Research Statement

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Introduction

I am interested in Symplectic Topology, and my recent work concerns the moduli spaces of J-holomorphic maps from a bordered Riemann surface to a symplectic manifold, with boundary mapping to a Lagrangian submanifold. The theory of J-holomorphic maps was introduced by Gromov and is one of the most profound methods in the study of symplectic manifolds. Applications to theoretical physics led to the development of Gromov-Witten invariants, which are invariants of the symplectic manifold and can be interpreted as a count of J-holomorphic maps from a closed Riemann surface passing through prescribed constraints in the manifold. Open String Theory motivated the study of J-holomorphic maps from a bordered Riemann surface with boundary mapping to a Lagrangian submanifold and predicts the existence of open Gromov-Witten invariants. Their proper mathematical definition, however, has proved to be a subtle problem. The two main obstacles are the question of orientability and the existence of boundary strata of the moduli space of maps from a bordered Riemann surface. My research concentrates on these difficulties.

I show that the local system of orientations on the moduli space of J-holomorphic maps from a bordered Riemann surface with a fixed complex structure is isomorphic to the pull-back of a local system defined on the product of the Lagrangian and its free loop space. The latter system is defined using only the first and second Stiefel-Whitney classes of the Lagrangian and the isomorphism allows us to determine whether the moduli space is orientable or not. The proof builds on the work of [FOOO], in which the authors provide a canonical orientation of the moduli space of disks when the Lagrangian is relatively spin, as well as [Sol], where this result is extended to surfaces of higher genus with a fixed complex structure and a relatively pin$^{\pm}$ Lagrangian. My work generalizes these results to an arbitrary Lagrangian.

I then define open Gromov-Witten disk invariants for orientable Lagrangians which appear as the fixed locus of an anti-symplectic involution and whose second Stiefel-Whitney class is a square of a class. The proof builds upon the works of [Cho] and [Sol], who define such invariants in dimension 4 and 6 using only point constraints. I generalize this to higher dimensions and any type of constraints. The idea is to use the anti-symplectic involution to glue together the boundaries of several moduli spaces, in order to obtain a space without a boundary. I determine the local system of orientations on this space and define the invariants by adapting the ideas from the closed case.

Currently I am working on extending the invariants to non-orientable Lagrangians. My future research plans include developing ψ-classes in this setting, generalizing the invariants to higher genus and defining the Fukaya category for non relatively spin Lagrangians.

Results

1. The local system of orientations on the moduli space of bordered Riemann surfaces. Moduli spaces are used to define invariants of symplectic manifolds such as Floer Homology or Gromov-Witten invariants. In essence, this is done by counting with a sign elements in a zero dimensional moduli space, and to do this we need an orientation on this space.

Let (M,ω) be a symplectic manifold and $L \subset M$ a Lagrangian submanifold. To define the moduli space, we will fix homology classes $\mathbf{b} = (b, b_1, .., b_h) \in H_2(M,L) \oplus H_1(L)^{\oplus h}$, a bordered Riemann surface $(\Sigma, \partial\Sigma; j)$ with a fixed complex structure j and ordering of the boundary components $\partial\Sigma = \coprod_{i=1}^{h} (\partial\Sigma)_i \cong \coprod_{i=1}^{h} S^1$, and finally integers $l \in \mathbb{N}$, $\vec{k} = (k_1, .., k_h) \in \mathbb{N}^h$. Then the moduli space $\mathfrak{M}_{l,\vec{k}}(\Sigma, \mathbf{b})$ is the space of unparametrized pseudo-holomorphic maps u from the marked bordered Riemann surface Σ to M with boundary $\partial\Sigma$ mapping to L, which represent the class b in the relative homology $H_2(M,L)$, and for which the restriction $u_{|(\partial\Sigma)_i}$ represents the class b_i in the homology group $H_1(L)$. The numbers l and k_i control the numbers of interior marked points and marked points on the i-th boundary component. We will denote by $\overline{\mathfrak{M}}_{l,\vec{k}}(\Sigma,\mathbf{b})$ the Gromov compactification of $\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b})$.

The orientability question in the case $\Sigma = D^2$ was studied in [FOOO]. The authors showed that the moduli space $\overline{\mathfrak{M}}_{l,k}(D^2,\mathbf{b})$ is not always orientable. However, in the case of a relatively spin Lagrangian they proved that it is orientable, and that a choice of a relatively spin structure determines a canonical orientation. This result was extended in [Sol] to relatively pin$^\pm$ Lagrangians and Riemann surfaces of higher genus with a fixed complex structure. The author constructed a canonical isomorphism between the determinant line bundle of $\overline{\mathfrak{M}}_{l,\vec{k}}(\Sigma,\mathbf{b})$ and the pull-back by the evaluation maps of a certain number of copies of $det(TL)$.

In my work I extend these results to any Lagrangian independent of any condition on orientability or $w_2(L)$. I show that the first Stiefel-Whitney class $w_1(\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b}))$ of the moduli space evaluated on a loop γ is equal to

$$w_1(\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b}))\cdot\gamma = \sum_{i=1}^{h}(w_1(TL)\cdot b_i + 1)\cdot(w_1(TL)\cdot\alpha_i) + \sum_{i=1}^{h} w_2(TL)\cdot\beta_i \tag{1.1}$$

where α_i is the loop a marked point on $\partial\Sigma_i$ traces in L and β_i is the torus $\partial\Sigma_i$ traces in L. When L is relatively spin or pin$^\pm$ one can show the term involving $w_2(TL)$ vanishes, and moreover the formula becomes that of [Sol] and [FOOO]. The presence of $w_2(TL)$ in the formula above means that the local system of orientations on $\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b})$ is not a pull-back of a system on L. I construct a local system $\mathcal{Z}_{(w_1,w_2)}$ on $L^h\times\mathcal{L}(L)^h$ which traces the twisting coming from the right-hand side in (1.1) and I show its pull-back is isomorphic to the local system of orientations on $\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b})$. Here $\mathcal{L}(L)$ denotes the free loop space of L.

More precisely, let $\mathfrak{B}(\Sigma,\mathbf{b})$ be the space of maps from the bordered Riemann surface Σ to M with boundary $\partial\Sigma$ mapping to L, which represent the class $b\in H_2(M,L)$, and for which the restriction $u_{|(\partial\Sigma)_i}$ represents the class $b_i\in H_1(L)$. There are canonical maps

$$ev_0^i : \mathfrak{B}(\Sigma,\mathbf{b})\to L\,,\ \text{assigning to a map } u\in\mathfrak{B}(\Sigma,\mathbf{b}) \text{ its evaluation at } 0\in S^1\cong\partial\Sigma_i\,,$$
$$ev^i_{\mathcal{L}(L)} : \mathfrak{B}(\Sigma,\mathbf{b})\to\mathcal{L}(L)\,,\ \text{assigning to a map } u\in\mathfrak{B}(\Sigma,\mathbf{b}) \text{ the element } u_{|(\partial\Sigma)_i} : (\partial\Sigma)_i\to L,$$
$$\text{a projection }\ \pi : \mathfrak{B}(\Sigma,\mathbf{b})\to\mathfrak{B}(\Sigma,\mathbf{b})/Aut(\Sigma,\partial\Sigma,j)\supset\mathfrak{M}(\Sigma,\mathbf{b})\,,$$
$$\text{and a forgetful map }\ f:\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b})\to\mathfrak{M}(\Sigma,\mathbf{b})\ \text{ given by forgetting the marked points.}$$

Combining the evaluation maps gives us maps $ev_0^{\vec{h}}$ and $ev^{\vec{h}}_{\mathcal{L}(L)}$ to the h-fold products L^h and $\mathcal{L}(L)^h$.

Theorem 1.1. *There is a local system $\mathcal{Z}_{(w_1,w_2)}$ on $L^h\times\mathcal{L}(L)^h$ such that the local system of orientations of $\mathfrak{M}_{l,\vec{k}}(\Sigma,\mathbf{b})$ is isomorphic to $f^*\circ\pi_!\circ(ev_0^{\vec{h}}\times ev^{\vec{h}}_{\mathcal{L}(L)})^*\mathcal{Z}_{(w_1,w_2)}$. The isomorphism is canonical once we choose a trivialization of TL over a basepoint in L, and trivializations of $TL\oplus det(TL)$ and $det(TL)\oplus det(TL)$ over those loops in L corresponding to a choice of a basepoint in each component of $\mathcal{L}(L)$.*

The idea of the proof is to consider $\bar\partial$-operators with values in $(E,F) = (TM\oplus 3det_{\mathbb{C}}TM, TL\oplus 3det(TL))$ and $(E^1,F^1) = (det_{\mathbb{C}}TM, det(TL))$. The advantage of this approach is that now we are looking at an orientable bundle F and one dimensional bundle F^1, and we can compute the first Stiefel-Whitney classes of $Ind(\bar\partial_{(E,F)})$ and $Ind(\bar\partial_{(E^1,F^1)})$ for these bundles. Then using $Ind(\bar\partial_{(E,F)})\oplus Ind(\bar\partial_{(E^1,F^1)})\cong Ind(\bar\partial_{(E\oplus E^1,F\oplus F^1)})\cong Ind(\bar\partial_{(TM,TL)})\oplus Ind(\bar\partial_{(4E^1,4F^1)})$, we can find the first Stiefel-Whitney class of $Ind(\bar\partial_{(TM,TL)})$ which, after suitable perturbation, is the tangent space of $\mathfrak{M}(\Sigma,\mathbf{b})$.

2. Open Gromov-Witten disk invariants in the presence of an anti-symplectic involution. The closed Gromov-Witten invariants are defined using the moduli space $\overline{\mathfrak{M}}_l(\Sigma^{cl}, A)$ of unparametrized pseudo-holomorphic maps from a marked, closed Riemann surface Σ^{cl} to a symplectic manifold M representing a fixed class $A \in H_2(M)$. This is done as follows. There are canonical maps $ev_i : \overline{\mathfrak{M}}_l(\Sigma^{cl}, A) \to M$ given by the evaluation at the i-th marked point and $\mathfrak{f} : \overline{\mathfrak{M}}_l(\Sigma^{cl}, A) \to \overline{\mathfrak{M}}_l(\Sigma^{cl})$ given by forgetting the target M and leaving just the domain. Here $\overline{\mathfrak{M}}_l(\Sigma^{cl})$ denotes the Deligne-Mumford moduli space of marked closed Riemann surfaces. Since $\overline{\mathfrak{M}}_l(\Sigma^{cl}, A)$ is orientable and all of its singular strata are of codimension greater than or equal to 2, it has a (virtual) fundamental class. The Gromov-Witten invariants are defined by evaluating cohomology classes of $M^l \times \overline{\mathfrak{M}}_l(\Sigma^{cl})$ on the push-forward of the fundamental class in $H_*(M^l \times \overline{\mathfrak{M}}_l(\Sigma^{cl}))$.

If we consider the moduli space of bordered Riemann surfaces we cannot in general define a fundamental class as this space is not necessarily orientable and it has codimension 1 boundary strata. Several results have been obtained in this setting. In [KL] and [Liu] the authors define invariants for an S^1-equivariant pair (M, L). In [Fuk] and [Iac] disk invariants are defined for the Calabi-Yau threefold.

More generally, in the presence of an anti-symplectic involution $\tau : M \to M$ with fixed locus L, disk invariants using point constraints have been defined in [Cho] and [Sol] when the dimension of L is less than or equal to 3, and they were computed for the quintic threefold in [PSW]. The idea is to consider all moduli spaces $\overline{\mathfrak{M}}_{l,k}(b) = \overline{\mathfrak{M}}_{l,k}(D^2, b)$, such that the connected sum $b\# - \tau_*(b)$ is equal to a fixed class $A \in H_2(M)$, and glue the boundary component $\overline{\mathfrak{M}}_{l_1,k_1}(b_1) \times_{ev} \overline{\mathfrak{M}}_{l_2,k_2}(b_2) \subset \partial\overline{\mathfrak{M}}_{l,k}(b)$ to the boundary component $\overline{\mathfrak{M}}_{l_1,k_1}(b_1) \times_{ev} \overline{\mathfrak{M}}_{l_2,k_2}(-\tau_*(b_2)) \subset \partial\overline{\mathfrak{M}}_{l,k}(b')$. We glue using a map inv, which acts as the identity on the first factor, and maps an element $(u, \vec{x}, \vec{z})$ to an element $(\tau \circ u \circ c, c(\vec{x}), c(\vec{z}))$ on the second factor (here $c : D^2 \to D^2$ is the complex conjugation on D^2). In this way we obtain a space $\widetilde{\mathcal{M}}(A)$ without boundary. The result in [Cho] says that if we restrict ourselves to using only real point constraints, and to relatively spin Lagrangian of dimension less than or equal to 3, then the space $\widetilde{\mathcal{M}}(A)$ is orientable and we can define the corresponding numbers as before. In [Sol] the author shows that when L is relatively pin$^\pm$, $dim(L) \leq 3$, and we are using only point constraints, the determinant line bundle of $\widetilde{\mathcal{M}}(A)$ is isomorphic to a certain number of copies of $det(TL)$. The author also notes that even when $\widetilde{\mathcal{M}}(A)$ is not orientable it will have a fundamental class with twisted coefficients and as long as this system is a pull-back of a system on L, as is in the above case, we can define the invariants as before.

In my work I define disk invariants in higher dimensions with any type of constraints by adapting the idea above. I construct a moduli space $\widetilde{\mathcal{M}}_{l+1,k}(A)$ without boundary in a similar way. The difference comes from having a preferred interior marked point z_0, which determines which bubble is not flipped, and from a decoration with $+$ or $-$ on the rest of the interior marked points. This decoration allows us to modify the evaluation maps ev_{z_i} to obtain a continuous map $\widetilde{ev}_{z_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to M$. I also construct a moduli space $\widetilde{\mathcal{M}}_{l+1,k}$ which keeps track of the domains (it is an appropriate version of the Deligne-Mumford space). Since $\widetilde{\mathcal{M}}_{l+1,k}(A)$ has no boundary it will have a fundamental class with coefficients in the local system of orientations $\mathcal{Z}_{w_1(\widetilde{\mathcal{M}}_{l+1,k}(A))}$. When this system is a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$, we can define the open Gromov-Witten invariants as in the closed case. I give sufficient conditions for this (Theorem 2.2) and I express the first Stiefel-Whitney class of $\widetilde{\mathcal{M}}_{l+1,k}(A)$ in terms of pulled-back classes and duals of boundary divisors (Theorem 2.3). The latter is helpful in determining the necessary conditions for $\mathcal{Z}_{w_1(\widetilde{\mathcal{M}}_{l+1,k}(A))}$ to be a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$. Finally I provide a connection to real algebraic geometry by showing $\widetilde{\mathcal{M}}_{l+1,k}(A)$ is isomorphic to the space of real spheres as considered in [Wel05] and [Wel08].

In more detail, assume the moduli spaces $\mathfrak{M}_{l,k}(b)$ are smooth and of expected dimension. The decoration with $+$ and $-$ on the interior marked points (except z_0) can be described as an element of

$\overline{\mathfrak{M}}_{l+1,k}(b) \times \mathbb{Z}_2^l = \widetilde{\mathfrak{M}}_{l+1,k}(b)$. We have a map *inv* defined as before which acts on the decoration of the second bubble by switching the signs and we can use it to identify the boundaries of $\widetilde{\mathfrak{M}}_{l+1,k}(b)$ for various b satisfying $b\# - \tau_*(b) = A$ for a fixed class A in $H_2(M)$.

Theorem 2.1. *Let* $\widetilde{\mathcal{M}}_{l+1,k}(A) = \coprod_{b:b\#-\tau_*(b)=A} \widetilde{\mathfrak{M}}_{l+1,k}(b)/\sim_{inv}$ *be the space obtained by identifying the boundaries of different* $\widetilde{\mathfrak{M}}_{l+1,k}(b)$ *using the map inv. The space* $\widetilde{\mathcal{M}}_{l+1,k}(A)$ *has no boundary provided the classes* $\partial b \in \pi_1(L)$ *are not zero or there is at least one boundary marked point.*

Moreover there exist continuous maps $ev_{x_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to L$ *given by evaluation at the i-th boundary point,* $\widetilde{ev}_{z_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to M$ *given by* $u(z_i)$ *on* $(u, z_i, +)$ *and by* $\tau \circ u(z_i)$ *on* $(u, z_i, -)$ *and* $\mathfrak{f} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to \widetilde{\mathcal{M}}_{l+1,k}$ *given by forgetting the target M.*

Remark. $\widetilde{\mathcal{M}}_{l+1,k}$ is defined similarly by gluing the boundaries of the space of marked disks $\widetilde{\mathfrak{M}}_{l+1,k}$ using the conjugation on D^2. It has no boundary but may be non-orientable.

The space $\widetilde{\mathcal{M}}_{l+1,k}(A)$ is not necessarily orientable but it will have a fundamental class with coefficients in the local system of orientations $\mathcal{Z}_{w_1(\widetilde{\mathcal{M}}_{l+1,k}(A))}$. We would like to know when this system is a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$, as in these cases, we can define the open Gromov-Witten invariants as in the closed case - we just have to consider cohomology classes with coefficients in the corresponding local system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$. Under certain topological conditions one can prove that this is the case. For example:

Theorem 2.2. *Suppose L is orientable,* $w_2(L) = \alpha^2$ *for some* $\alpha \in H_1(L)$ *and either the Maslov index* $\mu(b) = 4s$ *for every* $b \in \pi_2(M, L)$ *or otherwise there is no* $\mathbb{Z}_2$ *torsion in* $\pi_2(M, L)$*. Then, if there are no bubbles with only two real marked points, the local system of orientations on* $\widetilde{\mathcal{M}}_{l+1,k}(A)$ *is a pull-back of a system on* $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$*. In fact it is the pull-back of the local system of orientations on* $\widetilde{\mathcal{M}}_{l+1,k}$*.*

Examples of such manifolds are $\mathbb{C}P^{4s-1}$ and the Calabi-Yau manifolds, when we consider no boundary marked points or when all domains are stable. The next result is useful in determining when the local system of orientations on $\widetilde{\mathcal{M}}_{l+1,k}(A)$ is a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$.

Theorem 2.3. *Suppose* $w_2(L) = \alpha^2$ *for some* $\alpha \in H_1(L)$ *and that all domains are stable. Then*

$$w_1(\widetilde{\mathcal{M}}_{l+1,k}(A)) = (w_1(L) \cdot (\partial b) + 1) \cdot ev_{x_1}^*(w_1(L)) + \mathfrak{f}^*(w_1(\widetilde{\mathcal{M}}_{l+1,k})) + D_1^\vee + D_2^\vee$$

where $D_i^\vee \in H^1(\widetilde{\mathcal{M}}_{l+1,k}(A), \mathbb{Z}_2)$ *is the dual of the divisor* D_i *formed by the codimension 1 strata* $\widetilde{\mathfrak{M}}_{l_1+1,k_1}(b_1) \times_{ev} \widetilde{\mathfrak{M}}_{l_2,k_2}(b_2)$ *with Maslov index* $\mu(b_2) \equiv i$ *mod* 4*.*

Remark. The value of $(w_1(L) \cdot \partial b)$ is independent of the choice of b for which $b\# - \tau_*(b) = A$.

Finally, let $\mathbb{R}\mathfrak{M}_{l+1,k}(\mathbb{C}P^1, A)$ be the space of unparametrized pseudo-holomorphic maps from $\mathbb{C}P^1$ to M with $l+1$ complex and k real marked points such that $\tau \circ u \circ conj = u$, where $conj$ is the conjugation on $\mathbb{C}P^1$. This is the space of real spheres as considered in [Wel08] and [Wel05] where the author defines enumerative invariants which can be interpreted as counting the number of real spheres passing through prescribed points. The following gives a connection between these invariants and the open ones.

Theorem 2.4. *There is map* $\widetilde{\mathcal{M}}_{l+1,k}(A) \to \mathbb{R}\mathfrak{M}_{l+1,k}(A)$ *given by reflecting the disks and their images. This map is an isomorphism when* $k \neq 0$*, or* $0 \neq \partial b \in \pi_1(L)$ *for every b with* $b\# - \tau_*(b) = A$*.*

Current and Future Research

Open Gromov-Witten invariants for non-orientable Lagrangians. Currently I am working on extending the invariants defined above to non-orientable Lagrangians and Lagrangians with $\mathbb{Z}_2$ torsion in $\pi_2(M, L)$. In light of Theorem 2.3, it is easy to see that in general for these manifolds the local system of orientations on $\widetilde{\mathcal{M}}_{l+1,k}(A)$ will not be a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$. In order to catch the additional contribution to $w_1(\widetilde{\mathcal{M}}_{l+1,k}(A))$ we need to consider an enhanced version $\widetilde{\mathcal{M}}^{\bar{c}_1(A)}_{l+1,k}$ of $\widetilde{\mathcal{M}}_{l+1,k}$. If all domains are stable, $\widetilde{\mathcal{M}}^{\bar{c}_1(A)}_{l+1,k}$ is a finite branched cover of $\widetilde{\mathcal{M}}_{l+1,k}$ and away from the singular strata of $\widetilde{\mathcal{M}}_{l+1,k}$ it is actually isomorphic to $\widetilde{\mathcal{M}}_{l+1,k}$. Here $\bar{c}_1(A)$ is the the first Chern class of M evaluated on A mod 4. Moreover there is a local system $\mathcal{Z}_{br}$ on $\widetilde{\mathcal{M}}^{\bar{c}_1(A)}_{l+1,k}$ and a map $f : \widetilde{\mathcal{M}}_{l+1,k}(A) \to \widetilde{\mathcal{M}}^{\bar{c}_1(A)}_{l+1,k}$ such that the local system of orientations on $\widetilde{\mathcal{M}}_{l+1,k}(A)$ is a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$ tensored with the pull-back of $\mathcal{Z}_{br}$, and we can define the invariants as before.

ψ-classes. Let $\overline{\mathfrak{M}}_l(\Sigma^{cl})$ be the moduli space of closed Riemann surfaces with l marked points. There is a forgetful map $f : \overline{\mathfrak{M}}_{l+1}(\Sigma^{cl}) \to \overline{\mathfrak{M}}_l(\Sigma^{cl})$ given by forgetting the last marked point. There are l canonical sections $\sigma_i : \overline{\mathfrak{M}}_l(\Sigma^{cl}) \to \overline{\mathfrak{M}}_{l+1}(\Sigma^{cl})$ mapping $\overline{\mathfrak{M}}_l(\Sigma^{cl})$ to the boundary divisor $D_{i,l+1}$ having only z_i and z_{l+1} on a disk bubble. Let L_i be the pull-back by σ_i of the relative cotangent bundle of $D_{i,l+1} \subset \overline{\mathfrak{M}}_{l+1}(\Sigma^{cl})$. We define ψ_i to be the first Chern class of L_i.

These classes play an important role in the study of $\overline{\mathfrak{M}}_l(\Sigma^{cl})$ and in Gromov-Witten theory in general. For example, they satisfy a recursive relation, which allows one to compute all products $(\psi_1^{\alpha_1} \cup .. \cup \psi_l^{\alpha_i}).\left[\overline{\mathfrak{M}}_l(\mathbb{C}P^1)\right]$. In Gromov Witten theory one defines gravitational descendants, which are a mixture of ψ-classes and classes pulled-back from the target, and again using a sophisticated recursive relation, one can compute many Gromov-Witten invariants.

A natural question is whether there are similar classes for the moduli space of bordered surfaces. If we consider $\overline{\mathfrak{M}}_{l,k}(D^2)$, no interesting classes can exist as this space is contractible. $\widetilde{\mathcal{M}}_{l+1,k}$, however, has much more topology. It also has forgetful maps $f_b : \widetilde{\mathcal{M}}_{l+1,k} \to \widetilde{\mathcal{M}}_{l+1,k-1}$ given by forgetting the last boundary marked point, and $f_i : \widetilde{\mathcal{M}}_{l+1,k} \to \widetilde{\mathcal{M}}_{l,k}$ given by forgetting the last interior point. We can then consider similar sections and try to find any interesting cohomology classes.

Invariants for bordered surfaces of higher genus. Assuming the transversality issues in this case are resolved, we can try to define the open invariants for surfaces in a similar fashion. We again have the map *inv* defined on the components of the boundary formed as a fiber product of two moduli spaces. There will be an additional contribution to the sign of *inv* coming from the topological type of the bubble which is flipped, and from the dimension of L. The space $\widetilde{\mathcal{M}}^{\bar{c}_1(A)}_{l+1,k}$ should be capable of catching this addition. A more serious issue will come from having boundary on which *inv* is not defined. In the case of disks this kind of boundary is avoided by requiring either that there is a boundary marked point, or that ∂b is non-zero in $\pi_1(L)$. We will need stronger assumptions in the case of surfaces.

Fukaya Category with coefficients in $\mathcal{Z}_{(w_1,w_2)}$. The Fukaya category plays a prominent role in many recent developments of the subject, especially those involving mirror symmetry. It is an A_∞ category associated to a relatively spin Lagrangian. The operations m_k are defined using the fact that the moduli spaces of disks $\overline{\mathfrak{M}}_{k+1}(b)$ are orientable and therefore represent a $\mathbb{Q}$-chain in L. It is natural to ask whether we can extend its definition to an arbitrary Lagrangian. As I describe in the first part, in general for non relatively spin Lagrangians the moduli spaces are not orientable and will not define a $\mathbb{Q}$-chain in L. They will, however, define a $\mathcal{Z}_{(w_1,w_2)}$-chain in $L \times \mathcal{L}(L)$ and we can consider the appropriate enlargement of the $\mathbb{Q}$-chain complex to define the operations m_k.

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