NOTES ON

OPEN GROMOV-WITTEN DISK INVARIANTS IN THE PRESENCE OF AN ANTI-SYMPLECTIC INVOLUTION

PENKA GEORGIEVA

The closed Gromov-Witten invariants are defined using the moduli space $\overline{\mathfrak{M}}_l(\Sigma^{cl}, A)$ of unparametrized pseudo-holomorphic maps from a marked, closed Riemann surface Σ^{cl} to a symplectic manifold M representing a fixed class $A \in H_2(M)$. This is done as follows. There are canonical maps $ev_i : \overline{\mathfrak{M}}_l(\Sigma^{cl}, A) \to M$ given by the evaluation at the i-th marked point and $\mathfrak{f} : \overline{\mathfrak{M}}_l(\Sigma^{cl}, A) \to \overline{\mathfrak{M}}_l(\Sigma^{cl})$ given by forgetting the target M and leaving just the domain. Here $\overline{\mathfrak{M}}_l(\Sigma^{cl})$ denotes the Deligne-Mumford moduli space of marked closed Riemann surfaces. Since $\overline{\mathfrak{M}}_l(\Sigma^{cl}, A)$ is orientable and all of its singular strata are of codimension greater than or equal to 2, it has a (virtual) fundamental class. The Gromov-Witten invariants are defined by evaluating cohomology classes of $M^l \times \overline{\mathfrak{M}}_l(\Sigma^{cl})$ on the push-forward of the fundamental class in $H_*(M^l \times \overline{\mathfrak{M}}_l(\Sigma^{cl}))$.

If we consider the moduli space of bordered Riemann surfaces we cannot in general define a fundamental class as this space is not necessarily orientable and it has codimension 1 boundary strata. Several results have been obtained in this setting. Katz and Liu defined invariants for an S^1-equivariant pair (M, L), and disk invariants for the Calabi-Yau threefold are defined independently by Fukaya and Iacovino. More generally, in the presence of an anti-symplectic involution $\tau : M \to M$ with fixed locus L, disk invariants using point constraints are defined by Cho and Solomon (independently), when the dimension of L is less than or equal to 3, and they were computed for the quintic threefold. I will describe a generalization to higher dimension of this approach, by constructing a moduli space $\widetilde{\mathcal{M}}_{l+1,k}(A)$ without boundary with evaluation maps to the target and an appropriate version of Deligne-Mumford. This space will not necessarily be orientable but it will have a fundamental class in its orientation local system. When this system is a pull-back of a system on the target and Deligne-Mumford, we can push-forward the fundamental class and define the invariants similarly to the closed case, only using cohomology classes with coefficients in the corresponding local system. I will explain when this is indeed the case, as well as discuss when we do not expect it to be.

What makes the anti-symplectic involution useful.

In the presence of an anti-symplectic involution, we can consider the moduli space $\overline{\mathbb{R}\mathfrak{M}}_{l+1,k}(\mathbb{C}P^1, A)$ of unparametrized pseudo-holomorphic maps from $\mathbb{C}P^1$ to M with $l+1$ complex and k real marked points such that $\tau \circ u \circ conj = u$, where $conj$ is the conjugation on $\mathbb{C}P^1$. This is the space of real spheres used by Welschinger to define a count of the number of real spheres passing through prescribed points. It has the nice property that its codimension 1 strata does not form a boundary. Over its open part, there is a 2 to 1 covering by moduli spaces of disks, formed using the upper and lower half of $\mathbb{C}P^1$. The classes these disks represent in $H_2(M, L)$ have the property $b\# - \tau_*(b) = A$, where b is the class of a disk. Over different components of $\mathbb{R}\mathfrak{M}_{l+1,k}(A)$ these disks represent different classes, and as we approach the codimension 1 strata their moduli spaces form a boundary. The underlying moduli space of spheres, however, will not form a boundary and it will hint us how to glue the boundaries of the moduli spaces of disks in order to eliminate them.

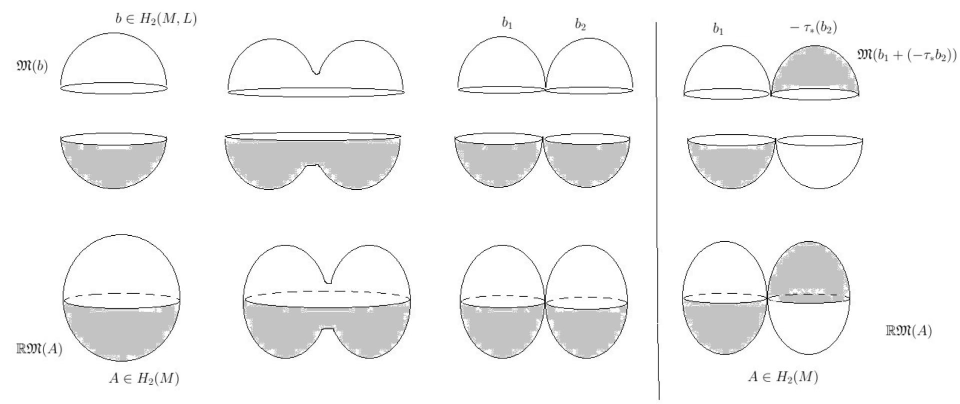

FIGURE 1. Deformation of the spheres and their covers

Construction of $\widetilde{\mathcal{M}}_{l+1,k}(A)$.

Let $\widetilde{\mathfrak{M}}_{l+1,l}(b) = \overline{\mathfrak{M}}_{l+1,k}(b) \times \mathbb{Z}_2^l$ be the moduli space of disks which represent the class $b \in H_2(M,L)$, with a decoration of $+$ and $-$ on the interior marked points except z_0. Let $\mathcal{M}_{l+1,k}(A) = \coprod_{b: b\# - \tau_*(b) = A} \widetilde{\mathfrak{M}}_{l+1,k}(b)$. There is a map inv : $\partial \mathcal{M}_{l+1,k}(A) \to \partial \mathcal{M}_{l+1,k}(A)$, which on a particular component is given by

$$inv : \widetilde{\mathfrak{M}}_{l_1+1,k_1}(b_1) \times_{ev} \widetilde{\mathfrak{M}}_{l_2,k_2}(b_2) \to \widetilde{\mathfrak{M}}_{l_1+1,k_1}(b_1) \times_{ev} \widetilde{\mathfrak{M}}_{l_2,k_2}(-\tau_*(b_2))$$
$$((u_1, \vec{x}_1, \vec{z}_1; \sigma_1), (u_2, \vec{x}_2, \vec{z}_2; \sigma_2)) \mapsto ((u_1, \vec{x}_1, \vec{z}_1; \sigma_1), (\tau \circ u_2 \circ c, c(\vec{x}_2), c(\vec{z}_2), -id_{\mathbb{Z}_2^{l_2}}(\sigma_2))$$

Here c is the conjugation on the disk. The subscript $l_1 + 1$ means that z_0 landed on that particular bubble - in other words we always flip the disk that does not contain z_0.

Let $\widetilde{\mathcal{M}}_{l+1,k}(A) = \mathcal{M}_{l+1,k}(A)/\sim_{inv}$ be the space obtained by identifying the boundaries of the moduli spaces $\widetilde{\mathfrak{M}}_{l+1,k}(b)$ using the map inv. $\widetilde{\mathcal{M}}_{l+1,k}(A)$ has no boundary since the map inv has no fixed points. We can see this as follows. If $u = \tau \circ u \circ c$, this would imply the image of u is in L and therefore represents zero in the relative homology. In order for u to be stable, its domain must have either at least 3 boundary marked points, on the order of which inv will act nontrivially, or a real and a complex marked point and inv will act nontrivially on the decoration of the complex point.

Remark. There are two possible types of boundary - one having two components on which *inv* is defined and one having a single component on which *inv* will not be defined. The latter is avoided by either requiring to have a boundary marked point or by having ∂b be non zero in $\pi_1(L)$.

Similarly we construct the equivalent $\widetilde{\mathcal{M}}_{l+1,k}$ of Deligne-Mumford by using the conjugation on the disk. The map *inv* defined on the boundary of $\widetilde{\mathfrak{M}}_{l+1,k}$ just conjugates the points and reverses the decoration on the second bubble. This space has no boundary as *inv* does not preserve any boundary component.

We have a forgetful map $\widetilde{\mathcal{M}}_{l+1,k}(A) \to \widetilde{\mathcal{M}}_{l+1,k}$ and the evaluation maps at a boundary marked point x_i glue together to form a continuous evaluation map $ev_{x_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to L$. We will use the decoration at an interior marked point z_i to define a continuous evaluation map $\widetilde{ev}_{z_i} : \widetilde{\mathcal{M}}_{l+1,k}(A) \to M$. If the decoration is +, we use the usual evaluation at z_i, if the decoration is -, we use the usual evaluation followed by τ.

Orientability of $\widetilde{\mathcal{M}}_{l+1,k}(A)$.

Theorem 1.1. *Suppose L is orientable, $w_2(L) = \alpha^2$ for some $\alpha \in H_1(L)$ and either the Maslov index $\mu(b) = 4s$ for every $b \in \pi_2(M,L)$ or otherwise there is no $\mathbb{Z}_2$ torsion in $\pi_2(M,L)$. Then, if there are no bubbles with only two real marked points, the local system of orientations on $\widetilde{\mathcal{M}}_{l+1,k}(A)$ is a pull-back of a system on $L^k \times M^{l+1} \times \widetilde{\mathcal{M}}_{l+1,k}$. In fact it is the pull-back of the local system of orientations of $\widetilde{\mathcal{M}}_{l+1,k}$.*

Idea: The orientability of L, together with $w_2(L) = \alpha^2$, implies that each of the moduli spaces $\widetilde{\mathfrak{M}}_{l+1,k}(b)$ is orientable. The sign of *inv* is $(-1)^{\frac{\mu(b_2)}{2}+k_2+1}$ when L is orientable, where $\mu(b_2)$ is the Maslov index of the second bubble, k_2 is the number of real marked points that landed on the second bubble and +1 is for the node. On the other hand $(-1)^{k_2+1}$ is precisely the sign of *inv* on the Deligne-Mumford $\widetilde{\mathcal{M}}_{l+1,k}$ and if $\mu(b_2) = 4s$, it will be the only contribution to the sign. Therefore, if $\mu(b) = 4s$ for every $b \in \pi_2(M,L)$, then $w_1(\widetilde{\mathcal{M}}_{l+1,k}(A)) = \mathfrak{f}^* w_1(\widetilde{\mathcal{M}}_{l+1,k})$.

If there is no $\mathbb{Z}_2$-torsion in $\pi_2(M,L)$, one can show that every loop we will hit the boundary divisor having $\mu(b_2) = 2 \bmod 4$ an even number of times, and therefore again the first Stiefel-Whitney class is a pullback from Deligne-Mumford. This follows from the invariance of the sign of *inv* under splitting, together with the fact that if we flip a class we need to flip it again in order to close the loop.

Theorem 1.2. *Suppose $w_2(L) = \alpha^2$ for some $\alpha \in H_1(L)$ and that all domains are stable. Then*

$$w_1(\widetilde{\mathcal{M}}_{l+1,k}(A)) = (w_1(L) \cdot (\partial b) + 1) \cdot ev_{x_1}^*(w_1(L)) + \mathfrak{f}^*(w_1(\widetilde{\mathcal{M}}_{l+1,k})) + D_1^\vee + D_2^\vee$$

where $D_i^\vee \in H^1(\widetilde{\mathcal{M}}_{l+1,k}(A), \mathbb{Z}_2)$ is the dual of the divisor D_i formed by the codimension 1 strata $\widetilde{\mathfrak{M}}_{l_1+1,k_1}(b_1) \times_{ev} \widetilde{\mathfrak{M}}_{l_2,k_2}(b_2)$ with Maslov index $\mu(b_2) \equiv i$ mod 4.

Idea: The sign of *inv* when L is not necessarily orientable is $(-1)^{\frac{\mu(b_2)+\widetilde{w}_1(b_2)}{2}+k_2+1}$ where $\widetilde{w}_1(b_2)$ is the evaluation of the first Stiefel-Whitney class of L on ∂b_2, as an element of $\mathbb{Z}$ (it is either 0 or 1). Again $(-1)^{k_2+1}$ will be the sign coming from Deligne-Mumford. There is an additional contribution to the sign only when $\mu(b_2) + \widetilde{w}_1(b_2) = i \bmod 4$ for $i = 1, 2$ which implies the result. (the expression $(w_1(L) \cdot (\partial b) + 1) \cdot ev^*_{x_1}(w_1(L)))$ comes from the nonorientability of $\widetilde{\mathfrak{M}}_{l+1,k}(b)$ when L is not orientable).

Remark on the sign of ***inv*****.** We give the orientation on $\mathfrak{M}(b)$ by orienting $Ind(\bar{\partial}_{(E,F)})$ and $Ind(\bar{\partial}_{(E^1,F^1)})$, where $(E, F) = (TM \oplus 3\det_{\mathbb{C}} TM, TL \oplus 3\det TL)$ and $(E^1, F^1) = (\det_{\mathbb{C}} TM, \det TL)$. To orient $Ind(\bar{\partial}_{(E,F)})$ we push up the Maslov index and pinch the disk to obtain a sphere attached to a disk. The index over the disk is isomorphic to $Ind(\bar{\partial}_{(\mathbb{C}^{n+3},\mathbb{R}^{n+3})}) \cong \mathbb{R}^{n+3}$ and τ and c do not act on it. However, τ will act on the canonical orientation of the index on the sphere and on the incident condition with a sign $(-1)^{c_1(E)\cdot\mathbb{C}P^1+n}$ and $(-1)^n$ respectively. On the other hand $c_1(E) \cdot \mathbb{C}P^1 = \frac{\mu(E)}{2} = \frac{4\mu(b)}{2}$ and therefore there is no contribution to the sign of inv. To orient $Ind(\bar{\partial}_{(E^1,F^1)})$ we push all of the Maslov index to the sphere when $w_1(\partial b) = 0$ so that on the disk we would have $Ind \cong ev^*_{x_1} det(TL)$ and this will stay unchanged under τ and c. Again, inv will act on the complex orientation of the index on the sphere with a sign equal to (-1) to the dimension of the index $c_1(M) \cdot (\mathbb{C}P^1) + 1$ and on the incident condition by (-1). The overall sign will be $(-1)^{c_1(M)\cdot(\mathbb{C}P^1)=\frac{\mu(b)}{2}}$. When $w_1(b) = 1$ we push up $\mu(b) - 1$ of the Maslov index and similarly to the above we will have a sign $(-1)^{c_1(M)\cdot(\mathbb{C}P^1)=\frac{\mu(b)-1}{2}}$. We will however have an additional (-1), coming from the index over the disk, because we need two points to evaluate in this case, and we transport the orientation of $det(TL)$ from one point to the other using the orientation of the boundary of the disk. Since c will reverse the orientation of the boundary of the disk, we get an additional (-1). We combine all of the above to obtain the formula $(-1)^{\frac{\mu(b_2)+\tilde{w}_1(b_2)}{2}}$. Finally c acts on the marked points by -1. When we construct $\widetilde{\mathfrak{M}}_{l+1,k}(b)$, we orient the D^2 carrying a marked point with a decoration + as usual but we reverse the orientation on the D^2 carrying a marked point with a decoration -. In this way c only changes the orientation of the boundary points and the formula follows.

Theorem 1.3. *There is map* $\widetilde{\mathcal{M}}_{l+1,k}(A) \to \overline{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)$ *given by reflecting the domains and their images in the target. This map is an isomorphism when* $k \neq 0$ *or* $0 \neq \partial b \in \pi_1(L)$ *for all* b *with* $b\# - \tau_*(b) = A$.

Idea: Let $PHolMaps = \{u : \mathbb{C}P^1 \to M | u = \tau \circ u \circ conj\}/Aut^+(\mathbb{C}P^1)$ be the space of pseudo-holomorphic maps invariant under τ and the conjugation on $\mathbb{C}P^1$, mod out by reparametrizations which preserve the orientation of $\mathbb{R}P^1 \subset \mathbb{C}P^1$. Identify boundaries by flipping the sphere not containing z_0 and denote this space by $\widehat{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)$. Then $\overline{\mathbb{R}\mathfrak{M}}_{l+1,k}(A) = \widehat{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)/\mathbb{Z}_2$ and there is a map back $\overline{\mathbb{R}\mathfrak{M}}_{l+1,k}(A) \to \widehat{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)$ sending an element to the sphere containing z_0 in the upper half. Therefore the cover is 2 sheeted with each sheet isomorphic to $\overline{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)$. The map from $\widetilde{\mathcal{M}}_{l+1,k}(A) \to \widehat{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)$ is given by reflecting the domain and its image in the target to obtain a sphere, and having z_0 and all points with decoration + on the upper half and all points with decoration $-$ on the lower half of the sphere. This map is an isomorphism onto one of the sheets and hence to $\overline{\mathbb{R}\mathfrak{M}}_{l+1,k}(A)$.