Stanford University EPGY Summer Institutes 2008
Math Olympiad Problem Solving

For each of these problems, experiment numerically with the given problem, and try to come up with conjectures. Then, try to prove that your conjectures are correct. Some of your ideas may work, and some of your ideas may not. The goals of this problem set are for you to work on some difficult problems, to get used to brainstorming, and to think about various strategies for approaching problems. For example, try small cases and look for patterns. Try the problem with different numbers. Don't be afraid of doing some algebra. It's ok if you can't prove all of your results.

1. (Putnam 1990) Let
$$T_0 = 2,\ T_1 = 3,\ T_2 = 6,$$
and for $n \geq 3$,
$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$
The first few terms are
$$2, 3, 6, 14, 40, 152, 784, 5158, 40576, 363392.$$
Find a formula for T_n of the form
$$T_n = A_n + B_n,$$
where (A_n) and (B_n) are well-known sequences.

2. For each integer $n > 1$, find *distinct* positive integers x and y such that
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}.$$

3. For each positive integer n, find positive integer solutions $x_1, \ldots, x_n$ of the equation
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1.$$

4. Define $s(n)$ to be the number of ways that the positive integer n can be written as an ordered sum of at least one positive integer. For example,
$$4 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1 = 2+1+1 = 1+1+1+1,$$
so $s(4) = 8$. Conjecture a general formula for $s(n)$.

5. Let $g(n)$ be the number of odd terms in the row of Pascal's Triangle which starts with $1, n, \ldots$. For example, $g(6) = 4$ since the row

$$1, 6, 15, 20, 15, 6, 1$$

contains 4 odd numbers. Conjecture a formula for (or an easy way of computing) $g(n)$.

6. A group of n people are standing in a circle, numbered consecutively clockwise from 1 to n. Starting with person #2, we remove every other person, proceeding clockwise. For example, if $n = 6$, the people are removed in the order 2,4,6,3,1, and the last person remaining is #5. Let $j(n)$ denote the last person remaining (e.g. $j(6) = 5$).

   (a) Compute $j(n)$ for $n = 2, 3, \ldots, 25$.

   (b) Conjecture an easy way of computing $j(n)$. You may not get a nice formula, but try to find an algorithm which is easy to implement.

7. Observe that

$$6 = 1^2 - 2^2 + 3^2$$

and

$$7 = -1^2 + 2^2 + 3^2 - 4^2 - 5^2 + 6^2.$$

Investigate this pattern, and make a conjecture about a more general result..

8. (Putnam 1983) Let $f(n) = n + \lfloor\sqrt{n}\rfloor$. Prove that, for every positive integer m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \ldots$$

contains the square of an integer. You should begin this problem by experimenting with some numerical values. Make tables of the sequence $m, f(m), f(f(m)), f(f(f(m))), \ldots$ for various positive integers m.

9. Lockers in a row are numbered $1, 2, 3 \ldots, 1000$. At first, all of the lockers are closed. A person walks by, and opens every other locker, starting with locker #2. Thus, lockers $2, 4, 6, \ldots, 998$ are open. Another person walks by, and changes the "state" (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, starting with #3. Then another person changes the state of every fourth locker, starting with #4. This process continues until no more lockers can be altered. Which lockers will be closed? Hint: Start doing some experimentation with a smaller number of lockers.

10. (1985 AIME) The numbers in the sequence

$$101, 104, 109, 116, \ldots$$

are of the form

$$a_n = 100 + n^2,$$

where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

11. (Russia, 1995) The sequence $a_0, a_1, a_2, \ldots$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all integers $m, n \geq 0$ with $m \geq n$. If $a_1 = 1$, find a_{1995}.

12. Into how many regions is the plane divided by n lines in **general position** (no two lines parallel; no three lines meet in a point)?

13. A **great circle** is a circle drawn on a sphere that is an "equator," i.e. its center is also the center of the sphere. Suppose that there are n great circles on a sphere, no three of which meet at any point. Into how many regions do they divide the sphere?

14. What is the first time after 12:00 at which the hour and minute hands meet?

15. Let $\mathbb{N}$ denote the natural numbers $\{1, 2, 3, 4, \ldots\}$. Consider a function $f : \mathbb{N} \to \mathbb{N}$ which satisfies

$$f(1) = 1, \ f(2n) = f(n), \ f(2n+1) = f(2n) + 1$$

for all $n \in \mathbb{N}$. Find a nice simple algorithm for $f(n)$. Your algorithm should be a single sentence long, at most.

16. Define the function $f(x)$ by

$$f(x) = \frac{1}{1-x}$$

and denote r iterations of the function f by f^r, i.e.

$$\begin{aligned} f^2(x) &= f(f(x)) \\ f^3(x) &= f(f(f(x))) \\ f^4(x) &= f(f(f(f(x)))). \end{aligned}$$

Compute $f^{1999}(2000)$.

17. (1997 IMO) An $n \times n$ square matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that there is no silver matrix for $n = 1997$.

18. (Taiwan, 1995) Consider the operation which transforms the 8-term sequence $x_1, x_2, \ldots, x_8$ into the new 8-term sequence

$$|x_2 - x_1|, |x_3 - x_2|, \ldots, |x_8 - x_7|, |x_1 - x_8|.$$

Find all 8-term sequences of integers which have the property that after finitely many applications of this operation, one is left with a sequence, all of whose terms are equal.

19. There are 25 people sitting around a table, and each person has two cards. One of the numbers $1, 2, 3 \ldots, 25$ is written on each card, and each number occurs on exactly two cards. At a signal, each person passes one of her cards–the one with the smaller number–to her right-hand neighbor. Prove that, sooner or later, one of the players will have two cards with the same number.

20. For positive integers n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

as the $a_1, a_2, \ldots, a_n$ range through all positive values such that

$$a_1 + a_2 + \cdots + a_n = 17.$$

Find S_{10}.