We would like to show that

\[(1) \sum_{n=1}^{\infty} (-1)^n \frac{2n^2 + 3}{(n+1)(n+3)} \text{ is divergent}\]

\[(2) \sum_{n=1}^{\infty} \frac{\ln(n)}{n+1} \text{ is divergent}\]

\[(3) \sum_{n=1}^{\infty} e^{-n} \text{ is convergent}\]

\[(4) \sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \text{ is absolutely convergent. Is it convergent?}\]

\[(5) \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2} \text{ is convergent but not absolutely convergent}\]

\[(6) \sum_{n=1}^{\infty} (-1)^n \frac{n^3 n!}{3^n} \text{ is divergent}\]

using one or several of the following tests:

**The Divergence Test:** If \(\lim_{n \to \infty} a_n \neq 0\) or does not exist then \(\sum_{n=1}^{\infty} a_n\) is divergent.

**The Integral Test:** If \(f(x)\) is continuous, positive and decreasing on \([1, \infty]\) and \(f(n) = a_n\) then \(\sum_{n=1}^{\infty} a_n\) is convergent if and only if \(\int_1^{\infty} f(x) \, dx\) is convergent.

**The Comparison Test:** If \(a_n \geq 0\) and \(b_n \geq 0\) for all \(n\) then:
(a) if \(\sum b_n\) is convergent and \(a_n \leq b_n\) for all \(n\), then \(\sum a_n\) is also convergent
(b) if \(\sum b_n\) is divergent and \(a_n \geq b_n\) for all \(n\), then \(\sum a_n\) is also divergent

**The Limit Comparison Test:** If \(a_n \geq 0\) and \(b_n \geq 0\) for all \(n\) and \(\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0\) then either both \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) are convergent or both are divergent.

**The Alternating Series Test:** Suppose we have \(\sum_{n=1}^{\infty} (-1)^n b_n\) and \(b_n > 0\). If \(b_{n+1} \leq b_n\) for all \(n\) and \(\lim_{n \to \infty} b_n = 0\) then the series \(\sum_{n=1}^{\infty} (-1)^n b_n\) is convergent.

**The Ratio Test:** If

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \begin{cases} 
L > 1 & \text{the series } \sum_{n=1}^{\infty} a_n \text{ is divergent}, \\
L < 1 & \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}, \\
= 1 & \text{the Ratio Test is inconclusive}
\end{cases}
\]

Two things good to know:

**Def.** We say \(\sum_{n=1}^{\infty} a_n\) is **absolutely convergent** if \(\sum_{n=1}^{\infty} |a_n|\) is convergent.

**Thm.** If \(\sum_{n=1}^{\infty} a_n\) is absolutely convergent then it is convergent.
Example (1) $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2+3}{(n+1)(n+3)}$

$$\lim_{n \to \infty} \frac{2n^2+3}{(n+1)(n+3)} = \lim_{n \to \infty} \frac{n^2(2 + \frac{3}{n})}{n(1 + \frac{1}{n})n(1 + \frac{2}{n})} = 2 \neq 0 \Rightarrow \lim_{n \to \infty} (-1)^n \frac{2n^2+3}{(n+1)(n+3)} \text{ does not exist}$$

From the Divergence test we can conclude the series $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2+3}{(n+1)(n+3)}$ is divergent.

Example (2) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n+1}$

$$\ln(n) \geq \ln(2) \text{ for } n \geq 2 \Rightarrow \frac{\ln(n)}{n+1} \geq \frac{\ln(2)}{n+1} \text{ } (\ast)$$

Now we want to know if the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n+1}$ is divergent.

We will use the Limit Comparison Test with $a_n = \frac{\ln(2)}{n+1}$ and $b_n = \frac{1}{n}$

$$\lim_{n \to \infty} \frac{\ln(2)}{n+1} \cdot \frac{1}{n} = \lim_{n \to \infty} \frac{\ln(2)n}{n+1} = \ln(2) > 0 \text{ and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}$$

The LCT tells us then that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n+1}$ is also divergent. This together with $(\ast)$ allows us to apply the Comparison Test and conclude that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n+1}$ is divergent.

Example (3) $\sum_{n=1}^{\infty} e^{-n}$

We can use several tests for this example - it is geometric series; you can also apply the integral test; here I will use the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{e^{-(n+1)}}{e^{-n}} = \lim_{n \to \infty} \frac{e^{-n}e^{-1}}{e^{-n}} = e^{-1} < 1$$

The Ratio Test tells us then that the series $\sum_{n=1}^{\infty} e^{-n}$ is convergent.

Example (4) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

To test for absolute convergence we form the series of the absolute values ie we consider $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$. This series has positive coefficients which allows us to use the comparison test.

$$|\sin(n)| \leq 1 \Rightarrow \left| \frac{\sin(n)}{n^2} \right| \leq \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent as } p\text{-series with } p = 2 > 1$$

The Comparison Test then tells us that $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right|$ is also convergent. Therefore $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is absolutely convergent. From the theorem at the bottom of page 1 we can conclude that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is convergent.
Example (5) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$

Let’s test for absolute convergence first. We form the series of the absolute values $\sum_{n=1}^{\infty} |(-1)^n \frac{n+1}{n^2}| = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$. This series has positive coefficients - we can use the comparison test.

$$\frac{n+1}{n^2} \geq \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{is divergent}$$

The Comparison test then tells us that $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ is also divergent and therefore the series $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ does not absolutely converge.

To see if $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ is convergent we need to apply the Alternating Series Test - our $b_n = \frac{n+1}{n^2} > 0$ here. There are two things we need to check $b_{n+1} \leq b_n$ and $\lim_{n \to \infty} b_n = 0$. To show the first one we can consider the function $f(x) = \frac{x+1}{x^2}$ - since $f(n) = b_n$ if we can show that $f$ is decreasing on $[1, \infty)$ this will give $b_{n+1} = f(n + 1) \leq f(n) = b_n$. To show $f$ is decreasing we need to show $f'(x) \leq 0$ on $[1, \infty)$

$$f'(x) = \frac{x^2 - 2x(x + 1)}{x^4} = \frac{-x^2 - 2x}{x^4} \leq 0 \quad \text{on} \quad [1, \infty) \quad (\ast)$$

Now we need to consider $\lim_{n \to \infty} \frac{n+1}{n^2} = \lim_{n \to \infty} \frac{n(1 + \frac{1}{n})}{n^2} = 0$. This and (\ast) allows us to conclude (using the Alt. Series Test) that $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$ is convergent.

Example (6) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{3^n}$ The Ratio Test

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n + 1)^3(n + 1)!}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n!} \right| = \lim_{n \to \infty} \frac{(n + 1)^3(n + 1)}{3^{n+1}} = \lim_{n \to \infty} \frac{n^3(1 + \frac{1}{n})^3(n + 1)}{3^{n+1}} = \infty$$

Therefore the series is divergent.