RESONANCE EXPANSIONS FOR TENSOR-VALUED WAVES ON ASYMMETRICALLY KERR-DE SITTER SPACES

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Abstract. In recent joint work with András Vasy [19], we analyze the low energy behavior of differential form-valued waves on black hole spacetimes. In order to deduce asymptotics and decay from this, one in addition needs high energy estimates for the wave operator acting on sections of the form bundle. The present paper provides these on perturbations of Schwarzschild-de Sitter spaces in all spacetime dimensions \( n \geq 4 \). In fact, we prove exponential decay, up to a finite-dimensional space of resonances, of waves valued in any finite rank subbundle of the tensor bundle, which in particular includes differential forms and symmetric tensors. As the main technical tool for working on vector bundles that do not have a natural positive definite inner product, we introduce pseudodifferential inner products, which are inner products depending on the position in phase space.

1. Introduction

We continue the analysis of (linear) aspects of the black hole stability problem in the spirit of earlier works by Dyatlov, Vasy and the author [10, 17, 18, 27] by studying linear tensor-valued wave equations on perturbations of Schwarzschild-de Sitter spaces with spacetime dimension \( n \geq 4 \); in particular, this includes wave equations for differential forms and symmetric 2-tensors. In our main result, we establish exponential decay up to a finite-dimensional space of resonances:

**Theorem 1.** Let \((M, g)\) denote a Kerr-de Sitter spacetime in \( n \geq 4 \) spacetime dimensions, with small angular momentum. Let \( E \subset T_k \) be a subbundle of the bundle \( T_k \) of (covariant) rank \( k \) tensors on \( M \), so that the tensor wave operator \( \Box_g = -\text{tr} \nabla^2 \) acts on sections of \( E \); for instance, one can take \( E \) to be equal to \( T_k \), symmetric rank \( k \)-tensors or differential forms of degree \( k \). Let \( \Omega \) denote a small neighborhood of the domain of outer communications, bounded beyond but close to the cosmological and the black hole horizons by spacelike boundaries, and let \( t^* \) be a smooth time coordinate on \( \Omega \). See Figure 1 for the setup.

Then for any \( f \in C^\infty_c(\Omega, E) \), the wave equation \( \Box_g u = f \) has a unique global forward solution (supported in the causal future of \( \text{supp} f \)) \( u \in C^\infty(\Omega, E) \), and \( u \) has an asymptotic expansion

\[
    u = \sum_{j=1}^{N} \sum_{m=0}^{m_j-1} \sum_{\ell=1}^{d_j} e^{-it^* \sigma_j} t^*_m u_{j m \ell}(x) + u',
\]

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where \( u_{jm\ell} \in \mathbb{C} \), the resonant states \( a_{jm\ell} \), only depending on \( \Box_g \), are smooth functions of the spatial coordinates and \( \sigma_j \in \mathbb{C} \) are resonances with \( \text{Im} \sigma_j > -\delta \) (whose multiplicity is \( m_j \geq 1 \) and for which the space of resonant states has dimension \( d_j \)), while \( u' \in e^{-\delta t} L^\infty(\Omega, \mathcal{E}) \) is exponentially decaying, for \( \delta > 0 \) small; we measure the size of sections of \( \mathcal{E} \) by means of a \( t_* \)-independent positive definite inner product.

The same result holds true if we add any stationary 0-th order term to \( \Box \), and one can also add stationary first order terms which are either small or subject to natural, but somewhat technical condition, which we explain in Remark 4.9. In fact, we can even work on spacetimes which merely approach a stationary perturbation of Schwarzschild-de Sitter space exponentially fast. See Section 2 for the form of the Schwarzschild-de Sitter metric and the precise assumptions on regularity and asymptotics of perturbations, for details on the setup, and Theorem 2.1 for the full statement of Theorem 1.

![Figure 1](setup.png)

**Figure 1.** Setup for Theorem 1 and Theorem 2.1 below. Shown are the black hole horizon \( \mathcal{H}^+ \) and cosmological horizon \( \overline{\mathcal{H}}^+ \), beyond which we put an artificial spacelike hypersurface \( H_2 \) with two connected components. The hypersurface \( H_1 \) plays the role of a Cauchy hypersurface, and the forcing as well as the solution to the wave equation are supported in its causal future. The domain \( \Omega \) is bounded by the hypersurfaces \( H_1 \) and \( H_2 \). The ‘point at future infinity’ in the usual Penrose diagrammatic representation is shown blown-up here, since the wave operator is well-behaved (namely, a \( b \)-operator in the sense of Melrose [23]) on the blown-up space, and the asymptotic information is encoded on the front face \( \mathcal{F} \) of the blow-up.

The resonances and resonant states depend strongly on the precise form of the operator and which bundle one is working on. In the case of the trivial bundle, thus considering scalar waves, they were computed in the Kerr-de Sitter setting by Dyatlov [10], following work by Sá Barreto and Zworski [3] as well as Bony and Häfler [5]. In recent work with Vasy [19], we compute the resonances for the Hodge d’Alembertian on differential forms, which equals the tensor wave operator plus a zeroth order curvature term: We show that there is only one resonance \( \sigma_1 = 0 \) in \( \text{Im} \sigma \geq 0 \), with multiplicity \( m_1 = 1 \), and we canonically identify the 0-resonant states with cohomological information of the underlying spacetime. Note however that [19] deals with a very general class of warped product type spacetimes with asymptotically hyperbolic ends, while the present paper is only concerned with (perturbations of) Schwarzschild-de Sitter spacetimes. We remark that in general one expects that \( \Box_g = -\text{tr} \nabla^2 \) on a bundle \( \mathcal{E} \) as in Theorem 1 has resonances in \( \text{Im} \sigma > 0 \), thus causing linear waves to grow exponentially in time.
We point out that if there are no resonances for $\Box_g$ (plus lower order terms) in $\Im \sigma \geq 0$, thus solutions decay exponentially, we can combine Theorem 1 with the framework for quasilinear wave-type equations developed by the author [16] and in collaboration with Vasy [18] and immediately obtain the **global solvability of quasilinear equations**. This also works if there is merely a simple resonance at $\sigma = 0$ which is annihilated by the nonlinearity.

The point of view from which we approach the proof of Theorem 1 was originally developed by Vasy [27] and extended by Vasy and the author [17]. In the context of scalar waves, more general and precise versions of Theorem 1 are known, see the references below. Thus, the main novelty is that we give a conceptually transparent framework that allows us to deal with tensor-valued waves on black hole spacetimes, where the natural inner product on the tensor bundle induced by the spacetime metric is not positive definite. The central motivation for the study of such waves is the **black hole stability problem**, see the lecture notes by Dafermos and Rodnianski [8] for details. Notice that in order to obtain energy estimates for waves, one needs to work with positive inner products on the tensor bundle, relative to which however $\Box$ is in general not well-behaved: Most severely, it is in general far from being symmetric at the trapped set, which prevents the use of estimates at normally hyperbolic trapping. In the context of black hole spacetimes, such estimates were pioneered by Wunsch and Zworski [29] and Dyatlov [13, 14]. On a pragmatic level, we show that one can conjugate $\Box$ by a suitable 0-th order pseudodifferential operator so as to make the conjugated operator (almost) symmetric at the trapped set with respect to a positive definite inner product, and one can then directly apply Dyatlov’s results [14]. The conceptually correct point of view to accomplish this conjugation is that of **pseudodifferential inner products**, which we introduce in this paper.

Roughly speaking, pseudodifferential inner products replace ordinary inner products $\int \langle B_0(u), v \rangle \, dg$, where $B_0$ is an inner product on the fibers of $\mathcal{E}$, mapping $\mathcal{E}$ into its anti-dual $\mathcal{E}^*$, by ‘inner products’ of the form $\int \langle B(x,D)u, v \rangle \, dg$, where $B \in \Psi^0$ is a zeroth order pseudodifferential operator mapping sections of $\mathcal{E}$ into sections of $\mathcal{E}^*$. Thus, we gain a significant amount of flexibility, since we can allow the inner product to **depend on the position in phase space**, rather than merely on the position in the base: Indeed, the principal symbol $b = \sigma_0(B)$ is an inner product on the vector bundle $\pi^*\mathcal{E}$ over $T^*M \setminus 0$, where $\pi: T^*M \setminus 0 \rightarrow M$ is the projection.

One can define adjoints of operators $P \in \Psi^m(M,\mathcal{E})$ (e.g. $P = \Box_g$), acting on sections of $\mathcal{E}$, relative to a pseudodifferential inner product $B$, denoted $P^*B$, which are well-defined modulo smoothing operators. Moreover, there is an invariant symbolic calculus involving the subprincipal operator $S_{\text{sub}}(P)$, which is a first order differential operator on $T^*M \setminus 0$ acting on sections of $\pi^*\mathcal{E}$ that invariantly encodes the subprincipal part of $P$, for computing principal symbols of commutators and imaginary parts of such operators. In the case that $P$ is principally scalar and real, the principal symbol of $P - P^{*B} \in \Psi^{m-1}(M,\mathcal{E})$ then vanishes in some conic subset of phase space $T^*M \setminus 0$ if and only if $S_{\text{sub}}(P) - S_{\text{sub}}(P)^{\text{sub}}$ does, which in turn can be reinterpreted as saying that the principal symbol of $QPQ^{-1} - (QPQ^{-1})^{*B_0}$ vanishes there, where $B_0$ is an ordinary inner product on $\mathcal{E}$, and $Q \in \Psi^0(M,\mathcal{E})$ is a suitably chosen elliptic operator. In the case considered in Theorem 1 then, it

\footnote{In other words, we reduce the high frequency analysis of tensor-valued waves to an essentially scalar problem.}
turns out that the subprincipal operator of $\Box_g$ on tensors, decomposed into parts acting on tangential and normal tensors according to the product decompositions $M = \mathbb{R} \times X_t$ and $X = (r_-, r_+) \times S^{n-2}$, at the trapped set equals the derivative along the Hamilton vector field $H_G$, $G$ the dual metric function, plus a nilpotent zeroth order term. This then enables one to choose a positive definite inner product $b$ on $\pi^*E$ relative to which $\mathcal{S}_{\text{sub}}(\Box_g)$ is arbitrarily close to being symmetric at the trapped set; thus with $B = b(x,D)$, the operator $\Box_g$ is arbitrarily close to being symmetric with respect to the pseudodifferential inner product $B$. Hence, one can indeed appeal to Dyatlov’s results on spectral gaps by considering a conjugate of $\Box_g$, which is the central ingredient in the proof of Theorem 1.

1.1. Related work. The study of non-scalar waves on black hole backgrounds has focused primarily on Maxwell’s equations: Sterbenz and Tataru [25] showed local energy decay for Maxwell’s equations on a class of spherically symmetric asymptotically flat spacetimes including Schwarzschild. Blue [4] established conformal energy and pointwise decay estimates in the exterior of the Schwarzschild black hole; Andersson and Blue [1] proved similar estimates on slowly rotating Kerr spacetimes. These followed earlier results for Schwarzschild by Ingelese and Nicolo [22] on energy and pointwise bounds for integer spin fields in the far exterior of the Schwarzschild black hole, and by Bachelot [2], who proved scattering for electromagnetic perturbations. Finster, Kamran, Smoller and Yau [15] proved local pointwise decay for Dirac waves on Kerr. There are further works which in particular establish bounds for certain components of the Maxwell field, see Donniger, Schlag and Soffer [9] and Whiting [28]. Dafermos [6], [7] studied the non-linear Einstein-Maxwell-scalar field system under the assumption of spherical symmetry.

The framework in which we describe resonances was introduced by Vasy [27]. In the scalar setting, this can directly be combined with estimates at normally hyperbolic trapping [13, 14, 29] to obtain resonance expansions for scalar waves. On exact Kerr-de Sitter space, Dyatlov proved a significant strengthening of this in [11], obtaining a full resonance expansion for scalar waves, improving on the result of Bony and H¨afner [5] in the Schwarzschild-de Sitter setting, which in turn followed S´a Barreto and Zworski [3]. Vasy [26] proved the meromorphic continuation of the resolvent of the Laplacian on differential forms on asymptotically hyperbolic spaces, and the fact that the underlying analysis of [27] works on sections of vector bundles just as it does on functions is fundamental for the present paper.

1.2. Structure of the paper. In Section 2, we recall the Schwarzschild-de Sitter metric and its extension past the horizons, put it into the framework of [17, 27] for the study of asymptotics of waves, and establish the normally hyperbolic nature of its trapping. We proceed to sketch the proof of Theorem 1, leaving the discussion of high energy estimates at the trapped set to the subsequent sections, which comprise the central part of the paper: We introduce pseudodifferential inner products on vector bundles in full generality in Section 3, and we use the theory developed there in Section 4 to study pseudodifferential inner products for wave operators on tensor bundles, uncovering the nilpotent nature of the subprincipal operator of $\Box$ on Schwarzschild-de Sitter space at the trapping in Section 4.2 and thereby finishing the proof of Theorem 1.

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2. Detailed setup and proof of the main theorem

We recall the form of the \( n \)-dimensional Schwarzschild-de Sitter metric, \( n \geq 4 \): We equip \( M = \mathbb{R}_t \times X \), \( X = (r_-, r_+) \times S^{n-2}_\mu \), with \( r_{\pm} \) defined below, with the metric

\[
g_0 = \mu dt^2 - (\mu^{-1} dr^2 + r^2 d\omega^2),
\]

where \( d\omega^2 \) is the round metric on the sphere \( S^{n-2}_\mu \), and \( \mu = 1 - \frac{2M}{r^{n-3}_-} - \lambda r^2 \), \( \lambda = \frac{2\Lambda}{(n-2)(n-1)} \), with \( M_\star > 0 \) the black hole mass and \( \Lambda > 0 \) the cosmological constant \( \Lambda \). The assumption

\[
M^2_\star \lambda^{n-3} < \frac{(n-3)^{n-3}}{(n-1)^{n-1}}
\]

guarantees that \( \mu \) has two unique positive roots \( 0 < r_- < r_+ \). Indeed, let \( \tilde{\mu} = r^{-2} \mu = r^{-2} - 2M_\star r^{1-n} - \lambda \). Then \( \tilde{\mu}' = -2r^{-n}(r^{n-3} - (n-1)M_\star) \) has a unique positive root \( r_p = [(n-1)M_\star]^{1/(n-3)} \), \( \tilde{\mu}'(r) > 0 \) for \( r \in (0, r_p) \) and \( \tilde{\mu}'(r) < 0 \) for \( r > r_p \); moreover, \( \tilde{\mu}(r) < 0 \) for \( r > 0 \) small and \( \tilde{\mu}(r) \to -\lambda < 0 \) as \( r \to \infty \), thus the existence of the roots \( 0 < r_- < r_+ \) of \( \tilde{\mu} \) is equivalent to the requirement \( \tilde{\mu}(r_p) = \frac{n-3}{n-1}r^{-2}_p - \lambda > 0 \), which is equivalent to (2.2).

Define \( \alpha = \mu^{1/2} \), thus \( d\alpha = \frac{1}{2} \mu' \alpha^{-1} dr \), and let

\[
\beta_\pm(r) := \pm \frac{2}{\mu'(r)}
\]

near \( r_\pm \), so \( \beta_\pm(r_\pm) > 0 \) there. Then the metric \( g_0 \) can be written as

\[
g_0 = \alpha^2 dt^2 - h, \quad h = \alpha^{-2} dr^2 + r^2 d\omega^2 = \beta_\pm^2 d\alpha^2 + r^2 d\omega^2,
\]

We introduce a new time variable \( t_* = t - F(\alpha) \), with \( \partial_{t_*} F = -\alpha^{-1} \beta_\pm \) near \( r = r_\pm \). Then

\[
g_0 = \mu dt_*^2 - \beta_\pm dt_* d\mu - r^2 d\omega^2
\]

near \( r = r_\pm \), which extends as a non-degenerate Lorentzian metric to a neighborhood \( \bar{M} = \mathbb{R}_t \times \bar{X} \) of \( M \), where \( \bar{X} = (r_- - 2\delta, r_+ + 2\delta) \times S^{n-2} \). We will consider the Cauchy problem for the tensor wave equation in the domain \( \Omega \subset \bar{M} \),

\[
\Omega = [0, \infty)_{t_*} \times [r_- - \delta, r_+ + \delta]_r \times S^{n-2}.
\]

Thus, \( \Omega \) bounded by the Cauchy surface \( H_1 = \{ t_* = 0 \} \), which is spacelike, and by the hypersurface \( H_2 = \bigcup_{\pm} \{ r = r_{\pm} + \delta \} \), which has two spacelike components, one lying beyond the black hole \( (r_-) \) and the other beyond the cosmological \( (r_+) \) horizon; see Figure 1.

For the purpose of analysis on spacetimes close to (but not necessarily asymptotically equal to!) Schwarzschild-de Sitter space, we encode the uniform (asymptotically stationary) structure of the spacetime by working on a compactified model, which puts the problem into the setting of Melrose’s \textit{b-analysis}, see [23]: Define \( \tau := e^{-t_*} \), and bordify \( \bar{M} \) to a manifold \( \bar{\mathcal{M}} \) with boundary by adding \( \tau = 0 \) as the boundary at future infinity and declaring \( \tau \) to be a smooth boundary defining function. The metric \( g_0 \) becomes a smooth \textit{Lorentzian b-metric} on \( \bar{\mathcal{M}} \). If \( dx_i \) denotes coordinate differentials on \( \bar{X} \), then \( g_0 \) is a linear combination of \( 4x^2 \), \( \frac{dx_i}{dx^2} \otimes dx_i + dx_i \otimes \frac{dx_i}{dx^2} \) and \( dx_i \otimes dx_j \) with coefficients which are smooth on \( \bar{\mathcal{M}} \), and \( g_0 \), written in such coordinates, is a non-degenerate matrix (with Lorentzian signature) \textit{up to and including} \( \tau = 0 \). Invariantly, we have the Lie algebra \( V_0(\bar{\mathcal{M}}) \) of \textit{b-vector fields}, which are the vector fields tangent to the boundary, spanned by
$\tau \partial_\tau = -\partial_t$ and $\partial_x$; elements of $\mathcal{V}_b(\overline{M})$ are sections of a natural vector bundle $b^*T\overline{M}$, the b-tangent bundle, and we have the dual bundle $b^*T^*\overline{M}$, spanned by $dx_i$. Thus, $g$ is a smooth non-degenerate section of the symmetric second tensor power $S^{2b}T^*\overline{M}$.

Now, given a complex vector bundle $\mathcal{E} \to \overline{M}$ of finite rank, equip it with an arbitrary Hermitian inner product and any smooth $b$-connection, which gives a notion of differentiating sections of $\mathcal{E}$ along $b$-vector fields; over $\Omega$ (which has compact closure in $\overline{M}$), all choices of inner products are equivalent. We can then define the $b$-Sobolev space $H^s_b(\Omega, \mathcal{E})$ for $s \in \mathbb{Z}_{\geq 0}$ to consist of all sections of $\mathcal{E}$ over $\Omega$ which are square integrable (with respect to the volume density $|dq|$ induced by the metric $g$) together with all of its $b$-derivatives up to order $s$, and extend this to all $s \in \mathbb{R}$ by duality and interpolation, or via the use of $b$-pseudodifferential operators. For the forward problem for the wave equation, we work on spaces of functions which vanish in the past of $H^1_0$ and which extend across $H^2$. Thus, we work with the space $H^s_b(\Omega, \mathcal{E})^\bullet$ of distributions $u \in H^s_b(\Omega, \mathcal{E})$ which are extendible distributions at $H^2$ and supported distributions at $H^1$, i.e. they are restrictions to $\Omega$ of distributions on $\overline{M}$ which are supported in $t_* \geq 0$. See Hörmander [21, Appendix B] for details. We also have weighted $b$-Sobolev spaces $H^s_b(\Omega, \mathcal{E}) = \tau^r H^s_b(\Omega, \mathcal{E})$, likewise for spaces of supported/extendible distributions. Note that the $b$-Sobolev spaces $H^s_b$ are independent of the choice of boundary defining function $\tau$ in that the choice $\tau' = a\tau^\gamma$, $a = a(x)$ smooth, $\gamma > 0$, while it changes the smooth structure of $\overline{M}$, yields the same spaces $H^s_b$ with equivalent norms. The asymptotic behavior of waves will be encoded on the boundary $\partial_{\infty}\Omega$ at future infinity of $\Omega$, that is, on

$$\partial_{\infty}\Omega = \{\tau = 0\} \times [r_--\delta, r_+ + \delta] \times \mathbb{S}^{n-2},$$

which is a smooth manifold with boundary. Similarly to the above definitions, we can define Sobolev spaces (including semiclassical versions of these) with supported/extendible character at the boundary.

Suppose $g$ is a Lorentzian $b$-metric such that for some smooth Lorentzian $b$-metric $g'$, we have $g - g' \in H^{-\infty}_{b^*}(\Omega, S^{2b}T^*\overline{M})$ for some $r > 0$. Changing $g'$ so as to make it invariant under time translations does not affect this condition, so let us assume $g'$ is $t_*$-invariant. We will consider the wave operator $\Box_g$ acting on sections of the bundle $T_k$ of covariant tensors of rank $k$ over $\Omega$. We assume that $g'$ and $g_0$ are close (in the $C^k$ sense for sufficiently high $k$), so that the dynamical and geometric structure of $g$ is close to that of $g_0$. Most importantly, the nature of the trapping for $g'$ (and thus for $g$) is still normally hyperbolic, and the subprincipal operator (see Section 3.3) of $\Box_g$ at the trapped set, while not necessarily having the nilpotent structure alluded to in the introduction and explained in Section 4.2, has

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2By the discussion of $b$-Sobolev spaces above, this condition on $g$ is invariant, i.e. independent of the specific choice of the boundary defining function $e^{-t_*}$ of the spacetime at future infinity.

3In other words, the metric $g$ is exponentially approaching a stationary metric close to the Schwarzschild-de Sitter metric, so for instance perturbations (within this setting) of Kerr-de Sitter spaces are allowed.

4See [17, §3] and [18, §5] for details.

5We will show the $r$-normal hyperbolicity for every $r$ of the trapping for Schwarzschild-de Sitter in all spacetime dimensions below, and $r$-normal hyperbolicity (for large, but finite $r$) is structurally stable under perturbations of the metric, see Dyatlov [13] and Hirsch, Shub and Pugh [20].
small imaginary part relative to (the symbol of) a pseudodifferential inner product on $T_k$. We then have:

**Theorem 2.1.** In the above notation, if $g'$ is sufficiently close to the Schwarzschild-de Sitter metric $g_0$, then there exist $s_0 \in \mathbb{R}$ and $\delta > 0$ as well as a finite set $\{\sigma_j : j = 1, \ldots, N\} \subset \mathbb{C}$, Im $\sigma_j > -\delta$, integers $m_j \geq 1$ and $d_j \geq 1$, and smooth functions $a_{jm\ell \tau} \in C^\infty(\partial_\infty \Omega)$, $1 \leq j \leq N$, $0 \leq m \leq m_j - 1$, $1 \leq \ell \leq d_j$, such that the following holds: The equation

$$\Box_g u = f, \quad f \in H^{s,\delta}_b(\Omega, T_k)^{\bullet,-}, \quad s \geq s_0,$$

has a unique solution $u \in H^{-s,-\infty}_b(\Omega, T_k)^{\bullet,-}$, which has an asymptotic expansion

$$u = \chi(\tau) \sum_{j=1}^N \sum_{m=0}^{m_j-1} \sum_{\ell=1}^{d_j} \tau^{i\sigma_j} |\log \tau|^m u_{jm\ell a_{jm\ell \tau}} + u',$$

where $\chi$ is a cutoff function, i.e. $\chi(\tau) \equiv 1$ near $\tau = 0$ and $\chi(\tau) \equiv 0$ near the Cauchy surface $H_1$, and $u_{jm\ell a_{jm\ell \tau}} \in \mathbb{C}$, while the remainder term is $u' \in H^{s,\delta}_b(\Omega, T_k)^{\bullet,-}$.

The same result holds true if we restrict to a subbundle of $T_k$ which is preserved by the action of $\Box$, for instance the degree $k$ form bundle, or the symmetric rank $k$ tensor bundle.

If $V \in C^\infty(\overline{\Omega}, \text{End}(T_k)) + H^{\infty,r}_b(\Omega, \text{End}(T_k))$, $r > 0$, is a smooth (conormal) End$(T_k)$-valued potential (without restriction on its size), the analogous result holds for $\Box_g$ replaced by $\Box_g + V$. We may even change $\Box_g$ by adding a first order $b$-differential operator $L$ acting on $T_k$ with coefficients which are elements of $C^\infty + H^{\infty,r}_b$, provided either the coefficients of $L$ are small, or the subprincipal operator of $\Box_g + L$ is sufficiently close to being symmetric with respect to a pseudodifferential inner product on $T_k$, see Remark 4.9.

The numbers $\sigma_j$ are called resonances or quasinormal modes, and the functions $a_{jm\ell a_{jm\ell \tau}}$ resonant states. They have been computed in various special cases; see the discussion in the introduction for references. The threshold regularity $s_0$ is related to the dynamics of the flow of the Hamiltonian vector field $H_G$ of the dual metric function $G$ (i.e. $G(x, \xi) = |\xi|^2_{G(x)}$) with $G$ the dual metric of $g$ near the horizons which are generalized radial sets, see [17, Proposition 2.1]. Thus, $s_0$ can easily be made explicit, but this is not the point of the present paper.

The proof of Theorem 2.1 proceeds in the same way as the proof of [17, Theorem 2.20] in the scalar setting, so we shall be brief: Denote by $N(\Box_g)$ the normal operator of $\Box_g$. We freeze the coefficients of $\Box_g \in \text{Diff}^2(\Omega, T_k)$ at $\partial_\infty \Omega$ and thus obtain a dilation-invariant operator $N(\Box_g)$, with $\Box_g - N(\Box_g)$ being an operator whose coefficients decay exponentially (in $t_+$) by assumption on the structure of $g$. Denote by $\widehat{\Box_g}(\sigma) \in \text{Diff}^2(\partial_\infty \Omega, T_k)$ the Mellin transformed normal operator family, depending holomorphically on $\sigma \in \mathbb{C}$, which we obtain from $N(\Box_g)$ by replacing $D_{t_+} \sigma$ by $-\sigma$.\footnote{Changing the boundary defining function $\tau$ to $a(x)\tau^\gamma$, we can express the normal operator with respect to the new defining function in terms of the normal operator with respect to $\tau$, namely it equals $a(x)^{-1} \overline{\partial}_\gamma a(\tau)a(x)$.}

Once we show high energy estimates for $\widehat{\Box_g}(\sigma)^{-1}$, which are polynomial bounds on its operator norm between suitable Sobolev spaces\footnote{These are semiclassical Sobolev spaces with extendible character at the boundary of $\partial_\infty \Omega$, see in particular [27] and the proof of [17, Theorem 2.20].} as $|\text{Re}\sigma| \to \infty$ in Im $\sigma > -\delta$, we can use a contour shifting argument to iteratively improve on

\begin{align*}
\Box_g u = f, \quad f \in H^{s,\delta}_b(\Omega, T_k)^{\bullet,-}, \quad s \geq s_0, \quad (2.4)
\end{align*}
decay of \( u \), picking up contributions of the poles of \( \hat{\Box}(\sigma)^{-1} \) which give rise to the resonance expansion.\(^8\) The fact that the remainder term \( u' \) has the same regularity as the forcing term \( f \), thus \( u' \) loses 2 derivatives relative to the elliptic gain of 2 derivatives, comes from the high energy estimate losing a power of 2, see [18, Theorem 5.5], which in turn is caused by the same loss for high energy estimates at normally hyperbolic trapping, see [14, Theorem 1], or [18, Theorem 4.5] for a microlocalized version of Dyatlov’s estimate.

Thus, the crucial point is to obtain high energy estimates at the trapped set for the operator \( \Box \) acting on \( T_k \) in \( \text{Im} \sigma > -\delta \). Dyatlov’s result [14, Theorem 1] (see also the discussion preceding [18, Theorem 5.5]) shows that a sufficient condition for these to hold is

\[
|\sigma|^{-1}\sigma_{b,1}\left(\frac{1}{2\ell}(\Box - \Box^*)\right) < \nu_{\text{min}}/2
\]

at the trapped set \( \Gamma \),\(^9\) where \( \nu_{\text{min}} \) is the minimal normal expansion rate of the Hamilton flow at the trapping, see [14] and the computation below. Here, the adjoint is taken with respect to a positive definite inner product on \( T_k \); note that the inner product induced by \( g \), with respect to which \( \Box \) is of course symmetric, is not positive definite, except when \( k = 0 \), i.e. for the scalar wave equation. Since \( g \) is close to the Schwarzschild-de Sitter metric, it suffices (by the dynamical stability of the trapping) to obtain such a bound for the Schwarzschild-de Sitter metric \( g_0 \). While this bound is impossible to obtain directly for the full range of Schwarzschild-de Sitter spacetimes, we show in Section 4.2 how it can be obtained if we use pseudodifferential products. Prosaically, this means that we consider a conjugated operator \( P := Q \Box Q^* \), where \( Q \in \Psi^0(\overline{M}, T_k) \) is elliptic with parametrix \( Q^* \), and for any \( \epsilon > 0 \), we can arrange \( |\sigma|^{-1}\sigma_{b,1}(\frac{1}{2\ell}(P - P^*)) < \epsilon \) (with the adjoint taken relative to an ordinary positive definite inner product on \( T_k \)), thus (2.5) holds for \( \Box \) replaced by \( P \); we will prove this in Theorem 4.8. Hence [14, Theorem 1] applies to \( P \), establishing a spectral gap; indeed, by the remark following [14, Theorem 1], Dyatlov’s result applies for operators on bundles as well, as soon as one establishes (2.5). Arranging (2.5) in a natural fashion lies at the heart of Sections 3 and 4.

It remains to establish the \( r \)-normal hyperbolicity for all \( r \) for the Schwarzschild-de Sitter metric. The dynamics at the trapping only depend on properties of the (scalar!) principal symbol \( g_0 \) of \( \Box \). For easier comparison with [12, 27, 29], we consider the operator \( P = -r^2 \Box \) instead. We take the Fourier transform in \( -t \) and rescale to a semiclassical operator on \( X \) (this amounts to multiplying \( \hat{P} \) by \( h^2 \), giving a second order semiclassical differential operator \( P_h \), with \( h = |\sigma|^{-1} \), and we then define \( z = h\sigma \)). Introducing coordinates on \( T^*X \) by writing 1-forms as \( \xi \, dr + \eta \, d\omega \), and letting

\[
\Delta_r = r^2 \mu = r^2(1 - \lambda r^2) - 2M_r^5 r^{5-n},
\]

\(^8\)As shown by Vasy [27, §7], these estimates in \( \text{Im} \sigma \gg 0 \) are automatic if the boundary defining function of future infinity is timelike; our choice does not satisfy this, but changing \( t_\ast \) by a smooth function of the spatial variables, this can easily be arranged, see [27, §6], and in fact we can arrange \( t_\ast = t \) away from the black hole and cosmological horizons.

\(^9\)We work in the b-setting, which via the Mellin transform is equivalent (on the normal operator level, which is all that matters) to the semiclassical setting considered in Dyatlov’s work, see the discussion in [18, §5].
\(P_h\) has semiclassical principal symbol
\[
p = \Delta_r \xi^2 - \frac{r^4}{\Delta_r} z^2 + |\eta|^2,
\]
and correspondingly the Hamilton vector field is
\[
H_p = 2\Delta_r \xi \partial_r - \left( \partial_r \Delta_r \xi^2 - \partial_r \left( \frac{r^4}{\Delta_r} \right) z^2 \right) \partial_\xi + H_{|\eta|^2}
\]
We work with real \(z\), hence \(z = \pm 1\). First, we locate the trapped set: If \(H_p r = 2\Delta_r \xi = 0\), then \(\xi = 0\), in which case \(H_p^2 r = 2\Delta_r H_p^2 \xi = 2\Delta_r \partial_r (r^4/\Delta_r) z^2\). Recall the definition of the function \(\tilde{\mu} = \mu/r^2 = \Delta_r/r^4\), then we can rewrite this as \(H_p^2 r = -2\Delta_r \tilde{\mu}^{-2} (\partial_r \tilde{\mu}) z^2\). We have already seen that \(\partial_r \tilde{\mu}\) has a single root \(r_p \in (r_-, r_+)\), and \((r - r_p) \partial_r \tilde{\mu} < 0\) for \(r \neq r_p\). Therefore, \(H_p^2 r = 0\) implies (still assuming \(H_p r = 0\)) \(r = r_p\). We rephrase this to show that the only trapping occurs in the cotangent bundle over \(r = r_p\): Let \(F(r) = (r - r_p)^2\), then \(H_p F = 2(r - r_p) H_p r\) and \(H_p F = 2(H_p r)^2 + 2(r - r_p) H_p^2 r\). Thus, if \(H_p F = 0\), then either \(r = r_p\), in which case \(H_p^2 F = 2(H_p r)^2 > 0\) unless \(H_p r = 0\), or \(H_p r = 0\), in which case \(H_p^2 F = 2(r - r_p) H_p^2 r > 0\) unless \(r = r_p\). So \(H_p F = 0, p = 0\) implies either \(H_p^2 F > 0\) or \(r = r_p, H_p r = 0\), i.e.
\[
(r, \omega; \xi, \eta) \in \Gamma_h := \left\{(r_p, \omega; 0, \eta) : \frac{r^4}{\Delta_r} z^2 = |\eta|^2\right\},
\]
so \(\Gamma_h\) is the only trapping in \(T^* X\), and \(F\) is an escape function. We compute the linearization of the \(H_p\)-flow at \(\Gamma_h\) in the normal coordinates \(r - r_p\) and \(\xi\), to wit
\[
H_p \begin{pmatrix} r - r_p \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 2(n - 3) r_p^{-4} (\tilde{\mu}|_{r=r_p})^{-2} z^2 \\ 2(n - 3) r_p^{-4} (\tilde{\mu}|_{r=r_p})^{-2} z^2 & 0 \end{pmatrix} \begin{pmatrix} r - r_p \\ \xi \end{pmatrix} + \mathcal{O}(|r - r_p|^2 + |\xi|^2),
\]
where we used \(\partial_r \tilde{\mu}|_{r=r_p} = -2(n - 3) r_p^{-4}\), which gives \(\partial_r \tilde{\mu} = -2(n - 3) r_p^{-4} (r - r_p) + \mathcal{O}(|r - r_p|^2)\). The eigenvalues of the linearization are therefore
\[
\pm 2r_p \left( \frac{n - 1}{1 - n^{-1} r_p^2 \lambda} \right)^{1/2},
\]
which reduces to the expression given in [27, p. 85] in the case \(n = 4\), where \(r_p = 3M_\bullet = \frac{3}{2} r_s\) with \(r_s = 2M_\bullet\) and \(\lambda = \Lambda/3\). In particular, the minimal expansion rate for the semiclassical rescaling of \(\Box\) at the trapping \(\Gamma_h\) is
\[
\nu_{\min} = 2r_p^{-1} \left( \frac{n - 1}{1 - n^{-1} r_p^2 \lambda} \right)^{1/2} > 0.
\]
The expansion rate of the flow within the trapped set is \(0\) by spherical symmetry; note that integral curves of \(H_p\) on \(\Gamma_h\) are simply unit speed geodesics of the round unit sphere \(S^{n-2}\). This shows the normal hyperbolicity (in fact, \(r\)-normal hyperbolicity for every \(r\)) of the trapping and finishes the proof of Theorem 2.1.

For later reference, we note that the spacetime trapped set, i.e. the set of points in phase space that never escape through either horizon along the Hamilton flow, is given by
\[
\Gamma = \{(t, r = r_p, \omega; \sigma, \xi = 0, \eta) : \sigma^2 = \Psi^2 |\eta|^2\},
\]
where \(\Psi = \alpha r^{-1}, \Psi'(r_p) = 0\).
3. Pseudodifferential inner products

We now develop a general theory of pseudodifferential inner products, which we apply to the setting of Theorem 2.1 in Section 4.

We work on a complex rank $N$ vector bundle $\mathcal{E}$ over the smooth compact $n$-dimensional manifold $X$ without boundary. We will define pseudodifferential inner products on $\mathcal{E}$, which are inner products depending on the position in phase space $T^*X$, rather than merely the position in the base $X$. As indicated in the introduction, we achieve this by replacing ordinary inner products by pseudodifferential operators whose symbols are inner products on the bundle $\pi^*\mathcal{E} \to T^*X \setminus 0$, where $\pi: T^*X \setminus 0 \to X$ is the projection.

3.1. Notation. Let $\mathcal{V}$ be a complex $N$-dimensional vector space. We denote by $\overline{\mathcal{V}}$ the complex conjugate of $\mathcal{V}$, i.e. $\overline{\mathcal{V}} = \mathcal{V}$ as sets, and the identity map $\iota: \mathcal{V} \to \overline{\mathcal{V}}$ is antilinear, so $\iota(\lambda v) = \overline{\lambda} \iota(v)$ for $v \in \mathcal{V}$, $\lambda \in \mathbb{C}$, which defines the linear structure on $\overline{\mathcal{V}}$. A Hermitian inner product $H$ on $\mathcal{V}$ is thus a linear map $H: \mathcal{V} \otimes \overline{\mathcal{V}} \to \mathbb{C}$ such that $H(u, \iota(v)) = \overline{H(v, u)}$ for $u, v \in \mathcal{V}$, and $H(u, \iota(u)) > 0$ for all non-zero $u \in \mathcal{V}$. This can be rephrased this in terms of the linear map $B: \mathcal{V} \to \overline{\mathcal{V}}^*$ defined by $B(u) = H(u, \cdot)$ and the natural dual pairing of $\overline{\mathcal{V}}$ with $\mathcal{V}$, namely $\langle Bu, \iota(v) \rangle = \overline{\langle v, Bu \rangle}$, and $\langle Bu, \iota(u) \rangle > 0$ for $u \in \mathcal{V}$ non-zero.

A map $A: \mathcal{V} \to \overline{\mathcal{V}}^*$ has a transpose $A^T: \overline{\mathcal{V}}^* \to \mathcal{V}^*$, which satisfies $\langle Au, \iota(v) \rangle = \langle u, A^T \iota(v) \rangle$ for all $u, v \in \mathcal{V}$, and an adjoint $A^*: \overline{\mathcal{V}}^* \to \mathcal{V}^*$ satisfying $\langle Au, \iota(v) \rangle = \overline{\langle A^*v, \iota(u) \rangle}$. Concretely, defining the antilinear map

$$j: \mathcal{V}^* \to \overline{\mathcal{V}}^*, \quad \langle j(\ell), \iota(v) \rangle = \overline{\langle \ell, v \rangle},$$

we have $A^* = j A^T j$. The symmetry of a Hermitian inner product $B$ as above is simply expressed by $B = B^*$. Similarly, a map $P: \mathcal{V} \to \mathcal{V}$ has a transpose $P^T: \mathcal{V}^* \to \overline{\mathcal{V}}^*$ and an adjoint $P^*: \overline{\mathcal{V}}^* \to \mathcal{V}^*$ defined by $\langle P\overline{\ell}, \iota(v) \rangle = \langle \overline{P^*\ell}, \iota(v) \rangle$ for $\overline{\ell} \in \overline{\mathcal{V}}^*$ and $v \in \mathcal{V}$, and one easily finds $P^* = j P^T j^{-1}$. We point out that the definitions of adjoints of maps $A: \mathcal{V} \to \overline{\mathcal{V}}^*$ and $P: \mathcal{V} \to \mathcal{V}$ are compatible in the sense that $(AP)^* = P^* A^*$. Furthermore, if $B: \mathcal{V} \to \overline{\mathcal{V}}^*$ is a Hermitian inner product and $Q: \mathcal{V} \to \mathcal{V}$ is invertible, then $B_1 = Q^* B Q$ defines another Hermitian inner product, $\langle B_1 u, \iota(v) \rangle = \overline{\langle B Q u, \iota(Qv) \rangle}$.

Now, given an inner product $B$ on $\mathcal{V}$ and any map $P: \mathcal{V} \to \mathcal{V}$, the adjoint $P^*B$ of $P$ with respect to $B$ is the unique map $P^*B: \mathcal{V} \to \mathcal{V}$ such that $\langle B Pu, \iota(v) \rangle = \langle Bu, \iota(P^*Bv) \rangle$ for all $u, v \in \mathcal{V}$. We find a formula for $P^*B$ by computing

$$\langle B Pu, \iota(v) \rangle = \overline{\langle B^* (B^*)^{-1} P^* B^* v, \iota(u) \rangle} = \langle Bu, \iota((BPB^{-1})^* v) \rangle,$$

i.e. $P^*B = (BPB^{-1})^* = B^{-1} P^* B$. The self-adjointness of $P$ with respect to $B$ is thus expressed by the equality $P = B^{-1} P^* B$.

If $\mathcal{E}$ is a complex rank $N$ vector bundle, we can similarly define the complex conjugate bundle $\overline{\mathcal{E}}$ as well as adjoints of vector bundle maps $\mathcal{E} \to \mathcal{E}$ and $\overline{\mathcal{E}} \to \overline{\mathcal{E}}^*$. We can also define adjoints of pseudodifferential operators mapping between these bundles: For convenience, we remove the dependence of adjoints on a volume density on $X$ by tensoring all bundles with the half-density bundle $\Omega^2$ over $X$, and

---

10We prefer to write $\iota(v)$ rather than $\overline{v}$ to prevent possible confusion with taking complex conjugates in complexifications of real vector spaces.
we have a natural pairing
\[(E^* \otimes \Omega^\frac{1}{2})_x \times (E \otimes \Omega^\frac{1}{2})_x \ni (\ell, \iota(v)) \mapsto \langle \ell, \iota(v) \rangle \in \Omega^1_x, \quad x \in X,\]
l likewise for the complex conjugate of \(E\). Thus, an operator \(A \in \Psi^m(X, E \otimes \Omega^\frac{1}{2}, E^* \otimes \Omega^\frac{1}{2})\) has an adjoint \(A^* \in \Psi^m(X, E \otimes \Omega^\frac{1}{2}, E^* \otimes \Omega^\frac{1}{2})\) defined by
\[
\int_X \langle A^*u, \iota(v) \rangle = \int_X \langle Au, \iota(u) \rangle,
\]
with principal symbol \(\sigma_m(A^*) = \sigma_m(A)^\ast \in S^m(T^*X \setminus 0, \pi^\ast \text{Hom}(E, E^*))\), and likewise \(P \in \Psi^m(X, E \otimes \Omega^\frac{1}{2})\) has an adjoint \(P^* \in \Psi^m(X, E^* \otimes \Omega^\frac{1}{2})\) with \(\sigma_m(P^*) = \sigma_m(P)^\ast\).

### 3.2. Definition of pseudodifferential inner products; adjoints.

We work with classical, i.e. one-step polyhomogeneous, symbols and operators, and denote by \(S^m_{\text{hom}}(T^*X \setminus 0)\) symbols which are homogeneous of degree \(m\) with respect to dilations in the fibers of \(T^*X \setminus 0\).

**Definition 3.1.** A pseudodifferential inner product (or \(\Psi\)-inner product) on the vector bundle \(E \to X\) is a pseudodifferential operator \(B \in \Psi^0(X; E \otimes \Omega^\frac{1}{2}, E^* \otimes \Omega^\frac{1}{2})\) satisfying \(B = B^\ast\), and such that moreover the principal symbol \(\sigma^0(B) = b \in S^0_{\text{hom}}(T^*X \setminus 0; \pi^\ast \text{Hom}(E, E^*))\) of \(B\) satisfies
\[
\langle b(x, \xi)u, \iota(u) \rangle > 0
\]
for all non-zero \(u \in E_x\), where \(x \in X\), \(\xi \in T^*_xX \setminus 0\). If the context is clear, we will also call the sesquilinear pairing
\[
C^\infty(X, E \otimes \Omega^\frac{1}{2}) \times C^\infty(X, E \otimes \Omega^\frac{1}{2}) \ni (u, v) \mapsto \int_X \langle B(x, D)u, \iota(v) \rangle
\]
the pseudodifferential inner product associated with \(B\).

In particular, the principal symbol \(b\) of \(B\) is a Hermitian inner product on \(\pi^\ast E\). Conversely, for any \(b \in S^0_{\text{hom}}(T^*X \setminus 0; \pi^\ast \text{Hom}(E, E^*))\) satisfying \(b = b^\ast\) and (3.1), there exists a \(\Psi\)-inner product \(B\) with \(\sigma^0(B) = b\); indeed, simply take \(\tilde{B}\) to be any quantization of \(b\) and put \(B = \frac{1}{2}(\tilde{B} + \tilde{B}^\ast)\).

**Remark 3.2.** While we will develop the theory of \(\Psi\)-inner products only in the standard calculus on a closed manifold, everything works \textit{mutatis mutandis} in other settings as well. Thus, in the \(b\)-calculus of Melrose [23], \(\Psi_b\)-inner products on a manifold with boundary are defined similarly to \(\Psi\)-inner products, except that adjoints are defined on the space \(C^\infty\) of functions vanishing to infinite order at the boundary, and the space of ‘trivial’, smoothing operators is now \(\Psi^\infty_b\), likewise for the scattering calculus [24], replacing ‘\(b\)’ by ‘\(sc\)’. In the semiclassical calculus on a closed manifold, adjoints are again defined on \(C^\infty\), but the space of ‘trivial’ operators is now \(h^\infty\Psi^{-\infty}_h\), and suitable factors of \(h\) need to be put in for computations involving subprincipal symbols.

We next discuss adjoints of \(\Psi\)DOs relative to \(\Psi\)-inner products.

**Definition 3.3.** Let \(B\) be a \(\Psi\)-inner product, and let \(P \in \Psi^m(X, E \otimes \Omega^\frac{1}{2})\), then \(P^+B \in \Psi^m(X, E \otimes \Omega^\frac{1}{2})\) is called an adjoint of \(P\) with respect to \(B\) if there exists an
Lemma 3.6. Suppose \( R \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2, \mathcal{E}^* \otimes \Omega^2) \) such that
\[
\int \langle Bu, \iota(v) \rangle = \int \langle Bu, \iota(P^*Bv) \rangle + \int \langle Ru, \iota(v) \rangle
\]
for all \( u, v \in C^\infty(X, \mathcal{E} \otimes \Omega^2) \).

Proof. Fix a coordinate system of \( u,v \) for \( u,v \in C^\infty(X, \mathcal{E} \otimes \Omega^2) \), modulo \( \Psi \)-inner product on \( \mathcal{F} \), provided a \( \Psi \)-inner product on \( \mathcal{F} \), modulo smoothing operators, suppose that \( \tilde{B} \) is self-adjoint by replacing it by \( \frac{1}{2}(B - B^*) \) (which changes \( B \) by an operator in \( \Psi^{-\infty} \)). Then the second claim follows from
\[
(P^*B)^* = (BPB^*)^* = B^-1BPB^-B = P
\]
modulo \( \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2) \).

Remark 3.4. This definition and the following lemma have straightforward generalizations to the case that \( P \) maps section of \( \mathcal{E} \) into sections of another vector bundle \( \mathcal{F} \), provided a \( \Psi \)-inner product on \( \mathcal{F} \) is given.

Lemma 3.5. In the notation of Definition 3.3, the adjoint of \( P \) with respect to \( B \) exists and is uniquely determined modulo \( \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2) \). In fact, \( P = (BPB^-)^* \), where \( B^- \) is a parametrix for \( B \). Moreover, \( (P^*B)^*B = P \) modulo \( \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2) \). In particular, \( \text{Im}^B P = \frac{1}{2}(P - P^*B) \) is self-adjoint with respect to \( B \) (i.e. its own adjoint modulo \( \Psi^{-\infty} \)).

Proof. Let \( B^- \) be a parametrix of \( B \) and put \( R_L = I - B^- B \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2) \). Then
\[
\int \langle Bu, \iota(v) \rangle = \int \langle Bu, \iota(P^*Bv) \rangle + \int \langle Ru, \iota(v) \rangle
\]
hence (3.2) holds with \( P^*B = (BPB^-)^* \) and \( R = BPR_L \). To show the uniqueness of \( P^*B \) modulo smoothing operators, suppose that \( \tilde{P} \) is another adjoint of \( P \) with respect to \( B \), with error term \( \tilde{R} \) (i.e. (3.2) holds with \( P^*B \) and \( R \) replaced by \( \tilde{P} \) and \( \tilde{R} \)). Then
\[
\int \langle (B(P^*B - \tilde{P})v, \iota(u) \rangle = \int \langle Bu, \iota((P^*B - \tilde{P})v) \rangle = \int \langle \tilde{R} - R \rangle u, \iota(v) \rangle
\]
for \( u, v \in C^\infty(X, \mathcal{E} \otimes \Omega^2) \), so \( B(P^*B - \tilde{P}) = (\tilde{R} - R)^* \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2, \mathcal{E}^* \otimes \Omega^2) \), and the ellipticity of \( B \) implies \( P^*B - \tilde{P} \in \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2) \), as claimed.

Since \( B \) is self-adjoint, we can assume that \( B^- \) is self-adjoint by replacing it by \( \frac{1}{2}(B^- + (B^-)^*) \) (which changes \( B^- \) by an operator in \( \Psi^{-\infty} \)). Then the second claim follows from
\[
(P^*B)^*B = (BPB^-)^*B^-BPB^-B = P
\]
modulo \( \Psi^{-\infty}(X, \mathcal{E} \otimes \Omega^2) \).

Lemma 3.6. Suppose \( P \in \Psi^m(X, \mathcal{E} \otimes \Omega^2) \) is self-adjoint with respect to \( B = \sigma^0(B) \), i.e.
\[
\langle bx, \xi \rangle p(x, \xi)u, \iota(v) \rangle = \langle bx, \xi \rangle u, \iota(p(x, \xi)v) \rangle, \quad x \in X, \xi \in T_xX, \ iu, v \in \mathcal{E}. \]

Proof. The hypothesis on \( P \) means \( (BPB^-)^* = P \) modulo \( \Psi^{-\infty} \), thus on the level of principal symbols, \( p = b^{-1}p^*b = p^b \), which proves the claim.

We now specialize to the case that \( P \in \Psi^m(X, \mathcal{E} \otimes \Omega^2) \) has a real, scalar principal symbol. Fix a coordinate system of \( X \) and a local trivialization of \( \mathcal{E} \), then the full symbol of \( P \) is a sum of homogeneous symbols \( p \sim p_m + p_{m-1} + \ldots \), with \( p_j \).
homogeneous of degree \( j \) and valued in complex \( N \times N \) matrices. Recall from \([21, §18]\) that the subprincipal symbol

\[
\sigma_{\text{sub}}(P) = p_{m-1}(x, \xi) - \frac{1}{2i} \sum_j \partial_{x_j} \xi_j p_m(x, \xi) \in S^{m-1}_{\text{hom}}(T^* X \setminus 0, \mathbb{C}^{N \times N})
\]  

(3.3)

is well-defined under changes of coordinates; however, it does depend on the choice of local trivialization of \( \mathcal{E} \). We compute the principal symbol of

\[
\text{Im}^B P := \frac{1}{2i} (P - P^{*B})
\]

for such \( P \) in a local trivialization of \( \mathcal{E} \); we will give an invariant formulation in Proposition 3.10 below.

**Lemma 3.7.** Let \( P \in \Psi^m(X, \mathcal{E} \otimes \Omega^1) \) be a principally real and scalar, and let \( B = b(x, D) \) be a \( \Psi \)-inner product on \( \mathcal{E} \). Then \( \text{Im}^B P \in \Psi^{m-1}(X, \mathcal{E} \otimes \Omega^2) \) has the principal symbol

\[
\sigma^{m-1}(\text{Im}^B P) = \text{Im}^b \sigma_{\text{sub}}(P) + \frac{1}{2} b^{-1} H_p(b),
\]

(3.4)

where \( \text{Im}^b \sigma_{\text{sub}}(P) = \frac{1}{2i} (\sigma_{\text{sub}}(P) - (P^{*B})^{*b}) \). Here, we interpret \( b \) and \( \sigma_{\text{sub}}(P) \) as \( N \times N \) matrices of scalar-valued symbols using a local frame of \( \mathcal{E} \) and the corresponding dual frame of \( \mathcal{E}^* \), and the action of \( H_p \) is component-wise.

**Proof.** We compute in a local coordinate system over which \( \mathcal{E} \) and \( \mathcal{E}^* \) are trivialized by a choice of \( N \) linearly independent sections \( e_1, \ldots, e_N \), and \( \mathcal{E}^* \) and \( \mathcal{E}^{*2} \) are trivialized by the dual sections \( e_1^*, \ldots, e_N^* \in \mathcal{E}^* \) satisfying \( e_i^*(e_j) = \delta_{ij}, \) extended linearly as linear functionals on \( \mathcal{E} \), resp. on \( \mathcal{E}^{*2} \), in the case of \( \mathcal{E}^* \), resp. \( \mathcal{E}^{*2} \). We trivialize \( \Omega^2 \) using the section \( |dx|^2 \). Let \( b_{ij}(x, \xi) = \langle b(x, \xi) e_j, \iota(e_i) \rangle \), then \( b(x, \xi) = (b_{ij}(x, \xi))_{i,j=1,...,N} \) a linear map from the fibers of \( \mathcal{E} \) to the fibers of \( \mathcal{E}^{*2} \), is the symbol of \( B \) in local coordinates: If \( u = \sum_j u_j e_j |dx|^2 \) and \( v = \sum_j v_j e_j |dx|^2 \), we have

\[
\langle b(x, \xi) u, v \rangle = \sum_{ij} b_{ij}(x, \xi) u_j \cdot \overline{v_i} |dx|,
\]

thus

\[
\int \langle Bu, v \rangle = \sum_{ij} \int (b_{ij}(x, D) u_j) \cdot \overline{v_j} dx.
\]

Note that \( b(x, \xi) \) is a Hermitian matrix, i.e. \( b_{ij}(x, \xi) = \overline{b_{ji}(x, \xi)} \), and in fact \( B = b(x, D) \) is self-adjoint (with respect to the standard Hermitian inner product on \( \mathbb{C}^N \)). The adjoint of \( P = p(x, D) \), which in local coordinates is simply an \( N \times N \) matrix of scalar \( \Psi \)DOs, with respect to \( B \) is the operator \( \overline{P} = \overline{p}(x, D) \) such that

\[
\int b(x, D)p(x, D)u \cdot \overline{v} \, dx = \int b(x, D)u \cdot \overline{\overline{p}(x, D)}v \, dx + \int R u \cdot \overline{v} \, dx, \quad R \in \Psi^{-\infty}.
\]

Let \( B^{-1} := b^{-1}(x, D) \) be a parametrix for \( b(x, D) \), in particular \( b^{-1}(x, \xi) = b(x, \xi)^{-1} \) modulo \( S^{-1} \); we may assume \( B^{-1}(x, D)^* = B^{-1}(x, D) \). We then have

\[
\overline{p}(x, D) = b^{-1}(x, D)p(x, D)^* b(x, D)
\]
by Lemma 3.5. Write \( p(x, \xi) = p_m(x, \xi) + p_{m-1}(x, \xi) + \ldots \), then the full symbol of \( P - \tilde{P} = B^*(B P - P^* B) \) (where \( P^* \) is the adjoint of \( P \) with respect to the standard Hermitian inner product on \( \mathbb{C}^N \)) is given, modulo \( S^{m-2} \), by

\[
\begin{align*}
\tilde{b}^{-1} \left( b p_m + \frac{1}{i} \sum_j \partial_{\xi_j} b \partial_{x_j} p_m + b p_{m-1} \right) \\
- p_m^* b - \frac{1}{i} \sum_j (\partial_{\xi_j} p_m^*) b - \frac{1}{i} \sum_j \partial_{\xi_j} p_m^* \partial_{x_j} b - p_{m-1}^* b \\
= \left( p_{m-1} - \frac{1}{2i} \sum_j \partial_{x_j, \xi_j} p_m \right) - \tilde{b}^{-1} \left( p_{m-1} - \frac{1}{2i} \sum_j \partial_{x_j, \xi_j} p_m \right)^* b + ib^{-1} H_{p_m}(b),
\end{align*}
\]

where we used that \( p_m \) is scalar and real. The claim follows. \( \square \)

### 3.3. Invariant formalism for subprincipal symbols of operators acting on bundles.

We continue to denote by \( P \in \Psi^m(X, \mathcal{E} \otimes \Omega_2) \) a principally scalar \( \Psi \text{DO} \) acting on the vector bundle \( \mathcal{E} \), with principal symbol \( p \).\(^{11}\) We will show how to modify the definition (3.3) of the subprincipal symbol of \( P \), expressed in terms of a local trivialization of \( \mathcal{E} \), in an invariant fashion, i.e. in a way that is both independent of the choice of local trivialization and of local coordinates on \( X \). This provides a completely invariant formulation of Lemma 3.7.

Let \( U \subset X \) be an open subset over which \( \mathcal{E} \) is trivial, and pick a frame \( e(x) = \{e_1(x), \ldots, e_N(x)\} \) trivializing \( \mathcal{E} \) over \( U \). Let us write \( P^e \) for \( P \) in the frame \( e \), i.e. \( P^e = (P^e_{jk})_{j,k = 1, \ldots, N} \) is the \( N \times N \) matrix of operators \( P^e_{jk} \in \Psi^m(U, \Omega_2) \) defined by \( P^e(\sum_k u_k(x)e_k(x)) = \sum_j P^e_{jk}(u_k)e_j(x), \ u_k \in \mathcal{C}^\infty(U, \Omega_2) \). Then \( \sigma^e_{\text{sub}}(P) \) as defined in (3.3), with the superscript making the choice of frame explicit, is simply an \( N \times N \) matrix of scalar symbols:

\[
\sigma^e_{\text{sub}}(P) = (\sigma_{\text{sub}}(P^e_{jk}))_{j,k = 1, \ldots, N}.
\]

We will consider the effect of a change of frame on the subprincipal symbol (3.3). Thus, let \( C \in \mathcal{C}^\infty(U, \text{End}(\mathcal{E})) \) be a change of frame, i.e. \( C(x) \) is invertible for all \( x \in X \). Then \( e_j'(x) = C(x)e_j(x) \) defines another frame \( e'(x) = \{e_1'(x), \ldots, e_N'(x)\} \) of \( \mathcal{E} \) over \( U \). One easily computes

\[
\sigma^e_{\text{sub}}(C^{-1} PC) = (C^{-1})^* \sigma^e_{\text{sub}}(P) C^{-1} - (C^{-1})^{-1} H_p(C^{-1}),
\]

with \( H_p \) interpreted as the diagonal \( N \times N \) matrix \( 1_{N \times N} H_p \) of first order differential operators, and \( C^{-1} \) is the matrix of \( C \) in the frame \( e' \). Now note that \( (C^{-1} PC)^e = P^e \) and \( (C^{-1})^{-1} H_p(C^{-1}) = (C^{-1})^{-1} H_p C^{-1} - H_p \); thus, we obtain

\[
\sigma^{e'}_{\text{sub}}(P) - iH_p = (C^{-1})^{-1} \left( \sigma^e_{\text{sub}}(P) - iH_p \right) C^{-1}.
\]

Thus, viewing \( \sigma^e_{\text{sub}}(P) - iH_p \) as the \( N \times N \) matrix (in the frame \( e' \)) of a differential operator acting on \( \mathcal{C}^\infty(T^*X \setminus 0, \pi^* \mathcal{E}) \), the right hand side of (3.5) is the matrix of the same differential operator, but expressed in the frame \( e \). Notice that the principal symbol \( p \) of \( P \) as a scalar, i.e. diagonal, \( N \times N \) matrix of symbols, is well-defined independently of the choice of frame. To summarize:

\(^{11}\)The discussion until Proposition 3.8 works for principally non-scalar operators as well with mostly notational changes.
Definition 3.8. For $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^\frac{1}{2})$ with scalar principal symbol $p$, there is a well-defined subprincipal operator $S_{\text{sub}}(P) \in \text{Diff}^1(T^*X \setminus 0, \pi^*\mathcal{E})$, homogeneous of degree $m - 1$ with respect to dilations in the fibers of $T^*X \setminus 0$, defined as follows: If $\{e_1(x), \ldots, e_N(x)\}$ is a local frame of $\mathcal{E}$, define the operators $P_{jk} \in \Psi^m(X, \Omega^\frac{1}{2})$ by $P(\sum_k u_k(x)e_k(x)) = \sum_{jk} P_{jk}(u_k)e_j(x)$, $u_k \in \mathcal{C}^{\infty}(X, \Omega^\frac{1}{2})$. Then

$$S_{\text{sub}}(P)\left(\sum_k q_k(x, \xi)e_k(x)\right) := \sum_{jk}(\sigma_{\text{sub}}(P_{jk})q_k)e_j - i\sum_k(\mathcal{H}_p q_k)e_k.$$ 

In shorthand notation, $S_{\text{sub}}(P) = \sigma_{\text{sub}}(P) - i\mathcal{H}_p$, understood in a local frame as a matrix of first order differential operators. We emphasize the dependence on the order of the operator by writing $S_{\text{sub}, m}(P)$, so that for $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^\frac{1}{2})$, we have $S_{\text{sub}, m+1}(P) = \sigma_{m}(P)$.

We shall compute the subprincipal operator of the Laplace-Beltrami operator acting on sections of the tensor bundle in Section 4.

Remark 3.9. For $\Psi_b$-inner products, the subprincipal operator of an operator $P \in \Psi^m_b(X, \mathcal{E} \otimes \Omega^\frac{1}{2})$ acting on $\mathcal{E}$-valued $b$-half-densities is an element of $\text{Diff}_b^1(bT^*X \setminus 0, \pi_b^*\mathcal{E})$, where $\pi_b : bT^*X \setminus 0 \to X$ is the projection. In the semiclassical setting, $P \in \Psi^m_b(X, \mathcal{E} \otimes \Omega^\frac{1}{2})$, we have $S_{\text{sub}}(P) \in \text{Diff}^1(T^*X, \pi^*\mathcal{E})$.

We can now express the symbols of commutators and imaginary parts in a completely invariant fashion:

Proposition 3.10. Let $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^\frac{1}{2})$ be a $\Psi$DO with scalar principal symbol $p$.

1. Suppose $Q \in \Psi^{m'}(X, \mathcal{E} \otimes \Omega^\frac{1}{2})$ is an operator acting on $\mathcal{E}$-valued half-densities, with principal symbol $q$. (We do not assume $Q$ is principally scalar.) Then
   $$\sigma^{m+m'}(Q) = [S_{\text{sub}}(P), q].$$
   If $Q$ is elliptic with parametrix $Q^-$, then
   $$S_{\text{sub}}(QPQ^-) = QS_{\text{sub}}(P)q^{-1}. \quad (3.6)$$

2. Suppose in addition that $p$ is real. Let $B$ be a $\Psi$-inner product on $\mathcal{E}$ with principal symbol $b$, then
   $$\sigma^{-1}(\mathcal{H}(b, P)) = S_{\text{sub}}(P), \quad (3.7)$$
   where $\mathcal{H}(b, P) = \frac{1}{2}\left(S_{\text{sub}}(P) - S_{\text{sub}}(P)^{b}\right)$; we take the adjoint of the differential operator $S_{\text{sub}}(P)$ with respect to the inner product $b$ on $\pi^*\mathcal{E}$ and the symplectic volume density on $T^*X$.

Proof. We verify this in a local frame $\xi(x) = \{e_1(x), \ldots, e_N(x)\}$ of $\mathcal{E}$. We compute

$$S_{\text{sub}}(P)\left(\sum_{jk} q_{jk}(x, \xi)e_k(x)\right)$$

$$= \sum_{jk}(\sigma_{\text{sub}}(P)_{jk}q_{\xi} - i\mathcal{H}_p(q_{\xi}))u_\xi e_j - iq_{\xi}u_\xi e_j - iq_{\xi}u_\xi e_jH_p,$$

while

$$qS_{\text{sub}}(P)\left(\sum_\xi u_\xi e_\xi(x)\right)$$
Proof. In order to shorten the notation, fix a global trivialization of $\Omega^1$ for any $Q$ noting that $Q$ appearing in the statement. Denote the principal symbol of $\sigma$ and use it to identify $E \otimes $ and $\Omega^1(T^*X \setminus 0)$ hence $S_{sub}(P)q - qS_{sub}(P) = [\sigma_{sub}(P), q] - iH_p(q)$ as an endomorphism (a zeroth order differential operator acting on sections of $\mathcal{E}$) of $\mathcal{E}$ in the frame $e$, which equals $\sigma^{m+m'-1}(P, Q)$ according to the usual (full) symbolic calculus.

Furthermore,

$$S_{sub, m}(Q P Q^{-1}) = S_{sub, m}(P) + S_{sub, m}(Q[P, Q^{-1}])$$

$$= S_{sub, m}(P) + q\sigma_{m+m'-1}([P, Q^{-1}]) = S_{sub, m}(P) + q[S_{sub, m}(P), q^{-1}]$$

$$= qS_{sub, m}(P)q^{-1},$$

noting that $Q[P, Q^{-1}]$ is of order $m - 1$.

For the second part, we have $S_{sub}(P)^{eb} = \sigma_{sub}(P)^{eb} - (iH_p)^{eb} = b^{-1}\sigma_{sub}(P)^{*b} + ib^{-1}(H_p)^{eb}$, where $(H_p)^*$ is the adjoint of $H_p$ as an operator acting on $C^\infty_c(T^*X \setminus 0)$, and we equip $T^*X$ with the natural symplectic volume density $|dx \, d\xi|$. We have $(H_p)^* = -H_p = -H_p$ since $p$ is real. Therefore,

$$S_{sub}(P) - S_{sub}(P)^{eb} = \sigma_{sub}(P) - \sigma_{sub}(P)^{*b} - iH_p + ib^{-1}H_p b$$

$$= \sigma_{sub}(P) - \sigma_{sub}(P)^{*b} + ib^{-1}H_p(b),$$

which indeed gives (3.4) upon division by $2i$.

In particular, (3.7) provides a very elegant point of view for understanding the imaginary part of a principally scalar and real (pseudo)differential operator with respect to a $\Psi$-inner product $B$, as already indicated in the introduction: For instance, the principal symbol of the imaginary part $\text{Im} P$ vanishes (or is small relative to $b = \sigma^0(B)$) in a subset of phase space if and only if the imaginary part of the first order differential operator $S_{sub}(P)$ on $T^*X \setminus 0$ has vanishing (or small with respect to the fiber inner product $b$ of $\pi^*\mathcal{E}$) coefficients in this subset.

### 3.4. Interpretation of pseudodifferential inner products in traditional terms.

We now show how to interpret the imaginary part $\text{Im}^B P$ of an operator $P$ with respect to a $\Psi$-inner product $B$ in terms of the imaginary part of a conjugated version of $P$ with respect to a standard inner product:

**Proposition 3.11.** Let $B$ be a $\Psi$-inner product on $\mathcal{E}$. Then for any positive definite Hermitian inner product $B_0 \in C^\infty(X, \text{Hom}(\mathcal{E} \otimes \Omega^\sharp, \mathcal{E}^* \otimes \Omega^\sharp))$ on $\mathcal{E}$, there exists an elliptic operator $Q \in \Psi^0(X, \text{End}(\mathcal{E} \otimes \Omega^\sharp))$ such that $B - Q^* B_0 Q \in \Psi^{-\infty}(X, \text{Hom}(\mathcal{E} \otimes \Omega^\sharp, \mathcal{E}^* \otimes \Omega^\sharp))$.

In particular, denoting by $Q^{-} \in \Psi^0(X, \text{End}(\mathcal{E} \otimes \Omega^\sharp))$ a parametrix of $Q$, we have for any $P \in \Psi^m(X, \mathcal{E} \otimes \Omega^\sharp)$ with real and scalar principal symbol:

$$Q(\text{Im}^B P)Q^{-} = \text{Im}^{B_0}(Q P Q^{-}),$$

(3.8) and $\sigma^{m-1}(\text{Im}^B P)$ and $\sigma^{m-1}(\text{Im}^{B_0}(Q P Q^{-}))$ (which are self-adjoint with respect to $\sigma^0(B)$ and $B_0$, respectively, hence diagonalizable) have the same eigenvalues.

**Proof.** In order to shorten the notation, fix a global trivialization of $\Omega^\sharp$ over $X$ and use it to identify $\mathcal{E} \otimes \Omega^\sharp$ with $\mathcal{E}$, likewise for all other half-density bundles appearing in the statement. Denote the principal symbol of $B$ by $b \in S^0_{\text{hom}}(T^*X \setminus 0)$.

\[\text{On a symbolic level, this is the same as equation (3.6).}\]
0, $\pi^* \text{Hom}(\mathcal{E}, \mathbb{R}^n)$). We similarly put $b_0 := B_0$, which is an inner product on $\pi^* \mathcal{E}$ that only depends on the base point.

We start with on the symbolic level by constructing an elliptic symbol $q_1 \in S^0_{\text{hom}}(T^*X \setminus 0, \pi^* \text{End}(\mathcal{E}))$ such that $b = q_1^* b q_1$; recall that $q_1^* \in S^0_{\text{hom}}(T^*X \setminus 0, \pi^* \text{End}(\mathcal{E}^*))$. For $t \in [0, 1]$, define the Hermitian inner product $b_t := (1-t)b_0 + tb$.

We will construct a differentiable family $q_t$ of symbols such that $b_t = q_t^* b q_t$ for $t \in [0, 1]$. Observe that for any such family, we have $\partial_0 b_t = B_0 = (\partial_0 q_t)^* b_0 q_t + q_t^* b_0 \partial_t q_t$, which suggests requiring $\partial_t q_t = \frac{1}{2} b_0^{-1}(q_t^*)^{-1}(b - b_0)$, which we can write as a linear expression in $q_t$ by noting that $(q_t^*)^{-1} = b_0 q_t b_t^{-1}$. Moreover, $q_0 = id$ is a valid choice for $q_t$ at $t = 0$. Thus, we are led to define $q_t$, $t \in [0, 1]$, as the solution of the ODE

$$\partial_t q_t = \frac{1}{2} q_t b_t^{-1}(b - b_0), \quad q_0 = id.$$ 

Reversing these arguments, for the solution $q_t$ we then have $q_t^* b_0 q_t = b_t$ for $t = 0$, and both $q_t^* b_0 q_t$ and $b_t$ are solutions of the same ODE, namely

$$\partial_t b_t = \frac{1}{2} ((b - b_0)b_t^{-1} b_t + b_t b_t^{-1}(b - b_0)), \quad b_0 = b_0,$$

hence $q_t^* b_0 q_t = b_t$ for all $t \in [0, 1]$.

Let $Q_1 \in \Psi^0(X, \text{End}(\mathcal{E}))$ be a quantization of $q_1$, then we conclude that $B - Q_1^* B_0 Q_1 \in \Psi^{-1}$. We iteratively remove this error to obtain a smoothing error: Suppose $Q_k \in \Psi^0(X, \text{End}(\mathcal{E}))$ is such that $B - Q_k^* B_0 Q_k \in \Psi^{-k}$ for some $k \geq 1$.

We will find $D_k \in \Psi^{-k}$, a quantization of $d_k \in S^{-k}_{\text{hom}}(T^*X \setminus 0, \pi^* \mathcal{E})$, such that $Q_{k+1} := Q_k + D_k$ satisfies $B - Q_{k+1}^* B_0 Q_{k+1} \in \Psi^{-k-1}$. This is equivalent to the equality of symbols

$$r_k := \sigma^{-k}(B - Q_k^* B_0 Q_k) = \sigma^{-k}(D_k b_0 Q_k + Q_k^* B_0 D_k) = d_k^* b_0 q_1 + (b_0 q_1^* )^* d_k,$$

which in view of $r_k^* = r_k$ is satisfied for $d_k = \frac{1}{2} ((b_0 q_1^* )^* )^{-1} r_k$. We define $Q \in \Psi^0(X, \text{End}(\mathcal{E}))$ to be the asymptotic limit of the $Q_k$ as $k \to \infty$, i.e. $Q \sim Q_1 + \sum_{k=1}^{\infty} D_k$, which thus satisfies $B - Q^* B_0 Q \in \Psi^{-\infty}$. This proves the first part of the proposition.

For the second part, denote parametrices of $B$ and $Q$ by $B^-$ and $Q^-$, respectively. Then, modulo operators in $\Psi^{-\infty}$, we have

$$(P^* B^-)^* = (Q^* B_0 Q P Q^- B_0^{-1} (Q^-)^* )^* = Q^- (Q P Q^-)^* B_0 Q,$$

hence

$$Q(P - P^* B) Q^- = (Q P Q^-) - (Q P Q^-)^* B_0 Q.$$

modulo $\Psi^{-\infty}$.

3.5. A simple example. On $\mathbb{R}^n_+ = \mathbb{R}^{x_1} \times \mathbb{R}^{n-1}_+$, we consider the operator $P = D_{x_1} + A \in \Psi^1(\mathbb{R}^n, \mathbb{C}^N)$, where $A = A(x, D) \in \Psi^0(\mathbb{R}^n, \mathbb{C}^N)$ is independent of $x_1$. Trivializing the half-density bundle over $\mathbb{R}^n$ via $|dx|^\frac{1}{2}$, we can consider $P$ as an operator in $\Psi^1(\mathbb{R}^n, \mathbb{C}^N \otimes \Omega^\frac{1}{2})$. Its principal symbol is $\sigma_1(P)(x, \xi) = \xi_1$, where we use the standard coordinates on $T^* \mathbb{R}^n$, i.e. writing covectors as $\xi \, dx$, so the Hamilton vector field is $H_{x_1}(P) = \partial_{x_1}$; moreover, in the trivialization of $\mathbb{C}^N$ by means of its standard basis, $\sigma_{\text{sub}}(P)(x, \xi) = A(x, \xi)$. Thus, the subprincipal operator of $P$ is

$$S_{\text{sub}}(P)(x, \xi) = A(x, \xi) - i \partial_{x_1} \in \text{Diff}^1(T^* \mathbb{R}^n \setminus 0, \pi^* \mathbb{C}^N),$$

with $A$ homogeneous of degree 0 in the fiber variables. Suppose we are interested in bounding $\frac{1}{2} (P - P^*)$ on $Z := T^*_{\{x_1 = 0\}} \mathbb{R}^n \setminus 0$ relative to a suitably chosen inner
product. Let us assume that \( A(0, \xi) \) is nilpotent for all \( |\xi| = 1 \), and that in fact at \( x = 0 \) and \( |\xi| = 1 \), we can choose a smooth frame \( e_1(\xi), \ldots, e_N(\xi) \) of the bundle \( \pi^*\mathbb{C}^N \rightarrow T^*\mathbb{R}^n \setminus 0 \) so that \( A(0, \xi) \), written in the basis \( e_1(\xi), \ldots, e_N(\xi) \), is a single Jordan block with zeros on the diagonal and ones directly above. Extend the \( e_j \) by homogeneity (of degree 0) in the fiber variables, and define them to be constant in the \( x_1 \)-direction along \( Z \), i.e. \( e_j(x_1, 0; \xi) = e_j(0, 0; \xi) \), and extend them in an arbitrary manner to a neighborhood of \( Z \).

Now, on \( Z \) we have \( A e_j = e_{j-1} \), writing \( e_0 := 0 \). Introduce a new frame \( e'_j := e_j' \) with \( \epsilon > 0 \) fixed, then \( A e'_j = \epsilon e'_{j-1} \). Define the inner product \( b \) on \( \pi^*\mathbb{C}^N \) by \( \langle b(x, \xi)(e'_i(x, \xi)), (e'_j(x, \xi)) \rangle = \delta_{ij} \), that is, \( \{e'_1, \ldots, e'_N\} \) is an orthonormal frame for \( b \). Then on \( Z \), we find that \( \operatorname{Im}^b S_{\text{sub}}(P) \) (which is of order 0) in the frame \( \{e'_1, \ldots, e'_N\} \) is given by the matrix which is zero apart from entries \( \epsilon / 2^i \) directly above and \( -\epsilon / 2^i \) directly below the diagonal. Thus, defining the \( \Psi \)-inner product \( B = b(x, D) \), we have arranged that \( \|\sigma_0(\operatorname{Im}P)(x, \xi)\|_b \leq \epsilon \) on \( Z \). Since \( \sigma_0(\operatorname{Im}P) \) is self-adjoint with respect to \( B \), this is really the statement that its eigenvalues are bounded from above and below by \( \epsilon \) and \( -\epsilon \), respectively.

Using Proposition 3.11, we can rephrase this as follows: If \( v_j \) denotes the standard basis of \( \mathbb{C}^N \) and \( \langle B_0(v_i), (v_j) \rangle = \delta_{ij} \) the standard inner product on \( \mathbb{C}^N \) (the particular choice of an ordinary inner product being irrelevant, see the statement of Proposition 3.11), define the map \( q(x, \xi) \in \mathbb{S}^0_{\text{hom}}(T^*\mathbb{R}^n \setminus 0, \pi^*\mathbb{C}^N) \) by \( q(x, \xi)e'_j(x, \xi) = v_j \). Let \( Q = q(x, D) \) and denote by \( Q^{-} \) a parametrix of \( Q \), then we find that \( Q P Q^{-} \in \Psi^1(\mathbb{R}^n, \mathbb{C}^N) \) satisfies \( \|\sigma_0(\operatorname{Im}P) Q P Q^{-} - B_0\| \leq \epsilon \).

If \( A \) has several Jordan blocks not all of which are nilpotent, one can (under the assumption of the existence of a smooth family of Jordan bases) similarly construct a \( \Psi \)-inner product so that the imaginary part of \( A \) relative to it is bounded by the maximal imaginary part of the eigenvalues of \( A \) (plus \( \epsilon \)) from above, and by the minimal imaginary part (minus \( \epsilon \)) from below.

### 4. Subprincipal operators of tensor Laplacians

Let \( (M, g) \) be a smooth manifold equipped with a metric tensor \( g \) of arbitrary signature. Denote by \( T_k M = \otimes^k T^* M, k \geq 1 \), the bundle of (covariant) tensors of rank \( k \) on \( M \). The metric \( g \) induces a metric (which we also call \( g \)) on \( T_k M \). We study the symbolic properties of \( \Delta_k = -\operatorname{tr} \nabla^2 \in \operatorname{Diff}^2(M, T_k M) \), the Laplace-Beltrami operator on \( M \) acting on the bundle \( T_k M \). Denote by \( G \in C^\infty(T^* M) \) the metric function, i.e. \( G(x, \xi) = |\xi|^2_{G(x)} \), where \( G \) is the dual metric of \( g \).

**Proposition 4.1.** The subprincipal operator of \( \Delta_k \) is

\[
S_{\text{sub}}(\Delta_k)(x, \xi) = -i \nabla^*_{\nabla^2 T_k M} \in \operatorname{Diff}^1(T^* M \setminus 0, \pi^* T_k M),
\]

where \( \nabla^*_{\nabla^2 T_k M} \) is the pullback connection, with \( \pi: T^* M \setminus 0 \rightarrow M \) being the projection.

**Proof.** Since both sides of (4.1) are invariantly defined, it suffices to prove the equality in an arbitrary local coordinate system. At a fixed point \( x_0 \in M \), introduce normal coordinates so that \( \partial_k g_{ij} = 0 \) at \( x_0 \). Then we schematically have

\[
(\Delta_k u)_{i_1 \ldots i_k} = -g^{jk} u_{i_1 \ldots i_k,j} = -g^{jk} (\partial_k u_{i_1 \ldots i_k,j} + \Gamma \cdot \partial u) = -g^{jk} \partial_j u_{i_1 \ldots i_k} + \partial(\Gamma \cdot u) + \partial \Gamma \cdot \partial u
\]
where \( \Gamma \) denoting Christoffel symbols. This suffices to see that the full symbol of \( \Delta_k \) in the local coordinate system is given by

\[
\sigma(\Delta_k)(x, \xi) = g^{jk}(x)\xi_j \xi_k + (x^j - x_0^j)\ell_j(x, \xi) + e(x),
\]

where \( \ell_j(x, \xi) \) is a linear map in \( \xi \) with values in \( \text{End}(\pi^*\mathcal{T}_kM) \), and \( e(x) \) is an endomorphism of \( \pi^*\mathcal{T}_kM \). Therefore, \( \sigma_{\text{sub}}(\Delta_k)(x_0, \xi) = 0 \), since \( \partial_k g^{jk}(x_0) = 0 \). Thus,

\[
S_{\text{sub}}(\Delta_k)(x_0, \xi) = -iH|_{\xi^i}^j = -2i g^{jk} \xi_k \partial x^j. \tag{4.2}
\]

We now compute the right hand side of (4.1). First, writing \( dx^I = dx^{i_1} \otimes \cdots \otimes dx^{i_k} \) for multiindices \( I = (i_1, \ldots, i_k) \), we note that sections of \( \pi^*\mathcal{T}_kM \) are of the form \( u_I(x, \xi) \) \( dx^I \), while pullbacks (under \( \pi \)) of sections of \( \mathcal{T}_kM \) are of the form \( u_I(x) \) \( dx^I \).

By definition, the pullback connection \( \nabla^\pi \mathcal{T}_kM \) is given by

\[
\nabla_{\partial_x^j} \pi \mathcal{T}_kM(u_I(x) \, dx^I) = \nabla_{\partial_x^j} \pi \mathcal{T}_kM(u_I(x) \, dx^I), \quad \nabla_{\partial_{\xi_k}} \pi \mathcal{T}_kM(u_I(x) \, dx^I) = 0.
\]

on pulled back sections and extended to sections of the pullback bundle using the Leibniz rule; thus,

\[
\nabla_{\partial_x^j} \pi \mathcal{T}_kM(u_I(x, \xi) \, dx^I) = \nabla_{\partial_x^j} \pi \mathcal{T}_kM(u_I(\cdot, \xi) \, dx^I)(x),
\]

\[
\nabla_{\partial_{\xi_k}} \pi \mathcal{T}_kM(u_I(x, \xi) \, dx^I) = \partial_{\xi_k} u_I(x, \xi) \, dx^I.
\]

Thus, in normal coordinates at \( x_0 \in M \), we simply have \( \nabla_{\partial_x^j} \pi \mathcal{T}_kM = \partial x^j \) and \( \nabla_{\partial_{\xi_k}} \pi \mathcal{T}_kM = \partial_{\xi_k} \), therefore

\[
\nabla_{\partial_{\xi_k}} \pi \mathcal{T}_kM = 2g^{jk} \xi_k \partial x^j
\]

at \( x_0 \), which verifies (4.1) in view of (4.2).

To simplify the study of the pullback connection on \( \pi^*\mathcal{T}_kM \) for general \( k \), we observe that there is a canonical bundle isomorphism \( \pi^*\mathcal{T}_kM \cong \bigotimes^k \pi^*TM \); hence the connection \( \nabla^\pi \mathcal{T}_kM \) is simply the product connection on \( \bigotimes^k \pi^*TM \). Therefore, if we understand certain properties of \( S_{\text{sub}}(\Delta_k) \), we can easily deduce them for \( S_{\text{sub}}(\Delta_k) \) for any \( k \). In our application, we will need to choose a positive definite pseudodifferential inner product \( B_k = b_k(x, D) \) on the bundle \( \mathcal{T}_kM \) with respect to which \( \Delta_k \) is arbitrarily close to being symmetric in certain subsets of phase space. Concretely, this means that we want the operator \( S_{\text{sub}}(\Delta_k) \) to be (almost) symmetric with respect to the inner product \( b_k \) on \( \pi^*\mathcal{T}_kM \). The following lemma shows that it suffices to accomplish this for \( k = 1 \):

**Lemma 4.2.** Let \( U \subset T^*M \setminus \emptyset \) be open, and let \( f \in \mathcal{C}^\infty(U) \) be real-valued. Fix a Hermitian inner product \( b \) (antilinear in the second slot) on \( \pi^*T^*M \), and define \( R \in \text{End}(\pi^*T^*M) \) by requiring that

\[
\int_U \langle i \nabla^\pi_{\partial_x^j} T^*M u, v \rangle_b \, d\sigma - \int_U \langle u, i \nabla^\pi_{\partial_x^j} T^*M v \rangle_b \, d\sigma = \int_U \langle u, Rv \rangle_b \, d\sigma
\]

for all \( u, v \in \mathcal{C}^\infty(U, \pi^*T^*M) \), where \( d\sigma \) is the natural symplectic volume density on \( T^*M \). There exists a constant \( C_k > 0 \), independent of \( U, f \) and \( b \), such that the following holds: If \( \sup_U \|R\|_b \leq \epsilon \) (using \( b \) to measure the operator norm of \( R \))
acting on each fiber) for some $\epsilon > 0$, then the inner product $b_k = \otimes^k b$ induced by $b$ on $\bigotimes^k \pi^* T^* M \cong \pi^* T^*_k M$ satisfies

$$
\int_U \langle \{ i \nabla^\pi T^*_k M \, u, v \} \rangle_{b_k} \, d\sigma - \int_U \langle \{ u, i \nabla^\pi T^* M \, v \} \rangle_{b_k} \, d\sigma = \int_U \langle \{ u, R_k v \} \rangle_{b_k} \, d\sigma,
$$

$u, v \in C_c^\infty (U, \pi^* T^*_k M)$, for $R_k \in \text{End} (\pi^* T^*_k M)$ satisfying $\sup_U \| R_k \|_{b_k} \leq k \epsilon$.

**Proof.** We show this for $k = 2$, the proof for general $k$ being entirely analogous. Denote $S = i \nabla^\pi T^* M$, then $S_2 = i \nabla^\pi T^*_2 M$ acts by $S_2(u_1 \otimes u_2) = Su_1 \otimes u_2 + u_1 \otimes Su_2$. Hence using $S(au) = aSu + iH_f (a)u$ for sections $u$ of $\pi^* T^* M$ and functions $a$ on $U$, we calculate

$$
\int_U \langle S_2(u_1 \otimes u_2), v_1 \otimes v_2 \rangle_{b_2} \, d\sigma = \int_U \langle Su_1, v_1 \rangle_b \langle u_2, v_2 \rangle_b + \langle u_1, v_1 \rangle_b \langle Su_2, v_2 \rangle_b \, d\sigma
$$

$$
= \int_U \langle u_1, S(v_1 \langle u_2, v_2 \rangle_b) \rangle_b + \int_U \langle u_2, S(v_2 \langle u_1, v_1 \rangle_b) \rangle_b \, d\sigma
$$

$$
+ \int_U \langle u_1 \otimes u_2, (R \otimes \text{id} + \text{id} \otimes R)(v_1 \otimes v_2) \rangle_{b_2} \, d\sigma
$$

$$
= \int_U \langle v_1 \otimes u_2, S_2(v_1 \otimes v_2) \rangle_{b_2} \, d\sigma - i \int_U H_f (\langle u_1, v_1 \rangle_b \langle u_2, v_2 \rangle_b) \, d\sigma
$$

$$
+ \int_U \langle u_1 \otimes u_2, R_2(v_1 \otimes v_2) \rangle_{b_2} \, d\sigma
$$

$$
= \int_U \langle v_1 \otimes u_2, S_2(v_1 \otimes v_2) \rangle_{b_2} \, d\sigma + \int_U \langle u_1 \otimes u_2, R_2(v_1 \otimes v_2) \rangle_{b_2} \, d\sigma
$$

with $R_2 = R \otimes \text{id} + \text{id} \otimes R$, where we used that $\int_U H_f u \, d\sigma = - \int_U u H_f 1 \, d\sigma = 0$ for $u \in C_c^\infty (U)$. From the explicit form of $R_2$, we see that $\| R_2 \|_{b_2} \leq 2 \epsilon$ indeed. \hfill \Box

### 4.1. Warped product spacetimes

Let $X$ be an $(n - 1)$-dimensional manifold equipped with a smooth Riemannian metric $h = h(x, dx)$, and let $\alpha \in C^\infty (X)$ be a positive function. We consider the manifold $M = \mathbb{R}_t \times X$, equipped with the Lorentzian metric

$$
g = \alpha^2 dt^2 - h. \quad (4.3)
$$

On such a spacetime, we have a natural splitting of 1-forms into their tangential and normal part relative to $\alpha dt$, i.e.

$$
u = \nu_T + u_N \alpha \, dt. \quad (4.4)
$$

In this section, we will compute the form of $\nabla^\pi T^* M$ as a $2 \times 2$ matrix of differential operators with respect to this decomposition. For brevity, we will use the notation $\nabla^M := \nabla^\pi T^* M$, similarly $\nabla^X := \nabla^\pi T^* X$, and we will moreover use the abstract index notation, fixing $x^0 = t$, and $x^i = (x^1, \ldots, x^{n-1})$ are coordinates on $X$ (independent of $t$). We let Greek indices $\mu, \nu, \lambda, \ldots$ run from 0 to $n - 1$, Latin indices $i, j, k, \ldots$ from 1 to $n - 1$. Moreover, the canonical dual variables $\xi_0 = : \sigma$ and $\xi_i = (\xi_1, \ldots, \xi_{n-1})$ on the fibers of $T^* M$ are indexed by decorated Greek indices $\tilde{\mu}$ (running from 0 to $n - 1$) and Latin indices $\tilde{i}, \tilde{j}, \ldots$ (running from 1 to $n - 1$). If an index appears both with and without tilde in one expression, it is summed.

---

13Thus, once we discuss Schwarzschild-de Sitter space in the next section, in the region where $t_* = t$ (which we can in particular arrange near the trapped set), $\sigma$ in the present notation is equal to $- \sigma$ in the notation of Section 2.
Proposition 4.4. which immediately gives (4.5).

The Christoffel symbols of \( M \) are given by:

\[
\nabla^M \mu u_\nu = \nabla^M u_\nu, \quad \nabla^M \mu u_\nu = \partial_\mu u_\nu,
\]

where we interpret \( \nabla^M \mu \) as acting on \( u \) for fixed values of the fiber variables, i.e., viewing \( u \) as a family of sections of \( T^*M \) depending on the fiber variables. As before, we denote by \( G \) the metric function on \( T^*M \), and we let \( H \) denote the metric function on \( T^*X \), interpreted as a \((t,\sigma)\)-independent function on \( T^*M \). Lastly, we denote the Christoffel symbols of \((M,g)\) by \( M\Gamma^{\kappa}_{\mu\nu} \), and those of \((X,h)\) by \( X\Gamma^{ij}_{kl} \).

**Lemma 4.3.** The Christoffel symbols of \( M \) are given by:

\[
M\Gamma^0_{00} = 0, \quad M\Gamma^0_{0i} = \alpha^{-1} \alpha_i, \quad M\Gamma^0_{ij} = 0,
\]

\[
M\Gamma^k_{00} = \alpha \kappa^k \alpha, \quad M\Gamma^k_{i0} = 0, \quad M\Gamma^k_{ij} = X\Gamma^{ij}_{kl}.
\]

**Proof.** We have \( g_{00} = \alpha^2 \), \( g_{0i} = g_{00} = 0 \) and \( g_{ij} = -h_{ij} \), and \( g \) is \( t \)-independent, thus \( \partial_0 g_{\mu\nu} = 0 \). Using \( M\Gamma^0_{\kappa\nu} = \frac{1}{2} (\partial_\kappa g_{\nu\sigma} + \partial_\nu g_{\kappa\sigma} - \partial_\sigma g_{\kappa\nu}) \), we then compute

\[
M\Gamma^0_{00} = 0, \quad M\Gamma^0_{0i} = \alpha \alpha_i, \quad M\Gamma^0_{ij} = 0,
\]

\[
M\Gamma^k_{00} = -\alpha \alpha_k, \quad M\Gamma^k_{i0} = 0, \quad M\Gamma^k_{ij} = -X\Gamma^{ij}_{kl},
\]

which immediately gives (4.5). \( \square \)

**Proposition 4.4.** For the metric \( g \) as in (4.3), the subprincipal operator of \( \Box^1 \) (the tensor wave operator acting on 1-forms on \( M \)) in the decomposition (4.4) of 1-forms is given by

\[
i S_{\sub(\Box^1)}(t,x',\sigma,\xi') = \begin{pmatrix}
2 \alpha^{-2} \sigma \partial_t + \sigma^2 \nabla_X^M H_{\alpha^{-2}} - \nabla_{\nabla_X^M}^M H_{\alpha^{-2}} & -2 \alpha^{-2} \sigma \partial_\alpha \\
-2 \alpha^{-2} \sigma \nabla_X^M \alpha & 2 \alpha^{-2} \sigma \partial_t + \sigma^2 H_{\alpha^{-2}} - H_{\sigma}
\end{pmatrix}.
\]

**Proof.** We start by computing the form of \( \nabla^M\mu u_\nu \) and \( \nabla^M\mu u_\nu \) for tangential and normal 1-forms. For tangential forms \( u = u_\mu dx^\mu \) with \( u_0 = 0 \), we have

\[
\nabla^0_0 u_0 = -M\Gamma^0_{00} u_\lambda = -\alpha \langle d\alpha, u \rangle_H, \quad \nabla^0_0 u_i = \partial_0 u_i,
\]

\[
\nabla^0_0 u_0 = 0, \quad \nabla^0_0 u_i = \nabla^X_0 u_i, \quad \nabla^M_0 u_0 = 0, \quad \nabla^M_0 u_i = \partial_i u_i,
\]

while for normal forms \( u = u_\mu dx^\mu \) with \( u_i = 0 \) and \( u_0 = \alpha v \), we compute

\[
\nabla^0_0 u_0 = \alpha \partial_0 v, \quad \nabla^0_0 u_i = -\alpha v,
\]

\[
\nabla^0_0 u_0 = \partial_j (\alpha v) - \alpha_j v = \alpha \partial_j v, \quad \nabla^0_0 u_i = 0, \quad \nabla^M_0 u_0 = \alpha \partial_0 v, \quad \nabla^M_0 u_i = 0.
\]

Since \( G = \alpha^{-2} \sigma^2 - H \), we find \( H_G = 2 \alpha^{-2} \sigma \partial_t + \sigma^2 H_{\alpha^{-2}} - H_H \). Using \( \langle d\alpha, \cdot \rangle_H = i_{\nabla^X_\alpha} \), we obtain

\[
\nabla^M_{\partial_\alpha} = \begin{pmatrix}
\partial_t & -d\alpha \\
-i_{\nabla^X_\alpha} & \partial_t
\end{pmatrix}.
\]

Moreover, for any \( f \in C^\infty(T^*X) \) (we will take \( f = \alpha^{-2} \) and \( f = H \)), viewed as a \((t,\sigma)\)-independent function on \( T^*M \), we have \( H_f = f_j \partial_j - f_j \partial_j \). Hence on tangential forms,

\[
\nabla^M_H u_0 = 0, \quad \nabla^M_H u_i = f_j \nabla^X_j u_i - f_j \partial_j u_i = \nabla^X_H u_i,
\]

and
while on normal forms as above,
\[ \nabla^M_{H_f} u_0 = \alpha f_f \partial_j v - \alpha f_f \partial_j v = \alpha H_f v, \quad \nabla^M_{H_f} u_i = 0. \]

Thus,
\[ \nabla^M_{H_f} = \begin{pmatrix} \nabla^X_{H_f} & 0 \\ 0 & H_f \end{pmatrix}. \]

The claim follows. \( \square \)

4.2. **Schwarzschild-de Sitter space.** We stay in the setting of the previous section, and now the spatial metric \( h \) has a decomposition
\[ h = \alpha^{-2} dr^2 + r^2 d\omega^2, \]
where \( d\omega^2 \) is the round metric on the unit sphere \( Y = \mathbb{S}^{n-2} \), with dual metric denoted \( \Omega \); see (2.1). Thus, writing \( \xi, \eta \), resp. \( \eta, \xi \), for the dual variables of \( r \), resp. \( \omega \in \mathbb{S}^{n-2} \), we have \( H = \alpha^2 \xi^2 + r^{-2} |\eta|^2 \). Write 1-forms on \( X \) as
\[ u = u_r + u_N \alpha^{-1} dr. \quad (4.6) \]

Abbreviate the derivative of a function \( f \) with respect to \( r \) by \( f' \). Since \( da = \alpha' dr \) and \( \nabla^X \alpha = \alpha^2 \alpha' \partial_r \), we have, in the decomposition (4.6),
\[ da = \begin{pmatrix} 0 \\ \alpha' \end{pmatrix}, \quad i_{\nabla^X \alpha} = \begin{pmatrix} 0 & \alpha' \end{pmatrix}. \]

We will need the Christoffel symbols of \( h \). We continue using the notation to the previous section, except now \( x^1 = r \) and \( \xi_1 = \xi \), while \( x^2, \ldots, x^n \) are \( r \)-independent coordinates on \( \mathbb{S}^{n-2} \), and moreover the lower bound for Greek indices is 1, and 2 for Latin indices.

**Lemma 4.5.** The Christoffel symbols of \( X \) are given by:
\[
\begin{align*}
X \Gamma^1_{11} &= -\alpha^{-1} \alpha', \quad X \Gamma^1_{i1} = 0, \quad X \Gamma^1_{ij} = -r \alpha^2 (d\omega^2)_{ij}, \\
X \Gamma^k_{i1} &= 0, \quad X \Gamma^k_{ij} = r^{-1} \delta^k_i, \quad X \Gamma^k_{ij} = Y \Gamma^k_{ij}.
\end{align*}
\]

*Proof.* We have \( h_{11} = \alpha^{-2}, \quad h_{ii} = h_{i1} = 0 \) and \( h_{ij} = r^2 (d\omega^2)_{ij} \), and \( (d\omega^2)_{ij} \) is \( r \)-independent. We then compute
\[ X \Gamma_{111} = -\alpha^{-3} \alpha', \quad X \Gamma_{1i1} = 0, \quad X \Gamma_{1ij} = -r (d\omega^2)_{ij}, \]
\[ X \Gamma_{kk1} = 0, \quad X \Gamma_{kki} = r (d\omega^2)_{ki}, \quad X \Gamma_{kij} = r^2 Y \Gamma_{kij}, \]
which immediately gives (4.7). \( \square \)

We are only interested in the subprincipal operator of \( \Box_1 \) at the trapped set, which we recall from (2.6) to be the set
\[ \Gamma = \{ r = r_p, \xi = 0, \sigma^2 = \Psi^2 |\eta|^2 \}, \quad \text{where } \Psi = \alpha r^{-1}, \Psi'(r_p) = 0. \quad (4.8) \]

Thus, at \( \Gamma \), we have
\[ H_H = 2 \alpha^2 \xi \partial_r - 2 \alpha \xi' \xi^2 \partial_k + 2 r^{-3} |\eta|^2 \partial_k + r^{-2} H_{|\eta|^2} = 2 r^{-3} |\eta|^2 \partial_k + r^{-2} H_{|\eta|^2}, \]
while \( \sigma^2 H_{\alpha^{-2}} = 2 \sigma^2 \alpha^{-3} \alpha' \partial_k \). Now \( \alpha^{-1} \alpha' = (r \Psi)^{-1} (r \Psi)' = r^{-1} \) at \( r = r_p \), therefore \( \sigma^2 \alpha^{-3} \alpha' = r^{-3} |\eta|^2 \), and we thus obtain
\[ \sigma^2 H_{\alpha^{-2}} - H_H = -r^{-2} H_{|\eta|^2} \text{ at } \Gamma. \quad (4.9) \]

Notice that \( |\eta|^2 \in C^\infty(T^* Y) \) is independent of \( (r, \xi) \).
Lemma 4.6. For a function \( f \in C^\infty(T^*Y) \), viewed as an \((r, \xi)\)-independent function on \( X \), we have
\[
\tilde{\nabla}^X_{H_f} = \begin{pmatrix} \tilde{\nabla}^Y_{H_f} & \alpha r(i_{H_f} d\omega^2) \end{pmatrix},
\]
in the decomposition (4.6) of 1-forms on \( X \).

Proof. On tangential forms \( u, i.e. \ u_1 = 0 \), we have
\[
\tilde{\nabla}^X_J u_1 = -r^{-1} u_j, \quad \tilde{\nabla}^Y_J u_1 = \nabla u_i, \quad \tilde{\nabla}^X_J u_1 = 0, \quad \tilde{\nabla}^Y_J u_1 = \nabla u_i.
\]
thus using \( H_f = f_j \partial_j - f_j \partial_j \), we get
\[
\tilde{\nabla}^X_J u_1 = -r^{-1} f_j u_j = -r^{-1} u(H_f) = -r^{-1} i_{H_f} u, \quad \tilde{\nabla}^Y_J u_1 = \nabla u_i.
\]
On normal forms \( u, i.e. \ u_1 = \alpha^{-1} v, \ u_1 = 0 \), we compute
\[
\tilde{\nabla}^X_J u_1 = \alpha^{-1} \partial_j v, \quad \tilde{\nabla}^Y_J u_1 = r \alpha(d\omega^2)_{ij} v, \quad \tilde{\nabla}^X_J u_1 = \alpha^{-1} \partial_j v, \quad \tilde{\nabla}^Y_J u_1 = 0,
\]
hence
\[
\tilde{\nabla}^X_J u_1 = \alpha^{-1} f_j \partial_j v - \alpha^{-1} f_j \partial_j v = \alpha^{-1} H_f v, \quad \tilde{\nabla}^Y_J u_1 = f_j r \alpha(d\omega^2)_{ij} v = \alpha r(i_{H_f} d\omega^2)v.
\]
The claim follows immediately. \( \square \)

Combining Proposition 4.4 and Lemma 4.6, we can thus compute the subprincipal operator of \( \Box_1 \) acting on 1-forms (sections of the pullback of \( T^*M \) to \( T^*M \setminus 0 \)) decomposed as
\[
u = u_{TT} + u T N \alpha^{-1} dr + u_N \alpha \ dt. \hspace{1cm}(4.10)
\]
In view of (4.9), we merely need to apply Lemma 4.6 to \( f = |\eta|^2 \), in which case \( H_f = 2\Omega^k \eta_j \partial_k - \partial_k \Omega^k \eta_j \partial_j \), so \( i_{H_f} = 2i_\eta \) on 1-forms (identifying the 1-form \( \eta \) with a tangent vector using the metric \( d\omega^2 \)), while \( i_{H_f} d\omega^2 = 2\eta \). Thus, we obtain:

Proposition 4.7. In the decomposition (4.10), the subprincipal operator of \( \Box_1 \) on Schwarzschild-de Sitter space at the trapped set \( \Gamma \) is given by
\[
i S_{\text{sub}}(\Box_1) = \begin{pmatrix} 2\alpha^{-2} \sigma \partial_t - r^{-2} \tilde{\nabla}^Y_{H_f} \eta^2 & -2\alpha r^{-1} \eta \sigma & 0 \\
2\alpha r^{-3} \eta & 2\alpha^{-2} \sigma \partial_t - r^{-2} |\eta|^2 & -2r^{-1} \sigma \\
0 & -2r^{-1} \sigma & 2\alpha^{-2} \sigma \partial_t - r^{-2} |\eta|^2 \end{pmatrix}.
\hspace{1cm}(4.11)
\]

Since \( \Box_1 \) is symmetric with respect to the natural inner product \( G \) on the 1-form bundle, which in the decomposition (4.10) is an orthogonal direct sum of inner products, \( G = (-r^{-2} \Omega) \oplus (-1) \oplus 1 \), the operator \( S_{\text{sub}}(\Box_1) \) is a symmetric operator acting on sections of \( \pi^* T^* M \) over \( T^* M \setminus 0 \) if we equip \( \pi^* T^* M \) with the fiber inner product \( G \) and use the symplectic volume density on \( T^* M \setminus 0 \).

The matrix \(-2 r^{-2} s\), with
\[
s = \begin{pmatrix} 0 & \Psi r^2 \eta & 0 \\
-\Psi r \eta & 0 & r \sigma \\
0 & r \sigma & 0 \end{pmatrix},
\]
\[
14 We use that \( \pi^* T^* X \) can be canonically identified with the horizontal subbundle of \( T^*(T^* X) \).
of 0-th order terms of $S_{\text{sub}}(\square_1)$ is nilpotent, which suggests in analogy to the discussion in Section 3.5 that the imaginary part of $S_{\text{sub}}(\square_1)$ with respect to a Riemannian fiber inner product can be made arbitrarily small. Indeed, for any fixed $\epsilon > 0$, define the ‘change of basis matrix’

$$q = \begin{pmatrix}
\text{id} & 0 & 0 \\
0 & \epsilon^{-1}\Psi r^2 & 0 \\
-\epsilon^{-2}|\eta|^{-1}\Psi r^2 i_\eta & 0 & \epsilon^{-2}|\eta|^{-1}\Psi r^3 \sigma
\end{pmatrix},$$

then

$$qsq^{-1} = \begin{pmatrix}
0 & \epsilon \eta & 0 \\
0 & 0 & \epsilon|\eta| \\
0 & 0 & 0
\end{pmatrix}.$$ 

In order to compute $qS_{\text{sub}}(\square_1)q^{-1}$, we note that the diagonal matrix of $t$-derivatives in (4.11) commutes with $q$, and it remains to study the derivatives along $H_{|\eta|^2}$; more specifically, $q$ has a block structure, with the columns and rows 1, 3 being the first block and the $(2, 2)$ entry the second, and the $(2, 2)$ block is an $\eta$-independent multiple of the identity, hence commutes with the relevant $(2, 2)$ entry $ir^{-2}H_{|\eta|^2}$ of $S_{\text{sub}}(\square_1)$. For the 1, 3 block, we compute

$$\begin{pmatrix}
\nabla^Y_{H_{|\eta|^2}} & 0 \\
0 & H_{|\eta|^2}
\end{pmatrix} \begin{pmatrix}
\text{id} & 0 \\
0 & \epsilon^{-1}\Psi r^2 & 0 \\
-\epsilon^{-2}|\eta|^{-1}\Psi r^2 i_\eta & 0 & \epsilon^{-2}|\eta|^{-1}\Psi r^3 \sigma
\end{pmatrix} = \epsilon^{-2}\Psi r^2|\eta|^{-1} \begin{pmatrix}
\nabla^Y_{H_{|\eta|^2}} & 0 \\
0 & H_{|\eta|^2} i_\eta
\end{pmatrix}.$$

Now $\nabla^Y_{H_{|\eta|^2}}$ and $H_{|\eta|^2}$ are the restrictions of the pullback connection $\nabla^{\pi^*\Lambda^{n-2}}_{H_{|\eta|^2}}$ of the full form bundle to 1-forms and functions, respectively, and the latter commutes with $i_\eta$, since by Proposition 3.10,

$$0 = S_{\text{sub}}([\square, \delta]) = -i[S_{\text{sub}}(\square), i_\eta] = -[\nabla^{\pi^*\Lambda^{n-2}}_{H_{|\eta|^2}}, i_\eta],$$

where $\square$ denotes the Hodge d’Alembertian on the form bundle and $\delta$ is the codifferential. Thus, (4.12) in fact vanishes, and therefore

$$qS_{\text{sub}}(\square_1)q^{-1} = -i \begin{pmatrix}
2\alpha^{-2}\sigma \partial_t - r^{-2}\nabla^Y_{H_{|\eta|^2}} & -2r^2 \epsilon \eta \\
0 & 2\alpha^{-2}\sigma \partial_t - r^{-2}H_{|\eta|^2} & -2r^2 \epsilon|\eta|
\end{pmatrix}.$$ 

Equip the 1-form bundle over $M$ in the decomposition (4.10) with the Hermitian inner product

$$B_0 = \Omega \oplus 1 \oplus 1,$$

then $qS_{\text{sub}}(\square_1)q^{-1}$ has imaginary part (with respect to $B_0$) of size $O(\epsilon)$. Put differently, $S_{\text{sub}}(\square_1)$ has imaginary part of size $O(\epsilon)$ relative to the Hermitian inner product $b := B_0(q, q)$, which is the symbol of a pseudodifferential inner product on $\pi^*T^* M$. We can now invoke Lemma 4.2 on a neighborhood of $\Gamma \cap \{|\sigma| = 1\}$ and use the homogeneity of $q, b$ and $S_{\text{sub}}(\square_1)$ to obtain:

**Theorem 4.8.** For any $\epsilon > 0$, there exists a (positive definite) $t_*$-independent pseudodifferential inner product $B = b(x, D)$ on $T_k M$ (thus, $b$ is an inner product
on \( \pi^* T_k M \), homogeneous of degree 0 with respect to dilations in the base \( T^* M \setminus 0 \), such that
\[
\sup_{\Gamma} |\sigma|^{-1} \left\| \frac{1}{2t} \left( S_{\text{sub}}(\Box_k) - S_{\text{sub}}(\Box_k^{\text{b}}) \right) \right\|_b \leq \epsilon,
\]
where \( \Gamma \) is the trapped set (4.8). Put differently, there is an elliptic \( \Psi DO \) \( Q \), invariant under \( t_* \)-translations, acting on sections of \( T_k M \), with parametrix \( Q^- \), such that relative to the ordinary positive definite inner product (4.13), we have
\[
\sup_{\Gamma} |\sigma|^{-1} \left\| \sigma_1 \left( \frac{1}{2t} \left( Q \Box_k Q^- - (Q \Box_k Q^-)^{\text{b}0} \right) \right) \right\|_{\text{sup}^1} \leq \epsilon.
\]
By restriction, the analogous statements are true for \( \Box \) acting on subbundles of the tensor bundle on \( M \), for instance differential forms of all degrees and symmetric 2-tensors.

By the \( t_* \)-translation invariance of the involved symbols, inner products and operators, this is really a statement about \( \Psi_0 \)-inner products, and \( Q \) is a \( b \)-pseudo-differential operator; see the discussion preceding Theorem 2.1 for the relationship of the stationary and the \( b \)-picture.

**Remark 4.9.** Adding a 0-th order term to \( \Box \) does not change \( \Box \) or its imaginary part at the principal symbol level, thus does not affect the subprincipal operator of \( \Box \) either; therefore, Theorem 4.8 holds in this case as well.

Adding a first order operator \( L \) (acting on sections of \( T_k M \)), which we assume to be \( t \)-independent for simplicity, does affect the subprincipal operator, more specifically its 0-th order part, since \( S_{\text{sub}}(\Box + L) = S_{\text{sub}}(\Box) + \sigma_1(L) \). Thus, if \( \sigma_1(L) \) is small at \( \Gamma \), we can use the same \( \Psi \)-inner product as for \( \Box \) and obtain a bound on \( \text{Im}^b S_{\text{sub}}(\Box + L) \) which is small, but no longer arbitrarily small. However, the bound merely needs to be smaller than \( \nu_{\text{min}}/2 \), see (2.5), which does hold for small \( L \).

If we do not restrict the size of \( L \), we can still obtain a spectral gap, provided one can choose a \( \Psi \)-inner product as in Theorem 4.8, again with \( \epsilon > 0 \) sufficiently (but not necessarily arbitrarily) small. This is the case if the 0-th order part of \( S_{\text{sub}}(\Box + L) \) is nilpotent (or has small eigenvalues) and can be conjugated in a \( t \)-independent manner to an operator which is sufficiently close to being symmetric, in the sense that it satisfies the bound (2.5) with \( \Box \) replaced by \( \Box + L \).

We remark that the subprincipal operator \( i S_{\text{sub}}(\Box) = H_c + i \sigma_{\text{sub}}(G) \) induces a notion of parallel transport on \( \pi^* T_k M \) along the Hamilton flow of \( H_c \). As a consequence of the nilpotent structure of \( S_{\text{sub}}(\Box) \) at the trapped set, parallel sections along the trapped set grow only polynomially in size (with respect to a fixed \( t \)-invariant positive definite inner product), rather than exponentially. Parallel sections as induced by \( S_{\text{sub}}(\Box + L) \), with \( L \) as in Remark 4.9, may grow exponentially, with their size bounded by \( C e^{\kappa|t|} \) for some constants \( C > 0 \) and \( \kappa \), where the additional factor of \( |\sigma| \) in the exponent accounts for the homogeneity of the parallel transport. If such a bound does not hold for any \( \kappa < \nu_{\text{min}}/2 \), the dispersion of waves concentrated at the trapped set caused by the normally hyperbolic nature of the trapping is expected to be too weak to counteract the exponential growth caused by the subprincipal part of \( \Box + L \), and correspondingly one does not expect a spectral gap. Notice that the growth of parallel sections is an averaged condition in that it involves the behavior of the parallel transport for large times, while the
choice of Ψ-inner products as explained above is a local condition and depends on the pointwise structure of $S_{\text{sub}}(\square)$; thus, establishing spectral gaps only using averaged data is an interesting open problem, even in the scalar setting.

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