The Comparison Test: if $a_n \geq 0$ and $b_n \geq 0$ for all $n$ then:
(a) if $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent
(b) if $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is also divergent

You should always check that your series have positive terms and that you have the correct inequalities.

We often use this test with the $p$-series - given a series $\sum a_n$ we would like to test for convergence, we try to find a $p$-series to compare it with.

The $p$-series: $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and it is divergent for $p \leq 1$.

Let’s see how this works:

We want to know whether the following series converge or diverge. For all of them we will use the comparison test.

(1) $\sum_{n=1}^{\infty} \frac{2n}{n^3 + 3}$
(2) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{2n-1}$
(3) $\sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt[n]{n}}$
(4) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2 + 3}$

Example (1) $\sum_{n=1}^{\infty} \frac{2n}{n^3 + 3}$: Here we have series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{2n}{n^3 + 3} \geq 0$ and we want to use the comparison test. For this we need to find another series $\sum_{n=1}^{\infty} b_n$ with all $b_n$ either less then $a_n$ for all $n$ or greater then $a_n$ for all $n$. This is NOT enough to give the final answer but it is a start. The series $\sum_{n=1}^{\infty} b_n$ will usually be $p$-series for some $p$.

Any time you have a polynomial in the denominator try to compare it with the term having the highest power of $n$

$$n^3 + 3 \geq n^3 \Rightarrow \frac{1}{n^3 + 3} \leq \frac{1}{n^3} \Rightarrow$$

$$\Rightarrow \frac{2n}{n^3 + 3} \leq \frac{2n}{2n^2} = \frac{1}{n^2} \text{ (these will be our } b_n \text{’s, they are all positive)}$$

This shows $a_n = \frac{2n}{n^3 + 3} \leq \frac{1}{n^2} = b_n$. The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent because it is p-series with $p = 2 > 1$. Now we have $a_n \leq b_n$ for all $n$ and $\sum_{n=1}^{\infty} b_n$ is convergent, so from (a) in the comparison test we can conclude that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n}{n^3 + 3}$ is also convergent.

Example (2) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{2n-1}$: As before we want to find $p$-series to compare it with. Let’s
start with the denominator:

\[ 2n - 1 \leq 2n \Rightarrow \frac{1}{2n - 1} \geq \frac{1}{2n} \Rightarrow \]

\[ \Rightarrow \frac{\sqrt[3]{n}}{2n - 1} \geq \frac{\sqrt[3]{n}}{2n} = \frac{n^{\frac{1}{3}}}{2n} = \frac{1}{2n^{\frac{2}{3}}} = b_n \geq 0 \]

The series \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2n^{\frac{2}{3}}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}} \) is divergent as p-series with \( p = \frac{2}{3} \leq 1 \).

So we have \( a_n \geq b_n \sum_{n=1}^{\infty} b_n \) is divergent. We can conclude using (b) in the comparison test that

\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{2n - 1} \] is also divergent.

**Example (3)** \( \sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}} \): First \( a_n = \frac{\ln(n)}{\sqrt{n}} \geq 0 \) for \( n \geq 1 \). If we want to compare \( a_n = \frac{\ln(n)}{\sqrt{n}} \) with the elements of some p-series we need to find a bound for the function \( \ln(n) \) in this case. When \( n = 1, \ln(1) = 0 \) so the \( a_1 \) term in this series is 0 and does not affect the sum ie \( \sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}} \). Now \( \ln(n) \geq \ln(2) \) for \( n \geq 2 \) and therefore

\[ \frac{\ln(n)}{\sqrt{n}} \geq \frac{\ln(2)}{\sqrt{n}} = \frac{\ln(2)}{n^{\frac{1}{2}}} = b_n \geq 0 \]

The series \( \sum_{n=2}^{\infty} \frac{\ln(2)}{n^{\frac{1}{2}}} = \ln(2) \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}} \) are divergent as p-series with \( p = \frac{1}{2} \leq 1 \). So we have \( a_n \geq b_n \) and \( \sum b_n \) is divergent - we can conclude from (b) that \( \sum a_n \) is also divergent.

**Example (4)** \( \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^{2}+3} \): Again \( a_n = \frac{\sin^2(n)}{n^{2}+3} \geq 0 \) and we need a bound for the function \( \sin^2(n) \). Since \( \sin(n) \leq 1 \Rightarrow \sin^2(n) \leq 1 \) and therefore

\[ \frac{\sin^2(n)}{n^{2}+3} \leq \frac{1}{n^{2}+3} \leq \frac{1}{n^{2}} = b_n \geq 0 \]

The series \( \sum_{n=1}^{\infty} \frac{1}{n^{2}} \) is convergent as p-series with \( p = 2 > 1 \). We have \( a_n \leq b_n \) and \( \sum b_n \) is convergent, therefore by (a) we conclude \( \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^{2}+3} \) is also convergent.

What about \( \sum_{n=1}^{\infty} \frac{\sin(n)}{n^{2}+3} \)? Since \( \sin(n) \) will attain both positive and negative values, we do NOT have \( a_n \geq 0 \) which is a necessary condition for the comparison test and therefore we cannot use the test for this series.

What about \( \sum_{n=1}^{\infty} \frac{1}{n^{2}+1} \)? We may try to play the same game: Start with the denominator \( n + 1 \geq n \Rightarrow \frac{1}{n+1} \leq \frac{1}{n} = b_n \). The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent as p-series with \( p = 1 \leq 1 \). Now we have \( a_n \leq b_n \) with \( \sum b_n \) divergent - the comparison test DOES NOT give you an answer in this case, because the inequality is \( \leq \) rather then \( \geq \). Always make sure you have the correct inequality to make a conclusion. So what do we do with this problem? Check the notes for the Limit Comparison Test that I’ve posted.