THE LOCAL SYSTEM OF ORIENTATIONS ON $\overline{\mathfrak{M}}_{l,\vec{k}}(\Sigma, \partial\Sigma; M, L)$

PENKA GEORGIEVA

ABSTRACT. We construct a canonical isomorphism of the local system of orientations on the moduli space of J-holomorphic maps from a bordered Riemann surface, with a pull-back of a local system on the product of the Lagrangian and its free loop space.

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1. INTRODUCTION

In this paper, we show that the local system of orientations on the moduli space of J-holomorphic maps from a bordered Riemann surface with a fixed complex structure is isomorphic to the pull-back of a local system defined on the product of the Lagrangian and its free loop space. The latter system is defined using only the first and second Stiefel-Whitney classes of the Lagrangian and the isomorphism allows us to determine whether the moduli space is orientable or not.

Previously, the orientability question in the case $\Sigma = D^2$ was studied in [FOOO]. The authors showed that the moduli space $\overline{\mathfrak{M}}_{l,k}(D^2, \mathbf{b})$ is not always orientable. However, in the case of a relatively spin Lagrangian they proved that it is orientable, and that a choice of a relatively spin structure determines a canonical orientation. This result was extended in [Sol] to relatively pin$^{\pm}$ Lagrangians and Riemann surfaces of higher genus with a fixed complex structure. The author constructed a canonical isomorphism between the determinant line bundle of $\overline{\mathfrak{M}}_{l,\vec{k}}(\Sigma, \mathbf{b})$ and the pull-back by the evaluation maps of a certain number of copies of $\det(TL)$. Here we extend these results to any Lagrangian.

The paper is organized as follows. In section 2, we show that the first Stiefel-Whitney class of $\mathrm{Ind}(D)$ evaluated on a loop γ is equal to

$$w_1(\mathrm{Ind}(D)) \cdot \gamma = \sum_{i=1}^{h} (w_1(TL) \cdot b_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^{h} w_2(TL) \cdot \beta_i \tag{1.1}$$

where α_i is the loop a marked point on $\partial\Sigma_i$ traces in L and β_i is the torus $\partial\Sigma_i$ traces in L. When L is relatively spin or pin$^{\pm}$ one can show the term involving $w_2(TL)$ vanishes, and moreover the formula becomes that of [Sol] and [FOOO]. The presence of $w_2(TL)$ in the formula above means that the local system of orientations on $\mathfrak{M}_{l,\vec{k}}(\Sigma, \mathbf{b})$ is not a pull-back of a system on L. In section 3, we construct a local system $\mathcal{Z}_{(w_1,w_2)}$ on $L^h \times \mathcal{L}(L)^h$ which traces the twisting coming from the right-hand side in (1.1) and in section 4, we show its pull-back is canonically isomorphic to the local system twisted by the first Stiefel-Whitney class of $\operatorname{Ind}(D)$. Here $\mathcal{L}(L)$ denotes the free loop space of L. Lastly, assuming all domains are stable, we show the system pushes-down to the system of local orientations on $\mathfrak{M}_{l,\vec{k}}(\Sigma, \mathbf{b})$ and we discuss its compatibility with the local system over the fiber product.

2. THE FIRST STIEFEL-WHITNEY CLASS OF $Ind(D)$

Let (M, ω) be a symplectic manifold, $L \subset M$ a Lagrangian submanifold and Σ a bordered Riemann surface with fixed complex structure j and ordered boundary components $\partial\Sigma_i \cong S^1$ for $i = 1, .., h$. For a fixed tuple of homology classes $\mathbf{b} = (b, b_1, .., b_h) \in H_2(M, L) \oplus H_1(L)^{\oplus h}$, denote with $\mathfrak{B}(\Sigma, \mathbf{b})$ the Banach manifold of $W^{1,p}$-maps from Σ to M with boundary $\partial\Sigma$ mapping to L, which represent the class b in the relative homology group $H_2(M, L)$, with boundary component $\partial\Sigma_i$ representing the class b_i in the homology group $H_1(L)$ for every $i = 1, .., h$.

Let J be an ω-compatible almost complex structure on M and consider the linearization D of $\bar{\partial}(u) = du + J \circ du \circ j$ for $u \in \mathfrak{B}(\Sigma, \mathbf{b})$. The family of operators D is Fredholm and thus defines a virtual vector bundle $\operatorname{Ind}(D)$ over $\mathfrak{B}(\Sigma, \mathbf{b})$, with fiber over u given by $\operatorname{Ind}(D_u\bar{\partial}) = \operatorname{Ker}(D_u\bar{\partial}) \ominus \operatorname{Coker}(D_u\bar{\partial})$. In this section we will express the first Stiefel-Whitney class of $\operatorname{Ind}(D)$ in terms of the first and second Stiefel-Whitney classes of the Lagrangian L.

For simplicity of notation assume that Σ has only one boundary component. The case of several boundary components is a direct extension and is discussed at the end of the section.

Theorem 2.1. *Let γ be a loop in $\mathfrak{B}(\Sigma, \mathbf{b})$. Then the following formula for the first Stiefel-Whitney of $\operatorname{Ind}(D)_{|\gamma}$ holds*

$$w_1(\operatorname{Ind}(D)_{|\gamma}) = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha) + w_2(TL) \cdot \beta$$

where b_1 is the homology class $\partial\Sigma$ represents in $H_1(L)$, α is the loop in L that a point on $\partial\Sigma$ traces along γ, and β is the torus in L that $\partial\Sigma$ traces along γ.

Proof. A connection on TM induces a family of operators $\bar{\partial}_{(TM,TL)}$, which at a point $u \in \mathfrak{B}(\Sigma, \mathbf{b})$ is the restriction of a complex linear operator with values in the pull-back bundle (cf. [MS]). We can homotope D to $\bar{\partial}_{(TM,TL)}$ by homotoping the zeroth-order term to zero. This gives an isomorphism of the indexes $\operatorname{Ind}(D) \cong \operatorname{Ind}(\bar{\partial}_{(TM,TL)})$. Let $(E, F) = (TM \oplus 3\det_{\mathbf{C}} TM, TL \oplus 3\det TL)$ and $(E^1, F^1) = (\det_{\mathbf{C}} TM, \det TL)$. The connection on TM together with a connection on $\det_{\mathbf{C}} TM$ induces connections on E, E^1 and $4E^1$ and we can consider the families of operators $\bar{\partial}_{(E,F)}$, $\bar{\partial}_{(E^1,F^1)}$ and $\bar{\partial}_{(4E^1,4F^1)}$. These families of operators are Fredholm and their indexes form virtual vector bundles over $\mathfrak{B}(\Sigma, \mathbf{b})$. Moreover, using the invariance of the index under the Whitney sum of bundles, we have

$$\begin{aligned}&\mathrm{Ind}(\bar{\partial}_{(E,F)}) \oplus \mathrm{Ind}(\bar{\partial}_{(E^1,F^1)}) \cong \mathrm{Ind}(\bar{\partial}_{(E\oplus E^1, F\oplus F^1)}) = \\ &= \mathrm{Ind}(\bar{\partial}_{(TM\oplus 4E^1, TL\oplus 4F^1)}) \cong \mathrm{Ind}(\bar{\partial}_{(TM,TL)}) \oplus \mathrm{Ind}(\bar{\partial}_{(4E^1,4F^1)})\end{aligned}$$

Therefore

$$\begin{aligned}&w_1(\mathrm{Ind}(D)) = w_1(\mathrm{Ind}(\bar{\partial}_{(TM,TL)})) = \\ &= w_1(\mathrm{Ind}(\bar{\partial}_{(E,F)})) + w_1(\mathrm{Ind}(\bar{\partial}_{(E^1,F^1)})) + w_1(\mathrm{Ind}(\bar{\partial}_{(4E^1,4F^1)}))\end{aligned}$$

We show in Proposition 2.2 that $w_1(\mathrm{Ind}(\bar{\partial}_{(E,F)})_{|\gamma}) = w_2(TL)\cdot\beta$ and as a corollary we obtain $w_1(\mathrm{Ind}(\bar{\partial}_{(4E^1,4F^1)})) = 0$. In Proposition 2.3, we show $w_1(\mathrm{Ind}(\bar{\partial}_{(E^1,F^1)})_{|\gamma}) = (w_1(TL)\cdot b_1 + 1)\cdot(w_1(TL)\cdot\alpha)$. Combining them gives the desired formula

$$w_1(\mathrm{Ind}(D)_{|\gamma}) = (w_1(TL)\cdot b_1 + 1)\cdot(w_1(TL)\cdot\alpha) + w_2(TL)\cdot\beta$$

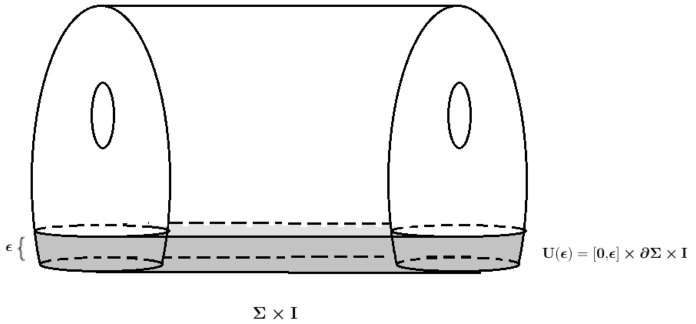
□

Proposition 2.2. *For every loop γ in $\mathfrak{B}(\Sigma, \mathbf{b})$, the first Stiefel-Whitney class of $\mathrm{Ind}(\bar{\partial}_{(E,F)})$ restricted to γ satisfies*

$$w_1(\mathrm{Ind}(\bar{\partial}_{(E,F)})_{|\gamma}) = w_2(TL)\cdot\beta$$

where β is the torus in L the boundary $\partial\Sigma$ traces along γ.

Proof. The loop γ is a map $\Phi : (\Sigma, \partial\Sigma)\times S^1 \to (M, L)$. Note that the bundle F is orientable with $w_2(F) = w_2(TL)$.



Over $U(\epsilon) = [0,\epsilon]\times\partial\Sigma\times I$, we can trivialize the pair $(\Phi^*(E), \Phi^*(F))_{|(U(\epsilon),U(0))} \cong (\mathbb{C}^{n+3}\times U(\epsilon), \mathbb{R}^{n+3}\times U(0))$. We then glue the bundle pair over the two ends of I to obtain a bundle isomorphic to $(\Phi^*(E), \Phi^*(F))$: glue $\mathbb{R}^{n+3}\times 0\times\partial\Sigma\times 1$ and $\mathbb{R}^{n+3}\times 0\times\partial\Sigma\times 0$ with a map $g\in\pi_1(SO(n+3))$ and extend g to a trivial map $\mathbb{C}^{n+3}\times\epsilon\times\partial\Sigma\times 1\to\mathbb{C}^{n+3}\times\epsilon\times\partial\Sigma\times 0$. In this way we have a trivialization of $\Phi^*(E)$ over $\epsilon\times\partial\Sigma\times S^1$. We can pinch Σ along $\epsilon\times\partial\Sigma$ to obtain a nodal surface formed by a closed Riemann surface and a disk joint together at single interior point. The bundle pair descends to the nodal surface (the bundle on the smooth surface is a pull-back of the bundle on the nodal surface). The index of $\bar{\partial}$ on Σ is isomorphic to the sum of the indexes of the $\bar{\partial}$'s on the closed surface and the disk minus the incident condition at the interior point (cf. [FOOO],[WW]). Since both the $\mathrm{Ind}(\bar{\partial})$ on the closed surface and the incident condition are complex vector spaces they both have canonical orientations. Therefore the question of the orientability of the index on Σ reduces to that on D^2.

It is shown in [FOOO] that the index of $\bar{\partial}$ for the bundle over the loop of disks is non orientable iff $g \in \pi_1(SO(n+3))$ is stably nontrivial. This is equivalent to $w_2(\Phi^*(F)) \cdot (\partial D^2 \times S^1) \neq 0$. In other words $w_1(\operatorname{Ind}(\bar{\partial}_{(E,F)})) \cdot \gamma = w_1(\operatorname{Ind}(\bar{\partial}_{(\Phi^*(E),\Phi^*(F))})) = w_2(\Phi^*(F)) \cdot (\partial D^2 \times S^1) = w_2(TL) \cdot \Phi(\partial D^2 \times S^1)$. □

The same proof applies to $\operatorname{Ind}(\bar{\partial}_{(4E^1,4F^1)})$ as $4F^1$ is an orientable bundle. Moreover, the second Stiefel-Whitney class of $4F^1$ is zero, and thus we obtain:

Corollary 2.3. *The first Stiefel-Whitney class of* $\operatorname{Ind}(\bar{\partial}_{(4E^1,4F^1)})_{|\gamma}$ *is zero for any loop* γ *in* $\mathfrak{B}(\Sigma, \mathbf{b})$.

Proposition 2.4. *For every loop* γ *in* $\mathfrak{B}(\Sigma, \mathbf{b})$, *the first Stiefel-Whitney class of* $\operatorname{Ind}(\bar{\partial}_{(E^1,F^1)})$ *restricted to* γ *satisfies*

$$w_1(\operatorname{Ind}(\bar{\partial}_{(E^1,F^1)})_{|\gamma}) = (w_1(TL) \cdot b_1 + 1) \cdot (w_1(TL) \cdot \alpha)$$

where b_1 *is the homology class* $\partial\Sigma$ *represents in* $H_1(L)$ *and* α *is the loop in* L *that a point on the boundary* $\partial\Sigma$ *traces along* γ.

Proof. When $w_1(\Phi^* F^1) \cdot \partial\Sigma = 0$, in the same way as above, we trivialize the pair $(\Phi^*(E^1), \Phi^*(F^1))_{|(U(\epsilon),U(0))} \cong (\mathbb{C} \times U(\epsilon), \mathbb{R} \times U(0))$ over $U(\epsilon) = [0,\epsilon] \times \partial\Sigma \times I$. We then glue the bundle pair over the two ends of I to obtain a bundle isomorphic to $(\Phi^*(E^1), \Phi^*(F^1))$: glue $\mathbb{R} \times 0 \times \partial\Sigma \times 1$ and $\mathbb{R} \times 0 \times \partial\Sigma \times 0$ with a map $g \in \pi_1(O(1))$ and extend g to a trivial map $\mathbb{C} \times \epsilon \times \partial\Sigma \times 1 \to \mathbb{C} \times \epsilon \times \partial\Sigma \times 0$. In this way we have a trivialization of $\Phi^*(E^1)$ over $\epsilon \times \partial\Sigma \times S^1$. We can pinch Σ along $\epsilon \times \partial\Sigma$ to obtain a nodal surface formed by a closed Riemann surface and a disk joint together at single interior point. The bundle pair descends to the nodal surface (the bundle on the smooth surface is a pull-back of the bundle on the nodal surface). The index of $\bar{\partial}$ on Σ is isomorphic to the sum of the indexes of the $\bar{\partial}$'s on the closed surface and the disk minus the incident condition at the interior point. Since both the $\operatorname{Ind}(\bar{\partial})$ on the closed surface and the incident condition are complex vector spaces they both have canonical orientations. Therefore the question of the orientability of the index on Σ reduces to that on D^2.

The index of $\bar{\partial}$ with values in a one dimensional vector bundle in the case of disks is isomorphic to the pullback by $\mu + 1$ boundary evaluation maps of the bundle (here μ is the Maslov index). Since we pushed all of the Maslov index up to the closed surface, the bundle over the disk in this case has Maslov 0. Therefore the index is isomorphic to the restriction of $\Phi^*(F^1)$ over $x_0 \times S^1$ for $x_0 \in \partial\Sigma$. In particular $w_1(Ind(\bar{\partial}_{(E^1,F^1)})) \cdot \gamma = w_1(\operatorname{Ind}(\bar{\partial}_{(\Phi^*(E^1),\Phi^*(F^1))})) = w_1(\Phi^* F^1_{|x_0 \times S^1}) = w_1(TL) \cdot \Phi(x_0 \times S^1)$.

When $w_1(\Phi^* F^1) \cdot \partial\Sigma \neq 0$, over $U(\epsilon)$ instead of trivializing, we consider the bundle pair isomorphism $(\Phi^*(E^1), \Phi^*(F^1))_{|(U(\epsilon),U(0))} \cong (\mathbb{C} \times U(\epsilon), e^{\frac{i\theta}{2}}\mathbb{R} \times U(0))$. We again glue $e^{\frac{i\theta}{2}}\mathbb{R} \times 0 \times \partial\Sigma \times 1$ and $e^{\frac{i\theta}{2}}\mathbb{R} \times 0 \times \partial\Sigma \times 0$ with a map $g \in \pi_1(O(1))$ and extend g to a trivial map $\mathbb{C} \times \epsilon \times \partial\Sigma \times 1 \to \mathbb{C} \times \epsilon \times \partial\Sigma \times 0$. Just like above, we pinch along $\epsilon \times \partial\Sigma$ and the question reduces to that on D^2. Now, however, we have pushed $\mu - 1$ of the Maslov index to the closed surface and we have Maslov 1 over the disk. Hence, the index over the loop of disks is isomorphic to the sum of the restrictions of $\Phi^*(F^1)$ to $x_0 \times S^1$ and $x_1 \times S^1$ for $x_0, x_1 \in \partial\Sigma$. Thus

$$\begin{gathered} w_1(Ind(\bar{\partial}_{(E^1,F^1)})) \cdot \gamma = w_1(\mathrm{Ind}(\bar{\partial}_{(\Phi^*(E^1),\Phi^*(F^1))})) = \\ = w_1(\Phi^*(F^1)_{|x_0 \times S^1}) + w_1(\Phi^*(F^1)_{|x_1 \times S^1}) = \\ = w_1(TL) \cdot \Phi(x_0 \times S^1) + w_1(TL) \cdot \Phi(x_1 \times S^1) \\ (= 0, \text{ in this case the index is orientable}). \end{gathered}$$

Combining the two cases gives $w_1(Ind(\bar{\partial}_{(E^1,F^1)})) \cdot \gamma = (w_1(TL) \cdot \Phi(\partial\Sigma) + 1) \cdot (w_1(TL) \cdot \Phi(x_0 \times S^1))$. □

When the Riemann surface Σ has several boundary components, we pinch near each one of them. In this way we obtain a nodal surface formed by a closed Riemann surface and several disks attached at an interior point to the surface. Since both $\mathrm{Ind}(\bar{\partial})$ on the closed surface and the incident conditions are complex vector spaces, they have canonical orientations. Thus, the orientability question reduces to that on the disks. Applying Proposition 2.2 and Proposition 2.3 we obtain

$$w_1(\mathrm{Ind}(D)_{|\gamma}) = \sum_{i=1}^{h} (w_1(TL) \cdot b_i + 1) \cdot (w_1(TL) \cdot \alpha_i) + \sum_{i=1}^{h} w_2(TL) \cdot \beta_i \tag{2.1}$$

where $b_i \in H_1(L)$ is the class the i-th boundary component of Σ represents in L, α_i is the loop a point on the i-th boundary component traces in L along γ, and β_i is the torus the i-th boundary component traces in L along γ. This is Theorem 2.1 in the case of several boundary components.

3. THE LOCAL SYSTEM $\mathcal{Z}_{(w_1,w_2)}$ ON $L^h \times \mathcal{L}(L)^h$

In this section we state the necessary definitions and theorems regarding local systems as discussed in [Ste] and we construct the system $\mathcal{Z}_{(w_1,w_2)}$ over $L^h \times \mathcal{L}(L)^h$. In the following section we will show its pull-back is canonically isomorphic to the local system twisted by the first Stiefel-Whitney class of $\mathrm{Ind}(D)$.

Definition 3.1. We have a *system of local groups* in a path connected topological space L if
1) for every $x \in L$ we are given a group G_x
2) for every class of paths α_{xy} we are given a group isomorphism $\alpha_{xy} : G_x \to G_y$
3) the composition $\beta_{yz} \circ \alpha_{xy}$ is the isomorphism corresponding to the path $\alpha_{xy}\beta_{yz}$

Remark 3.2. The pullback system is defined in the obvious way.

Theorem 3.3. *Let $x_0 \in L$, G a group and $\psi : \pi_1(L, x_0) \to Aut(G)$ a homomorphism. Then there is a unique system G_x of local groups in L, such that G_0 is a copy of G and the operations of $\pi_1(L, x_0)$ in G_0 are those determined by ψ.*

Definition 3.4. The *local system of orientations* is the system arising via the homomorphism $\psi : \pi_1(L, x_0) \to Aut(\mathbb{Z}) = \mathbb{Z}_2$, assigning to $\alpha \in \pi_1(L, x_0)$ the value of the first Stiefel-Whitney class of L evaluated on the class of α.

Remark 3.5. In [Ste] this system is called the system of twisted integer coefficients. It is the system twisted by the first Stiefel-Whitney class of L.

To construct the local system $\mathcal{Z}_{(w_1,w_2)}$ on $L^h \times \mathcal{L}(L)^h$, we first construct a system over every component of $L \times \mathcal{L}(L)$. This defines a system over $L \times \mathcal{L}(L)$ and we pull-back h copies of it to the product $L^h \times \mathcal{L}(L)^h$ via the projection maps. The system $\mathcal{Z}_{(w_1,w_2)}$ is defined as the tensor product of the pulled-back systems.

Let $(x_i \times \gamma_j)$ be a basepoint for the component $L_i \times \mathcal{L}(L)_j \subset L \times \mathcal{L}(L)$. Define a local system over $L_i \times \mathcal{L}(L)_j$ via the homomorphism:

$$\begin{gathered}\psi : \pi_1(L_i \times \mathcal{L}(L)_j, x_i \times \gamma_j) = \pi_1(L_i, x_i) \times \pi_1(\mathcal{L}(L)_j, \gamma_j) \to \operatorname{Aut}(\mathbb{Z})\\ (\alpha,\beta) \mapsto \psi(\alpha,\beta)(t) = (-1)^\epsilon t\\ \text{where } \epsilon = (w_1(TL)\cdot\gamma_j + 1)\cdot(w_1(TL)\cdot\alpha) + w_2(TL)\cdot\beta\end{gathered}$$

Remark 3.6. We can use twisted $\mathbb{Q}$ or $\mathbb{R}$ coefficients instead as well, if required by the method for achieving transversality.

To summarize, the system $\mathcal{Z}_{(w_1,w_2)}$ over a component of $L^h \times \mathcal{L}(L)^h$ with a basepoint $\vec{x} \times \vec{\gamma} = (x_1,..,x_h,\gamma_1,..,\gamma_h)$ is given by the homomorphism

$$\psi : \pi_1(L^h \times \mathcal{L}(L)^h, \vec{x} \times \vec{\gamma}) \to \operatorname{Aut}(\mathbb{Z}) \tag{3.1}$$

$$(\alpha_1,..,\beta_h) \mapsto \{t \mapsto (-1)^\epsilon t\} \tag{3.2}$$

$$\text{where } \epsilon = \sum_{i=1}^{h}(w_1(TL)\cdot\gamma_i + 1)\cdot(w_1(TL)\cdot\alpha_i) + \sum_{i=1}^{h} w_2(TL)\cdot\beta_i \tag{3.3}$$

4. CANONICAL ISOMORPHISM OF $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ WITH THE PULL-BACK OF $\mathcal{Z}_{(w_1,w_2)}$

In this section we construct a canonical isomorphism between the local system on $\mathfrak{B}(\Sigma,\mathbf{b})$, twisted by the first Stiefel-Whitney class of $\mathrm{Ind}(D)$, and the pull-back of $\mathcal{Z}_{(w_1,w_2)}$. The former system is essentially the local system of orientations on the moduli space.

Definition 4.1. The local system $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ over a component $\mathfrak{B}(\Sigma,\mathbf{b})_0 \subset \mathfrak{B}(\Sigma,\mathbf{b})$ with a basepoint u_0 is given by the homomorphism

$$\begin{gathered}\psi : \pi_1(\mathfrak{B}(\Sigma,\mathbf{b})_0, u_0) \to \operatorname{Aut}(\mathbb{Z})\\ \gamma \mapsto \left\{t \mapsto (-1)^{w_1(\mathrm{Ind}(D\bar{\partial})\cdot\gamma} t\right\}\end{gathered}$$

The group $\mathbb{Z}$ at the basepoint is fixed by a choice of a trivialization of $\det(\mathrm{Ind}(D))$ over the basepoint.

There are canonical evaluation maps

$$\begin{aligned}&ev_0^i : \mathfrak{B}(\Sigma,\mathbf{b}) \to L, \text{ assigning to a map } u \text{ its evaluation at } 0 \in S^1 \cong (\partial\Sigma)_i,\\ &ev_{\mathcal{L}(L)}^i : \mathfrak{B}(\Sigma,\mathbf{b}) \to \mathcal{L}(L), \text{ assigning to a map } u \text{ the loop } u_{|(\partial\Sigma)_i} : (\partial\Sigma)_i \to L.\end{aligned}$$

Combining them yields a map $\overrightarrow{ev}_L \times \overrightarrow{ev}_{\mathcal{L}(L)} = (ev_0^1 \times .. \times ev_{\mathcal{L}(L)}^h) : \mathfrak{B}(\Sigma,\mathbf{b}) \to L^h \times \mathcal{L}(L)^h$.

Theorem 4.2. *The local system $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ is isomorphic to the pull-back system $(\overrightarrow{ev}_L \times \overrightarrow{ev}_{\mathcal{L}(L)})^*\mathcal{Z}_{(w_1,w_2)}$.*

Proof. We restrict ourselves to a particular component. Let $u_0 \in \mathfrak{B}(\Sigma,\mathbf{b})$ map to the basepoint $\vec{x}_0 \times \vec{\gamma}_0 \in L^h \times \mathcal{L}(L)^h$. It is enough to show that the action of $\pi_1(\mathfrak{B}(\Sigma,\mathbf{b}),u_0)$ on the group $\mathbb{Z}_{u_0}$ induced by $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ is the same as the one induced by the pull-back system $(\overrightarrow{ev}_L \times \overrightarrow{ev}_{\mathcal{L}(L)})^* \mathcal{Z}_{(w_1,w_2)}$.

By definition the action induced by $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ is $(-1)^\epsilon$, where $\epsilon = w_1(\mathrm{Ind}(D))\cdot\gamma$ for $\gamma \in \pi_1(\mathfrak{B}(\Sigma,\mathbf{b}),u_0)$. By Theorem 2.1 (and its extension (2.1)) we have

$$w_1(\mathrm{Ind}(D))\cdot\gamma = \sum_{i=1}^{h}(w_1(TL)\cdot\gamma_i + 1)\cdot(w_1(TL)\cdot\alpha_i) + \sum_{i=1}^{h} w_2(TL)\cdot\beta_i$$

The action of $\gamma \in \pi_1(\mathfrak{B}(\Sigma,\mathbf{b}),u_0)$ induced by the pull-back system is by definition the action of the image $\widetilde{\gamma} \in \pi_1(L^h \times \mathcal{L}(L)^h, \vec{x}_0 \times \vec{\gamma}_0)$ of γ under the evaluation maps. By (3.3) this is define to be $(-1)^\epsilon$, where

$$\epsilon = \sum_{i=1}^{h}(w_1(TL)\cdot\gamma_i + 1)\cdot(w_1(TL)\cdot\alpha_i) + \sum_{i=1}^{h} w_2(TL)\cdot\beta_i$$

This shows the two actions are the same. □

To make the isomorphism canonical, we need to choose a trivialization of the determinant line of $\mathrm{Ind}(D)$ over u_0. This fixes the group at u_0 and by uniqueness (Theorem 3.3) the two systems must be the same.

In section 2, we showed that $\mathrm{Ind}(D) \cong \mathrm{Ind}(\bar{\partial}_{(TM,TL)})$ using a fiberwise homotopy. Since the space of Fredholm operators is contractible, a trivialization of $\mathrm{Ind}(\bar{\partial}_{(TM,TL)})_{|u_0}$ induces one on $\mathrm{Ind}(D)_{|u_0}$ in a canonical fashion. We also have $\mathrm{Ind}(\bar{\partial}_{(TM,TL)}) \oplus \mathrm{Ind}(\bar{\partial}_{(4E^1,4F^1)}) \cong \mathrm{Ind}(\bar{\partial}_{(E,F)}) \oplus \mathrm{Ind}(\bar{\partial}_{(E^1,F^1)})$, and therefore a choice of trivializations over u_0 of $\mathrm{Ind}(\bar{\partial}_{(4E^1,4F^1)}), \mathrm{Ind}(\bar{\partial}_{(E,F)})$ and $\mathrm{Ind}(\bar{\partial}_{(E^1,F^1)})$ will induce one on $\mathrm{Ind}(\bar{\partial}_{(TM,TL)})$.

In Proposition 2.2 we showed $\mathrm{Ind}(\bar{\partial}_{(E,F)})$ is isomorphic to the index of $\bar{\partial}$ over a disk plus a complex vector bundle. The complex bundle has a canonical orientation, and therefore we only need to choose a trivialization of the index over the disk. Since the bundle pair over the disk is trivial, we can choose an isomorphism to the bundle pair $(\mathbb{C}^{n+3},\mathbb{R}^{n+3})$, thus $\mathrm{Ind}(\bar{\partial}) \cong \mathbb{R}^{n+3}$ by evaluation at a point. This means that a choice of a trivialization of F over the image under u_0 of the boundary of Σ, fixes the determinant line of $\mathrm{Ind}(\bar{\partial}_{(E,F)})_{u_0}$. Similarly, a choice of a trivialization of $4F^1$ over the image of the boundary fixes the determinant line of $\mathrm{Ind}(\bar{\partial}_{(4E^1,4F^1)})$.

In Proposition 2.3, we showed $\mathrm{Ind}(\bar{\partial}_{(E^1,F^1)})$ is isomorphic to the index of a $\bar{\partial}$ over a disk plus a complex vector bundle, and again we only need to fix the determinant line of the index over the disk. When $w_1(F^1)\cdot b_1 = 0$, the index is isomorphic to $F^1_{|u_0(x_0)}$ and hence a choice of a trivialization of $F^1_{|u_0(x_0)}$ fixes the determinant line. When F^1 is not orientable along b_1, the index is isomorphic to F^1 restricted to the images of two points $x_0, x_1 \in \partial\Sigma$. We can use the orientation of the boundary of Σ to transport a choice of a trivialization of $F^1_{|u_0(x_0)}$ to $F^1_{|u_0(x_1)}$. In this way, a choice of a trivialization of $F^1_{|u_0(x_0)}$ canonically determines an orientation of $\mathrm{Ind}(\bar{\partial}_{(E^1,F^1)})_{|u_0}$.

To summarize, a choice of a trivialization of F and $4F^1$ over the image of the boundary of Σ, and a choice of a trivialization of F^1 over the image of a point on the boundary, fixes an orientation of $\mathrm{Ind}(D)_{|u_0}$. Changing any of these trivializations, changes the orientation of $\mathrm{Ind}(D)_{|u_0}$. Changing the orientation of the boundary of Σ, when F^1 is not orientable over the image of the boundary, also changes the orientation of $\mathrm{Ind}(D)_{|u_0}$.

5. The system of local orientations on the moduli space of stable maps

There is a forgetful map $\mathfrak{f} : \mathfrak{B}(\Sigma, \mathbf{b}) \times (S^1)^k \times (\Sigma)^l \to \mathfrak{B}(\Sigma, \mathbf{b})$, and we can pull-back the system $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$. Over the product $\mathfrak{B}(\Sigma, \mathbf{b}) \times (S^1)^k \times (\Sigma)^l$, the system $\mathfrak{f}^*\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ is canonically isomorphic to $\mathfrak{f}^*\mathcal{Z}_{w_1(\mathrm{Ind}(D))} \otimes \det[(TS^1)^k \times (T\Sigma)^l]$ since Σ is oriented. There is a projection map

$$\pi : \mathfrak{B}(\Sigma, \mathbf{b}) \times (S^1)^k \times (\Sigma)^l \to (\mathfrak{B}(\Sigma, \mathbf{b}) \times (S^1)^k \times (\Sigma)^l)/Aut(\Sigma, \partial\Sigma; j)$$

When the domain is stable the map π is a fibration with fiber $Aut(\Sigma, \partial\Sigma, j)$. The fact that the system $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ is trivial along the fiber and that the fiber is connected, means the system pushes-down to the factor space. This, together with a choice of an orientation of $Aut(\Sigma, \partial\Sigma, j)$, canonically determines the system $\pi_! \circ \mathfrak{f}^*\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ over $(\mathfrak{B}(\Sigma, \mathbf{b}) \times (S^1)^k \times (\Sigma)^l)/Aut(\Sigma, \partial\Sigma; j)$. After perturbation to achieve transversality, this system is the system of local orientations on the moduli space $\mathfrak{M}_{l,k}(\Sigma, \mathbf{b})$. In section 4, we showed that after certain choices of trivializations, the system $\mathcal{Z}_{w_1(\mathrm{Ind}(D))}$ is canonically isomorphic to $(\overrightarrow{ev}_0 \times \overrightarrow{ev}_{\mathcal{L}(L)})^*\mathcal{Z}_{(w_1,w_2)}$. This gives

Theorem 5.1. *There is a local system $\mathcal{Z}_{(w_1,w_2)}$ on $L^h \times \mathcal{L}(L)^h$ such that the local system of orientations on $\mathfrak{M}_{l,\vec{k}}(\Sigma, \mathbf{b})$ is isomorphic to $\pi_! \circ \mathfrak{f}^* \circ (\overrightarrow{ev}_0 \times \overrightarrow{ev}_{\mathcal{L}(L)})^*\mathcal{Z}_{(w_1,w_2)}$. The isomorphism is canonical once we choose a trivialization of TL over a basepoint in L, and trivializations of $TL \oplus det(TL)$ and $det(TL) \oplus det(TL)$ over those loops in L corresponding to a choice of a basepoint in each component of $\mathcal{L}(L)$.*

5.1. The codimension one strata.

The fiber product $\mathfrak{B}(\Sigma, \mathbf{b}') \times_{ev} \mathfrak{B}(D^2, \mathbf{b}'') \subset \mathfrak{B}(\Sigma, \mathbf{b}') \times S^1 \times \mathfrak{B}(D^2, \mathbf{b}'') \times S^1$ is defined as the pull-back of the diagonal in $L \times L$ by the map

$$ev : \mathfrak{B}(\Sigma, \mathbf{b}') \times S^1 \times \mathfrak{B}(D^2, \mathbf{b}'') \times S^1 \to L \times L$$

$$(u, \phi, v, \theta) \mapsto (u(\phi), v(\theta))$$

There is also a map

$$\mathfrak{B}(\Sigma, \mathbf{b}') \times_{ev} \mathfrak{B}(D^2, b'') \to \mathfrak{B}(\Sigma, \mathbf{b})$$

for $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$. We can consider two local systems over the fiber product - the pull-back of the system twisted by $w_1(\mathrm{Ind}(D^b))$, and the tensor product of $ev^*(\det(TL))$ with the systems twisted by $w_1(\mathrm{Ind}(D^{b'})$ and $w_1(\mathrm{Ind}(D^{b''}))$. The short exact sequence

$$0 \to TL \to \mathrm{Ind}(D^{b'}) \oplus \mathrm{Ind}(D^{b''}) \to \mathrm{Ind}(D^b) \to 0$$

implies that the two systems are isomorphic.

If the distinguished loop $\gamma_0 \subset L$ is chosen equal to $\gamma_0' \# \gamma_0''$, we can define the relative sign between the two systems. First, trivializations of the bundle F over γ_0' and γ_0'' determines a trivialization over γ_0. We can compare it with the chosen trivialization of F over γ_0: this defines an element $\epsilon \in \pi_1(SO(n+3)) = \mathbb{Z}_2$. Then, with this choice of γ_0, a path between u_0 and $u_0' \# u_0''$ defines a loop α_{u_0} in

$L^h \times \mathcal{L}(L)^h$. The local system $\mathcal{Z}_{(w_1,w_2)}$ associates to α_{u_0} an element $\delta \in \mathrm{Aut}(\mathbb{Z}) = \mathbb{Z}_2$. The relative sign between the two system is defined to be $(-1)^{\epsilon+\delta}$.

When $\Sigma = D^2$, following section 8.3 in [FOOO], we find that the relative sign between the local system on $\partial\mathfrak{M}_{l,k}(D^2, \mathbf{b})$ and the local system on $\mathfrak{M}_{l',k'}(D^2, \mathbf{b}') \times_{ev} \mathfrak{M}_{l'',k''}(D^2, \mathbf{b}'')$ is $(-1)^s$, where $s = \epsilon + \delta + (k')(k'' + \mu'') + (n + \mu' + k' + 1)$.

References

[FOOO] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, *Lagrangian Floer Theory: Anomaly and Obstruction*, AMS, Providence, RI, 2009

[MS] D. McDuff, D. Salamon, *J-holomorphic curves and symplectic topology*, volume 52 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2004.

[Sol] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, preprint 2006, arXiv.org:math/0606429

[Ste] N. E. Steenrod, *Homology with local coefficients*, The Annals of Mathematics, Second Series, Vol. 44, No. 4 (Oct., 1943), pp. 610-627

[WW] K. Wehrheim, C. Woodward, *Orientations for pseudoholomorphic quilts*, preprint 2007, available at http://math.mit.edu/ katrin/

Department of Mathematics, Stanford University, Stanford CA 94305
E-mail address: `penkag@math.stanford.edu`
URL: `http://math.stanford.edu/∼penkag`