WF}^\hat{s} + m - 1/r(u)$. As in the real principal type setting (see equation (6.8) in particular), one can also rewrite the wavefront set statement as an estimate on the $L^2_\Sigma$ norm of an operator of order $\hat{s} + m - 1$, elliptic at $L_\pm$, applied to $u$. In particular, we will see that it suffices to have only microlocal $H^\hat{s}_b$-$\text{membership}$ of $Pu$ near the part of the radial set that we propagate to/from, and local membership in $H^{\hat{s} - 1}_b$, which comes from a use of elliptic regularity (Theorem 5.1) in our argument.

Moreover, as the proof will show, the theorem also holds for operators $P$ which are perturbations of those for which it directly applies: Indeed, even though the dynamical assumptions (1)-(4) are (probably) not stable under perturbations, the estimates derived from these are. Here, perturbations are to be understood in the sense that $P'_m$ and $P''_{m - 1}$ may be changed by operators whose norms in the respective spaces they belong to are sufficiently small, and $P'_m, P''_{m - 1}$ and $R$ may be changed arbitrarily, with the estimate corresponding to the wavefront set statement of the theorem being locally uniform.

**Proof of Theorem 6.10.** We again drop the bundle $E$ from the notation. The proof is an adaptation of the proof of [18, Proposition 2.1], see also [34, Propositions 2.10 and 2.11] for a related result, to our non-smooth setting. Since $Ru \in H^\hat{s}_b$ by the a priori regularity on $u$, we can absorb $Ru$ into $f = Pu$ and thus assume $R = 0$. Finally, let us assume $m \geq 2$ and $r = 0$ for now; these conditions will be eliminated at the end of the proof.

Define the regularizer $\varphi_t(\rho) = (1 + t\rho)^{-1}$ for $t \geq 0$ as in the proof of Theorem 6.6, put $p''_m = \rho^{-m} p''_m$ and $\sigma = \hat{s} + m - 1$, and consider the commutant
\begin{equation}
a_t = \varphi_t(\rho) \psi(\rho_0) \psi_0(p''_m) \psi_1(x) \rho^{\sigma - (m - 1)/2},
\end{equation}
where $\psi, \psi_0, \psi_1 \in C^\infty_c(\mathbb{R})$ are equal to 1 near 0 and have derivatives which are $\leq 0$ on $[0, \infty)$; we will be more specific about the supports of $\psi, \psi_0, \psi_1$ below. Let us

\textsuperscript{40}In fact, condition (6.37) is insensitive to changes both of the b-density and the metric on $E$ by the radiality of $H_{p''_m}$ at $L_\pm$; see [34, Footnote 19] for details.
also assume that $\sqrt{-\psi_t^2}$ and $\sqrt{-\psi_t^2}$ are smooth in a neighborhood of $[0, \infty)$. As usual, we put $H_{p_m} = p^{1-m} H_{p_m}$. We then compute, using $H_{p_m}^\bullet \psi = -t \bar{\psi}_{t} H_{p_m}^\bullet$:

$$
a_t H_{p_m} a_t = \phi^2 \rho^{2\sigma} \psi \psi_0 \psi_1 \left((\sigma - (m - 1)/2 - t \rho \bar{\psi}_t) (\rho^{-1} H_{p_m}^\bullet \psi) \psi_0 \psi_1ight) + (x^{-1} H_{p_m}^\bullet x) x \psi \psi_0 \psi_1' + (H_{p_m}^\bullet \rho_0) \psi' \psi_0 \psi_1 + H_{p_m}^\bullet (p_m') \psi_0 \psi_1',
$$

and to compute $a_t H_{p_m}^\bullet a_t$, we can use (6.34) to simplify the resulting expression.

To motivate the next step, recall that the objective is to obtain an estimate similar to (6.25); however, since in our situation, the weight $\rho^{\sigma -(m-1)/2}$ can only give a limited amount of positivity at $L_\pm$, we need to absorb error terms, in particular the ones involving $P - P^*$, into the commutator $a_t H_{p_m} a_t$. Thus, consider

$$
a_t H_{p_m} a_t \pm \rho^{m-1} a_t^2 \tilde{\beta} \hat{\beta} = \pm \phi^2 \rho^{2\sigma} \psi \psi_0 \psi_1
$$

$$
\times \left( [\beta_0 (\sigma - (m - 1)/2 - t \rho \bar{\psi}_t + \tilde{\beta}) \pm (\sigma - (m - 1)/2 - t \rho \bar{\psi}_t) (\rho^{-1} H_{p_m}^\bullet \psi) \psi_0 \psi_1,
\right.
$$

the second condition is satisfied close to $Y = \{x = 0\}$, since $\rho^{-1} H_{p_m}^\bullet \rho = o(1)$ as $x \to 0$ by Riemann-Lebesgue. The first condition is satisfied at $L_\pm$ by assumption, and the second condition is satisfied close to $Y = \{x = 0\}$, since $x^{-1} H_{p_m}^\bullet x = o(1)$ as $x \to 0$ by Riemann-Lebesgue. The second condition is satisfied close to $Y$, since $x^{-1} H_{p_m}^\bullet x = o(1)$ as $x \to 0$.

- On supp $d(\psi_1 \circ x) \cap$ supp $d(\psi_0 \circ \rho_0) \cap$ supp $d(\psi_0 \circ p_m')$:

$$
\tilde{\beta} \beta_0 \geq c_1 > 0, \quad |x^{-1} H_{p_m}^\bullet x| \leq c_1/2.
$$

The second condition is satisfied close to $Y$, since $x^{-1} H_{p_m}^\bullet x = o(1)$ as $x \to 0$.

- On supp $d(\psi_0 \circ \rho_0) \cap$ supp $d(\psi_1 \circ x) \cap$ supp $d(\psi_0 \circ p_m')$:

$$
\pm H_{p_m}^\bullet \rho_0 \geq \beta_0, \quad |H_{p_m}^\bullet \rho_0| \leq c_2/2.
$$

- On supp $d(\psi_0 \circ p_m') \cap$ supp $d(\psi_0 \circ \rho_0) \cap$ supp $d(\psi_1 \circ x)$:

$$
|p_m' | \geq c_3 > 0.
$$

This can be arranged as follows: First, note that we can ensure

$$
p_m' \geq 2 c_3
$$

there; then, since $|p_m' | = o(1)$ as $x \to 0$, shrinking the support of $\psi_1$ if necessary guarantees (6.41).

We can ensure that all these assumptions are satisfied by first choosing $\psi_1$, localizing near $S_+^\bullet M$, then $\psi$, localizing near $L_\pm$ within the characteristic set $(p_m')^{-1}(0)$ of $P_m''$, such that the inequalities in (6.39) and (6.40) are strict on $(p_m')^{-1}(0)$, then choosing $\psi_0$ (localizing near $(p_m')^{-1}(0)$) such that strict inequalities hold in (6.39),
and (6.42), and finally shrinking the support of \( \psi_1 \), if necessary, such that all inequalities hold.

We can then write

\[
\alpha_t H_{P_m} a_t \pm \rho^{m-1} a_t^2 \beta_0 \beta = \pm \left( \frac{c_0}{8} \rho^{m-1} a_t^2 + b_{1,t}^2 + b_{2,t}^2 - b_{3,t}^2 + f_t + g_t \right),
\]

(6.43)

where, with a mollifier \( J \) as in Lemma 6.5,

\[
b_{1,t} = \varphi_t \rho^2 \psi \psi_0 \psi_1 \left[ \beta_0 (\sigma - (m-1)/2) - t \varphi_t \right],
\]

\[
b_{2,t} = \varphi_t \rho^2 \psi \psi_0 \psi_1 \sqrt{-\psi_1^* - \psi_1^2} \left[ \pm (H_{P_m} \rho_0 + J H_{P_m} \rho_0) - \frac{c_2}{4} \right]^{1/2},
\]

\[
b_{3,t} = \varphi_t \rho^2 \psi \psi_0 \psi_1 \left[ (\beta \beta_0 \pm x^{-1} J H_{P_m} x + \frac{c_1}{4}) x \right]^{1/2},
\]

\[
g_t = \varphi_t^2 \rho^2 \sigma \psi^2 \psi_0 \psi_1^2 (m \beta \sigma P_m^2 + H_{P_m} P_m^2),
\]

and \( f_t = f_{1,t} + f_{2,t} + f_{3,t} \) with

\[
f_{1,t} = \varphi_t^2 \rho^2 \sigma \psi^2 \psi_0^2 \varphi_1 \left[ \pm (\sigma - (m-1)/2) - t \varphi_t \right] (\rho^{-1} (H_{P_m} - J H_{P_m}) \rho) + \frac{3c_0}{8},
\]

\[
f_{2,t} = \varphi_t^2 \rho^2 \sigma \psi^2 \psi_0^2 \varphi_1 \left[ \pm (H_{P_m} - J H_{P_m}) \rho_0 - \frac{c_2}{4} \right],
\]

\[
f_{3,t} = \varphi_t^2 \rho^2 \sigma \psi^2 \psi_0^2 \varphi_1 \left[ \pm x^{-1} (H_{P_m} - J H_{P_m}) x - \frac{c_1}{4} \right].
\]

In particular, \( b_{1,t}, b_{2,t}, b_{3,t} \in S^\sigma, b_{4,t} \in x^{1/2} S^\sigma, f_t \in S^{2\sigma, \infty} H_{-1}^{b_2} + S^{2\sigma}, g_t \in H_{-1}^{b_2} S^{2\sigma} + S^{2\sigma} \) uniformly, with the symbol orders one lower if \( t > 0 \) for \( b_{1,t}, \) \( j = 1, 2, 3, \) and two lower for \( f_t, g_t. \) The term \( b_{2,t}^2 \) will give rise to an operator which is elliptic at \( L_{\pm}. \) The term \( b_{2,t}^2 \) (which has the same, ‘advantageous,’ sign as \( b_{1,t} \)) can be discarded, and the term \( -b_{3,t}^2, \) with a ‘disadvantageous’ sign, will be bounded using the a priori regularity assumptions on \( u. \) An important point here is that the non-smooth symbol \( f_t \) is non-negative if we choose the mollifier \( J \) to be close enough to \( I; \) in fact, we then have \( f_{j,t} \geq 0 \) for \( j = 1, 2, 3. \) Lastly, we will be able to estimate the contribution of the term \( g_t \) using elliptic regularity, noting that its support is disjoint from the characteristic set \( p_{m-1}^{-1}(0) \) of \( P_m. \)

Let \( A_t \in \Psi_b^{-\sigma-(m-1)/2}, B_{1,t}, B_{2,t}, B_{3,t} \in \Psi_b^\sigma \) denote quantizations with uniform \( b \)-microsupport contained in the support of the respective full symbols \( a_t, b_{1,t}, b_{2,t} \) and \( b_{3,t}. \) Then we compute as in the proof of real principal type propagation (see equation (6.23) there):

\[
\text{Re}(i A_t^* [P_m, A_t] u, u) = -\left\langle \frac{1}{2i} (P_m - P_m^*) A_t u, A_t u \right\rangle - \text{Re}(i A_t f, A_t u) + \text{Re}(i A_t P_{m-1} u, A_t u)
\]

We split the first term on the right hand side into two pieces corresponding to the decomposition \( P_m = P_m^o + P_m^o. \) The piece involving \( P_m^o \) will be dealt with later. For the other piece, note that \( P_m^o \) is a sum of terms of the form \( \tau^\omega w Q_m, \) where \( w \in H_b^0 \) is real-valued and \( Q_m = q_m(z, \cdot \partial) \in \Psi_b^m \) has a real principal symbol. Now,

\[
\tau^\omega w Q_m - (\tau^\omega w Q_m)^* = \tau^\omega (Q_m - Q_m^*) + \tau^\omega (w Q_m^* - Q_m w) + \tau^\omega (Q_m^* - \tau^{-\alpha} Q_m^* \tau^\alpha) w,
\]
thus, using Theorem \(3.12\) \((3)\) with \(k = 1, k' = 0\) (applicable because we are assuming \(m \geq 1\)) to compute \(Q^*_m w\) and with \(k = 0, k' = 0\) to compute the last term, we get

\[
i(P_m' - (P_m^*)^*) = R_1 + R_2 + R_3,
\]

where

\[
R_1 \in H_b^{s-1, \alpha} \Psi_b^{m-1}, R_2 \in \Psi_b^{m-1} \circ \Psi_b^{\delta, 0} H_b^{s-1, \alpha}, R_3 \in \Psi_b^{m-1, 0} H_b^{s-1, \alpha}.
\]

Let \(\chi \in C_0^\infty(\mathbb{R}_+), \chi \equiv 1\) near 0. Writing \(R_1\) as the sum of terms of the form \(w'Q',\)
where \(w' \in H_b^{s-1, \alpha} \) and \(Q' \in \Psi_b^{m-1},\) we have for \(\epsilon > 0,\) which we can choose to be as small as we like provided we shrink the support of the Schwartz kernel of \(A_t:\)

\[
\langle w'(z)Q'A_t u, A_t u \rangle = \langle \chi(x/\epsilon)w'(z)Q'A_t u, A_t u \rangle;
\]

by Lemma 6.4, this can be bounded by \(c_{\epsilon \epsilon'}\|A_t u\|^2_{H_b^{(m-1)/2}},\) which is the case if \(s - 1 \geq (m - 1)/2.\)

In a similar manner, we can treat the terms involving \(c\) by Lemma 6.4, this can be bounded by \(c_{\epsilon \epsilon'}\|A_t u\|^2_{H_b^{(m-1)/2}},\) where \(c_{\epsilon \epsilon'} \rightarrow 0\) as \(\epsilon \rightarrow 0.\)

\[41\]

In a similar manner, we can treat the terms involving \(R_2\) and \(R_3.\) Hence, under the assumption that the Schwartz kernel of \(A_t\) is localized sharply enough near \(\partial M \times \partial M,\) we have

\[
|\langle (P_m' - (P_m^*)^*)A_t u, A_t u \rangle| \leq C_\delta + \delta \|A_t u\|^2_{H_b^{(m-1)/2}}
\]

for an arbitrarily small, but fixed \(\delta > 0.\)

Next, for \(\delta > 0,\) we estimate

\[
|\langle A_t f, A_t u \rangle| \leq C_\delta + \delta \|A_t u\|^2_{H_b^{(m-1)/2}},
\]

using that \(\|A_t f\|_{H_b^{(m-1)/2}}\) is uniformly bounded.

Finally, we can bound the term \(\langle A_t P_m' A_t u \rangle\) as in the proof of Theorem 6.6, thus obtaining

\[
|\langle A_t P_m' A_t u \rangle| \leq C_\delta + \delta \|A_t u\|^2_{H_b^{(m-1)/2}}.
\]

Therefore, writing \(P'' := P_m'' + P_m''\) and \(Q := \frac{1}{2}(P'' - (P')^*) \in \Psi_b^{m-1},\) we get

\[
\pm \text{Re} \langle (iA_t^*[P_m, A_t] + A_t^* Q A_t)u, u \rangle \leq C_\delta + \delta \|A_t u\|^2_{H_b^{(m-1)/2}}
\]

Now, using that \(|\langle B_{1,3} B_{1,3} u, u \rangle| = \|B_{1,3} u\|^2_{L^2_b}\) is uniformly bounded because of the assumed a priori control of \(u\) in a neighborhood of \(L_{\pm} \in L_{\pm} \cap \{x > 0\},\) we deduce, using the operator \(\Lambda = \Lambda_{(m-1)/2}:\)

\[
\text{Re} \left( \left( \pm iA_t^*[P_m, A_t] \pm A_t^* Q A_t - \frac{c_0}{8} (AA_t)^* (AA_t) 
- B_{1,1} B_{1,1} - B_{2,1} B_{2,1} + B_{3,1} B_{3,1} \right) u, u \right) \leq C_\delta + \left( \delta - \frac{c_0}{8} \right) \|A_t u\|^2_{H_b^{(m-1)/2}} - \|B_{1,1} u\|^2_{L^2_b},
\]

where we discarded the negative term \(\langle B_{2,1} B_{2,1} u, u \rangle\) on the right hand side. If we choose \(\delta < c_0/8,\) then we can also discard the term on the right hand side involving \(A_t u,\) hence

\[
\|B_{1,1} u\|^2_{L^2_b} \leq C - \text{Re} \left( \left( \pm iA_t^*[P_m, A_t] \pm A_t^* Q A_t - \frac{c_0}{8} (AA_t)^* (AA_t) 
- B_{1,1} B_{1,1} - B_{2,1} B_{2,1} + B_{3,1} B_{3,1} \right) u, u \right).
\]
We now exploit the commutator relation (6.43) in the same way as in the proof of Theorem 6.6: If we introduce operators

\[ J^+ \in \Psi_b^{-\sigma-(m-1)/2-1}, \quad J^- \in \Psi_b^{-\sigma+(m-1)/2+1} \]

with real principal symbols \( j^+, j^- \), satisfying \( J^+ J^- = I + \tilde{R}, \tilde{R} \in \Psi_b^{-\infty} \), we obtain, keeping in mind (6.37),

\[
\text{Re} \left( J^- \left( \pm i A^*_1 [P_m, A_t] \pm A^*_t QA_t - \frac{c_0}{\rho} (AA_t)^*(AA_t) \right) \right.
\]

\[
- B^*_1, t B_1, t - B^*_2, t B_2, t + B^*_3, t B_3, t \big) u, (J^*)^u \big) \right)
\]

\[
\geq \text{Re} \left[ \left( \pm a_t H_{p_m}, a_t + \rho^{-m-1} a_t^2 \beta \rho - \frac{c_0}{\rho} \rho^{-m-1} a_t^2 \right) \right.
\]

\[
- b^2_1, t - b^2_2, t + b^2_3, t \big) \big) (z, bD) u, (J^*)^u \big) - C
\]

\[
= \text{Re}((j^- f_t)(z, bD) u, (J^*)^u) + \text{Re}((j^- g_t)(z, bD) u, (J^*)^u) - C,
\]

where we absorbed various error terms in the constant \( C \); see the discussion around equation (6.29) for details. The term involving \( f_t \) is uniformly bounded from below as explained in the proof of Theorem 6.6 after equation (6.29). It remains to bound the term involving \( g_t \). Note that we can write \((j^- g_t)(z, \psi) \) as a sum of terms of the form \( w(z) \psi_1(c)^2 s(z, \zeta) \), where \( w \in H_b^{-1} \), or \( w \in C^\infty \), and \( s \in S^{\sigma+(m-1)/2+1} \), and we can assume

\[
(bS^* M \cap \text{supp} \ s) \cap p_m^{-1}(0) = \emptyset,
\]

since this holds for \( g_t \) in place of \( s \). Thus, on \( bS^* M \cap \text{supp} \ s \), we can use elliptic regularity, Theorem 5.1, to conclude that \( \text{WF}^{\sigma+1}(u) \cap (bS^* M \cap \text{supp} \ s) = \emptyset \); but this implies that

\[
(w\varphi_1^2 s)(z, bD) u \in H_b^{-(m-1)/2}
\]

is uniformly bounded. Therefore, we finally obtain from (6.45) a uniform bound on \( \|B_{1, t} u\|_{L^2} \), which implies \( B_{1, t} u \in L^2 \) and thus the claimed microlocal regularity of \( u \) at \( L_{\pm} \), finishing the proof of the first part of the theorem in the case \( m \geq 1 \), \( r = 0 \).

The proof of the second part is similar, only instead of requiring (6.39), we require

\[
\beta_0 (\sigma - (m-1)/2 + \bar{\beta}) \leq -c_0 < 0
\]
on \( \text{supp} \psi \cap \rho \) \cap \text{supp} \psi_0 \circ \rho_m'' \cap \text{supp} \psi_1 \circ x \), and we correspondingly define

\[
b_{1, t} = \varphi_t \rho^\sigma \psi \psi_0 \psi_1 \left[ -\beta_0 (\sigma - (m-1)/2 - 2 \rho \bar{\rho} + \bar{\beta}) 
\right]
\]

\[
\left[ \pm (\sigma - (m-1)/2 - 2 \rho \bar{\rho} + \bar{\beta}) (\rho^{-1} JH_{p_m} \rho + \frac{c_0}{2}) \right]^{1/2}.
\]

We also redefine

\[
b_{3, t} = \varphi_t \rho^\sigma \psi \psi_0 \psi_1 \left[ \left( \beta \beta_0 \pm x^{-1} JH_{p_m} \rho \right) - \frac{c_1}{4} \right]^{1/2},
\]

\[
f_{1, t} = \varphi_t^2 \rho^2 \bar{\varphi}_{t} \psi_0 \psi_1^2 \left[ \pm (\sigma - (m-1)/2 - 2 \rho \bar{\rho} + \bar{\beta}) (\rho^{-1} (H_{p_m} - JH_{p_m}) \rho + \frac{3c_0}{8}) \right],
\]

\[
f_{3, t} = \varphi_t^2 \rho^2 \bar{\varphi}_{t} \psi_0 \psi_1 \left[ \pm x^{-1} (H_{p_m} - JH_{p_m}) \rho \right] \left( x \rho^{-1} (H_{p_m} - JH_{p_m}) \rho + \frac{c_1}{4} \right).
\]
Equation (6.43) then becomes
\[ a_{t} \mathcal{H}_{p,u} a_{t} + \rho^{m-1}a_{t}^{2}b_{0}\dot{\beta} = \mp \left( \frac{c_{0}}{8} \rho^{m-1}a_{t}^{2} + b^{2}_{1\dot{t}} - b^{2}_{2\dot{t}} + b_{3\dot{t}} + f_{t} + g_{t} \right), \]
and the rest of the proof proceeds as before, the most important difference being that now the term \( b^{2}_{2\dot{t}} \) has an advantageous sign (namely, the same as \( b^{2}_{1\dot{t}} \)), whereas \(-b^{2}_{2\dot{t}}\) does not, which is the reason for the microlocal regularity assumption on \( u \) in a punctured neighborhood of \( L_{\pm} \) within \( b^{*}_{\sigma_{0}} \).

The last step in the proof is to remove the restrictions on \( m \) (the order of the operator) and \( r \) (the growth rate of \( u \) and \( f \)). We accomplish this by rewriting the equation \( Pu = f \) (without restrictions on \( m \) and \( r \)) as
\[ (x^{-r}\mathcal{P}A^{\lambda}x^{r})(x^{-r}\Lambda^{-}u) = x^{-r}f + x^{-r}PRx^{r}(x^{-r}u), \]
where \( \Lambda \in \Psi(T^{m-m_{0}}), m_{0} \geq 2 \), have principal symbols \( \rho^{+(m-m_{0})} \) and satisfy \( \Lambda^{+}\Lambda^{-} = I + R, R \in \Psi_{b}^{\infty} \). Then \( x^{-r}\mathcal{P}A^{+}x^{r} \) has order \( m_{0} \), and, recalling \( \hat{s} = \sigma - m + 1 \),
\[
x^{-r}f \in H^{2}_{b}, \quad x^{-r}\Lambda^{-}u \in H^{2+m_{0}-3/2}_{b},
\]
lie in unweighted b-Sobolev spaces. The principal symbol of \( \hat{P}^{\prime} := x^{-r}(P^{m} + P^{m-1})\Lambda^{+}x^{r} \) is an elliptic multiple of the principal symbol of \( P^{m} = P^{m} + P^{m-1} \), hence the Hamilton vector fields of \( \hat{P}^{\prime} \) and \( P^{m} \) agree, up to a non-vanishing factor, on the characteristic set of \( P^{m} \); in particular, even though \( \beta_{0} \) in equation (6.34) may be different for \( P^{m} \) than for \( P^{m} \), \( \hat{\beta} \) does not change, at least on \( L_{\pm} \). However, the imaginary part of the subprincipal symbol, hence \( \beta \), does change, resulting in a shift of the threshold values in the statement of the theorem: Concretely, we claim
\[
\sigma^{m_{0}-1}_{b}\left( \frac{1}{2i}(\hat{P}^{\prime} - (\hat{P}^{\prime})^{*}) \right) = \pm \rho^{m_{0}-1}b_{0}\left( \beta + \frac{m-m_{0}}{2} - r\hat{\beta} \right) \quad \text{at} \quad L_{\pm}. \tag{6.46}
\]
Granted this, the threshold quantity is the sup, resp. inf, over \( L_{\pm} \) of
\[
\hat{s} + (m_{0} - 1)/2 + \hat{\beta} + (m - m_{0})/2 - r\hat{\beta} = \hat{s} + (m - 1)/2 + \hat{\beta} - r\hat{\beta}.
\]
To prove (6.46), we can assume that \( \Lambda^{+} \) and \( P^{m} \) are (formally) self-adjoint.\(^{42}\) We then compute
\[
\sigma^{m_{0}-1}_{b}\left( \frac{1}{2i}(x^{-r}P^{m-1}\Lambda^{+}x^{r} - x^{r}\Lambda^{+}(P^{m-1})^{*}x^{-r}) \right)
= \rho^{m_{0}-1}\rho^{1-m}\sigma^{m_{0}-1}_{b}\left( \frac{1}{2i}(P^{m-1} - (P^{m-1})^{*}) \right) = \pm \rho^{m_{0}-1}b_{0}\hat{\beta},
\]
and
\[
\sigma^{m_{0}-1}_{b}\left( \frac{1}{2i}(x^{-r}P^{m}\Lambda^{+}x^{r} - x^{r}\Lambda^{+}P^{m}x^{-r}) \right)
= \sigma^{m_{0}-1}_{b}\left( \frac{1}{2i}(P^{m}\Lambda^{+}) \right) + \sigma^{m_{0}-1}_{b}\left( \frac{1}{2i}(x^{-r}[P^{m}\Lambda^{+}, x^{r}] - x^{r}[\Lambda^{+}P^{m}, x^{-r}]) \right)
= \pm \frac{m - m_{0}}{2}\beta_{0}\rho^{m_{0}-1} - r\rho^{m_{0}-1}H_{m_{0}m},
\]
\[
= \pm \left( \frac{m - m_{0}}{2}\beta_{0} - r\hat{\beta}_{0} \right) \rho^{m_{0}-1} - r\rho^{m_{0}-1}H_{m_{0}m}.
\]
\(^{42}\)Indeed, write \( P^{m} = (P^{m} + (P^{m})^{*})/2 + (P^{m} - (P^{m})^{*})/2 \) and absorb \((P^{m} - (P^{m})^{*})/2\) into \( P^{m-1} \).
The last term on the right hand side involving \( p''_m \) vanishes at \( L_{\pm} \), proving (6.46).

Lastly, the regularities needed for the proof to go through are that the conditions in (6.31) hold for some \( m_0 \geq 1 \); thus, choosing \( m_0 = \max(1, 5 - 2s) = 1 + 2(2 - s)_+ \), we obtain the conditions given in the statement of Theorem 6.10. \( \square \)

## 7. Global solvability results for second order hyperbolic operators with non-smooth coefficients

Even though complex absorption is a useful tool to put wave equations on some classes of geometric spaces into a Fredholm framework, as done by Vasy [34] in various dilation-invariant settings, and microlocally easy to deal with, it is problematic in general non dilation-invariant situations if one wants to prove the existence of forward solutions, as pointed out by Vasy and the author [18]. We shall the strategy of [18] and use standard, non-microlocal, energy estimates for wave operators to show the invertibility of the forward problem on sufficiently weighted spaces; using the microlocal regularity results of Sections 5 and 6, we will in fact show higher regularity and the existence of partial expansions of forward solutions.

### 7.1. Energy estimates

Let \((M, g)\) be a compact manifold with boundary e-quipped with a Lorentzian b-metric \( g \) satisfying

\[
g \in C^\infty(M; Sym^2 bTM) + H^s_b(M; Sym^2 bTM) \tag{7.1}
\]

for \( \alpha > 0 \) and some \( s > n/2 + 1 \), where the b-Sobolev space here is defined using an arbitrary fixed smooth b-density on \( M \). Let \( U \subset M \) be open, and suppose \( t: U \to (t_0, t_1) \) is a proper function such that \( dt \) is timelike on \( U \). We consider the operator

\[
P = \square_g + L, \quad L \in (C^\infty + H^1_b)Diff^1_b + (C^\infty + H^1_b)^2.
\]

**Remark 7.1.** Although all arguments in this section are presented for \( P \) and \( \square_g \) acting on functions, the results are true for \( P \) and \( \square_g \) acting on natural vector bundles as well, e.g. the bundle of \( q \)-forms; only minor, mostly notational, changes are needed to verify this.

Since \( s > n/2 \), one obtains using Lemma 4.2 and Corollary 3.10 that in any coordinate system the coefficients \( G^{ij} \) of the dual metric \( G \) are elements of \( C^\infty + H^1_b \), and all Christoffel symbols are elements of \( C^\infty + H^1_b^{-1} \). Therefore, by definition of \( \square_g \), one easily obtains that

\[
\square_g \in (C^\infty + H^1_b)Diff^2_b + (C^\infty + H^1_b^{-1})Diff^1_b,
\]

thus

\[
P \in (C^\infty + H^1_b)Diff^2_b + (C^\infty + H^1_b^{-1})Diff^1_b + (C^\infty + H^1_b^{-2}). \tag{7.2}
\]

**Proposition 7.2.** Let \( t_0 < T_0 < T_0^\dagger < T_1 < t_1 \) and \( r \in \mathbb{R} \), and suppose \( s > n/2 + 2 \). Then there exists a constant \( C > 0 \) such that for all \( u \in H^{2r}_b(M) \), the following estimate holds:

\[
\|u\|_{H^{1-r}_{b}(t^{-1}([T_0, T_1])))} \leq C(|PU|_{H^{2r}_b(t^{-1}([T_0, T_1])))} + \|u\|_{H^{1-r}_{b}(t^{-1}([T_0, T_1])))}.
\]

This also holds with \( P \) replaced by \( P^* \). If one replaces \( C \) by any \( C' > C \), the estimate also holds for small perturbations of \( P \) in the space indicated in (7.2).
Proof. Let us work in a coordinate system $z_1 = x, z_2 = y_1, \ldots, z_n = y_{n-1}$, where $x$ is a boundary defining function in case we are working near the boundary. By piecing together estimates from coordinate patches, one can deduce the full result.

Write $^b\partial_j = \partial_{z_j}$ for $2 \leq j \leq n$, and $^b\partial_1 = x\partial_x$ if we are working near the boundary, $^b\partial_1 = \partial_x$ otherwise. Moreover, let us fix the Riemannian $b$-metric
\[ \tilde{g} = \frac{dx^2}{x^2} + dy^2 \]
near the boundary, $\tilde{g} = dx^2 + dy^2$ away from it. We adopt the summation convention in this proof.

We will imitate the proof of [34, Proposition 3.8], which proves a similar result in a smooth, semiclassical setting. Thus, consider the commutant $V = -iZ$, where $Z = x^{-2r} \chi(t) W$ with $\chi \in C^\infty(\mathbb{R})$, chosen later in the proof, and $W = G(-, b^l t)$, which is timelike in $U$. We will compute the ‘commutator’
\[ -i(V^*P - P^*V) = -i(V^*\Box_g - \Box_b^* V) - iV^*L + iL^*V, \]
(7.3)
where the adjoints are taken with respect to the (b-)metric $\tilde{g}$. First, we need to make sense of all appearing operator compositions. Notice that $V^* = -i^b \partial_j Z^j = V - i(\partial^b_j Z^j) \in x^{-2r}(C^\infty + H_b^s)\text{Diff}_b^1$, and writing $V = -iZ^j \partial_j$, we get
\[ V^* = -i^b \partial_j Z^j = V - i(\partial^b_j Z^j) \in x^{-2r}(C^\infty + H_b^s)\text{Diff}_b^1 + x^{-2r}(C^\infty + H_b^{s-1}), \]
similarly
\[ \Box_g, \Box_b^* \in (C^\infty + H_b^s)\text{Diff}_b^2 + (C^\infty + H_b^{s-1})\text{Diff}_b^1 + (C^\infty + H_b^{s-2}); \]
now, since
\[ (C^\infty + H_b^{s-j})\text{Diff}_b^i (C^\infty + H_b^{s-k})\text{Diff}_b^j \subseteq \sum_{i \leq j} (C^\infty + H_b^{s-j}H_b^{s-k-j})\text{Diff}_b^{i+j-k}, \]
it suffices to require $s > n/2 + 2$, since then $H_b^{s-j}H_b^{s-k-j} \subseteq H_b^{s-k-j}$ for $0 \leq j, k \leq 2$, $0 \leq j + k \leq 3$.

Returning to the computation of (7.3), we conclude that $-i(V^*\Box_g - \Box_g^* V) \in (C^\infty + H_b^{s-3})\text{Diff}_b^1$, and thus its principal symbol is defined. Since it is a formally self-adjoint (with respect to $\tilde{g}$) operator with real coefficients that vanishes on constants, it equals $^b\partial^* C^b_0 l$ provided the principal symbols are equal. To compute it, let us write
\[ -i(V^*\Box_g - \Box_g^* V) = -(^b\partial_k Z^k)\Box_g + i[\Box_g, V] - i(\Box_g - \Box_g^*)V. \]
We define $S^i \in C^\infty + H_b^{s-1}$ by
\[ \sigma_b^2(-i(\Box_g - \Box_g^*) V) = 2S^i Z^i \zeta_i \zeta_j = (S^i Z^j + S^j Z^i) \zeta_i \zeta_j. \]
Moreover, with $H_G$ denoting the Hamilton vector field of the dual metric of $g$,
\[ H_G = G^{ij} \zeta_i \zeta_j + G^{ij} \zeta_i \partial_k - (\partial_k G^{ij}) \zeta_i \zeta_j \partial_k, \]
we find $\sigma_b^2(-i(V^*\Box_g - \Box_g^* V)) = B_{ij} \zeta_i \zeta_j$ with
\[ B_{ij} = -^b\partial_k (Z^k G^{ij}) + G^{ik} (^b\partial_k Z^j) + G^{jk} (^b\partial_k Z^i) + S^i Z^j + S^j Z^i \in x^{-2r}(C^\infty + H_b^{s-1}), \]

\[ \text{See Vasy [33] for a similar computation.} \]
for

\[ \omega \]

Notice here that for a b-1-form \( B^{ij} \), the idea being that the remaining terms, considered error terms, can then be dominated by choosing \( \chi' \) large compared to \( \chi \). We get

\[
B^{ij} = x^{-2r}(b_0^b \chi)(G^{ik} W^j + G^{jk} W^i - G^{ij} W^k) \\
+ \chi(G^{ik}(b_0^b x^{-2r} W^j) + G^{jk}(b_0^b x^{-2r} W^i)) \\
- \frac{1}{2} \partial_b(x^{-2r}W^i G^{ij}) + x^{-2r}(S^i W^j + S^j W^i)).
\]

We obtain

\[
E_{W;\dot{a}b} (\omega) := \frac{1}{2} (b_0^b \partial_\chi)(G^{ik} W^j + G^{jk} W^i - G^{ij} W^k) \omega_i \omega_j \\
= \frac{1}{2} (\omega, b_0^b d\chi) G \omega(W) + \omega(W) (b_0^b d\chi, \omega)_G - \frac{1}{2} \partial_b(W)(\omega, \omega)_G \\
= \chi' (t) E_{W;\dot{a}b} (\omega)
\]

is related to the sesquilinear energy-momentum tensor

\[
E_{X,Y} (\omega) = \text{Re}(\omega(X) \omega(Y)) - \frac{1}{2} (X,Y) (\omega, \omega)_G,
\]

where \( (\cdot, \cdot)_G \) is the sesquilinear inner product on \( \mathbb{C}^b T^* M \). This quantity, rewritten in terms of b-vector fields as

\[
E_{X,Y} (\omega) = \text{Re}(\omega(X) \omega(Y)) - \frac{1}{2} (X,Y) (\omega, \omega)_G,
\]

is well-known to be positive definite provided \( X \) and \( Y \) are both future (or both past) timelike;\(^{44}\) in our setting, we thus have \( E_{W;\dot{a}b} = E_{W,W} > 0 \) by our definition of \( W \). Correspondingly,

\[
C = x^{-2r} \chi' A + x^{-2r} \chi R
\]

with \( A \) positive definite and \( R \) symmetric.

We obtain\(^{45}\)

\[
\langle -i(V^* P - P^* V) u, u \rangle = \langle C^b du, b^b du \rangle - \langle i Lu, V u \rangle + \langle i V u, Lu \rangle.
\]

\(^44\) Here is a short proof, following Alinhac [1]: Choose a null frame \( L, L_1, E_1, \ldots, E_n \), i.e. \( g(L, L) = g(L, L) = 0 \), \( g(L, L) = 1 \), and the \( E_i \) are an orthonormal basis of the orthogonal complement of \( L, L \). We can then write \( X = a L + b_1 L_1 Y = a' L + b' L_1 \) for \( a, a', b, b' > 0 \) (where, if necessary, we replace \( L \) and \( L \) by \( -L \) and \( -L ) \), and compute \( g(X, Y) = 2(ab + a'b) \). Moreover, let \( \omega_L = \omega(L), \omega_{L_1} = \omega(L), \omega_j = \omega(E_j) \), then one easily computes \( (\omega, \omega)_G = \text{Re}(\omega_L \omega_{L_1} - \sum_j |\omega_j|^2) \); therefore

\[
E_{X,Y} (\omega) = \text{Re}((a \omega_L + b \omega_{L_1})(a' \omega_L + b' \omega_{L_1})) - \frac{1}{2} g(X, Y)(\omega, \omega)_G \\
= aa' |\omega_L|^2 + bb' |\omega_{L_1}|^2 + (ab + a'b) \sum_j |\omega_j|^2 > 0.
\]

\(^45\) The integrations by parts here and further below are readily justified using \( s > n/2 + 2 \).

In fact, since we are assuming \( u \in H_b^{s+2} \), we have \( Vu \in H_b^{s-2} \) for \( s > n/2, s \geq 1 \), and then \( P^* Vu \in H_b^{s-1, r} \) provided multiplication with an \( H_b^{s-2} \) function is continuous \( H_b^{s-1, r} \) for \( j = 0, 1, 2 \), which is true for \( s > n/2 + 1 \); similarly, one has \( Pu \in H_b^{s-2, r} \) provided \( s > n/2, 2 \), and then \( V^* Pu \in H_b^{s-1, r} \) if multiplication by an \( H_b^{s-2} \) function is continuous \( H_b^{s-1} \) for \( j = 0, 1, 2 \), which holds for \( s > n/2, s \geq 1 \).
We now finish the proof by making \( \chi' \) large compared to \( \chi \) on \( t^{-1}([T'_0, T'_1]) \), as follows: Pick \( T'_1 \in (T_1, t_1) \) and let
\[
\hat{\chi}(s) = \hat{\chi}_1 \left( \frac{s - T_0}{T_0 - T_0} \right) \chi_0(-F^{-1}(s - T'_1)), \quad \chi(s) = \hat{\chi}(s) H(T_1 - s),
\]
where \( H \) is the Heaviside step function, \( \chi_0(s) = e^{-1/s} H(s) \in C^\infty(\mathbb{R}) \) (which satisfies \( \chi'_0(s) = s^{-2} \chi_0(s) \)) and \( \hat{\chi}_1 \in C^\infty(\mathbb{R}) \) equals 0 on \((-\infty, 0]\) and 1 on \([1, \infty)\); see Figure 2.

\[\text{Figure 2. Graph of the commutant } \chi. \text{ The dashed line is the graph of the part of } \hat{\chi} \text{ that is cut off using the Heaviside function in the definition of } \chi.\]

Then in \( (T'_0, T'_1) \),
\[
\chi'(s) = -F^{-1}(s - T'_1)H(T_1 - s) - \chi_0(-F^{-1}(T_1 - T'_1))\delta_{T_1}
= -F(s - T'_1)\chi(s) - \chi_0(-F^{-1}(T_1 - T'_1))\delta_{T_1},
\]
in particular \( \chi(s) = -F^{-1}(s - T'_1)^2 \chi'(s) \) on \( (T'_0, T'_1) \); hence any \( \gamma > 0 \), we can choose \( F > 0 \) so large that \( \chi \leq -\gamma \chi' \) on \( (T'_0, T'_1) \); therefore
\[
-\langle \chi'A + \chi R \rangle \geq \frac{1}{2} \chi' \hat{\chi}_1 A \text{ on } (T'_0, T'_1).
\]
Put \( \chi_1(s) = \hat{\chi}_1(s) H(T_1 - s) \), then
\[
-C^b du, b du \geq \frac{1}{2} (x^{-2r}(-\chi' \chi_1) A^b du, b du)
+ \chi_0(-F^{-1}(T_1 - T'_1))(x^{-2r} A \delta_{T_1} b du, b du) - C''\|b du\|_{H^r_{K_0}}^2 (t^{-1}((T_0, T'_0)))
\]
and the term on the right hand side involving \( \delta_{T_1} \) is positive, thus can be dropped. Hence, using equation (7.6) and the positivity of \( A \),
\[
c_0 \sqrt{-\chi' \chi_1^b du} \|b du\|_{H^r_{K_0}}^2 \leq \frac{1}{2} (x^{-2r}(-\chi' \chi_1) A^b du, b du)
\leq C''\|b du\|_{H^r_{K_0}}^2 (t^{-1}((T_0, T'_0))) + C''\|1/2 Pu\|_{H^r_{K_0}} \|1/2 b du\|_{H^r_{K_0}}
+ C''\|1/2 b du\|_{H^r_{K_0}}^2 + C''\|1/2 b du\|_{H^r_{K_0}} \|1/2 u\|_{H^r_{K_0}}
\leq C'''\|u\|_{H^r_{K_0}}^2 (t^{-1}((T_0, T'_0))) + C''\|1/2 Pu\|_{H^r_{K_0}}^2 + C''\gamma \sqrt{-\chi' \chi_1^b du} \|b du\|_{H^r_{K_0}}^2 \quad (7.7)
+ C'\gamma \sqrt{-\chi' \chi_1^b du} \|b du\|_{H^r_{K_0}}^2,
\]
where the norms are on \( t^{-1}([T_0, T_1]) \) unless otherwise specified. Choosing \( F \) large and thus \( \gamma \) small allows us to absorb the second to last term on the right into the
left hand side. To finish the proof, we need to treat the last term, as follows: We compute, using $Wx = \chi G(x^2t, bdt) = m\chi'$ with $m \in C^\infty + H^s_0$ positive,

$$
\langle (W^* x^{-2r} + x^{-2r} \chi W)u, u \rangle
$$

$$
= -\langle (Wx^{-2r} \chi - x^{-2r} \chi W)u, u \rangle - \langle (\nabla_3 W)x^{-2r} \chi u, u \rangle
$$

$$
\geq -\langle x^{-2r} m\chi' \chi u, u \rangle H^s_{0, r}(t^{-1}([T_0, T_1])) - \langle x^{-2r} w\chi u, u \rangle - \langle (\nabla_3 W)x^{-2r} \chi u, u \rangle
$$

$$
\geq \sqrt{-\chi' \chi m^{1/2} u}\|H^s_{0, r}(t^{-1}([T_0, T_1]))\| - \sqrt{|\gamma|} \|m^{1/2} u\|_{H^s_{0, r}(t^{-1}([T_0, T_1]))}^2
$$

$$
- C\sqrt{|\gamma|} \|u\|_{H^s_{0, r}(t^{-1}([T_0, T_1]))}^2
$$

where $w = x^{2r} Wx^{-2r} \in C^\infty + H^s_0$. Similarly as above, we now choose $F$ large to obtain

$$
\|\sqrt{-\chi' \chi^2 u}\|_{H^s_{0, r}(t^{-1}([T_0, T_1]))}^2 \leq C\sqrt{\|\chi\|_{L^2}^2} + C\sqrt{|\gamma|} \|u\|_{H^s_{0, r}(t^{-1}([T_0, T_1]))}^2.
$$

Using this estimate in (7.7) and absorbing one of the resulting terms, namely $\gamma\sqrt{\|\chi\|_{L^2}^2} \|u\|_{H^s_{0, r}(t^{-1}([T_0, T_1]))}^2$, into the left hand side finishes the proof of the estimate, since $\sqrt{-\chi' \chi m^{1/2} u}$ has a positive lower bound on $t^{-1}([T_0, T_1])$.

That the estimate holds for perturbations of $P$ follows simply from the observation that all constants in this proof depend on finitely many seminorms of the coefficients of $P$, hence the constants only change by small amounts if one makes a small perturbation of $P$. $\square$

7.2. Analytic, geometric and dynamical assumptions on non-smooth linear problems. The arguments of the first half of [18, §2.1.3] leading to a Fredholm framework for the forward problem for certain $P$, e.g. wave operators on non-smooth perturbations of the static model of de Sitter space, now go through with only minor technical modifications. Because there are large dimension-dependent losses in estimates for the adjoint of $P$ relative to the regularity of the coefficients of $P$, say $C^\infty + H^s_0$ for the highest order ones, the spaces that $P$ acts on as a Fredholm operator are roughly of the order $s - n/2$.

This can be vastly improved with a calculus for right quantizations of non-smooth symbols just like the one developed in this paper for left quantizations. Right quantizations have ‘good’ mapping properties on negative order (but lossy ones on positive order) $b$-Sobolev spaces. Correspondingly, all microlocal results (elliptic regularity, propagation of singularities, including at radial points) hold by the same proofs mutatis mutandis. Then, viewing $P^*$ as the right quantization of a non-smooth symbol gives estimates which allow one to put $P$ into a Fredholm framework on spaces with regularity $s - \epsilon$, $\epsilon > 0$.

Our focus here however is to prove the invertibility of the forward problem, whose discussion in the second half of [18, §2.1.3] (in the smooth setting) we follow.

Let us from now assume that the operator

$$
P = \Box_g + L, \quad L \in (C^\infty + H^{-1, r}_0)\text{Diff}_b + (C^\infty + H^{-1, r}_0),
$$

with $\alpha > 0$, and $g$ now satisfying

$$
g \in C^\infty(M; \text{Sym}^2 TM) + H^{\alpha, \alpha}_0(M; \text{Sym}^2 bTM),
$$

is such that.\(^{46}\)

\(^{46}\)An example to keep in mind for the remainder of the section is the wave operator on a perturbed static asymptotically de Sitter space.
(1) $P$ satisfies the dynamical assumptions of Theorem 6.10, i.e. has the indicated radial point structure. Let $L_{\pm}$, $\bar{\beta}$, $\hat{\beta}$ be defined as in the statement of Theorem 6.10.

(2) $P \in (C^\infty + H_0^{s,\alpha})^{\text{Diff}}_b + (C^\infty + H_0^{s-1,\alpha})^{\text{Diff}}_b + (C^\infty + H_0^{s-1,\alpha})$; note that the regularity of the lowest order term is higher than what we assumed before.

(3) The characteristic set $\Sigma$ of $P$ has the form $\Sigma = \Sigma_+ \cup \Sigma_-$ with $\Sigma_{\pm}$ a union of connected components of $\Sigma$, and $L_{\pm} \subset \Sigma_{\pm}$.

We denote by $t_1$ and $t_2$ two smooth functions on $M$ and put for $\delta_1, \delta_2$ small

$$\Omega_{\delta_1, \delta_2} := t_1^{-1}(\delta_1, \infty) \cap t_2^{-1}(\delta_2, \infty), \quad \Omega = \Omega_{0,0},$$

$$\Omega_{\delta_1, \delta_2}^2 := t_1^{-1}(\delta_1, \infty) \cap t_2^{-1}(\delta_2, \infty),$$

where we assume that:

(4) The differentials of $t_1$ and $t_2$ have the opposite timelike character near their respective zero sets within $\Omega_0$, more specifically, $t_1$ is future timelike, $t_2$ past timelike,

(5) putting $H_j := t_j^{-1}(0)$, the $H_j$ intersect the boundary $\partial M$ transversally, and $H_1$ and $H_2$ intersect only in the interior of $M$, and they do so transversally,

(6) $\Omega_{\delta_1, \delta_2}$ is compact.

Let us make two additional assumptions:

(7) Assume that there is a boundary defining function $x$ of $M$ such that $dx/x$ is timelike on $\Omega \cap \partial M$ with timelike character opposite to the one of $t_1$, i.e. $dx/x$ is past oriented.

(8) The metric $g$ is non-trapping in the following sense: All bicharacteristics in $\Sigma_\Omega := \Sigma \cap bS_\Omega^* M$ from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ flow (within $\Sigma_\Omega$) to $bS_{H_1}^* M \cup L_+$ in the forward direction (i.e. either enter $bS_{H_1}^* M$ in finite time or tend to $L_+$) and to $bS_{H_2}^* M \cup L_+$ in the backward direction, and from any point in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ to $bS_{H_2}^* M \cup L_-$ in the forward direction and to $bS_{H_1}^* M \cup L_-$ in the backward direction.

See Figure 3 for the setup.

**Figure 3.** The domain $\Omega$ on which we have a global energy estimate as well as solvability and uniqueness on appropriate weighted b-Sobolev spaces. The ‘artificial’ spacelike boundary hypersurfaces $H_1$ and $H_2$ are also indicated.

Conditions (1) and (8) are (probably) not stable under perturbations of $P$, and it will in fact be crucial later that they can be relaxed. Namely, we do not need to
Lemma 7.5. (solvability and higher regularity:)

Denote by $H^{s,r}_b(\Omega_{\delta_1,\delta_2})^{-\bullet}$ distributions which are supported ($\bullet$) at the ‘artificial’ boundary hypersurface $t^{-1}_1(\delta_1)$ and extendible ($-$) at $t^{-1}_2(\delta_2)$, and the other way around for $H^{s,r}_b(\Omega_{\delta_1,\delta_2})^{-\bullet}$. Then we have the following global energy estimate:

**Lemma 7.3.** (Cf. [18, Lemma 2.15].) Suppose $s > n/2 + 2$. There exists $r_0 > 0$ such that for $r \leq r_0$, $\tilde{r} \leq r_0$, there is $C > 0$ such that for $u \in H^{s,r}_b(\Omega_{\delta_1,\delta_2})^{-\bullet}$, $v \in H^{2,\tilde{r}}_b(\Omega_{\delta_1,\delta_2})^{-\bullet}$, one has

\[
\|u\| H^{s,r}_b(\Omega_{\delta_1,\delta_2})^{-\bullet} \leq C \|Pu\| H^{0,r}_b(\Omega_{\delta_1,\delta_2})^{-\bullet},
\]

\[
\|v\| H^{2,\tilde{r}}_b(\Omega_{\delta_1,\delta_2})^{-\bullet} \leq C \|P^*v\| H^{0,\tilde{r}}_b(\Omega_{\delta_1,\delta_2})^{-\bullet}.
\]

If one replaces $C$ by any $C' > C$, the estimates also hold for small perturbations of $P$ in the space indicated in assumption (2).

**Proof.** The proof uses [18, Lemma 2.4], adapted to the non-smooth setting as in Proposition 7.2, and then follows the proof of [18, Lemma 2.15], the point being that the terms in (7.4) with $x^{-2r}$ differentiated and thus possessing a factor of $r$ can be used to dominate the other, ‘error’, terms in (7.5). \qed

**Remark 7.4.** For this lemma we in fact only need to assume conditions (4)-(7).

By a duality argument and the propagation of singularities, we thus obtain solvability and higher regularity:

**Lemma 7.5.** (Cf. [18, Corollaries 2.10 and 2.16].) Let $0 \leq s' \leq s$ and assume $s > n/2 + 6$. There exists $r_0 > 0$ such that for $r \leq r_0$, there is $C > 0$ with the following property: If $f \in H^{s'-1,r}_b(\Omega)^{-\bullet}$, then there exists a unique $u \in H^{s',r}_b(\Omega)^{-\bullet}$ such that $Pu = f$, and $u$ moreover satisfies

\[
\|u\| H^{s',r}_b(\Omega)^{-\bullet} \leq C \|f\| H^{s'-1,r}_b(\Omega)^{-\bullet}.
\]

If one replaces $C$ by any $C' > C$, this result also holds for small perturbations of $P$ in the space indicated in assumption (2).

**Proof.** We follow the proof of [18, Corollary 2.10]. Choose $\delta_1 < 0$ and $\delta_2 < 0$ small, and choose an extension

\[
\tilde{f} \in H^{s'-1,r}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet} \subset H^{-1,r}_b(\Omega_{0,\bar{\delta}_2})^{-\bullet}
\]

satisfying

\[
\|\tilde{f}\| H^{s'-1,r}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet} \leq 2 \|f\| H^{s'-1,r}_b(\Omega)^{-\bullet}. \tag{7.8}
\]

By Lemma 7.3, applied with $\tilde{r} = -r$, we have

\[
\|\phi\| H^{1,\tilde{r}}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet} \leq C \|P^*\phi\| H^{0,\tilde{r}}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet}.
\]

for $\phi \in H^{2,\tilde{r}}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet}$. Therefore, by the Hahn-Banach theorem, there exists $\tilde{u} \in H^{0,\tilde{r}}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet}$ such that

\[
\langle P\tilde{u}, \phi \rangle = \langle \tilde{u}, P^*\phi \rangle = \langle f, \phi \rangle, \quad \phi \in H^{2,\tilde{r}}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet},
\]

and

\[
\|\tilde{u}\| H^{0,\tilde{r}}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet} \leq C \|\tilde{f}\| H^{s'-1,r}_b(\Omega_{\delta_0,\bar{\delta}_2})^{-\bullet}. \tag{7.9}
\]
We can view $\tilde{u}$ as an element of $H^0_b(\Omega_{s_1}, s_2)$ with support in $\Omega_{0, s_2}$, similarly for $\tilde{f}$; then $(P\tilde{u}, \phi) = (\tilde{f}, \phi)$ for all $\phi \in C^\infty_c(\Omega_{0, s_2})$ (with the dot referring to infinite order of vanishing at $\partial M$), i.e. $P\tilde{u} = \tilde{f}$ in distributions on $\Omega_{s_1, s_2}$.

Now, $\tilde{u}$ vanishes on $\Omega_{s_1, s_2} \setminus \Omega_{0, s_2}$, in particular is in $H^1_{b, \text{loc}}(\Omega_{s_1, s_2})$. Elliptic regularity and the propagation of singularities, Theorems 5.1, 6.6 and 6.10, imply that $\tilde{u} \in H^1_{b, \text{loc}}(\Omega_{s_1, s_2})$. Indeed, by Theorem 5.1 with $s = -1$, $\tilde{u}$ is in $H^{1/2}_b$ on the elliptic set of $P$ within $\Omega_{s_1, s_2}$; Theorem 6.6 with $\tilde{s} = -1/2$ gives $H^{1/2}_b$-control of $\tilde{u}$ on the characteristic set away from radial points, and then an application of Theorem 6.10 gives $H^{1/2}_b$-control of $\tilde{u}$ on all of $\Omega_{s_1, s_2}$. Iterating this argument gives $H^s_{b, \text{loc}}(\Omega_{s_1, s_2})$, and we in fact get an estimate

$$\|\chi\|_{H^s_{b, \text{loc}}(\Omega_{s_1, s_2})} \leq C\left(\|\tilde{P}\tilde{u}\|_{H^{s-1, r}_b(\Omega_{s_1, s_2})} + \|\tilde{\chi}u\|_{H^s_{b, \text{loc}}(\Omega_{s_1, s_2})}\right)$$

for appropriate $\chi, \tilde{\chi} \in C^\infty_c(\Omega_{s_1, s_2})$, $\chi \equiv 1$ on supp $\chi$. In view of the support properties of $\tilde{u}$, an appropriate choice of $\chi$ and $\tilde{\chi}$ gives that the restriction of $\tilde{u}$ to $\Omega$ is an element of $H^s_{b, \text{loc}}(\Omega)$, with norm bounded by $H^s_{b, \text{loc}}(\Omega)$-norm of $\tilde{f}$ in view of (7.9) and (7.8).

To prove uniqueness, suppose $u \in H^s_{b, \text{loc}}(\Omega)$ satisfies $Pu = 0$, then viewing $u$ as a distribution on $\Omega_{s_1, 0}$ with support in $\Omega$, elliptic regularity and the propagation of singularities, applied as above, give $u \in H^s_{b, \text{loc}}(\Omega_{s_1, 0}) \subset H^2_{b, \text{loc}}(\Omega_{s_1, 0})$; hence, for any $\tilde{\delta} > 0$, Lemma 7.3 applied to $u' = u|_{s_1 - \tilde{\delta}} \in H^s_{b, \text{loc}}(\Omega_{s_1, s_2})$ gives $u' = 0$, thus, since $\tilde{\delta} > 0$ is arbitrary, $u = 0$.

**Corollary 7.6.** (Cf. [18, Corollary 2.17].) Let $0 \leq s' \leq s$ and assume $s > n/2 + 6$. There exists $\varepsilon_0 > 0$ such that for $r \leq \varepsilon_0$, there is $C > 0$ with the following property: If $u \in H^{s, r}_b(\Omega)$ is such that $Pu \in H^{s-1, r}_b(\Omega)$, then

$$\|u\|_{H^{s, r}_b(\Omega)} \leq C\|Pu\|_{H^{s-1, r}_b(\Omega)}.$$

If one replaces $C$ by any $C' > C$, this result also holds for small perturbations of $Pu$ in the space indicated in assumption (2).

**Proof.** Let $u' \in H^{s, r}_b(\Omega)$ be the solution of $Pu' = Pu$ given by the existence part Lemma 7.5, then $P(u - u') = 0$, and the uniqueness part implies $u = u'$.

We also obtain the following propagation of singularities type result:

**Corollary 7.7.** Let $0 \leq s'' \leq s' \leq s$ and assume $s > n/2 + 6$; moreover, let $r \in \mathbb{R}$ be such that $s'' - 1 + \inf_{t < 0} (\beta - r\beta) > 0$. Then there is $C > 0$ such that the following holds: Any $u \in H^{s'', r}_b(\Omega)$ with $Pu \in H^{s-1, r}_b(\Omega)$ in fact satisfies $u \in H^{s', r}_b(\Omega)$, and obeys the estimate

$$\|u\|_{H^{s', r}_b(\Omega)} \leq C\|Pu\|_{H^{s-1, r}_b(\Omega)} + \|u\|_{H^{s'', r}_b(\Omega)}.$$

If one replaces $C$ by any $C' > C$, this result also holds for small perturbations of $Pu$ in the space indicated in assumption (2).

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47 The conditions of all theorems used here are satisfied because of $s > n/2 + 6$; if necessary, we need to make $r_0$ smaller, i.e. assume that $r \leq r_0$ is more negative, in order for the assumptions of Theorem 6.10 to be fulfilled.
Proof: As in the proof of Lemma 7.5, working on $\Omega_{\delta_1,0}$ for $\delta_1 < 0$ small, we obtain $u \in H^{s',r}_{b,\text{loc}}$ by iteratively using elliptic regularity, real principal type propagation and the propagation near radial points; the latter, applied in the first step with $\tilde{s} = s' - 1/2$, is the reason for the condition on $s'$. Thus, $u \in H^{s',r}_{b}(\Omega_{0,\delta})^*^-$ for $\delta > 0$. From here, arguing as in the proof of Proposition 2.13 in [18], we obtain the desired conclusion. 

Let us rephrase Lemma 7.5 and Corollary 7.6 as an invertibility statement:

**Theorem 7.8.** (Cf. [18, Theorem 2.18].) Let $0 \leq s' \leq s$ and assume $s > n/2 + 6$. There exists $r_0 < 0$ with the following property: Let $r \leq r_0$ and define the spaces

$$X^{s',r} = \{ u \in H^{s',r}_b(\Omega)^*^- : Pu \in H^{s-1,r}_b(\Omega)^*^- \}, \quad \mathcal{Y}^{s',r} = H^{s',r}_b(\Omega)^*^-.$$

Then $P: X^{s',r} \rightarrow \mathcal{Y}^{s-1,r}$ is a continuous, invertible map with continuous inverse.

Moreover, the operator norm of the inverse, as a map from $H^{s-1,r}_b(\Omega)^*^-$ to $H^{s',r}_b(\Omega)^*^-$, of small perturbations of $P$ in the space indicated in assumption (2) is uniformly bounded.

We can now apply the arguments of [18], see also [34] for the dilation-invariant case, to obtain more precise asymptotics of solutions $u$ to $Pu = f$ using the knowledge of poles of the inverse of the Mellin transformed normal operator family $\hat{P}(\sigma)$, where the normal operator $N(P)$ of $P$ is defined just as in the smooth setting by ‘freezing’ the coefficients of $P$ at the boundary $\partial M$. This makes sense in our setting since the coefficients of $P$ are continuous; also, the coefficients of $N(P)$ are then smooth, since all non-smooth contributions to $P$ vanish at the boundary.

**Theorem 7.9.** (Cf. [18, Theorem 2.20].) Let $s > n/2 + 6$, $0 < \alpha \leq 1$, and assume $g \in \mathcal{C}^\infty(\mathbb{M}; \text{Sym}^2 \mathbb{L}TM) + H^s_{\text{b,loc}}(\mathbb{M}; \text{Sym}^2 \mathbb{L}TM)$. Let

$$P = \Box_g + L, \quad L \in (\mathcal{C}^\infty + H^{s-1}_0(\mathbb{M}))\text{Diff}_0^1 + (\mathcal{C}^\infty + H^{s-1}_0(\mathbb{M})).$$

Further, let $t_1$ and $\Omega \subset \mathbb{M}$ be as above, and suppose $P$, $\Omega$ and $g$ satisfy the assumptions (1)-(8) above. Let $\sigma_j$ be the poles of $P^{-1}(\sigma)$, of which there are only finitely many in any half space $\text{Im} \sigma \geq -C$ by assumption (7).\footnote{See [34, §7] for an explanation.} Let $r \in \mathbb{R}$ be such that $r \neq \text{Im} \sigma_j$ and $r < -\text{Im} \sigma_j + \alpha$ for all $j$, and let $r_0 \in \mathbb{R}$. Moreover, let $1 \leq s_0 \leq s' \leq s$, and suppose that

$$s' - 2 + \inf_{L^\pm} (\tilde{\beta} - r\tilde{\beta}) > 0.$$

Finally, let $\phi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $\text{supp} \phi \subset (0, \infty)$ and $\phi \circ t_1 \equiv 1$ near $\partial M \cap \Omega$.

Then any solution $u \in H^{s,0}_b(\Omega)^*^-$ of $Pu = f$ with $f \in H^{s'-1,r}_b(\Omega)^*^-$ satisfies

$$u - \sum_j x^{i\sigma_j}(\phi \circ t_1)a_j = u' \in H^{s',r}_b(\Omega)^*^-$$

for some $a_j \in \mathcal{C}^\infty(\partial M \cap \Omega)$, where the sum is understood over the finite set of $j$ such that $-\text{Im} \sigma_j < r < -\text{Im} \sigma_j + \alpha$.

The result is stable under small perturbations of $P$ in the space indicated in assumption (2) in the sense that, even though the $\sigma_j$ might change, all $\mathcal{C}^\infty$-seminorms of the expansion terms $a_j$ and the $H^{s',r}_b(\Omega)^*^-$-norm of the remainder term $u'$ are
bounded by $C(\|u\|_{H^{s_0}_{b}\cdot\cdot\cdot} + \|f\|_{H^{s_{0} - 1, r}_{b}\cdot\cdot\cdot})$ for some uniform constant $C$ (depending on which norm we are bounding).

Proof. By making $r_0$ smaller (i.e. more negative) if necessary, we may assume that $r_0 \leq r$ and

$$s_0 - 1 + \inf_{L^2} (\hat{\beta} - r_0 \hat{\beta}) > 0.$$ 

First, assume $\sigma_* := \min_j \{- \Im \sigma_j\} > r$. Then $u \in H^{s_0, r_0}_{b}(\Omega)^{-}$ and $P u = f \in H^{s_{0} - 1, r}_{b}(\Omega)^{-}$ by Corollary 7.7. Since

$$P - N(P) \in (x C^\infty + H^{s, \alpha}_{b}) \mathrm{Diff}_{b}^2 + (x C^\infty + H^{s_{0} - 1, \alpha}_{b}) \mathrm{Diff}_{b}^1 + (x C^\infty + H^{s_{0} - 1, \alpha}_{b}),$$

we thus obtain $\hat{f} := (P - N(P)) u \in H^{s', r_{0} + \alpha}_{b}(\Omega)^{-}$, where we use $s' = 1 - 2$ and $s - 1 \geq s' - 1$; hence

$$N(P) u = f - \hat{f} \in H^{s_{0} - 2, r'}_{b}(\Omega)^{-}$$

with $r' = \min(r, r_0 + \alpha)$. Applying $^{49}$ [34, Lemma 3.1] gives $u \in H^{s'_{0} - 1, r'}_{b}(\Omega)^{-}$ in view of the absence of poles of $\hat{P}(\sigma)$ in $\Im \sigma \geq -r$; but then $P u \in H^{s'_{0} - 1, r}_{b}(\Omega)^{-}$ implies $u \in H^{s'_{0} - r'}_{b}(\Omega)^{-}$, again by Corollary 7.7, where we use

$$(s' - 1) - 1 + \inf (\hat{\beta} - r \hat{\beta}) \geq s' - 2 + \inf (\hat{\beta} - r \hat{\beta}) > 0.$$ 

If $r' = r$, we are done; otherwise, we iterate, replacing $r_0$ by $r_0 + \alpha$, and obtain $u \in H^{s', r}_{b}(\Omega)^{-}$ after finitely many steps.

If there are $\sigma_j$ with $- \Im \sigma_j < r$, then, assuming that $\sigma_* - \alpha < r_0 < \sigma_*$, in fact that $r_0$ is arbitrarily close to $\sigma_*$, as we may by the first part of the proof, the application of [34, Lemma 3.1] gives a partial expansion $u_1$ or $u$ with remainder $u' \in H^{s'_{0} - 1, r'}_{b}(\Omega)^{-}$, where $r' = \min(r, r_0 + \alpha)$; in fact, we may need to decrease $r'$ by an arbitrarily small amount to ensure that it does not equal $- \Im \sigma_j$ for some $j$. Now $N(P) u_1 = 0$ and $u_1 \in H^{\infty, r_0}_{b}(\Omega)^{-}$, thus

$$(P - N(P)) u_1 \in H^{s_{0} + 1, r_0}_{b}(\Omega)^{-} + H^{s_{0} - 1, r_0 + \alpha}_{b}(\Omega)^{-} \subset H^{s_{0} - 1, r'}_{b}(\Omega)^{-},$$

where the two terms correspond to the coefficients of $P - N(P)$ being sums of $x C^\infty$- and $H^{s_{0} - 1, \alpha}_{b}$-functions. Therefore,

$$P u' = P u - N(P) u_1 - (P - N(P)) u_1 \in H^{s'_{0} - 1, r'}_{b}(\Omega)^{-},$$

which by Corollary 7.7 implies $u' \in H^{s', r}_{b}(\Omega)^{-}$. \□

Remark 7.10. In the smooth setting, one can use the partial expansion $u_1$ to obtain better information on $\hat{f}$ for a next step in the iteration. This however relies on the fact that $P - N(P) \in x \mathrm{Diff}^2$ there (see the proof of [18, Theorem 2.20]); here, however, we also have terms in the space $H^{s_{0} - 1, \alpha}_{b} \mathrm{Diff}^2_{b} \subset P - N(P)$, and $H^{s_{0} - 1, \alpha}_{b}$-functions do not have a Taylor expansion at $x = 0$, hence the above iteration scheme does not yield additional information after the first step in which one gets a non-trivial part $u_1$ of the expansion of $u$.

Of course, if $\alpha > 1$, meaning that each iteration steps only produces a gain in decay of order 1 (as opposed to $\alpha$), then one can run $[\alpha - 1]$ more iteration steps.

---

$^{49}$ This requires $s' \geq 1$ in view of the supported/extendible spaces that we are using here; see also [18, Footnote 28].
following the first one which gave a non-trivial partial expansion. However, this is irrelevant for our applications, hence we omit the details here.

Combining Theorem 7.9 with Theorem 7.8 gives us a forward solution operator for \( P \) which, provided we understand the poles of \( \hat{P}(\sigma)^{-1} \), will be the key tool in our discussion of quasilinear wave equations in the next section.

8. Quasilinear wave and Klein-Gordon equations on the static model of de Sitter space

8.1. The static model of de Sitter space. We recall the form of the Lorentzian b-metric of a static patch \((M, g)\) of \(n\)-dimensional de Sitter space \([34, \S 4]\). We use the coordinates \( \mu \in (-\delta, 1), \omega \in S^{n-2}, \tau \in (0, \infty) \), for small \( \delta > 0 \), on physical space near the cosmological horizon \( \mu = 0 \) and the natural coordinates in the fiber of the b-cotangent bundle, which come from writing b-covectors as

\[
\xi \, d\mu + \eta \, d\omega + \sigma \, \frac{d\tau}{\tau}.
\]

Moreover, we write \( r^2 = 1 - \mu \), and \( K \) for the dual metric on the round sphere; in a coordinate system on the sphere, its components are denoted \( K^{ij} \). We shall also have occasion to use the coordinates \( Y = r\omega \in \mathbb{R}^{n-1} \) and \( \tau \), valid near \( r = 0 \), with b-covectors written

\[
\zeta \, dY + \sigma \, \frac{d\tau}{\tau}.
\]

Then the quadratic form associated with the dual metric \( G \) of the static de Sitter metric \( g \), which is the same as the b-principal symbol of \( P := \square_g \), is given by

\[
p = \sigma_k^2(P) = -4r^2\mu\xi^2 + 4r^2\sigma\xi + \sigma^2 - r^{-2}|\eta|^2_K = (Y \cdot \zeta - \sigma)^2 - |\zeta|^2. \tag{8.1}
\]

Correspondingly, the Hamilton vector field is

\[
H_p = (\partial_\xi \eta) \partial_\mu - (\partial_{\mu} p) \partial_\xi + (\partial_\sigma p) \tau \partial_\tau - (\tau \partial_\tau p) \partial_\sigma - r^{-2} H_{|\eta|^2_K}
= 4r^2(-2\mu\xi + \sigma)\partial_\mu - (4\xi^2(1-2r^2) - 4\sigma\xi - r^{-4}|\eta|^2_K)\partial_\xi
+ (4r^2\xi + 2\sigma)\tau \partial_\tau - r^{-2} H_{|\eta|^2_K}
= 2(Y \cdot \zeta - \sigma)(Y \partial_\nu - \zeta \partial_\tau - \sigma \partial_\tau) - 2\zeta \cdot \partial_\tau. \tag{8.2}
\]

We aim to show that \( P \) fits into the framework of Section 7, so that Theorems 7.8 and 7.9 apply to \( P \) and non-smooth perturbations of it. Since the metric \( g \) is \( \tau \)-independent, the computations are very similar to those performed by Vasy [34] in the Mellin transformed picture; also, in [18, \S 2], it is used, even if not explicitly stated, that \( P \) does fit into the smooth framework there, but we will provide all details here for the sake of completeness.

Denote the characteristic set of \( p \) by \( \Sigma = p^{-1}(0) \subset b^*T^*M \setminus o \).

**Lemma 8.1.** \( \Sigma \) is a smooth conic codimension 1 submanifold of \( b^*T^*M \setminus o \) transversal to \( b^*T^*_Y M \).

**Proof.** We have to show that \( dp \neq 0 \) whenever \( p = 0 \). We compute

\[
dp = (4\xi^2(1-2r^2) - 4\sigma\xi - r^{-4}|\eta|^2_K)\,d\mu + 4r^2(-2\mu\xi + \sigma)\,d\xi
+ (4(1-\mu)\xi + 2\sigma)\,d\sigma - r^{-2}d|\eta|^2_K.
\]
Thus if \( dp = 0 \), all coefficients have to vanish, thus \( \sigma = 2\mu \xi \) and \( \sigma = 2(\mu - 1)\xi \), giving \( \xi = 0 \) and thus \( \sigma = 0 \), hence also \( \eta = 0 \). Thus \( dp \) vanishes only at the zero section of \( bT^*M \) in this coordinate system. In the coordinates valid near \( r = 0 \), we compute

\[
dp = 2(Y \cdot \zeta - \sigma)\zeta \cdot dY + 2((Y \cdot \zeta - \sigma)Y - 2\zeta) \cdot d\zeta - 2(Y \cdot \zeta - \sigma) d\sigma,
\]

thus \( dp = 0 \) implies \( Y \cdot \zeta = \sigma \), hence \( \zeta = 0 \) and then \( \sigma = 0 \). Thus, the first half of the statement is proved. The transversality is clear since \( dp \) and \( d\tau \) are linearly independent at \( \Sigma \) by inspection. \( \square \)

We will occasionally also use \( \Sigma \) to denote the characteristic set viewed as a subset of the radially compactified b-cotangent bundle \( bT^*M \) or as a subset of the boundary \( bS^*M \) of \( bT^*M \) at fiber infinity.

8.1.1. \textit{Radial points.} Since \( g \) is a Lorentzian b-metric, the Hamilton vector field \( H_p \) cannot be radial except at the boundary \( Y = \partial M \) at future infinity, where \( \tau = 0 \). In the coordinate system near \( r = 0 \), one easily checks using (8.2) that there are no radial points over \( Y = 0 \). At radial points, we then moreover have \( H_p = 4r^2(-2\mu \xi + \sigma) = 0 \), thus \( \sigma = 2\mu \xi \). Further, we compute

\[
H_{|\eta|^2_{K}} = H_{(\eta, K^{ij}(\omega)\omega_j)} = 2\eta_j K^{ij}(\omega)\partial_{\omega_i} - 2\eta_i(\partial_{\omega_k} K^{ij})\eta_j \partial_{\omega_k}.
\]

The coefficient of \( \partial_{\omega_i} \) must vanish for all \( i \), which implies \( \eta = 0 \), since \( K \) is nondegenerate. Now, if \( \xi = 0 \), then \( \sigma = 0 \), i.e. all fiber variables vanish and we are outside the characteristic set \( \Sigma \); thus \( \xi \neq 0 \). At points where \( \sigma = 2\mu \xi, \eta = 0, \tau = 0 \), the expression for \( p \) simplifies to \( 4r^2\mu \xi^2 + 4\mu^2 \xi^2 = 4\mu \xi^2 \), which does not vanish unless \( \mu = 0 \). Hence, \( \mu = 0, \tau = 0, \eta = 0, \sigma = 0 \), and we easily check that at these conditions are also sufficient for a point in this coordinate patch to be a radial point. Thus:

\textbf{Lemma 8.2.} The set of radial points of \( \Box g \) is a disjoint union \( \mathcal{R} = \mathcal{R}_{+} \cup \mathcal{R}_{-} \), where

\[
\mathcal{R}_{\pm} = \{ \mu = 0, \tau = 0, \eta = 0, \sigma = 0, \pm \xi > 0 \} = \{ \tau = 0, \sigma = 0, Y = \mp \zeta/|\zeta| \} \subset \Sigma.
\]

To analyze the flow near \( L_{\pm} := \partial \mathcal{R}_{\pm} \subset bS^*M \), we introduce normalized coordinates

\[
\hat{\rho} = \frac{\xi}{\zeta}, \hat{\eta} = \frac{\eta}{\zeta}, \hat{\sigma} = \frac{\sigma}{\zeta}
\]

and consider the homogeneous degree 0 vector field \( H_p := |\hat{\rho}| H_p \). We get a good qualitative understanding of the dynamics near \( L_{\pm} \) by looking at the linearization \( W \) of \( \pm H_p = \hat{\rho}H_p \).

50 We follow the recipe of [6, §3].
point \( q \in \mathcal{I} \) of functions in \( C^\infty(\mathcal{I}) \) vanishing at \( q \) into itself. The linearization of \( \hat{q} \) at \( q \) is then the vector field \( \hat{q} \) acting on \( \mathcal{I}/\mathcal{I}^2 \cong T^*_q \mathcal{I} \mathcal{I}^2 \), where the isomorphism is given by \( f + \mathcal{I}^2 \mapsto df|_q \). Computing the linearization \( W \) of \( \hat{q} \) at \( q \) now amounts to ignoring terms of \( \hat{q} \) that vanish to at least second order at \( q \), which gives

\[
W = 4\left(-2\mu + \sigma\right)\partial_\mu - 4\left(\hat{\rho}\partial_\hat{\rho} + \hat{\eta}\partial_{\hat{\eta}} + \sigma\partial_\sigma\right) + 4\tau\partial_\tau - 2K^{ij}(\omega)\hat{\eta}_j\partial_{\omega_i}.
\]

We read off the eigenvectors and corresponding eigenvalues:

\[
\begin{align*}
&d\hat{\rho}, d\hat{\eta}, d\hat{\sigma} \text{ with eigenvalue } -4, \\
&d\mu - d\hat{\sigma} \text{ with eigenvalue } -8, \\
&\partial_\tau \text{ with eigenvalue } +4, \\
&d\omega_i - \frac{1}{2}K^{ij}\hat{\eta}_j \text{ with eigenvalue } 0.
\end{align*}
\]

Thus, \( L_+ (L_-) \) is a sink (source) of the Hamilton flow within \( S^*_p M \), with an unstable (stable) direction normal to the boundary. More precisely, the \( \tau \)-independence of the metric suggests the definition

\[
\mathcal{L}_\pm = \partial\{\mu = 0, \sigma = 0, \eta = 0, \pm\xi > 0\} \subset \mathbf{S}_p^* M,
\]

so that \( L_\pm = \mathbf{S}_p^* M \cap \mathcal{L}_\pm \); moreover \( \mathcal{L}_\pm \subset \Sigma \), and \( H_p \) is tangent to \( \mathcal{L}_\pm \); indeed,

\[
H_p = 4\xi^2\partial_\xi + 4\xi\tau\partial_\tau \text{ at } \mathcal{L}_\pm.
\]

Lastly, \( \mathcal{L}_+ (\mathcal{L}_-) \) is indeed the unstable (stable) manifold at \( L_\pm \). Now, going back to the full rescaled Hamilton vector field \( H_p \), we have at \( L_\pm \) (in fact, at \( \mathcal{L}_\pm \)):

\[
|\hat{\rho}|^{-1}H_p|\hat{\rho}| = \mp\beta_0, \quad -\tau^{-1}H_p\tau = \mp\tilde{\beta}\beta_0
\]

with \( \beta_0 = 4 \) and \( \tilde{\beta} = 1 \); furthermore, near \( \mathcal{L}_\pm \),

\[
\mp H_p\hat{\eta} = 4\hat{\eta}, \quad \mp H_p\hat{\sigma} = 4\hat{\sigma}, \quad \mp H_p(\mu - \hat{\sigma}) = 8(\mu - \hat{\sigma})
\]

modulo terms that vanish quadratically at \( \mathcal{L}_\pm \), hence, putting \( \beta_1 = 8 \), the quadratic defining function \( \rho_0 := \hat{\eta}^2 + \hat{\sigma}^2 + (\mu - \hat{\sigma})^2 \) of \( \mathcal{L}_\pm \) within \( \Sigma \) satisfies

\[
\mp H_p\rho_0 - \beta_1\rho_0 \geq 0
\]

modulo terms that vanish cubically at \( \mathcal{L}_\pm \).

We have thus verified the geometric and dynamical assumptions (1)-(5) in Section 6.4 regarding the characteristic set and the Hamilton flow of \( p \) near the radial set. Note that assumption (5) is automatic here with \( \tilde{\beta} = 0 \), since \( P \) is formally self-adjoint with respect to the metric \( \mathbf{b} \)-density. In other words, we have verified assumption (1) of Section 7.2.

8.1.2. Global behavior of the characteristic set. The next assumption to be checked is (3) in Section 7.2. This is easily accomplished: Indeed, from (8.1), we have

\[
p = (\sigma + 2r^2\xi)^2 - 4r^2\xi^2 - r^{-2}|\eta|^2_K,
\]

and thus \( \Sigma = \Sigma_+ \cup \Sigma_- \), where

\[
\Sigma_\pm = \{\pm(\sigma + 2r^2\xi) > 0\} \cap \Sigma = \{\pm(\sigma - Y \cdot \xi) > 0\}
\]

(8.5)

since \( p = 0 \), \( \sigma + 2r^2\xi = 0 \) implies \( \xi = \eta = 0 \), thus \( \sigma = 0 \), thus \( \{\sigma + 2r^2\xi = 0\} \) does not intersect the characteristic set \( p^{-1}(0) \), and similarly in the \((Y, \tau, \xi, \sigma)\) coordinates. Moreover, we have \( \mathcal{L}_\pm \subset \Sigma_\pm \) by definition of \( \mathcal{L}_\pm \).
We proceed with a description of the domain $\Omega \subset M$ with artificial boundaries $H_1$ and $H_2$, which have defining functions $t_1, t_2$, and check the assumptions (4)-(8) in Section 7.2. We first observe that $G\left(\frac{dt}{\tau}, \frac{dt}{\tau}\right) = 1 > 0$. Now, pick any $\delta > 0$ and $\tau_0 > 0$ and define

$$t_1 = \tau_0 - \tau, \quad t_2 = \mu + \delta.$$ 

Then

$$G(\eta dt_1, \eta dt_1)|_{t_1=0} = G\left(-\tau \frac{dt}{\tau}, -\tau \frac{dt}{\tau}\right)|_{t=\tau} = \tau_0^2 > 0,$$

$$G(\eta dt_2, \eta dt_2)|_{t_2=0} = G(d\mu, d\mu)|_{\mu = -\delta} = 4\delta(1 + \delta) > 0,$$

$$G(\eta dt_1, \eta dt_2)|_{t_1=t_2=0} = -4(1 + \delta)\tau_0 < 0,$$

thus $t_1$ and $t_2$ are timelike with opposite timelike character; indeed, with the usual time orientation on de Sitter space (namely where $-dt/\tau$ is future oriented), $t_1$ is future oriented and $t_2$ is past oriented, as is $dt/\tau$. Moreover, $dt_2$ and $dt$ are clearly linearly independent at $Y \cap H_2$, as are $dt_1$ and $dt_2$ at $H_1 \cap H_2$. Thus, the assumptions (4)-(7) are verified.

It remains to check the non-trapping assumption (8). Let us first analyze the flow in $bT_{\H}^* M \setminus bT_{\H}^* M$.\footnote{Notice that $H_p \tau = 0$ in $bT_{Y}^* M$, thus bicharacteristics that intersect $bT_{\H}^* M$ are in fact contained in $bT_{\H}^* M$, and correspondingly bicharacteristics containing points in $bT_{\H}^* M \setminus bT_{Y}^* M$ stay in $bT_{\H}^* M \setminus bT_{Y}^* M$.} There,

$$\pm H_p \tau = \pm 2(\sigma + 2r^2 \xi) \tau > 0 \text{ on } \Sigma_\pm, \quad (8.6)$$

In particular, in $\Sigma_\pm \setminus bT_{Y}^* M$, bicharacteristics reach $bT_{\H}^* M$ (i.e. $\tau = \tau_0$) in finite time in the forward (+), resp. backward (−), direction. We show that they stay within $bT_{\H}^* M$: For this, observe that $p = 0$ and $\mu < 0$, thus $r > 1$, imply

$$2|\xi| < 2|\sigma + 2r^2 \xi|$$

by equation (8.4). In fact, if $\xi \neq 0$, the first inequality is strict, and if $\xi = 0$, the second inequality is strict, and we conclude the strict inequality

$$2|\xi| < |\sigma + 2r^2 \xi| \text{ if } p = 0, \mu < 0.$$ 

Hence, on $(\Sigma_\pm \setminus bT_{Y}^* M) \cap \Sigma_0$, if $\mu < 0$, then

$$\pm H_p \mu = \pm 4r^2(\sigma + 2r^2 \xi - 2\xi) > 0, \quad (8.7)$$

thus in the forward (on $\Sigma_+$), resp. backward (on $\Sigma_-$), direction, bicharacteristics cannot cross $bT_{\H}^* M = \{\mu = -\delta\}$.

Next, backward, resp. forward, bicharacteristics in $L_\pm \setminus L_\pm$ tend to $L_\pm$ by equation (8.6), since $H_p$ is tangent to $L_\pm$, and $L_\mp = L_\pm \cap \{\tau = 0\}$; in fact, by equation (8.3), more is true, namely these bicharacteristics, as curves in $bT^* M \setminus \alpha$, tend to $L_\pm$ if the latter is considered a subset of the boundary $bS^* M$ of $bT^* M$ at fiber infinity. Now, consider a backward, resp. forward, bicharacteristic $\gamma$ in $(\Sigma_\pm \setminus L_\pm) \cap bT_{\H}^* M$.\footnote{Including those within $bT_{Y}^* M$.} By (8.6), $\tau$ is non-increasing along $\gamma$, and by (8.7), $\mu$ is strictly decreasing along $\gamma$ once $\gamma$ enters $\mu < 0$, hence it then reaches $bT_{\H}^* M$ in finite time, staying within $bT_{\H}^* M$. We have to show that $\gamma$ necessarily enters $\mu < 0$ in finite time. Assume this is not the case. Then observe that

$$\mp H_p(\sigma - Y \cdot \xi) = \mp 2|\xi|^2 = \mp 2(\sigma - Y \cdot \xi)^2 \text{ on } \Sigma_\pm, \quad (8.8)$$
thus $\sigma - Y \cdot \zeta$ converges to 0 along $\gamma$. Now on $\Sigma$, $|\zeta| = |\sigma - Y \cdot \zeta|$, thus, also $\zeta$ converges to 0, and moreover, on $\Sigma$, we have

$$|\sigma| \leq |Y \cdot \zeta| + |Y \cdot \zeta - \sigma| \leq (1 + |Y|)|\zeta|$$

since we are assuming $|Y| \leq 1$ on $\gamma$, hence $\sigma$ converges to 0 along $\gamma$. But $H_p \sigma = 0$, i.e. $\sigma$ is constant. Thus necessarily $\sigma = 0$, hence $p = 0$ gives $|Y \cdot \zeta| = |\zeta|$, and thus we must in fact have $|Y| = 1$ on $\gamma$, more precisely $Y = \mp \zeta/|\zeta|$, and thus $\gamma$ lies in $L_{\pm}$, which contradicts our assumption $\gamma \not\in L_{\pm}$. Hence, $\gamma$ enters $|Y| > 1$ in finite time, and thus, as we have already seen, reaches $bT_{H_2}^\ast M$ in finite time.

Finally, we show that forward, resp. backward, bicharacteristics $\gamma$ in $(\Sigma_{\pm} \cap bT_Y^0 M \setminus R_\pm) \cap \Sigma_\Gamma$ tend to $L_{\pm}$. By equation (8.8), $\pm(\sigma - Y \cdot \zeta) \to \infty$ (in finite time) along $\gamma$, hence $|\zeta| = |\sigma - Y \cdot \zeta|$ on $\gamma \subset \Sigma$ tends to $\infty$, and therefore

$$|Y| \geq |Y \cdot \zeta| / |\zeta| \geq |\sigma - Y \cdot \zeta| / |\zeta| - |\sigma| / |\zeta| \to 1$$

since $\sigma$ is constant along $\gamma$. On the other hand, at points on $\gamma$ where $|Y| > 1$, i.e. $\mu < 0$, we have $\pm H_p \mu > 0$ by (8.7). We conclude that $\gamma$ tends to $|Y| = 1$, i.e. $\mu = 0$. Moreover,

$$\left( Y \cdot \frac{\zeta}{|\zeta|} - \frac{\sigma}{|\zeta|} \right)^2 = 1$$

on $\Sigma$, thus $|Y \cdot \zeta/|\zeta|| \to 1$ along $\gamma$. Together with $|Y| \to 1$, this implies $Y \to \mp \zeta/|\zeta|$, and since $\sigma$ is constant and $|\zeta| \to \infty$, we conclude that $\gamma$ tends to $L_{\pm}$. Thus, assumption (8) in Section 7.2 is verified.

8.1.3. The normal operator. The Mellin transformed normal operator $\hat{P}(\sigma)$ of $P = \square_g$, with principal symbol (in the high energy sense, $\sigma$ being the large parameter) given by the right hand side of (8.1), fits into the framework of Vasy [34]. In the current setting, the poles of $\hat{P}(\sigma)^{-1}$ for $P$ acting on functions have been computed explicitly by Vasy [32]. In fact, if more generally $P_\lambda = \square_g - \lambda$, the only possible poles of $\hat{P}_\lambda(\sigma)^{-1}$ are in

$$i\delta_{\pm}(\lambda) - iN, \quad \delta_{\pm}(\lambda) = -\frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}, \quad (8.9)$$

and the pole with largest imaginary part is simple unless $\lambda = (n-1)^2/4$, in which case it is a double pole. Notice that for $\lambda = 0$, all non-zero resonances have imaginary part $\leq -1$.

8.2. Quasilinear wave equations. We are now prepared to discuss existence, uniqueness and asymptotics of solutions to quasilinear wave and Klein-Gordon equations for complex- and/or real-valued functions on the static model of de Sitter space, in fact on the domain $\Omega$ described in the previous section, with small data, i.e. small forcing. Keep in mind though that the methods work in greater generality, as explained in the introduction. In particular, we will prove Theorems 1.1 and 1.2.

Remark 8.3. The only reason for us to stick to the scalar case here as opposed to considering wave equations on natural vector bundles is the knowledge of the location of resonances in this case, see Section 8.1.3; the author is not aware of corresponding statements for bundle-valued equations. The general statement is that as long as there is no resonance or only a simple resonance at 0 in the closed upper half plane, the arguments presented in this section go through. Likewise, we
can work on the more general class of static asymptotically de Sitter spaces, since
the normal operator, hence the resonances are the same as on exact static de Sitter
space, and in fact on much more general spacetimes, provided the above resonance
condition as well as all assumptions in Section 7.2 are satisfied; examples of the
latter kind include perturbations (even of the asymptotic model) of asymptotically
de Sitter spaces, see also Footnote 54.

Let us from now on denote by \(g_{dS}\) the static de Sitter metric. We start with a
discussion of quasilinear wave equations.

**Definition 8.4.** For \(s, \alpha \in \mathbb{R}\), define the Hilbert space

\[
X^{s,\alpha} := \mathbb{C} \oplus H^{s,\alpha}_b(\Omega)^*\cdot
\]

with norm \(\|(c, v)\|_{X^{s,\alpha}}^2 = |c|^2 + \|v\|_{H^{s,\alpha}_b(\Omega)^*}^2\). We will identify an element \((c, v) \in X^{s,\alpha}\) with the distribution \((\phi \circ t_1) c + v\), where \(\phi \) and \(t_1\) are as in the statement of
Theorem 7.9.

**Theorem 8.5.** Let \(s > n/2 + 6\) and \(0 < \alpha < 1\). Let \(\epsilon_0 > 0\), and assume that for
\(\epsilon \in [0, \epsilon_0)\),

\[
g : X^{s-\epsilon,\alpha-\epsilon} \to (C^\infty + H^{s-\epsilon,\alpha-\epsilon}_b(M; \text{Sym}^2 TM)) (8.10)
\]

is a continuous map such that \(g(0) = g_{dS}\). 53 Furthermore, assume that

\[
g : X^{s-\epsilon,\alpha-\epsilon} \times H^{s-1-\epsilon,\alpha-\epsilon}_b(\Omega; \text{Sym}^2 TM)^* \to H^{s-1-\epsilon,\alpha-\epsilon}_b(\Omega)^* (8.11)
\]

is continuous for \(\epsilon \in [0, \epsilon_0)\) with \(q(0) = 0\) and satisfies

\[
\|q(u, bdu)\|_{H^{s-1-\epsilon,\alpha}_b(\Omega)^*} \leq L_q(R)\|u\|_{X^{s,\alpha}} (8.12)
\]

for all \(u \in X^{s,\alpha}\) with norm \(\leq R\), where \(L_q : \mathbb{R}_{\geq 0} \to \mathbb{R}\) is continuous and non-decreasing. Then there is a constant \(C_L > 0\) so that the following holds: If \(L_q(0) < C_L\), then for small \(R > 0\), there is \(C_f > 0\) such that for all \(f \in H^{s-1-\epsilon,\alpha}_b(\Omega)^*\) with
norm \(\leq C_f\), there exists a solution \(u \in X^{s,\alpha}\) of the equation

\[
\Box g(u) = f + q(u, bdu) (8.13)
\]

with norm \(\leq R\).

If instead \(s > n/2 + 7\) and for \(j = 0, 1\),

\[
g : X^{s-j,\alpha} \to (C^\infty + H^{s-j,\alpha}_b(M; \text{Sym}^2 TM)),
\]

\[
q : X^{s-j,\alpha} \times H^{s-1-j,\alpha}_b(\Omega; \text{Sym}^2 TM)^* \to H^{s-1-j,\alpha}_b(\Omega)^* (8.11)
\]

are continuous, \(g\) is locally Lipschitz, and

\[
\|q(u, bdu) - q(v, bdv)\|_{H^{s-1-j,\alpha}_b(\Omega)^*} \leq L_q(R)\|u - v\|_{X^{s,\alpha}} (8.13)
\]

for \(u, v \in X^{s-j,\alpha}\) with norm \(\leq R\), then there is a constant \(C_L > 0\) so that
the following holds: If \(L_q(0) < C_L\), then for small \(R > 0\), there is \(C_f > 0\) such that
for all \(f \in H^{s-1,\alpha}_b(\Omega)^*\) with norm \(\leq C_f\), there exists a unique solution \(u \in X^{s,\alpha}\)
of the equation (8.13) with norm \(\leq R\), and in the topology of \(X^{s-1,\alpha}\), \(u\) depends
continuously on \(f\).

---

53 It is sufficient to assume that \(g(0)\) is such that \(N(\Box g(0))\) is a small perturbation of \(N(g_{dS})\)
in \(\text{Diff}^2(Y \cap \Omega)\).
Remark 8.6. Of course, we require all sections $g(u)$ of $\text{Sym}^2 bTM$ to take values in symmetric 2-tensors with real coefficients. If we assume that $q$ and $f$ are real-valued, we may therefore work in the real Hilbert space

$$X_{\mathbb{R}}^{s,\alpha} := \mathbb{R} \oplus H^s_b(\Omega; \mathbb{R})^\bullet$$

and find the solution $u$ there. This remark also applies to all theorems later in this section.

Proof of Theorem 8.5. To not overburden the notation, we will occasionally write $H^s_b$ in place of $H^s_b(\Omega; \mathbb{R})^\bullet$ if the context is clear.

By assumption on $g$, there exists $R_S$ such that for $u \in X^{s,\alpha}$ with $\|u\|_{X^{s,\alpha}} \leq R_S$, the operator $\Box g(u)$ satisfies the relaxed versions of the assumptions (1)-(8) in Section 7.2 (see the discussion after assumption (8)), thus Theorem 7.8 is applicable, giving a continuous forward solution operator $S_{g(u)}$ on sufficiently weighted $b$-Sobolev spaces. For such $u$, the normal operator $N(\Box g(u))$ is a small perturbation of $N(\Box g(\epsilon))$ in $\text{Diff}^2(Y \cap \Omega)$, and since further $s - 2 - \alpha > 0$, we can apply Theorem 7.9 to conclude that the solution operator in fact maps continuously, with uniformly bounded operator norm

$$\|S_{g(u)}\| \leq C_S, \quad \|u\|_{X^{s,\alpha}} \leq R_S.$$  \hfill (8.15)

Let $C_L := C_S^{-1}$, and assume that $L_q(0) < C_L$, then $L_q(R_q) < C_L$ for $R_q > 0$ small. Put $R := \min(R_S, R_q)$ and $C_f = R(C_S^{-1} - L_q(R))$; let $f \in H^{-2-s,0-\alpha}_b(\Omega; \mathbb{R})^\bullet$ have norm $\leq C_f$. Define $u_0 := 0$ and iteratively $u_{k+1} \in X^{s,\alpha}$ by solving

$$\Box g(u_k) u_{k+1} = f + q(u_k, b du_k),$$

i.e. $u_{k+1} = S_{g(u_k)}(f + q(u_k, b du_k))$. For $u_{k+1}$ to be well-defined, we need to check that $\|u_k\|_{X^{s,\alpha}} \leq R$ for all $k$. For $k = 0$, this is clear; for $k > 0$, we deduce from (8.15) and (8.12) that

$$\|u_{k+1}\|_{X^{s,\alpha}} \leq C_S \left( \|f\|_{H^{-2-s,0-\alpha}_b} + L_q(R) \|u_k\|_{X^{s,\alpha}} \right) \leq C_S \left( R(C_S^{-1} - L_q(R)) + L_q(R) R \right) = R.$$

Since $\{u_k\} \subset X^{s,\alpha}$ is bounded in a separable Hilbert space, we can extract a subsequence, which we continue to call $u_k$, that converges weakly: $u_k \rightharpoonup u \in X^{s,\alpha}$. Since the inclusion $X^{s,\alpha} \to X^{s-0,\alpha-0}$ is compact, we then also have strong convergence $u_k \to u$ in $X^{s-0,\alpha-0}$. Then we have by the continuity assumption (8.10) of $g$

$$\Box g(u_k) \to \Box g(u) \text{ in } (C^\infty + H^{-2-s,0-\alpha}_b)\text{Diff}^2_b,$$

therefore for $\epsilon \in (0, \epsilon_0)$

$$\|\Box g(u_k) u_{k+1} - \Box g(u) u\|_{H^{-2-s,0-\alpha}_b} \leq \|(\Box g(u_k) - \Box g(u)) u_{k+1}\|_{H^{-2-s,0-\alpha}_b} + \|\Box g(u)(u_{k+1} - u)\|_{H^{-2-s,0-\alpha}_b} \to 0,$$

\footnote{Note that the poles of the meromorphic family $N(\Box g(u))^{-1}$ depend continuously on $u$ (see [34]), and the simple pole at 0, corresponding to the constant function 1 being annihilated by $N(\Box g(u))$, is preserved under perturbations.}
Moreover, by (8.11),
\[ q(u_k, b du_k) \xrightarrow{k \to \infty} q(u, b du) \in H_b^{s-1-\epsilon, \alpha-\epsilon}, \]
thus
\[ \Box g(u) = \lim_{k \to \infty} \Box g(u_k) = \lim_{k \to \infty} (f + q(u_k, b du_k)) = f + q(u, b du) \]
with limits taken in \( H_b^{s-2-\epsilon, \alpha-\epsilon} \), i.e. \( u \) satisfies the PDE (8.13).

We now show the well-posedness under the Lipschitz hypotheses. For \( u \in X^{s, \alpha} \), we have
\[ \Box g(u) = g^{ij}(u) b^i D_j + \tilde{g}^j(u, b du) b^j \]
with \( g^{ij}(u) \in \mathcal{C}^\infty + H_b^{s-1, \alpha}, \tilde{g}^j(u, b du) \in \mathcal{C}^\infty + H_b^{s-2, \alpha} \); using the explicit formula for the inverse of a metric, Corollary 3.10 and Lemma 4.2, we deduce from the
\[ \Box g(u) \]
are Lipschitz as well; hence, for some constant \( C_g(R) \), we obtain
\[ \| \Box g(u) - \Box g(v) \|_{\mathcal{L}(X^{s, \alpha}, H_b^{s-2, \alpha})} \leq C_g(R)\| u - v \|_{X^{s, \alpha}} \]
for \( u, v \in X^{s, \alpha} \) with norms \( \leq R \). Therefore, we get the following estimate for the difference of two solution operators \( S_g(u) \) and \( S_g(v) \), \( u, v \in X^{s, \alpha} \), with a loss of 2 derivatives relative to the elliptic setting, using a ‘resolvent identity’:
\[ \| S_g(u) - S_g(v) \|_{\mathcal{L}(H_b^{s-1, \alpha}, X^{s-1, \alpha})} = \| S_g(u)(\Box g(v) - \Box g(u))S_g(v) \|_{H_b^{s-1, \alpha}, X^{s-1, \alpha}} \]
\[ \leq C_S^2 \| \Box g(u) - \Box g(v) \|_{\mathcal{L}(X^{s, \alpha}, H_b^{s-2, \alpha})} \leq C_S^2 C_g(R)\| u - v \|_{X^{s, \alpha}}. \]
Here, we assumed \( C_S \) is such that \( \| S_g(u) \|_{\mathcal{L}(H_b^{s-2, \alpha}, X^{s-1, \alpha})} \leq C_S \) for small \( u \in X^{s, \alpha} \), which is where we use that \( s > n/2 + 6 \). We can now prove uniqueness and stability in one stroke: Suppose that \( u_1, u_2 \in X^{s, \alpha} \) have norm \( \leq R \) and satisfy
\[ \Box g(u_1) u_j = f_j + q(u_j, b du_j), \quad j = 1, 2, \]
where the \( f_j \in H_b^{s-1, \alpha}, j = 1, 2 \), have norm \( \leq C_f \). Then
\[ \Box g(u_1) (u_1 - u_2) = f_1 - f_2 + q(u_1, b du_1) - q(u_2, b du_2) - (\Box g(u_1) - \Box g(u_2)) u_2; \]
thus, writing \( u_2 = S_g(u_2)(f_2 + q(u_2, b du_2)) \) in the last term and using the estimate (8.17),
\[ \| u_1 - u_2 \|_{X^{s-1, \alpha}} \leq C_S \| f_1 - f_2 \|_{H_b^{s-2, \alpha}} + \]
\[ (L_q(R) + C_S C_g(R) (C_f + L_q(R)) R) \| u_1 - u_2 \|_{X^{s-1, \alpha}}. \]
Since \( C_S L_q(0) < 1 \), we can absorb the second term on the right into the left hand side for small \( R > 0 \), recalling that \( C_f = C_f(R) \to 0 \) as \( R \to 0 \). Hence
\[ \| u_1 - u_2 \|_{X^{s-1, \alpha}} \leq C' \| f_1 - f_2 \|_{H_b^{s-2, \alpha}}. \]
To complete the proof, we must show that the conditions on \( g \) and \( q \) guarantee the existence of a solution \( u \) of the PDE (8.13); but this follows from similar estimates: Using the same iterative procedure as before, we obtain a sequence \( (u_k)_k \) which is bounded in \( X^{s, \alpha} \) and Cauchy in \( X^{s-1, \alpha} \), hence converges in norm in \( X^{s-1, \alpha} \) to an element of \( X^{s, \alpha} \), the latter being a consequence of the weak compactness of the unit ball in \( X^{s, \alpha} \). \( \square \)
Remark 8.7. In the case that \( g(u) \equiv g \) is constant, see [18, Theorem 2.24] for a discussion of the corresponding semilinear equations. There, one in particular obtains more precise asymptotics in the case of polynomial non-linearities, see [18, Theorem 2.35]; see also Remark 7.10.

We next turn to a special case of Theorem 8.5 which is very natural and allows for a stronger conclusion.

Theorem 8.8. Let \( s > n/2 + 7 \) and \( 0 < \alpha < 1 \). Let \( N, N' \in \mathbb{N} \), and suppose \( c_k \in C^\infty(\mathbb{R}; \mathbb{R}) \), \( g_k \in (C^\infty + H_0^1)(M; \text{Sym}^2 bTM) \) for \( 1 \leq k \leq N \); define the map \( g: \mathcal{X}_R^{s,\alpha} \to (C^\infty + H_0^1)(M; \text{Sym}^2 bTM) \) by

\[
g(u) = \sum_{k=1}^N c_k(u)g_k,
\]

and assume \( g(0) = g_{\text{IS}} \).

Moreover, define

\[
q(u, \ln u) = \sum_{j=1}^{N'} u^{e_j} \prod_{k=1}^{N_j} X_{jk}u, \quad e_j + N_j \geq 2, N_j \geq 1, X_{jk} \in (C^\infty + H_0^{s-1})V_b.
\]

Then for small \( R > 0 \), there exists \( C_f > 0 \) such that for all \( f \in H_0^{s-1,\alpha}(\Omega; \mathbb{R}) \), with norm \( \leq C_f \), the equation

\[
\Box_g(u) = f + q(u, \ln u) \quad (8.18)
\]

has a unique solution \( u \in \mathcal{X}_R^{s,\alpha} \), with norm \( \leq R \), and in the topology of \( \mathcal{X}_R^{s-1,\alpha} \), \( u \) depends continuously on \( f \). If one in fact has \( f \in H_0^{s-1,\alpha}(\Omega; \mathbb{R}) \) for some \( s' \in (s, \infty) \), then \( u \in \mathcal{X}_R^{s',\alpha} \).

Remark 8.9. One could, for instance, choose the metrics \( g_k \) such that at every point \( p \in M \), the linear space \( \text{Sym}^2 bT_pM \) is spanned by the \( g_k(p) \), and in a similar manner the b-vector fields \( X_{jk} \).

Remark 8.10. The point of the last part of the theorem is that even though a priori the radius of the ball which is the set of \( f \in H_0^{s-1,\alpha}(\Omega) \) for which one has solvability in \( \mathcal{X}^{s,\alpha} \) according to Theorem 8.5 could shrink to 0 as \( s' \to \infty \), this does not happen in the setting of Theorem 8.8. We use a straightforward approach to proving this by differentiating the PDE; a somewhat more robust way could be to use Nash-Moser iteration, see e.g. [29], which however would require a more careful analysis of all estimates in Sections 3–7, as indicated, for instance, in Remark 4.1.

For the proof, we need one more definition:

Definition 8.11. (Cf. [7, Definition 1.1].) For \( s' > s, \alpha \in \mathbb{R} \) and \( \Gamma \subset bS^sM \), let

\[
H_b^{s,\alpha; s', \Gamma} := \{ u \in H_b^{s,\alpha} : \text{WF}^{s',\alpha}_b(u) \cap \Gamma = \emptyset \}.
\]

Proof of Theorem 8.8. The map \( g \) satisfies the requirements of Theorem 8.5 by Proposition 4.8, and \( q \) satisfies (8.11) and (8.22) with \( L_q(0) = 0 \), thus Theorem 8.5 implies the existence and uniqueness of solutions in \( \mathcal{X}^{s,\alpha} \) with small norm as well as their stability in the topology of \( \mathcal{X}^{s-1,\alpha} \). The uniqueness of \( u \) in all of \( \mathcal{X}_R^{s,\alpha} \), in fact

\footnote{\( g(0) \) can be more general; see Footnote 53.}
in $H^s_{b,\text{loc}}(\Omega^c)$, follows from local uniqueness for quasilinear symmetric hyperbolic systems, see e.g. Taylor [31, §16.3].

It remains to establish the higher regularity statement: by an iterative argument, it suffices to prove the following: If $s' > s$, $u \in \mathcal{X}^{s'-1/2,\alpha}_{\mathbb{R}}$, $\|u\|_{\mathcal{X}^{s,\alpha}} \leq R$, and $u$ solves (8.18) with $f \in H^{s'-1,\alpha}_{b}$, then $u \in \mathcal{X}^{s,\alpha}_{\mathbb{R}}$. We will use the summation convention for the remainder of the proof. Equation (8.18) in local coordinates reads

$$
(g^{ij}(u)\partial_j^2 + h^j(u, \partial u)\partial_j)u = f + q(u, \partial u),
$$

(8.19) where $g^{ij}(v)$, $h^j(v; z)$ and $q(v; z)$ are $C^\infty$-functions of $v$ and $z$. As is standard in ODEs to obtain higher regularity (and exploited in a similar setting by Beals and Reed [7, §4]), we will differentiate this equation with respect to certain b-vector field $V$: After differentiating and collecting/rewriting terms, one obtains an equation like (8.19) for $Vu$, where only the coefficients of first order terms are changed, and without $q$ and with a different forcing term; one can then appeal to the regularity theory for the equation for $Vu$, which is thus again a wave equation with lower order terms. Concretely, suppose $\Sigma \subset \tilde{\Sigma}$ is a closed subset of the characteristic set of $\Box_q(u)$, consisting of bicharacteristic strips and contained in the coordinate patch we are working in; we want to propagate $\mathcal{X}^{s,\alpha}$-regularity of $u$ into $\tilde{\Sigma}$, assuming we have this regularity on backward/forward bicharacteristics from $\tilde{\Sigma}$ or in a punctured neighborhood of $\tilde{\Sigma}$. With $\pi: \mathcal{B}^s M \to M$ denoting the projection to the base, choose $\chi, \chi_0 \in C^\infty_c(\mathbb{R}^n_+)$ so that $\chi$ is identically 1 near $\pi(\Sigma)$ and $\chi_0$ is identically 1 on supp $\chi$. Let $V_0 \in \mathcal{V}_b(\mathbb{R}^n_+)$ be a constant coefficient b-vector field which is non-characteristic (in the b-sense) on $\tilde{\Sigma}$, which is possible if $\tilde{\Sigma}$ is sufficiently small, and put $V = \chi_0 V_0$. Applying $V$ to (8.19), we obtain, suppressing the arguments $u, \partial u$,

$$(g^{ij}(u)\partial_j^2 + h^j(u, \partial u)\partial_j)Vu = Vf + (g^{ij}(V u)\partial_j^2 V u + g^{ij}[V, \partial_j^1]) u + (g^{ij})(V u)\partial_j u + g^{ij}[V, \partial_j^2] u$$

$$= Vf + (\partial_v q)V u + (\partial_v q)[V, \partial_j] u$$

$$- (\partial_v h)^j Vu \partial_j u - h^j(V, \partial_j) u - (\partial_v h^k)[V, \partial_k] u =: f_1.$$ 

Since $V_0$ annihilates constants, $Vu \in H^{s'-3/2,\alpha}_{b}$ locally near $\pi(\tilde{\Sigma})$. Similarly, $[V, \partial_j] u \in H^{s'-3/2,\alpha}_{b}$ locally near $\pi(\tilde{\Sigma})$, and $h^j(u, \partial u) \in C^\infty + H^{s'-3/2,\alpha}_{b}, q(u, \partial u) \in H^{s'-3/2,\alpha}_{b}$, similarly for derivatives of $h^j$ and $q$; lastly, $Vf \in H^{s'-2,\alpha}_{b}$, thus $f_1 \in H^{s'-2,\alpha}_{b}$ locally near $\pi(\tilde{\Sigma})$. We need to analyze the last two terms on the left hand side: Since $V$ is non-characteristic on supp $\chi \supset \pi(\tilde{\Sigma})$, we can write

$$\partial_j = (1 - \chi)^j \partial_j + Q_j V + \tilde{R}_j, \quad Q_j \in \Psi^0_{b}, \tilde{R}_j \in \Psi^1_{b}, \text{WF}_b(\tilde{R}_j) \cap \tilde{\Sigma} = \emptyset$$

put $R_j := (1 - \chi)^j \partial_j + \tilde{R}_j = \partial_j - Q_j V$. Note that $R_j$ annihilates constants. We can then write

$$\partial_j^2 u = \partial_j Q_j V u + \partial_j R_j u,$$

and the second term is in $H^{s,\alpha}_{b}$ microlocally near $\tilde{\Sigma}$. Thus, we have

$$(g^{ij}(V u)\partial_j^2 u = ((g^{ij}(V u)\partial_j Q_j) V u + (g^{ij})(V u)\partial_j R_j u;$$

\[56\]We only assume that the $\mathcal{X}^{s,\alpha}$-norm of $u$ is small – the reason for this assumption is that it ensures that $\Box_q(u)$ fits into our framework.
the second term on the right is a product of a function in $H^{s'-3/2,\alpha}_b$ with $b\partial_i R_j u$, the latter a priori being an element of $H^{s'-5/2,\alpha;\infty,\Sigma}_b$; we will prove below in Lemma 8.12 that this product is an element of $H^{s'-5/2,\alpha;\infty,\Sigma}_b$. Moreover, $[V, b\partial_i^2]$ is a second order $b$-differential operator, vanishing on constants, with coefficients vanishing near $\pi(\tilde{\Sigma})$; hence $g^{ij}[V, b\partial_i^2]u \in H^{s'-5/2,\alpha;\infty,\Sigma}_b$. We conclude that
\[
P_1(Vu) = f_2 \in H^{s'-5/2,\alpha;\infty,\Sigma}_b,
\]
where
\[
P_1 = \Box_{g(u)} + \tilde{P}, \quad \tilde{P} = [(\partial_j h^k) b\partial_k u - \partial_j g b\partial_j] + (g^{ij})'Vu b\partial_i Q_j.
\]
Since we are assuming $u \in \mathcal{X}^{s'-1/2,\alpha}$, and moreover $\tilde{P}$ is an element of $H^{s'-3/2,\alpha}_b\Psi^1_b$ near $\pi(\tilde{\Sigma})$, we see that, a fortiori,
\[
P_1 \in (C^\infty + H^{s'-1,\alpha}_b)\text{Diff}^2_b + (C^\infty + H^{s'-2,\alpha}_b)\Psi^1_b.
\]
Hence, we can propagate $H^{s'-1,\alpha}_b$-regularity of $Vu$ into $\Sigma$ by Theorems 6.6 and 6.10:\footnote{Recall that these two theorems only deal with the propagation of regularity which is 1/2 more than the a priori regularity of $Vu$, which is $H^{s'-3/2,\alpha}_b$.} the point here is that real principal type propagation only depends on the principal symbol of $\Box$, which is the same as the principal symbol of $\Box_{g(u)}$, and the propagation of $H^{s'-1,\alpha}_b$-regularity near radial points works for arbitrary $H^{s'-2,\alpha}_b\Psi^1_b$-perturbations of $\Box_{g(u)}$; see Remark 6.11. Therefore, writing $u = c + u'$ with $u' \in H^{s'-1/2,\alpha}_b$ a priori, we obtain $u' \in H^{s',\alpha}_b$ microlocally near $\Sigma$ by standard elliptic regularity, since $V$ is non-characteristic on $\Sigma$. Away from the characteristic set of $\Box_{g(u)}$,\footnote{Notice that $P_1$ and $\Box_{g(u)}$ have the same characteristic set.} we simply use $P_1 Vu \in H^{s'-5/2,\alpha}_b$ and elliptic regularity for $P_1 V$ to deduce that $u' \in H^{s'+1/2,\alpha}_b$ there;\footnote{Let us stress the importance of only using local rather than microlocal regularity information of $P_1 Vu$, since the proof of Theorem 5.1, giving elliptic regularity for $Vu$ solving $P_1(Vu) = f$, only works with local assumptions on $f$, see Remark 5.2.} here, we would choose $V$ such that it is non-characteristic on a set disjoint from $\Sigma$. Putting all such pieces of regularity information together by choosing finitely many such sets $\Sigma$, we obtain $u' \in H^{s',\alpha}_b(\Omega)^\bullet$.\footnote{To finish the proof, we need the following lemma, which we prove using ideas from [7, Theorem 1.3].}

**Lemma 8.12.** Let $\alpha \in \mathbb{R}$ and $s > n/2 + 1$. Then, in the notation of Definition 8.11, for $u \in H^s_b$ and $v \in H^{s-1,\alpha;s;\Gamma}_b$, we have $uv \in H^{s-1,\alpha;s;\Gamma}_b$.

**Proof.** Without loss, we may assume $\alpha = 0$. By Corollary 3.10, $w \in H^{s-1}_b$, and we must prove the microlocal regularity of $uv$. Using a partition of unity, it suffices to assume $\Gamma = (\mathbb{R}_+^n_x)_z \times K$ for a conic set $K \subset \mathbb{R}_+^n \setminus 0$; moreover, since the complement
of the wave front set is open, we can assume that $K$ is open. By assumption, we can then write 
\[ |\hat{u}(\xi)| = \frac{u_0(\xi)}{(\xi^s)^s}, u_0 \in L^2, \quad |\hat{v}(\xi)| = \left(\frac{\chi_K^{-1}(\xi)}{(\xi^s)^s} + \frac{\chi_K^c(\xi)}{(\xi^s)^s-1}\right) v_0(\xi), v_0 \in L^2, \]
where $\chi_K$ denotes the characteristic function of $K$, and $K^c$ the complement of $K$. Now, let $K_0 \subset K$ be closed and conic. Then
\[ \chi_{K_0}(\xi)|\hat{u}(\xi)||\xi|^s \leq \int \chi_{K_0}(\xi)\left(\frac{\chi_K^{-1}(\xi)}{(\xi^s)^s} + \frac{\chi_K^c(\xi)}{(\xi^s)^s-1}\right) u_0(\xi - \xi) v_0(\xi) \, d\xi \]
We want to use Lemma 3.7 to show that this is an element of $L^2$, thus finishing the proof. But we have
\[ \frac{|\xi|^s}{(\xi - \xi)^s(\xi^s)^s} \in L^\infty_{\xi} L^2_{\xi}, \]
and on the support of $\chi_{K_0}(\xi)\chi_K^c(\xi)$, we have $|\xi - \xi| \geq c|\xi|, c > 0$, thus
\[ \chi_{K_0}(\xi)\chi_K^c(\xi)\frac{1}{|\xi|^s|\xi^s|^{s-1}} \lesssim \frac{1}{|\xi|^s} \in L^\infty_{\xi} L^2_{\xi}, \]
since $s > n/2 + 1$. \hfill \Box

8.3. Conformal changes of the metric. Reconsidering the proof of Theorem 8.5, one cannot bound $\|S_{g(u)} - S_{g(v)}\|_{L(H^{s-1,0})}$ by $\|u - v\|_{\chi^{s,0}}$ in general,\(^{60}\) which however would immediately give uniqueness and stability of solutions to (8.13) in the space $\chi^{s,0}$. But there is a situation where we do have good control on $S_{g(u)} - S_{g(v)}$ as an operator from $H^{s-1,0}$ to $\chi^{s,0}$, namely when $\Box g(u)$ and $\Box g(v)$ have the same characteristic set, since in this case, in (8.17) the composition of $\Box g(v) - \Box g(u)$ with $S_{g(v)}$ loses no derivative (ignoring issues coming from the limited regularity of $g(u), g(v)$ for the moment – they will turn out to be irrelevant). This situation arises if $g(u) = \mu(u)g(0)$ for $\mu(u) \in C^\infty(M) + H^s_0(M)$; that this is in fact the only possibility is shown by a pointwise application of the following lemma.

**Lemma 8.13.** Let $d \geq 1$, and assume $g, g'$ are bilinear forms on $\mathbb{R}^{1+d}$ with signature $(1, d)$ such that the zero sets of the associated quadratic forms $q, q'$ coincide. Then $g = \mu g'$ for some $\mu \in \mathbb{R}^\times$.

**Proof.** By a linear change of coordinates, we may assume that $g'$ is the Minkowski bilinear form on $\mathbb{R}^{1+d}$. Let $g_{ij}$, $0 \leq i, j \leq d$, be the components of $g$, and let us write vectors in $\mathbb{R}^{1+d}$ as $(x_1, x') \in \mathbb{R} \times \mathbb{R}^d$. Since $g'(1, 0) \neq 0$, we have $g(1, 0) = g_00 \neq 0$. Dividing $g$ by $\mu := g_{00}$, we may assume $g_{00} = 1$; we now show that $g = g'$. For all $x' \in \mathbb{R}^d$, $|x'| = 1$ (Euclidean norm!), we have $q(1, x') = 0$ and $q(1, -x') = 0$, hence $q(1, x') - q(1, -x') = 0$, in coordinates
\[ 4 \sum_{i \geq 1} g_{0i} x_i' = 0, \quad |x'| = 1, \]

\(^{60}\)Indeed, consider a similar situation for scalar first order operators $P_a := \partial_t - a\partial_x$, $a \in \mathbb{R}$, on $[0, 1] \times \mathbb{R}_x$. The forward solution operator $S_a$ is constructed by integrating the forcing term along the bicharacteristics $s \rightarrow (s, x_0 - as)$ of $P_a$, and it is easy to see that $S_a \in L^2$, $L^2$. However, $S_a - S_b$ is constructed using the difference of integrals of the forcing $f$ along two different bicharacteristics, which one can naturally only bound using $df$, i.e. one only obtains the estimate $\|S_a - S_b\|_{H^1} \lesssim |a - b|\|f\|_{H^1}$, which is an estimate with a loss of 2 derivatives, similar to (8.17). The core of the problem is that there is no estimate of the form $\|f(a + t) - f\|_{H^1} \lesssim |a|\|f\|_{H^1}$, although such an estimate holds if the norm on the right is replaced by the $H^1$-norm.
and thus \( g_0 = 0 \) for all \( i \geq 1 \). Now let \( \tilde{q}(x') := q(0, x') \) and \( \tilde{q}'(x') := q'(0, x') \), then
\[
\tilde{q}(x') = -1 \iff q(1, x') = 0 \iff q'(1, x') = 0 \iff \tilde{q}'(x') = -1,
\]
thus by scaling \( \tilde{q} \equiv \tilde{q}' \) on \( \mathbb{R}^d \), hence by polarization \( g_{ij} = g_{ij}' \) for \( 1 \leq i, j \leq d \), and the proof is complete. \( \square \)

In this restricted setting, we have the following well-posedness result; notice that the topology in which we have stability is stronger than in Theorem 8.5, and we also allow more general non-linearities \( q \).

**Theorem 8.14.** Let \( s > n/2+6 \) and \( 0 < \alpha < 1 \). Let \( g_0 \in (C^\infty+H_b^{s,\alpha})(M; \text{Sym}^2bTM) \) be a metric satisfying the assumptions (1)-(8) in Section 7.2 on \( \Omega \), for example \( g_0 = g_{\text{dS}} \) and let \( \mu : X^{s,\alpha} \rightarrow X^{s,0}_R \) be\(^{61}\) a continuous map with \( \mu(0) = 1 \) and
\[
\|\mu(u) - \mu(v)\|_{X^{s,\alpha}} \leq L_\mu(R)\|u - v\|_{X^{s,\alpha}} \quad (8.21)
\]
for all \( u, v \in X^{s,\alpha} \) with norms \( \leq R \), where \( L_\mu : \mathbb{R}_\geq 0 \rightarrow \mathbb{R} \) is continuous and non-decreasing. Put \( g(u) := \mu(u)g_0 \).

1. Let
\[
q : X^{s,\alpha} \times H_b^{s-1,\alpha}(\Omega; b\text{Sym}^2bTM) \rightarrow H_b^{s-1,\alpha}(\Omega) \quad (8.22)
\]
be continuous with \( q(0) = 0 \), satisfying
\[
\|q(u, bdu) - q(v, bdv)\|_{H_b^{s-1,\alpha}(\Omega)} \leq L_q(R)\|u - v\|_{X^{s,\alpha}} \quad (8.23)
\]
for all \( u, v \in X^{s,\alpha} \) with norms \( \leq R \), where \( L_q : \mathbb{R}_\geq 0 \rightarrow \mathbb{R} \) is continuous and non-decreasing. Then there is a constant \( C_L > 0 \) so that the following holds: If \( L_q(0) < C_L \), then for small \( R > 0 \), there is \( C_f > 0 \) such that for all \( f \in H_b^{s-1,\alpha}(\Omega) \) with norm \( \leq C_f \), there exists a unique solution \( u \in X^{s,\alpha} \) of the equation
\[
\Box g(u)u = f + q(u, bdu) \quad (8.24)
\]
with norm \( \leq R \), which depends continuously on \( f \).

2. More generally, if
\[
q : X^{s,\alpha} \times H_b^{s-1,\alpha}(\Omega; b\text{Sym}^2bTM) \rightarrow X^{s,\alpha} \quad (8.25)
\]
is continuous with \( q(0) = 0 \) and satisfies
\[
\|q(u_1, bdu_1, w_1) - q(u_2, bdu_2, w_2)\|_{H_b^{s-1,\alpha}(\Omega)} \leq L_q(R)(\|u_1 - u_2\|_{X^{s,\alpha}} + \|w_1 - w_2\|_{H_b^{s-1,\alpha}(\Omega)}) \quad (8.26)
\]
for all \( u_j \in X^{s,\alpha} \), \( w_j \in H_b^{s-1,\alpha}(\Omega) \) with \( \|u_j\| + \|w_j\| \leq R \), then there is a constant \( C_L > 0 \) such that the following holds: If \( L_q(0) < C_L \), then for small \( R > 0 \), there is \( C_f > 0 \) such that for all \( f \in H_b^{s-1,\alpha}(\Omega) \) with norm \( \leq C_f \), there exists a unique solution \( u \in X^{s,\alpha} \) of the equation
\[
\Box g(u)u = f + q(u, bdu, \Box g(u)u) \quad (8.27)
\]
with \( \|u\|_{X^{s,\alpha}} + \|\Box g(u)u\|_{H_b^{s-1,\alpha}} \leq R \), which depends continuously on \( f \).

\(^{61}\)See Footnote 53.

\(^{62}\)\( X^{s,\alpha}_R \) was defined in (8.14).
Proof. First, note that \( N(\square g(u)) = \mu(u) N(\square g_0) \), which is a constant multiple of \( N(\square g_0) \) by the definition of the space \( \mathcal{X}^{s,\alpha} \). Thus, as in the proof of Theorem 8.5, there exists \( R_S > 0 \) such that

\[
S_{g(u)} : H^{s-1,\alpha}_b(\Omega)^* \rightarrow \mathcal{X}^{s,\alpha}
\]
is continuous with uniformly bounded operator norm

\[
\|S_{g(u)}\| \leq C_S;
\]
for \( \|u\|_{\mathcal{X}^{s,\alpha}} \leq R_S \); let us also assume that

\[
\mu(u) \geq c_0 > 0, \quad \|u\|_{\mathcal{X}^{s,\alpha}} \leq R_S.
\] (8.28)

We now prove the first half of the theorem. Let \( C_L := C_S^{-1} \), and assume that \( L_q(0) < C_L \), then \( L_q(R_q) < C_L \) for \( R_q > 0 \) small. Put \( \bar{R} := \min(R_S, R_q) \); let \( 0 < R \leq \bar{R} \), to be specified later, and put and \( C_f(R) = R(C_S^{-1} - L_q(R)) \); let \( f \in H^{s-1,\alpha}_b(\Omega)^* \) have norm \( \leq C_f(R) \). Let \( B(\bar{R}) \) denote the metric ball of radius \( R \) in \( \mathcal{X}^{s,\alpha} \), and define \( T : B(\bar{R}) \rightarrow B(\bar{R}) \),

\[
Tu := S_{g(u)}(f + q(u, bdu)).
\]
By the choice of \( R, C_L \) and \( C_f, T \) is well-defined by the same estimate as in the proof of Theorem 8.5. The crucial new feature here is that for \( R \) sufficiently small, \( T \) is in fact a contraction. This follows once we prove the existence of a constant \( C_i > 0 \) such that for \( u, v \in \mathcal{X}^{s,\alpha} \) with norms \( \leq R \), we have

\[
\|S_{g(u)} - S_{g(v)}\|_{L(H^{s-1,\alpha}_b, \mathcal{X}^{s,\alpha})} \leq C_S C_i L_\mu(R) \|u - v\|_{\mathcal{X}^{s,\alpha}}.
\] (8.29)

Indeed, assuming this, we obtain

\[
\|Tu - Tv\|_{\mathcal{X}^{s,\alpha}} \leq \|S_{g(u)}(q(u, bdu) - q(v, bdv))\|_{\mathcal{X}^{s,\alpha}} + \|(S_{g(u)} - S_{g(v)})(f + q(v, bdv))\|_{\mathcal{X}^{s,\alpha}}
\]
\[
\leq (C_S L_q(R) + C_S C_i L_\mu(R)(C_f(R) + L_q(R))\|u - v\|_{\mathcal{X}^{s,\alpha}};
\]
and since \( C_S L_q(R) \leq C_S L_q(\bar{R}) < \theta < 1 \) for \( R \leq \bar{R} \), we can choose \( R \) so small that

\[
C_S C_i L_\mu(R)(C_f(R) + L_q(R)) \leq \theta - C_S L_q(R),
\] (8.30)
where we use that \( C_f(R) \to 0 \) as \( R \to 0 \). With this choice of \( R, T \) is a contraction, thus has a unique fixed point \( u \in \mathcal{X}^{s,\alpha} \) which solves the PDE (8.24).

Continuing to assume (8.29), let us prove the continuous dependence of the solution \( u \) on \( f \). For this, let us assume that \( u_j \in \mathcal{X}^{s,\alpha}, j = 1, 2 \), solves

\[
\square_{g(u)}u_j = f_j + q(u_j, bdu_j),
\]
where \( f_j \in H^{s-1,\alpha}_b \) has norm \( \leq C_f \). Then, as in the proof of Theorem 8.5,

\[
\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}} \leq C_S \left( \|f_1 - f_2\|_{H^{s-1,\alpha}_b} + (L_q(R) + C_i L_\mu(R)(C_f + L_q(R)))\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}} \right).
\]
Because of (8.30), the prefactor of \( \|u_1 - u_2\| \) on the right hand side is \( \leq \theta < 1 \), hence we conclude

\[
\|u_1 - u_2\|_{\mathcal{X}^{s,\alpha}} \leq \frac{C_S}{1 - \theta} \|f_1 - f_2\|_{H^{s-1,\alpha}_b},
\]
as desired.
We now prove the crucial estimate (8.29) by using the identity in (8.17), as follows: By definition of ∂, we have

\[ \partial g(u) = \partial \mu(v)g_0 \frac{\mu(v)}{\mu(u)} = \frac{\mu(v)}{\mu(u)} \partial g(v) + E_{u,v}, \]  

(8.31)

where \( E_{u,v} \in H_b^{-1,\alpha} \mathcal{V}_b \) satisfies the estimate\(^{63}\)

\[ \|E_{u,v}\|_{H_b^{-1,\alpha} \mathcal{V}_b} \leq C \left\| \partial \left( \frac{\mu(v)}{\mu(u)} \right) \right\|_{H_b^{-1}} . \]

where the constant \( C \) is uniform for \( \|u\|_{\mathcal{X}^{s,\alpha}}, \|v\|_{\mathcal{X}^{s,\alpha}} \leq R \). Thus,

\[ \| (\partial g(v) - \partial g(u)) S_g(v) \|_{L(H_b^{-1,\alpha})} \]

\[ \leq \left\| 1 - \frac{\mu(v)}{\mu(u)} \right\|_{H_b^{-1,\alpha}} + \|E_{u,v}\|_{L(H_b^{-1,\alpha},H_b^{s,\alpha})} \|S_g(v)\|_{L(H_b^{s,\alpha},H_b^{-1,\alpha})} \]

\[ \leq \left\| 1 - \frac{\mu(v)}{\mu(u)} \right\|_{\mathcal{X}^{s,\alpha}} + C \|S_g(v)\|_{L(H_b^{s,\alpha},H_b^{-1,\alpha})} . \]

Now,

\[ \left\| 1 - \frac{\mu(v)}{\mu(u)} \right\|_{\mathcal{X}^{s,\alpha}} \leq C' \left\| \frac{1}{\mu(u)} \right\|_{\mathcal{X}^{s,\alpha}} \|\mu(u) - \mu(v)\|_{\mathcal{X}^{s,\alpha}} \leq C'L\mu(R) \|u - v\|_{\mathcal{X}^{s,\alpha}}, \]

(8.32)

where

\[ C'_i := C' \sup_{\|w\|_{\mathcal{X}^{s,\alpha}} \leq R} \left\| \frac{1}{\mu(u)} \right\|_{\mathcal{X}^{s,\alpha}} < \infty \]

by assumption (8.28) and Lemma 4.2. Likewise, since \( \partial d(\mu(v)/\mu(u)) = \partial d(1 - \mu(v)/\mu(u)) \),

\[ \left\| \partial d \left( \frac{\mu(v)}{\mu(u)} \right) \right\|_{H_b^{-1}} \leq \left\| 1 - \frac{\mu(v)}{\mu(u)} \right\|_{\mathcal{X}^{s,\alpha}} \leq C'L\mu(R) \|u - v\|_{\mathcal{X}^{s,\alpha}}; \]

therefore,

\[ \| (\partial g(v) - \partial g(u)) S_g(v) \|_{L(H_b^{-1,\alpha})} \leq C'L\mu(R) \|u - v\|_{\mathcal{X}^{s,\alpha}} \]  

(8.33)

for \( C_i = C'_j (1 + CC_S) \), and with \( \|S_g(u)\|_{L(H_b^{s,\alpha},H_b^{-1,\alpha})} \leq C_S \) and the identity in (8.17), we finally obtain the estimate (8.29).

We proceed to prove the second half of the theorem along the lines of the proof of [18, Theorem 2.24]. We work on the space

\[ Y^{s,\alpha} := \{ u \in \mathcal{X}^{s,\alpha} ; \partial g_0 u \in H_b^{s-1,\alpha} \} , \quad \|u\|_{Y^{s,\alpha}} = \|u\|_{\mathcal{X}^{s,\alpha}} + \|\partial g_0 u\|_{H_b^{s-1,\alpha}} , \]  

(8.34)

which is complete.\(^{64}\) The idea is that all operators \( \partial g(u) \) are (pointwise) multiples of each other modulo first order operators, thus \( \partial g_0 \) is as good as any other such operator, and therefore \( \partial g \) in the third argument of the non-linearity \( q \) acts as a first order operator on the successive approximations \( T^k(0) \) in the iteration scheme

\[^{63}\text{To define a norm of an element } E \in H_b^{s,\alpha} \mathcal{V}_b(M) \text{, use a partition of unity on } M \text{ to reduce this task to a local one, and as the norm of } E \in H_b^{s,\alpha} \mathcal{V}_b(X^+_b), \text{ take the sum of the } H_b^{s,\alpha}\text{-norms of the coefficients of } E.\]

\[^{64}\text{Indeed, if } (u_k)_k \text{ is a Cauchy sequence in } Y^{s,\alpha}, \text{ then } u_k \to u \in \mathcal{X}^{s,\alpha} \text{ and } \partial g_0 u_k \to \partial g_0 u \in H_b^{s-2,\alpha}, \text{ but } \partial g_0 u_k \text{ is also Cauchy in } H_b^{s-1,\alpha}, \text{ hence } \partial g_0 u_k \in \mathcal{X}^{s-1,\alpha} \text{ and } \partial g_0 u_k \to \partial g_0 u \in H_b^{s-1,\alpha}, \text{ and therefore } u_k \to u \in Y^{s,\alpha}.\]
implicit in the application of the Banach fixed point theorem used above to solve equation (8.24). Thus, let $B(R)$ denote the metric ball of radius $R \leq R_S$ in $\mathcal{Y}^{s,\alpha}$, and define $T: B(R) \to \mathcal{Y}^{s,\alpha}$,

$$Tu := S_{g(u)}(f + q(u))$$

where we write $q(u) := q(u, bdu, \Box_{g(u)} u)$ to simplify the notation. We will prove that for $R > 0$ small enough, the image of $T$ is contained in $B(R)$. We first estimate for $u \in B(R)$ and $w \in \mathcal{Y}^{s,\alpha}$, using (8.31) and an estimate similar to (8.32) (with $v = 0$):

$$\|\Box_{g(u)} w\|_{H^{s-1,\alpha}} \leq \|\Box_{g(u)} w\|_{H^{s-1,\alpha}} + \|\Box_{g(u)} - \Box_{g(0)} w\|_{H^{s-1,\alpha}}$$

$$\leq \|w\|_{\mathcal{Y}^{s,\alpha}} + \tilde{C}_i \|u\|_{X^{s,\alpha}} \|w\|_{\mathcal{Y}^{s,\alpha}} \leq (1 + \tilde{C}_i R)\|w\|_{\mathcal{Y}^{s,\alpha}}$$

for some constant $\tilde{C}_i > 0$. For convenience, we choose $R \leq \tilde{C}_i^{-1}$, thus

$$\|\Box_{g(u)} w\|_{H^{s-1,\alpha}} \leq 2\|w\|_{\mathcal{Y}^{s,\alpha}}, \quad w \in \mathcal{Y}^{s,\alpha}.$$ 

Using this, we obtain for $u, v \in B(R)$:

$$\|\Box_{g(u)} u - \Box_{g(v)} v\|_{H^{s-1,\alpha}} \leq \|\Box_{g(u)} (u - v)\|_{H^{s-1,\alpha}} + \|\Box_{g(u)} - \Box_{g(v)} v\|_{H^{s-1,\alpha}}$$

$$\leq 2\|u - v\|_{\mathcal{Y}^{s,\alpha}} + \left(\left|1 - \frac{\mu(u)}{\mu(v)}\right|\Box_{g(u)} \Box_{g(v)} + E_{u,v}\right)\|v\|_{H^{s-1,\alpha}}$$

$$\leq 2\|u - v\|_{\mathcal{Y}^{s,\alpha}} + C' L_\mu(R)\|u - v\|_{X^{s,\alpha}} \left(\|\Box_{g(u)} v\|_{H^{s-1,\alpha}} + \|v\|_{X^{s,\alpha}}\right)$$

$$\leq (2 + 3C'L_\mu(R)R)\|u - v\|_{\mathcal{Y}^{s,\alpha}} \leq 3\|u - v\|_{\mathcal{Y}^{s,\alpha}}$$

for sufficiently small $R$, where $C' = C'_i (1 + C)$. Thus, with $L'_q(R) := 3L_q(R)$, we have

$$\|q(u) - q(v)\|_{H^{s-1,\alpha}} \leq L'_q(R)\|u - v\|_{\mathcal{Y}^{s,\alpha}}$$

for $u, v \in \mathcal{Y}^{s,\alpha}$ with norm $\leq R$.

We can now analyze the map $T$: First, for $u \in B(R)$ and $f \in H^{s-1,\alpha}_b$, $\|f\| \leq C_f$, we have, recalling (8.33), here applied with $v = 0$,

$$\|Tu\|_{X^{s,\alpha}} \leq C_S(C_f + L'_q(R)R),$$

$$\|\Box_{g(0)} Tu\|_{H^{s-1,\alpha}} \leq \|\Box_{g(0)} - \Box_{g(u)} S_{g(u)}(f + q(u))\|_{H^{s-1,\alpha}} + \|f + q(u)\|_{H^{s-1,\alpha}}$$

$$\leq (1 + C_i L_\mu(R)R)\left(1 + L'_q(R)R\right).$$

Thus, if $L'_q(0) < (1 + C_S)^{-1}$, then

$$C_f(R) := R\left(1 + C_S + C_i L_\mu(R)R - L'_q(R)\right)$$

is positive for small enough $R > 0$. We conclude that for $f \in H^{s-1,\alpha}_b$ with norm $\leq C_f(R)$, the map $T$ indeed maps $B(R)$ into itself. We next have to check that $T$ is in fact a contraction on $B(R)$, where we choose $R$ even smaller if necessary. As in the proof of the first half of the theorem, we can arrange

$$\|Tu - Tv\|_{X^{s,\alpha}} \leq \theta\|u - v\|_{\mathcal{Y}^{s,\alpha}}, \quad u, v \in B(R)$$

for some fixed $\theta < 1$. Moreover, for $u, v \in B(R)$,

$$\|\Box_{g(0)}(Tu - Tv)\|_{H^{s-1,\alpha}}$$

$$\leq \|\Box_{g(0)} S_{g(u)}(q(u) - q(v))\|_{H^{s-1,\alpha}} + \|\Box_{g(0)} (S_{g(u)} - S_{g(v)})(f + q(v))\|_{H^{s-1,\alpha}}.$$
The first term on the right can be estimated by
\[
\|q(u) - q(v)\|_{H^{s-1,α}_b} + \|(\Box_{g(u)} - \Box_{g(0)})S_g(u)(q(u) - q(v))\|_{H^{s-1,α}_b} \\
\leq L'_q(R)(1 + C_iL_μ(R))\|u - v\|_{Y^{s,α}}.
\]
For the second term on the right hand side of (8.36), we use the algebraic identity
\[
\Box_{g(0)}(S_g(u) - S_g(v)) = (I + (\Box_{g(0)} - \Box_{g(u)}))S_g(u)(\Box_{g(u)} - \Box_{g(0)})S_g(v),
\]
which gives
\[
\|\Box_{g(0)}(S_g(u) - S_g(v))\|_{L^2(X^{s-1,α})} \leq (1 + C_iL_μ(R)C_iL_μ(R))\|u - v\|_{Y^{s,α}}.
\]
Plugging this into equation (8.36), we obtain
\[
\|\Box_{g(0)}(Tu - Tv)\|_{H^{s-1,α}_b} \leq C'(R)\|u - v\|_{Y^{s,α}}
\]
with
\[
C'(R) = (1 + C_iL_μ(R)R)(L'_q(R) + C_iL_μ(R)(C_f(R) + L'_q(R)R)).
\]
Now if \(L'_q(0)\) is sufficiently small, then since the second summand of the second factor of \(C'(R)\) tends to 0 as \(R \to 0\), we can choose \(R\) so small that \(C'(R) < 1 - \theta\), and we finally get with (8.35):
\[
\|Tu - Tv\|_{Y^{s,α}} \leq \theta\|u - v\|_{Y^{s,α}}, \quad u, v \in B(R),
\]
for some \(\theta' < 1\), which proves that \(T\) is a contraction on \(B(R)\), thus has a unique fixed point, which solves the PDE (8.27). The continuous dependence on \(f\) is shown as in the proof of the first half of the theorem.

\[\square\]

Remark 8.15. The space \(Y^{s,α}\) introduced in the proof of the second part, see equation (8.34), which the solution \(u\) of equation (8.27) belongs to, is a coisotropic space similar to the ones used in [34, 18], with the difference being that here \(\Box_{g_0}\) is allowed to have non-smooth coefficients. It still is a natural space in the sense that the space of elements of the form \(c(\phi \circ t_1) + w, \ c \in \mathbb{C}, w \in \hat{C}^∞\), is dense. Indeed, since \(\Box_{g_0}\) annihilates constants, it suffices to check that \(\hat{C}^∞\) is dense in \(Y^{s,α}_{0,\alpha} := \{u \in H^{s,α}_b; \ \Box_{g_0}u \in H^{s-1,α}_b\}\). Let \(J_ε\) be a mollifier as in Lemma 6.5. Given \(u \in Y^{s,α}_0\), put \(u_ε := J_εu\). Then \(u_ε \to u\) in \(H^{s,α}_b\), and
\[
\Box_{g_0}u_ε = J_ε\Box_{g_0}u + [\Box_{g_0}, J_ε]u;
\]
the first term converges to \(\Box_{g_0}u\) in \(H^{s-1,α}_b\). To analyze the second term, observe that we have
\[
\Box_{g_0}J_ε - J_ε\Box_{g_0} = \Box_{g_0}(J_ε - I) + (I - J_ε)\Box_{g_0} \to 0 \text{ strongly in } \mathcal{L}(H^{s+1,α}_b, H^{s-1,α}_b),
\]
and since \(H^{s+1,α}_b \subset H^{s,α}_b\) is dense, it suffices to show that \([\Box_{g_0}, J_ε]\) is a bounded family in \(\mathcal{L}(H^{s,α}_b, H^{s-1,α}_b)\). Write
\[
\Box_{g_0} = Q_1 + Q_2 + E, \quad Q_1 \in \text{Diff}^0, Q_2 \in H^{s,α}_b\text{Diff}^1, E \in (C^∞ + H^{s-1,α}_b)\text{Diff}^1.
\]
Then \([Q_1, J_ε]\) and \([E, J_ε]\) are bounded in \(\mathcal{L}(H^{s,α}_b, H^{s-1,α}_b)\). Now \(Q_2 J_ε\) can be expanded into a leading order term \(Q'_ε\) and a remainder \(R_{1,ε}\) which is uniformly bounded in \(H^{s,α}_b\); but also \(J_ε Q_2\) has an expansion by Theorem 3.12 (2a) (with \(k = k' = 1\)) into the same leading order term \(Q'_ε\) and a remainder \(R_{2,ε}\) which is uniformly bounded in \(H^{s,α}_b\). Hence \([Q_2, J_ε] = R_{1,ε} - R_{2,ε}\) is bounded in \(\mathcal{L}(H^{s,α}_b, H^{s-1,α}_b)\) by Proposition 3.9, finishing the argument.
### 8.4. Quasilinear Klein-Gordon equations

One has corresponding results to the theorems in the previous two sections for quasilinear Klein-Gordon equations, i.e. for Theorems 8.5, 8.8 and 8.14 with \( \Box \) replaced by \( \Box - m^2 \); only the function spaces need to be adapted to the situation at hand, as follows: Denote \( P := \Box_{g_{12}} - m^2 \) and let \( (\sigma_j)_{j \in \mathbb{N}} \) be the sequence of poles of \( \tilde{P}(\sigma)^{-1} \), with multiplicity, sorted by increasing \(- \text{Im} \sigma_j\).\(^65\) Let us assume that the ‘mass’ \( m \in \mathbb{C} \) is such that \( \text{Im} \sigma_1 < 0 \). A major new feature of Klein-Gordon equations as compared to wave equations is that non-linearities like \( q(u) = u^p \) can be dealt with, more generally

\[
q(u, bdu) = \sum_j u^{e_j} \prod_{i=1}^{N_j} X_{ji} u, \quad e_j + N_j \geq 2, X_{ji} \in V_b.
\]

See [18, Theorem 2.24] for the related discussion of semilinear equations. We give an (incomplete) short list of possible scenarios and the relevant function spaces; for concreteness, we work on exact de Sitter space, but our methods work in much greater generality.

1. If \( \text{Im} \sigma_1 \neq \text{Im} \sigma_2 \), as is e.g. the case for small mass \( m^2 < (n - 1)^2/4 \), let \( \alpha_0 = \min(1, \text{Im} \sigma_1 - \text{Im} \sigma_2) \), and for \(- \text{Im} \sigma_1 < \alpha < - \text{Im} \sigma_1 + \alpha_0 \), put

\[
X_a^{s, \alpha} := \mathbb{C}(\tau^{i\sigma_1}) \oplus H_{b_{a}}^{s, \alpha},
\]

We can then solve quasilinear equations of the form explained above with forcing in \( H_{b_{a}}^{s-1, \alpha} \) and get one term, \( e^{i\sigma_1} \), in the expansion of the solution. Notice that if the mass is real and small, then all \( \sigma_j \) are purely imaginary, hence the term in the expansion is real as well if all data are, which is necessary for an analogue of Theorem 8.8 to hold.

2. If \( \text{Im} \sigma_1 - \text{Im} \sigma_2 < 1 \), e.g. if \( m^2 \geq n(n - 2)/4 \), let \( \alpha_0 := \min(1, \text{Im} \sigma_1 - \text{Im} \sigma_3) \), and for \(- \text{Im} \sigma_2 < \alpha < - \text{Im} \sigma_1 + \alpha_0 \), put

\[
X_a^{s, \alpha} := \mathbb{C}(\tau^{i\sigma_1}) \oplus \mathbb{C}(\tau^{i\sigma_2}) \oplus H_{b_{a}}^{s, \alpha}, \quad \sigma_2 \neq \sigma_1,
\]

\[
X_1^{s, \alpha} := \mathbb{C}(\tau^{i\sigma_1}) \oplus \mathbb{C}(\tau^{i\sigma_1} \log \tau) \oplus H_{b_{1}}^{s, \alpha}, \quad \sigma_2 = \sigma_1,
\]

then we can solve equations as above with forcing in \( H_{b_{a}}^{s-1, \alpha} \) and obtain two terms in the expansion. For masses \( m^2 > (n - 1)^2/4 \), we have \( \text{Im} \sigma_1 = \text{Im} \sigma_2 =: -\sigma \) and \( \text{Re} \sigma_1 = -\text{Re} \sigma_2 =: \rho \), hence the terms in the expansion for real data are a linear combination of \( \tau^\sigma \cos(\rho \tau) \) and \( \tau^\sigma \sin(\rho \tau) \).

3. If the forcing decays more slowly than \( \tau^{i\sigma_1} \), then with \( 0 < \alpha < - \text{Im} \sigma_1 \), we can work on the space

\[
X_a^{s, \alpha} := H_{b_{a}}^{s-1, \alpha},
\]

with forcing in \( H_{b_{a}}^{s-1, \alpha} \).

To prove the higher regularity statement in Theorem 8.8 for quasilinear Klein-Gordon equations, one first obtains higher regularity \( H_0^{s, \alpha} \) with \( 0 \leq \alpha < - \text{Im} \sigma_1 \) and then, if the amount of decay of the forcing is high enough to allow for it, applies Theorem 7.9 to obtain a partial expansion of \( u \).

In the third setting, the assumption that the mass \( m \) is independent of the solution \( u \) can easily be relaxed: Namely, assuming that \( m = m(u) \) or \( m = m(u, bdu) \) with continuous (or Lipschitz) dependence on \( u \in X_a^{s, \alpha} \), the poles of the inverse of the normal operator family of \( \Box_{g(u)} - m(u)^2 \) depend continuously on \( u \), hence

\[^{65}\text{See equation (8.9) for the explicit formula. Also keep in mind that everything we do works in greater generality; we stick to the case of exact de Sitter space here for clarity.}\]
for small $u$, there is still no pole with imaginary part $\geq -\alpha$, therefore the solution operator produces an element of $H_b^{s,\alpha}$ for small $u$; thus, well-posedness results analogous to Theorems 8.5 and 8.14 continue to hold in this setting. If the forcing in fact does decay faster than $\tau^{\alpha_1}$, these results can be improved in many cases: Once one has the solution $u \in H_b^{s,\alpha}$, in particular the mass $m(u)$ is now fixed, one can apply Theorem 7.9 to obtain a partial expansion of $u$.

8.5. **Backward problems.** We briefly indicate how our methods also apply to backward problems on static patches of (asymptotically) de Sitter spaces; see Figure 4 for an exemplary setup.

Figure 4. Setup for a backward problem on static de Sitter space:
We work on spaces with high decay, consisting of functions supported at $H_2$ and extendible at $H_1$ (notice the switch compared to the forward problem). In the situation shown, we prescribe initial data at $H_2$ or, put differently, forcing in the shaded region.

We only state an analogue of Theorem 8.8, but remark that analogues of Theorems 8.5 and 8.14 also hold. For simplicity, we again only work on static de Sitter spaces. We use the notation from Section 7.2.

**Theorem 8.16.** Let $s > n/2 + 6$, $N, N' \in \mathbb{N}$, and suppose $c_k \in C^\infty(\mathbb{R}; \mathbb{R})$, $g_k \in (C^\infty + H_b^s)(M; \text{Sym}^2 b TM)$ for $1 \leq k \leq N$; for $r \in \mathbb{R}$, define the map

$$
g: H_b^{s,r}(\Omega)^{-\cdot} \to (C^\infty + H_b^{s,r})(M; \text{Sym}^2 b TM), \quad g(u) = \sum_{k=1}^N c_k(u) g_k,
$$

and assume $g(0) = g_{\text{dS}}$. Moreover, define

$$
q(u, b du) = \sum_{j=0}^{N'} u^{c_j} \prod_{k=1}^{N_j} X_{jk} u, \quad c_j + N_j \geq 2, X_{jk} \in \mathcal{V}_b(M),
$$

and let further $L \in \text{Diff}^1_b$ with real coefficients. Then there is $r_\ast \in \mathbb{R}$ such that for all $r > r_\ast$, the following holds: For small $R > 0$, there exists $C_f > 0$ such that for all $f \in H_b^{s-1,r}(\Omega; \mathbb{R})^{-\cdot}$ with norm $\leq C_f$, the equation

$$(\Box g(u) + L)u = f + q(u, b du)$$

has a unique solution $u \in H_b^{s,r}(\Omega; \mathbb{R})^{-\cdot}$ with norm $\leq R$, and in the topology of $H_b^{s-1,r}(\Omega)^{-\cdot}$, $u$ depends continuously on $f$. If one in fact has $f \in H_b^{s'-1,r}(\Omega; \mathbb{R})^{-\cdot}$ for some $s' \in (s, \infty)$, then $u \in H_b^{s,r}(\Omega; \mathbb{R})^{-\cdot}$.

**Remark 8.17.** Notice that the structure of lower order terms is completely irrelevant here! One could in fact let $L$ depend on $u$ in a Lipschitz fashion and still have well-posedness.
Proof of Theorem 8.16. Let \( r_0 < 0 \) as given by Lemma 7.3, and suppose \( r > -r_0 \). As in the proof of Lemma 7.5, we obtain for \( u \in H_b^{s,r}(\Omega)^{-}\bullet \) with \( \|u\| \leq R, R > 0 \) sufficiently small, a backward solution operator
\[
S_{g(u)} : H_b^{-1,r}(\Omega)^{-}\bullet \rightarrow H_b^{0,r}(\Omega)^{-}\bullet
\]
for \( \Box_g(u) + L \), with uniformly bounded operator norm. Now, if we take \( r > r_* \) with \( r_* \geq -r_0 \) sufficiently large, \( S_{g(u)} \) restricts to an operator
\[
S_{g(u)} : H_b^{s-1,r}(\Omega)^{-}\bullet \rightarrow H_b^{s,r}(\Omega)^{-}\bullet.
\]
Indeed, given \( v \in H_b^{0,r}(\Omega)^{-}\bullet \) solving \( \Box_g(u) v \in H_b^{s-1,r}(\Omega)^{-}\bullet \), we apply the propagation near radial points, Theorem 6.10, this time propagating regularity away from the boundary, and the real principal type propagation and elliptic regularity iteratively to prove \( v \in H_b^{s,r}(\Omega)^{-}\bullet \); the last application of the radial points result requires that \( r \) be larger than an \( s \)-dependent quantity, hence the condition on \( r_* \) in the statement of the theorem. From here, a Picard iteration argument, namely considering
\[
u \mapsto S_{g(u)}(f + g(u, bdu))
\]
gives existence and well-posedness. The higher regularity statement is proved as in the proof of Theorem 8.8. \( \square \)

A slightly more elaborate version of this theorem, applied to the Einstein vacuum equations, would enable us to make a ‘scattering theory’ construction of vacuum asymptotically de Sitter spacetimes as done in the Kerr setting in [9]. In fact, such a construction would automatically work in the Kerr-de Sitter setting as well, yielding the existence of dynamical vacuum black holes in de Sitter spacetimes; the point here is that for the backward problem, one works in decaying spaces, where one has non-trapping estimates in the smooth setting, as proved in [17] by a positive commutator argument, which, along the lines of the proofs in Section 6, should hold in the non-smooth setting as well. We will elaborate on this approach to such problems in future research.

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Department of Mathematics, Stanford University, CA 94305-2125, USA

E-mail address: phintz@math.stanford.edu