Stochastic Volatility, Smile & Asymptotics

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Abstract

We consider the pricing and hedging problem for options on stocks whose volatility is a random process. Traditional approaches, such as that of Hull & White, have been successful in accounting for the much observed smile curve, and the success of a large class of such models in this respect is guaranteed by a theorem of Renault & Touzi, for which we present a simplified proof. We also present new asymptotic formulas that describe the geometry of smile curves and can be used for interpolation of implied volatility data.

Motivated by the robustness of the smile effect to specific modelling of the unobserved volatility process, we present a new approach to stochastic volatility modelling starting with the Black-Scholes pricing PDE with a random volatility coefficient. We identify and exploit distinct time scales of fluctuation for the stock price and volatility processes yielding an asymptotic approximation that is a Black-Scholes type price or hedging ratio plus a Gaussian random variable quantifying the risk from the uncertainty in the volatility. These lead us to translate volatility risk into pricing and hedging bands for the derivative securities, without needing to estimate the market's value of risk. For some special cases, we can give explicit formulas.

We outline how this method can be used to save on the cost of hedging in a random volatility environment, and run simulations demonstrating its effectiveness. The theory needs estimation of some statistics of the volatility process, and we run experiments to obtain approximations to these from simulated smile curve data.

Keywords: Option pricing, Volatility, Stochastic Volatility models, Hedging, Smile Curve.

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1 Review

1.1 Introduction

Extensions of the Black-Scholes model for option pricing began appearing in the finance literature not long after publication of the original paper [7] in 1973. For example, Merton [24] generalised the Black-Scholes formula to account for a deterministic time-dependent rather than constant volatility later the same year, and in 1976, he incorporated jump diffusion models for the price of the underlying asset [25]. However, the resulting deviations from the Black-Scholes derivative price are typically quite small, at the cost of having to estimate more complex parameters (recall that the Black-Scholes model requires estimation only of a constant volatility parameter corresponding to a lognormal diffusion model approximation of the underlying's price process). For this reason, the original form of the model has remained robust (allowing for modifications for dividend payments and transactions costs): although its limitations are recognised by traders, no simple 'better' model has been universally accepted.

Indeed, there is an almost perverse pressure by practitioners for real prices to conform to those of the Black-Scholes world as demonstrated by prevalent use of the implied Black-Scholes volatility measure on market prices. That is, traders are given to buying and selling in units of volatility corresponding to options prices through the Black-Scholes formula (which is strictly increasing in the volatility parameter). This is often termed 'trading the skew' [35], where skew refers to a trader's estimate of future volatility. Empirical studies of implied volatility date back to Latane & Rendleman [22](1976), and include Beckers [4](1981), Rubinstein [30](1985) and Canina & Figlewski [8](1993). This measure has been used to express a significant discrepancy between market and Black-Scholes prices: the implied volatilities of market prices vary with strike price and time-to-maturity of the options contract whereas, in the Black-Scholes model, the volatility is assumed constant. In particular, the variation of implied volatility with strike price for options with the same time-to-maturity is often a U-shaped curve, the smile curve, with minimum at or near the current asset price.

This particular shortcoming is remedied by stochastic volatility models first studied by Hull & White [19], Scott [33] and Wiggins [40] in 1987. The underlying asset price is modelled as a stochastic process which is now driven by a random volatility Itô process that may or may not be independent. As noted by Taylor [38], there is no economic intuition behind this generalisation, although Nelson has shown stochastic volatility arises as the continuous time limit of discrete statistical models such as ARCH[1], based on time-series analysis of financial data [26]. As a result of the increase in complexity, estimation of the unobserved volatility process from market data is much harder than estimation of the constant Black-Scholes volatility. Nonetheless, it is believable that market volatility (in the intuitive sense of wild price swings) *is* random, and the fact that stochastic volatility prices produce the smile curve for *any* volatility process instantaneously uncorrelated with the price process, as we shall see below, gives this extension of Black-Scholes a little more tractability than earlier ones.

In the spirit of the smile result which is robust to different models of the volatility, we shall consider the scenario of the volatility process fluctuating on a different time scale to that of the price process: an asymptotic analysis will yield pricing and hedging bands which are *not* sensitive to how the volatility is modelled. In the special Black-Scholes type cases when volatility does not depend on stock price, we derive explicit formulas for the width of these bands, enabling a simple modification of the Black-Scholes formula.

We run simulations of stock prices with stochastic volatility and test the effectiveness of our modified hedging strategy against simple Black-Scholes delta hedging. Finally, we test the ease-of-use of this model by using simulated smile data to infer estimates of the necessary parameters. This is seen to work remarkably well.

Our results demonstrate the tools of a separation-of-scales modelling approach that can be applied to a wide variety of incomplete market problems. This is discussed in the conclusions.

1.2 Literature Survey

We begin with a brief review of the 'traditional' approach to modelling stochastic volatility that originates from 1987 (see [19, 33, 40]). The asset price $\{X_t\}$ is taken to be a diffusion process that is lognormal conditional on the path of a stochastic volatility process $\{\sigma_t\}$:

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t. \tag{1.1}$$

Thus X_t is driven by the process σ_t, and is subordinate to it. We shall denote $V_t = \sigma_t^2$, and look at models for V_t; some authors model σ_t, but there are no real advantages of one or other approach.

The price of a European option $\{C_t, 0 \leq t \leq T\}$ is a function of current price and volatility: $C_t = C(X_t, V_t, t)$. This assumes that we can find the present volatility although it is not directly observable. Then, the *no-arbitrage* price of a European call option with strike price K and expiration

[1] Auto-regressive conditional heteroskedasticity

date T, in a market with constant spot interest rate r, is given by

$$C_t = C(X_t, V_t, t) = E_t^{Q(\lambda^\sigma)}[e^{-r(T-t)}(X_T - K)^+], \tag{1.2}$$

where $Q(\lambda^\sigma)$ is an equivalent martingale measure[2] (under which the prices of the traded assets, bond, stock and option, are martingales) depending on $\lambda^\sigma = \lambda(X_t, \sigma_t, t)$, the volatility risk premium, or the market price of risk. The notation E_t means the conditional expectation given the past up to time t: $E_t(\cdot) = E(\cdot|\mathcal{F}_t)$, with $\mathcal{F}_t$ the σ-algebra of events generated by $\{(X_s, \sigma_s) : s \leq t\}$. The presence of a second non-traded factor that the derivative price depends upon, the stochastic volatility, gives rise to an incomplete market in which the risk inherent in holding an option cannot be completely hedged by trading in just the underlying asset. Thus there must be a parameter λ^σ which represents how the market values the unhedged risk, and this too must be estimated from data once the model is put into practice. The appearance of this parameter will be more apparent in the derivation of the next section.

The pricing formula (1.2) can be simplified under the following assumptions: if $\lambda^\sigma = \lambda(\sigma_t, t)$, (independent of X_t), *and* σ_t is uncorrelated with X_t, then by iterated expectations,

$$C(X_t, V_t, t) = E_t^{Q(\lambda^\sigma)}[C^{BS}(\sqrt{\overline{V}})], \tag{1.3}$$

where

$$\overline{V} = \frac{1}{T-t}\int_t^T \sigma_s^2 ds, \tag{1.4}$$

and $C^{BS}(\sigma)$ denotes the classical Black-Scholes formula written as a function of the volatility parameter σ. Thus $\sqrt{\overline{V}}$ is the root-mean-square time average of $\sigma(\cdot)$ over the remaining trajectory of each realisation. This is a generalisation of the Hull-White formula [19] which originally assumed that $\lambda^\sigma \equiv 0$, meaning that the risk from the volatility process is non-compensated (or can be diversified away). In that case, the pricing measure is just that defined by the volatility process with no modification, and the formula is $C(X_t, V_t, t) = E_t\{C^{BS}(\sqrt{\overline{V}})\}$.

1.3 Partial Differential Equation for Option Price

An alternative characterisation of the option price is available when the stochastic volatility is also an Itô process, namely as the solution of a parabolic partial differential equation similar to the Black-Scholes pricing PDE, but with an extra dimension representing the dependence on the volatility process. It is often described as a special case of the general asset pricing PDE presented by Garman [15]. We give here a direct derivation which demonstrates how the market price of risk function arises naturally, as well as the breakdown of the hedged and unhedged portions of the risk.

We suppose that the asset price $\{X_t\}$ satisfies

$$dX_t = \mu X_t dt + \sqrt{V_t} X_t dW_t, \tag{1.5}$$

where the square of the volatility process $V_t = \sigma_t^2$ is a continuous solution of

$$dV_t = \alpha(V_t, t)dt + \beta(V_t, t)dZ_t. \tag{1.6}$$

The Brownian motions W_t and Z_t have constant correlation $\rho \in [-1, 1]$: $d < W, Z >_t = \rho dt$. Then we look for the option price as a smooth function $C(X_t, V_t, t)$ by trying to construct a hedged

[2] Details of the theory of pricing by equivalent martingale measures in the classical Black-Scholes complete markets case can be found in [16] or [11, Chapter 6].

portfolio of assets which can be priced by the no-arbitrage principle. Unlike the Black-Scholes case, it is not sufficient to hedge solely with the underlying asset, since the dW_t term can be balanced, but the dZ_t term cannot. Thus we try and hedge with the underlying asset *and* another option which has a different expiration date.

Let $C^{(1)}(x,v,t)$ be the price of an option with expiration date T_1, and try to find processes $\{a_t, b_t, c_t\}$ such that

$$C^{(1)}_{T_1} = a_{T_1} X_{T_1} + b_{T_1}\hat{\beta}_{T_1} + c_{T_1} C^{(2)}_{T_1}, \tag{1.7}$$

where $\hat{\beta}_t$ is the price of a riskless bond under the prevailing short-term constant interest rate r, and $C^{(2)}_t$ is the price of a European option with the same strike price as $C^{(1)}$, but different expiration date $T_2 > T_1 > t$. In other words, the right-hand side of (1.7) is a portfolio whose payoff at time T_1 equals almost surely the payoff of $C^{(1)}$. In addition, the portfolio is to be self-financing so that

$$dC^{(1)}_t = a_t dX_t + b_t d\hat{\beta}_t + c_t dC^{(2)}_t. \tag{1.8}$$

If such a portfolio can be found, for there to be no arbitrage opportunities, it must be that

$$C^{(1)}_t = a_t X_t + b_t\hat{\beta}_t + c_t C^{(2)}_t, \tag{1.9}$$

for all $t < T_1$. Expanding (1.8) by Itô's formula,

$$\begin{aligned} &\left(\frac{\partial C^{(1)}}{\partial t} + \mathcal{L}_1 C^{(1)}\right) dt + \frac{\partial C^{(1)}}{\partial x} dX_t + \frac{\partial C^{(1)}}{\partial v} dV_t \\ = \;&\left(a_t + c_t \frac{\partial C^{(2)}}{\partial x}\right) dX_t + c_t \frac{\partial C^{(2)}}{\partial v} dV_t + \left[c_t \mathcal{L}_1 C^{(2)} + b_t r \hat{\beta}_t\right] dt, \end{aligned} \tag{1.10}$$

where

$$\mathcal{L}_1 := \frac{1}{2} v x^2 \frac{\partial^2}{\partial x^2} + \rho x \sqrt{v}\beta(v,t)\frac{\partial^2}{\partial x \partial v} + \frac{1}{2}\beta(v,t)^2 \frac{\partial^2}{\partial v^2},$$

and $C^{(1)}$, $C^{(2)}$ and its derivatives are evaluated at (X_t, V_t, t). The risk from the dZ_t terms can be eliminated by balancing the dV_t terms, which gives

$$c_t = \frac{\partial C^{(1)}/\partial v}{\partial C^{(2)}/\partial v}, \tag{1.11}$$

and to eliminate the dW_t terms associated with dX_t, we must have

$$a_t = \frac{\partial C^{(1)}}{\partial x} - c_t \frac{\partial C^{(2)}}{\partial x}. \tag{1.12}$$

Substituting for a_t, c_t and $b_t = C^{(1)}_t - a_t X_t - c_t C^{(2)}_t$ and comparing dt terms in (1.10) gives

$$\left(\frac{\partial C^{(1)}}{\partial v}\right)^{-1} \mathcal{L}_2 C^{(1)}(X_t, V_t, t) = \left(\frac{\partial C^{(2)}}{\partial v}\right)^{-1} \mathcal{L}_2 C^{(2)}(X_t, V_t, t), \tag{1.13}$$

where

$$\mathcal{L}_2 := \frac{\partial}{\partial t} + \mathcal{L}_1 + r(x\frac{\partial}{\partial x} - \cdot).$$

That is, $\mathcal{L}_2$ is the classical Black-Scholes differential operator with volatility parameter $\sqrt{v}$, plus second-order terms from the V_t diffusion process. Now, the left-hand side of (1.13) contains terms depending on T_1 but not T_2 and *vice versa* for the right-hand side. Thus both sides must be equal to a function that does not depend on expiration date. Denoting this function $\lambda(x,v,t)\beta(v,t)-\alpha(v,t)$, the pricing function $C(x,v,t)$, with the dependence on expiry date supressed, must satisfy the PDE

$$\begin{aligned}\frac{\partial C}{\partial t}+\frac{1}{2}vx^2C_{xx}+\rho x\sqrt{v}\beta(v,t)C_{xv}+\frac{1}{2}\beta^2(v,t)C_{vv}&\\ +(\alpha(v,t)-\lambda(x,v,t)\beta(v,t))C_v+r(xC_x-C)&=0.\end{aligned} \tag{1.14}$$

For a call option, the terminal condition is $C(x,v,T)=(x-K)^+$ and $C(0,v,t)=0$, $C\sim x-Ke^{-r(T-t)}$, as $x\to\infty$. The $v-$boundary conditions depend on the specifics of the process $\{V_t\}$. The function λ is termed the market price of volatility risk because

$$dC(X_t,V_t,t)=[(\mu-r)X_tC_x+\lambda\beta(V_t,t)C_v]\,dt+\sigma_tX_tC_xdW_t+\beta(V_t,t)C_vdZ_t.$$

From this expression, we see that an infinitesimal fractional increase in the volatility risk β increases the infinitesimal rate of return on the option by λ times that fraction.

The equation (1.14) can be considered as the Feynman-Kac PDE associated with the expectation in (1.2) when the volatility is an Itô process. In particular, the transformation of the measure is seen explicitly in the transformed drift coefficient in front of the C_v term.

A summary of specific volatility models proposed in the literature is given in the following table[3].

Authors	Correlation	Risk Premium	Volatility Process
Hull-White [19]	$\rho=0$	$\lambda\equiv 0$	$dV_t=\alpha V_tdt+\xi V_tdZ_t$, lognormal
Scott [33]	$\rho=0$	$\lambda\equiv 0$	$\log\sigma_t$ is mean-reverting OU
Stein-Stein [37]	$\rho=0$	λ =const.	σ_t is modulus of a mean-reverting OU
Ball-Roma [3]	$\rho=0$	$\lambda\equiv 0$	$dV_t=\alpha(m-V_t)+\zeta\sqrt{V_t}dZ_t$
Heston [17]	$\rho\neq 0$	$\lambda=\overline{\lambda}v$	$dV_t=\alpha(m-V_t)+\zeta\sqrt{V_t}dZ_t$

There is little intrinsic intuition behind these volatility models, beyond the need to enforce positivity and, sometimes, mean-reverting behaviour for the process. This and the fact that a whole unknown and utterly unobservable function $\lambda(x,v,t)$ has to be estimated in general in order to price consistently with this theory, has led us and other researchers [1, 13, 20] to new approaches to translating uncertain volatility risk into price. We address this in Section 3.

2 Smile Curve

Definition Given a price $C(K,\tau)$ for a European option with time-to-maturity $\tau=T-t$ and strike price K (from, say, market data or a different pricing model), we can calculate the *implied Black-Scholes volatility* $I(K;x,\tau)$, such that $C^{BS}(x,\tau;I,K)=C(K,\tau)$ where x is the present (time t) price of the underlying asset. It can be found uniquely because of the monotonicity of the Black-Scholes formula in the volatility parameter: $\partial C^{BS}/\partial\sigma>0$.

The most quoted phenomenon testifying to the limitations of the classical Black-Scholes model is the *smile effect*: that implied volatilities of market prices are not constant across the range of options, but vary with strike price and time-to-maturity of the contract. In particular, the graph of

[3]OU denotes an Ornstein-Uhlenbeck process which is of the form $dY_t=(aY_t+b)\,dt+kdZ_t$.

$I(K,\tau)$ against K for fixed τ obtained from market options prices is often observed to be $U-$shaped with minimum at or near the money ($K_{\min} \approx x$). This is called the smile curve. Empirical evidence for this observation can be found, for example, in [8, 10].

In this section, we show that under certain conditions, stochastic volatility models *do* give prices whose Black-Scholes implied volatilities plotted against strike price form a smile curve. The first is a simplified proof of a global result originally due to Renault & Touzi [28] which guarantees existence of a smile locally centred around the current forward price of the stock $xe^{r(T-t)}$. Then we derive new asymptotic results that are valid when the volatility process has a small stochastic component. These give more detailed information about the geometry of the smile curve in cases where the volatility of the volatility is small, and this can be used for estimation purposes when callibrating specific models from market data as we shall outline below. In addition, the more difficult correlated case ($\rho \neq 0$), for which a global result about the shape of implied volatility curves is not known, can be tackled by asymptotic methods.

2.1 Analytical Results

Theorem 1 Renault-Touzi *In a stochastic volatility model defined by (1.5), where $\{V_t\}$ is a stochastic process uncorrelated with $\{X_t\}$, suppose the risk premium process is a function of X_t and t but not of V_t: $\lambda_t^\sigma = \lambda(X_t, t)$. Then provided $\overline{V}$ defined by (1.4) is an L^2 random variable, the implied volatility curve $I(K,\tau)$ for fixed τ is a smile, that is, it is locally convex around the minimum $K_{\min} = xe^{r\tau}$, which is the forward price of the stock.*

Proof Let us fix t and T and start with $\overline{V}$ being a Bernoulli random variable

$$\begin{aligned} \overline{V} &= \sigma_1 \text{ with probability } \alpha \\ & \sigma_2 \text{ with probability } 1-\alpha, \end{aligned}$$

under the measure $Q(\lambda^\sigma)$. Then provided the assumptions listed hold, the Hull-White formula (1.3) applies:

$$C^{BS}(I(\alpha,K);K) = \alpha C^{BS}(\sigma_1;K) + (1-\alpha)C^{BS}(\sigma_2;K), \tag{2.15}$$

where C^{BS} denotes the classical Black-Scholes formula, and below we give only its volatility argument. We have written $I = I(\alpha, K)$ to stress the dependence on α and K with x and τ fixed. Differentiating (2.15) with respect to K gives

$$C_\sigma^{BS}(I(\alpha,K))\frac{\partial I}{\partial K} + C_K^{BS}(I(\alpha,K)) = \alpha C_K^{BS}(\sigma_1) + (1-\alpha)C_K^{BS}(\sigma_2),$$

so that

$$\operatorname{sign}\left(\frac{\partial I}{\partial K}\right) = \operatorname{sign}(f(\alpha)),$$

where, for fixed K,

$$f(\alpha) := \alpha C_K^{BS}(\sigma_1) + (1-\alpha)C_K^{BS}(\sigma_2) - C_K^{BS}(I(\alpha,K)),$$

because $C_\sigma^{BS} > 0$. Clearly if $\alpha = 0, 1$ then $I \equiv \sigma_2, \sigma_1$ respectively, so $f(0) = f(1) = 0$. The key (and somewhat surprising) step is to now differentiate f and (2.15) with respect to α

$$\begin{aligned} f'(\alpha) &= C_K^{BS}(\sigma_1) - C_K^{BS}(\sigma_2) - C_{K\sigma}^{BS}(I(\alpha,K))I_\alpha, & (2.16) \\ C_\sigma^{BS}(I(\alpha,K))I_\alpha &= C^{BS}(\sigma_1) - C^{BS}(\sigma_2), & (2.17) \end{aligned}$$

from which we can obtain an expression for I_α. Differentiating f again and substituting directly from the Black-Scholes formula then gives

$$f''(\alpha) = 2[C^{BS}(\sigma_1) - C^{BS}(\sigma_2)]\frac{\tilde{L}I_\alpha}{\tau I^3},$$

where $\tilde{L} = \log(xe^{r\tau}/K)$. Assuming, without loss of generality, that $\sigma_1 < \sigma_2$, monotonicity of C^{BS} in the volatility parameter gives $C^{BS}(\sigma_1) < C^{BS}(\sigma_2)$ and $I_\alpha < 0$ from (2.16). Since $I > 0$, we conclude that $\text{sign}(f''(\alpha)) = \text{sign}(\tilde{L})$. Thus for $K < K_{min} := xe^{r\tau}$, $\tilde{L} > 0$ and f can only achieve a minimum in $[0,1]$, implying that $f < 0$ for $0 < \alpha < 1$. Hence $\partial I/\partial K < 0$. Similarly for $K > K_{min}$, $\partial I/\partial K > 0$, and $\partial I/\partial K = 0$ at $K = K_{min}$. The implied volatility curve is locally convex around K_{min}.

This can be extended by induction on the number of possible values that $\overline{V}$ can take, and this step is given in Appendix A. The result then holds for all distributions of $\overline{V}$ that are the limit of finite state distributions, as the number of states goes to infinity. This includes all L^2 (square integrable) random variables.

Numerical computations show that as expiration is approached, the region of concavity around $xe^{r\tau}$ shrinks, and the curve becomes flat outside this region - the smile becomes a very sharp downward spike. This is illustrated in Figure 1. To our knowledge, there has been no discussion in the empirical literature as to how the smile curve evolves in time: it would be interesting to see if it does indeed become convex nearer to the money as $t \to T$ (from real data).

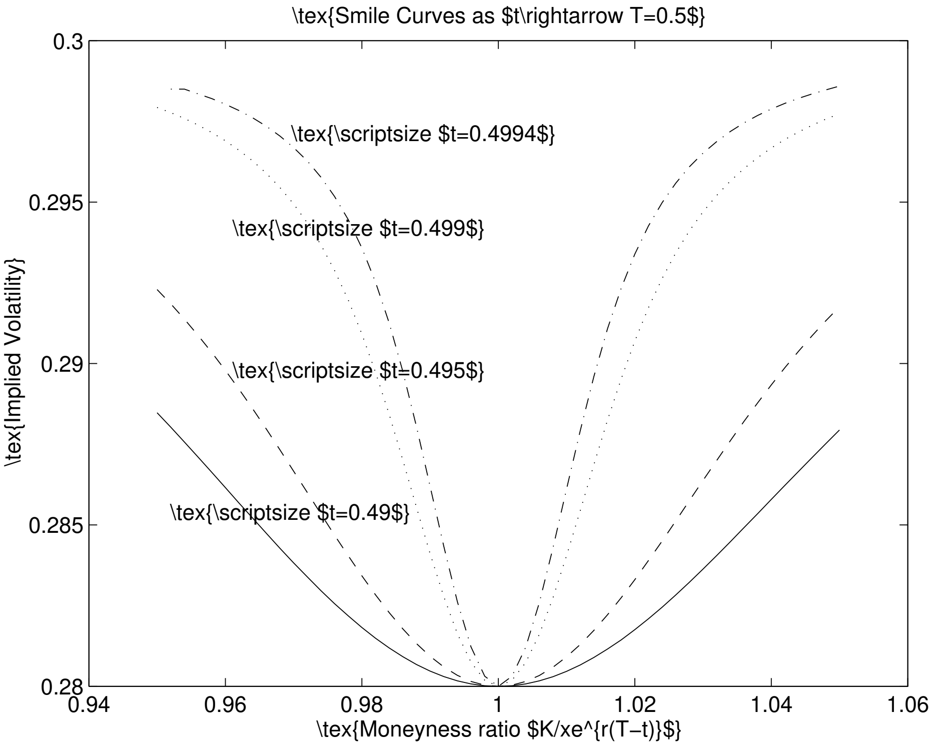

Figure 1: The implied volatility smile curves become spikier as $t \to T$. Here, $\overline{V}$ is Bernoulli with $\alpha = 0.2$, $\sigma_1 = 0.2$ and $\sigma_2 = 0.3$. The shifting base of the smile curve with time has been removed by the $xe^{r(T-t)}$ scaling.

Asymptotically, with $\overline{V}$ the Bernoulli distribution above, we can show that $I \to \max\{\sigma_1, \sigma_2\}$ as $t \uparrow T$, provided $|\tilde{L}| = |\log(xe^{r\tau}/K)| > \sqrt{T-t}$. At $K_{\min}$ ($\tilde{L} = 0$), $I \to \alpha\sigma_1 + (1-\alpha)\sigma_2 < \sigma_{\max}$, so the narrow spike behaviour is confirmed analytically. Note that I is not defined when $t = T$ because any value I matches the terminal payoff.

2.2 Asymptotic Smile

We utilise two approaches to demonstrate the smile effect when the randomness of the volatility process is small. For the uncorrelated case, a direct asymptotic expansion of the Hull-White formula is simplest. Similar Taylor series expansions were presented in [19, 3], but these are not generally valid without an asymptotic approximation. When $\rho \neq 0$, where the result is far more complex, we construct an asymptotic solution of the PDE satisfied by the stochastic volatility option pricing function. As described by Black [6], one often finds that $\rho < 0$ (rising volatility pushes stock prices down), so the latter is an important case.

2.3 Uncorrelated Case

Suppose $\sigma^\varepsilon(t) = \sigma_0 + \varepsilon z(t)$ for $\varepsilon << 1$ where $z(t)$ is some process with first and second moments. In this subsection, we shall assume for simplicity that volatility risk is non-compensated so that $\lambda \equiv 0$ and the measure $Q(\lambda)$ is denoted just Q. Then we can expand pathwise

$$\sqrt{V^\varepsilon} = \left[\frac{1}{T-t}\int_t^T (\sigma^\varepsilon(s))^2\, ds\right]^{\frac{1}{2}} = \sigma_0 + \varepsilon H(t;T) + \varepsilon^2 G(t;T) + \cdots, \tag{2.18}$$

where

$$\begin{aligned} H(t;T) &= \frac{1}{T-t}\int_t^T z(s)ds, \\ G(t;T) &= \frac{1}{2\sigma_0}\left[\frac{1}{T-t}\int_t^T z^2(s)ds - H^2(t)\right]. \end{aligned}$$

We look for an asymptotic expansion of the implied volatility I^ε defined by $C^{BS}(I^\varepsilon) = E_t^Q\left[C^{BS}(\sqrt{V^\varepsilon})\right]$ of the form $I^\varepsilon = \sigma_0 + \varepsilon I_1 + \varepsilon^2 I_2 + \cdots$. Expanding and comparing powers of ε reveals

$$\begin{aligned} I_1 &= E_t^Q\left[H(t;T)\right], \\ I_2 &= \frac{1}{2}\mathrm{var}_t[H(t;T)]\frac{C^{BS}_{\sigma\sigma}(\sigma_0)}{C^{BS}_{\sigma}(\sigma_0)} + E_t^Q[G(t;T)] \\ &= E_t^Q[G(t;T)] - \frac{1}{2\sigma_0}\mathrm{var}_t[H(t;T)]\left(\frac{1}{4}\sigma_0^2\tau - \frac{\tilde{L}^2}{\sigma_0^2\tau}\right), \end{aligned}$$

and so the plot of I^ε against K deviates from a flat line at $\mathcal{O}(\varepsilon^2)$ with a minimum at $\tilde{L} = 0$ (that is, $K = xe^{r\tau}$), and the smile becomes steeper as $t \uparrow T$(*i.e.* $\tau \downarrow 0$). Provided the expansion (2.18) is valid, which depends on τ and the process $z(\cdot)$, the expansion for I^ε as a function of K is asymptotic in the region

$$xe^{r\tau}(1 - \sigma_0\sqrt{\varepsilon\tau}) < K < xe^{r\tau}(1 + \sigma_0\sqrt{\varepsilon\tau}),$$

that is, when $\tilde{L}^2/\sigma_0^2\tau$ is smaller than ε. This is shown in Figure 2. It remains convex, in this approximation, for $K < xe^{r\tau + 1/(\sigma\sqrt{\tau})}$.

Thus the asymptotic analysis gives detailed information about the geometry of the smile curve when the stochastic component of the volatility is small, which is not available in the global result. A similar calculation will be presented in section (3.3) when the volatility is oscillating on a time

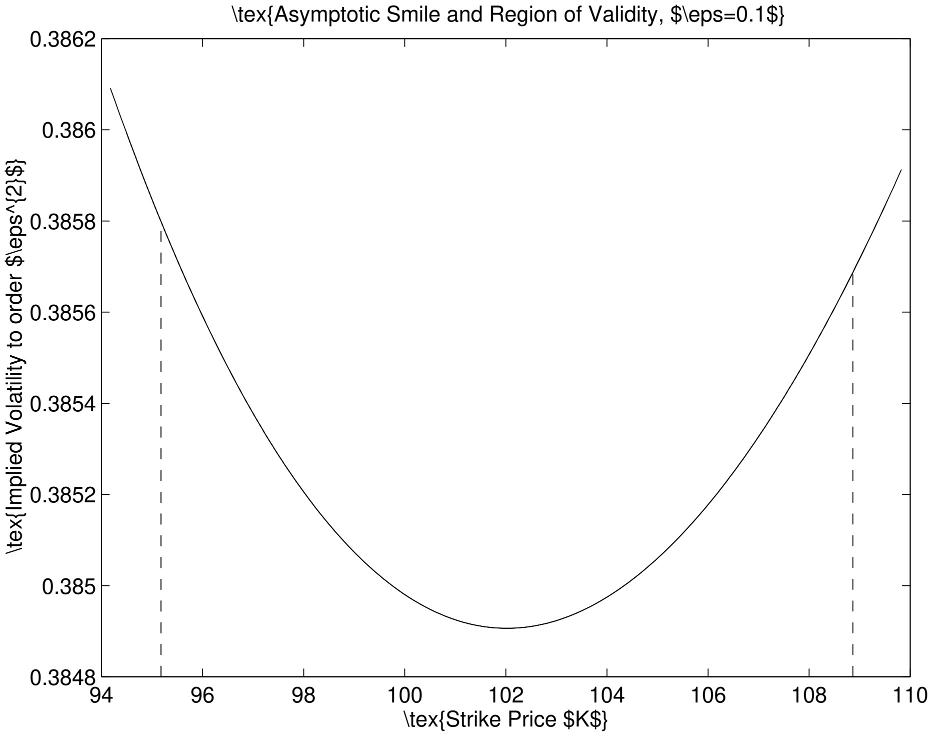

Figure 2: The dashed lines enclose the region of validity of the asymptotic approximation to the smile curve when the stochastic component of the volatility is small. Here $\sigma_0 = 0.3$, time to expiration $\tau = 0.5$, and the expectations in the expressions for I_1 and I_2 are taken to be of order 1.

scale that is short compared to the length of the derivative contract. This case will be important in connecting the traditional approach to stochastic volatility with the approach presented in Section 3.

In addition, the expansion shows that such stochastic volatility models do not all display the same time-to-maturity effect as they do the smile effect. To see this, we consider the variation of I^ε with T for fixed t, and as

$$\frac{\partial I^1}{\partial T} = \frac{1}{T-t} E_t^Q \left\{ z(T) - H(t;T) \right\},$$

it is clear that the variation of implied volatility with time-to-maturity will depend on the statistics of the volatility process. This is consistent with Rubinstein [30] who found different time-to-maturity effects in different time periods and for different moneyness ratios.

2.4 Correlated Case

When $\{V_t\}$ is an Itô process as in section (1.3), we model the small randomness of volatility by supposing that

$$dV_t^\varepsilon = \varepsilon\alpha(V_t^\varepsilon, t)dt + \sqrt{\varepsilon}\beta(V_t^\varepsilon, t)dZ_t$$

for some $\varepsilon << 1$, so that the option pricing function $C^\varepsilon(x, v, t)$ satisfies

$$\begin{aligned} C_t^\varepsilon + \frac{1}{2}vx^2C_{xx}^\varepsilon + \sqrt{\varepsilon}\rho x\sqrt{v}\beta(v,t)C_{xv}^\varepsilon + \frac{1}{2}\varepsilon\beta^2(v,t)C_{vv}^\varepsilon & & \quad (2.19)\\ + \left(\varepsilon\alpha(v,t) + \sqrt{\varepsilon}d(v,t)\right)C_v^\varepsilon + r(xC_x^\varepsilon - C^\varepsilon) &= 0, & \quad (2.20) \end{aligned}$$

with appropriate terminal and boundary conditions. We have assumed that λ is independent of x and defined $d(v,t) = -\lambda(v,t)\beta(v,t)$.

When $\varepsilon = 0$, (2.20) reduces to the Black-Scholes PDE with volatility parameter $\sqrt{v}$; the solution is $C^{BS}(\sqrt{v})$. Constructing an asymptotic solution

$$C^\varepsilon = C^{BS}(\sqrt{v}) + \sqrt{\varepsilon}C^1 + \varepsilon C^2 + \cdots,$$

we find

$$\mathcal{L}_{BS}(\sqrt{v})C^1 = -\frac{1}{2}\rho x\beta(v,t)C^{BS}_{x\sigma}(\sqrt{v}) - \frac{d(v,t)}{2\sqrt{v}}C^{BS}_\sigma(\sqrt{v}),$$

where $\mathcal{L}_{BS}(\sigma)u := u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + r(xu_x - u)$, and C^1 has zero terminal and boundary conditions. Because v is essentially a parameter in this equation, we can construct a solution of the form

$$C^1 = A(\sqrt{v},t)xC^{BS}_{x\sigma}(\sqrt{v}) + B(\sqrt{v},t)C^{BS}_\sigma(\sqrt{v}).$$

For simplicity, suppose α, β and λ do not explicitly depend on t. Then

$$\begin{aligned} A(\sigma,t) &= -\frac{\rho\sigma\beta(\sigma^2)(T-t)}{2(\sigma+1)}, &(2.21)\\ B(\sigma,t) &= \left[\frac{\rho\beta(\sigma^2)}{4}\left(1+\frac{\sigma}{\sigma+1}\right)+\frac{d(\sigma^2)}{2\sigma}\right](T-t) - \frac{\rho\beta(\sigma^2)\sigma(\frac{3}{2}\sigma-2)}{2(\sigma+1)}(T-t)^2. \end{aligned}$$

In any case, the first-order correction to the implied volatility $I^\varepsilon = \sqrt{v} + \sqrt{\varepsilon}I^1 + \varepsilon I^2 + \cdots$ is given by

$$I^1 = B(\sqrt{v},t) + A(\sqrt{v},t)\left(\frac{1}{2} - \frac{\tilde{L}}{v(T-t)}\right). \tag{2.22}$$

Since A vanishes and $B(\sigma,t) = -(T-t)d(\sigma^2)$ when $\rho = 0$, the implied volatility is still flat to $\mathcal{O}(\sqrt{\varepsilon})$ in the uncorrelated case, as was shown in the previous section. When $\rho \neq 0$, I^ε is linear to first-order in $\tilde{L}$, so there is no apparent local minimum. In the case of t-independent coefficients, if $\rho < 0$, then $A > 0$ and $\partial I^1/\partial K > 0$, so I^ε increases slowly upwards as K increases, within the region of validity $\tilde{L} = \mathcal{O}(1)$.

This ties in with the empirical observations of Rubinstein [30] and MacBeth & Merville [23] where monotonic implied volatility curves were reported: sometimes I was seen to increase with K and in other time periods decrease with K. Hull & White and Wiggins [40], from numerical experiments, related the slope of $\partial I/\partial K$ to the sign of the correlation ρ, and this is confirmed by the analytical formulas (2.21,2.22). However, as they noted, it is not clear why the correlation should change sign in different time periods. This behaviour holds independently of specific volatility modelling and choice of $\lambda(v,t)$, and the formulas (2.21-2.22) can be used to fit smile data and suggest tractable volatility models.

We can further construct C^2, and much calculation reveals that it is a quartic in $\tilde{L}$ if $\rho \neq 0$ and a quadratic in $\tilde{L}$ with minimum at $\tilde{L} = 0$ if $\rho = 0$, which is expected from the previous section's result.

It is worth noting that if α, β or λ depend on x, it will not in general be possible to construct asymptotic solutions from the Black-Scholes formula and its derivatives.

3 Time Scales and Asymptotics

We now take a somewhat different approach to the traditional view of stochastic volatility considered above. The various models of the volatility process in the literature exhibit differing degrees of success that might seem to commend or condemn them. For example, with Hull-White's log-normal model, the volatility stays positive, but it can grow unboundedly with time which does not reflect true behaviour. The square-root process of Heston [17] and Ball-Roma [3] also remains positive and reflects an intuitively appealing idea of volatility being mean-reverting; in addition, the option price has a closed-form solution, up to calculation of Fourier integrals. The same is also true for Stein's model [37]. What they all have in common is that they give prices that are slightly different from Black-Scholes prices[4], and that these prices always show the smile effect[5] (when the correlation is zero). The models can then be callibrated from market prices (for example by the method of moments outlined by Scott) at some expense. At the very least, two model-describing parameters plus the correlation ρ, today's value of the volatility and the market price of risk have to be estimated, so the complexity over Black-Scholes is considerably greater while the pricing biases are small. This at least suggests that we might take the random volatility to be something simpler than a continuous Itô process: if σ_t a two-state Markov chain gives the smile, why model it with a far more complicated process?

New approaches to modelling uncertain volatility have been initiated by Avellaneda *et al.* [1, 2] who construct worst case hedging strategies for complex portfolios of options assuming volatility lies in a known band, and Fournie *et al.*[14], who suggest simple, rapidly fluctuating volatility processes.

The approach we propose is to suppose that uncertainty in the volatility leads to uncertainty in the option price. Since the volatility cannot be directly observed and modelled, we shall restrict ourselves to finding pricing bands rather than exact prices, and hedging strategies that account for the risk inherent in a randomly fluctuating volatility environment, without assuming knowledge of the exact distribution. Furthermore, we shall exploit the fact that the asset price and its volatility may fluctuate on different time scales. The price is modelled as an Itô process as usual, whose driving Brownian motion oscillates on a scale that is "infinitely fast". Now, if we suppose the volatility is stochastic, to account for the smile effect, it is intuitive to expect that its time scale of variation is small compared with the lifetime of the derivative contract we are interested in (typically many months), but not as small as the infinitesimal scale of the Brownian motion. Thus the fastest random fluctuations of price are picked up by the Wiener process, and intermediate fluctuations by the volatility process. For example, it is reasonable that asset price variations are on the scale of minutes whilst volatility fluctuations are daily, which is small compared to a nine-month options contract.

Over the longer $\mathcal{O}(T)$ time scale, the fast fluctuations of the volatility give the appearance of a constant (in time) volatility function which would correspond to a Black-Scholes price; the randomness of the volatility would manifest itself as confidence bands around this price whose width can be calculated. Thus the fact that stock price and volatility change on different time scales allows us to develop an asymptotic pricing theory around Black-Scholes. The law of large numbers and the central limit theorem remove the dependence on the specifics of the volatility process, reflecting the robustness of this method.

[4]It is not obvious what the Black-Scholes price is in the context of a stochastic volatility model: Hull-White use $E(\overline{V})$ as the volatility (squared) parameter in the Black-Scholes formula, and Stein-Stein use the mean that σ_t reverts to in their model. But both of these, which are used to describe pricing biases of the new models from Black-Scholes are really measuring errors in approximating stochastic volatility options prices with Black-Scholes prices by adjusting the volatility parameter - it is like trying to guess the implied volatility curve and then inferring biases from the errors of the guess. A more logical measure would be to assume volatility is stochastic according to the models, simulate the asset price, and then calculate a historical Black-Scholes volatility parameter by fitting the generated data to a lognormal process.

[5]We shall loosely refer to the smile effect as meaning any non-constant variation of Black-Scholes implied volatility with strike price.

3.1 Model

The generalised Black-Scholes analysis tells us that the price $\overline{C}(\tau,x)$ of a European option on a stock whose price is modelled as the Itô process

$$dX_t = \mu(t,X_t)dt + \hat{\Sigma}(t,X_t)dW_t$$

satisfies the PDE

$$\overline{C}_\tau = \frac{1}{2}\Sigma^2(\tau,x)\overline{C}_{xx} + r(x\overline{C}_x - \overline{C}), \tag{3.23}$$

where $\tau = T - t$ is the time to expiration of the contract[6], and $\Sigma(\tau,x) := \hat{\Sigma}(T-\tau,x)$ is a deterministic function called the volatility. If we now suppose the volatility is random and rapidly varying in time, so that it is given by $\sigma^\varepsilon(\tau,x) = \sigma(\frac{\tau}{\varepsilon},x)$, where $\sigma(\tau,x) = \hat{\sigma}(T-\tau,x)$ for some stochastic process $\{\hat{\sigma}(t,x), t\geq 0\}$[7] that stays positive, taking values in a function space, and some small $\varepsilon > 0$, then we define the associated *stochastic* option price $C^\varepsilon(\tau,x)$ as the solution of the stochastic PDE

$$C^\varepsilon_\tau = \frac{1}{2}\sigma^2(\frac{\tau}{\varepsilon},x)C^\varepsilon_{xx} + r(xC^\varepsilon_x - C^\varepsilon), \tag{3.24}$$

with initial condition $C^\varepsilon(0,x) = (x-K)^+$ for a call option. We have allowed the volatility to depend on the stock price, but our results will not depend on specific modelling of such a function space valued process.

We are interested in the asymptotic behaviour of C^ε as $\varepsilon \downarrow 0$: this approximation will tell us how to deal with the risk from the randomness of the volatility.

3.2 Results

3.2.1 Pricing

Suppose that $\sigma(\tau,x)$ is an ergodic Markov process and that $\sigma^2(\tau,x)$ is stationary. For the moment, we shall restrict admissible processes to be bounded and continuous almost surely. Then, under further technical assumptions given in Appendix B[8], and with $\|\cdot\|$ denoting the space-time sup-norm defined there in (B.48), we have that as $\varepsilon\downarrow 0$,

$$\mathcal{P}\left(\|C^\varepsilon - \overline{C}\| > \delta\right) \to 0,$$

for any $\delta > 0$ and each $0\leq\tau\leq T$, where $\overline{C}(\tau,x;\sqrt{\overline{\sigma^2}}(x))$ denotes the solution to the generalised Black-Scholes PDE (3.23) with diffusion coefficient[9] $\overline{\sigma^2}(x) = E\{\sigma^2(\tau,x)\}$ which does not depend on τ by the stationarity assumption. This is an averaging principle result and will be derived in Appendix B by multiple scales and the ergodic theorem. It is analogous to the law of large numbers, and says that C^ε converges in probability to the Black-Scholes price with an averaged volatility function.

In the special case when the volatility process is separable in τ and x, $\sigma(\tau,x) = q(\tau)x$, where $\{q(\cdot)\}$ is an ergodic Markov real-valued process with $q^2(\cdot)$ stationary, the ergodic theorem implies that

$$\overline{V}^\varepsilon(\tau) = \frac{1}{\tau}\int_0^\tau q^2\left(\frac{s}{\varepsilon}\right)ds \to \overline{q^2} = E\{q^2(\cdot)\} \tag{3.25}$$

[6] We have changed time variable from t to τ so as to deal with forward problems for which 'forward-pointing' Itô integrals are well-defined.

[7] The process $\hat{\sigma}(t,x)$ defines an increasing filtration $\mathcal{F}_t\subset\mathcal{F}_s$ for $t<s$

[8] The necessary smoothing of the initial data for options is dealt with there.

[9] The diffusion coefficient is the coefficient in front of the second-derivative term.

as $\varepsilon \downarrow 0$ with probability 1, and so, by change of time variable,

$$C^{\varepsilon}(x,v,\tau) = C^{BS}(\tau,x;\sqrt{\overline{V}^{\varepsilon}}) \to \overline{C}(\tau,x;x\sqrt{\overline{q^2}}) \tag{3.26}$$

in probability. Thus only the statistical average of the stationary process is needed, rather than the path average.

We define the (scaled) error Z^{ε} in approximating C^{ε} by $\overline{C}$ by the following expression:

$$C^{\varepsilon}(\tau,x) = \overline{C}(\tau,x;\sqrt{\overline{\sigma^2}(x)}) + \sqrt{\varepsilon}Z^{\varepsilon}(\tau,x), \tag{3.27}$$

and consider the statistics of this fluctuation around the Black-Scholes average: the quantification of the stochastic volatility risk will be in this term. From the stochastic PDE (3.24) and the Black-Scholes PDE (3.23) satisfied by $\overline{C}(\tau,x;\sqrt{\overline{\sigma^2}(x)})$, Z^{ε} satisfies

$$Z^{\varepsilon}_{\tau} = \frac{1}{2}\sigma^2(\frac{\tau}{\varepsilon},x)Z^{\varepsilon}_{xx} + r(xZ^{\varepsilon}_x - Z^{\varepsilon}) + \frac{1}{2}\left(\frac{\sigma^2(\frac{\tau}{\varepsilon},x) - \overline{\sigma^2}(x)}{\sqrt{\varepsilon}}\right)\overline{C}_{xx}. \tag{3.28}$$

Writing this in time-integrated form, we can use the central limit theorem for Markov processes:

$$\frac{1}{\sqrt{\varepsilon}}\int_0^{\tau}\left\langle \sigma^2(\frac{s}{\varepsilon},\cdot) - \overline{\sigma^2}(\cdot), g(s,\cdot)\right\rangle ds \to \int_0^{\tau}\left\langle g(s,\cdot), d\tilde{W}(s,\cdot)\right\rangle, \tag{3.29}$$

as $\varepsilon \downarrow 0$ where $\langle\cdot,\cdot\rangle$ denotes the $L^2(\mathcal{R}^+)$ inner product and convergence is in the sense of distributions for all test functions $g(\tau,\cdot) \in \mathcal{S}(\mathcal{R}^+)$, the Schwarz space of rapidly decreasing $C^{\infty}(\mathcal{R}^+)$ functions. Here $\tilde{W}(\tau,x)$ is a Brownian motion[10] with covariance function $\gamma^2(x,y)$ defined by

$$\gamma^2(x,y) = 2\int_0^{\infty} R(s,x,y)ds, \tag{3.30}$$

$$R(|s_1-s_2|,x,y) = E\left\{(\sigma^2(s_1,x) - \overline{\sigma^2}(x))(\sigma^2(s_2,y) - \overline{\sigma^2}(y))\right\}. \tag{3.31}$$

Applying this to (3.28) tells us that the fluctuation Z^{ε} converges weakly to a process $Z(\tau,x)$ which satisfies

$$dZ = [\frac{1}{2}\overline{\sigma^2}(x)Z_{xx} + r(xZ_x - Z)]d\tau + \frac{1}{2}\overline{C}_{xx}d\tilde{W}, \tag{3.32}$$

and $Z(0,x) \equiv 0$. Further details are given in Appendix B. This is a linear stochastic PDE, and so $Z(\tau,x)$ is Gaussian with mean zero and covariance $U(\tau,x,y) := E\{Z(\tau,x)Z(\tau,y)\}$ that satisfies the following (Lyapunov) partial differential equation:

$$U_{\tau} = \frac{1}{2}\overline{\sigma^2}(x)U_{xx} + \frac{1}{2}\overline{\sigma^2}(y)U_{yy} + r(xU_x + yU_y - 2U) + \frac{1}{4}\overline{C}_{xx}(\tau,x)\gamma^2(x,y)\overline{C}_{xx}(\tau,y), \tag{3.33}$$

with $U(0,x,y) \equiv 0$. It gives the pricing bands or confidence intervals accounting for the unhedged volatility risk: for example,

$$\overline{C}(\tau,x;\sqrt{\overline{\sigma^2}(x)}) \pm 2\sqrt{\varepsilon\zeta(\tau,x)},$$

where $\zeta(\tau,x) = E\{Z^2(\tau,x)\} = U(\tau,x,x)$. Pricing within two standard deviations of the appropriate Black-Scholes mean provides roughly 95% protection against volatility fluctuations. These bands could, for example, be utilised to quote a bid-ask spread.

[10] Again, we mean that $\tilde{W}$ evolves forward with natural time and is evaluated at τ.

When $\sigma(\tau, x) = xq(\tau)$, corresponding to the 'traditional' separable stochastic volatility situation of X_t lognormal driven by the process $\{q(\cdot)\}$,

$$\sqrt{\zeta(\tau,x)} = \frac{\hat{\gamma}}{2\sqrt{2\pi\overline{q^2}}}\frac{K^{a+1}}{x^a}\exp\left(-\nu\tau - \frac{[\log(x/K)]^2}{2\overline{q^2}\tau}\right), \tag{3.34}$$

where $a = r/\overline{q^2} - 1/2$, $\nu = r/2 + \overline{q^2}/8 + r^2/2\overline{q^2}$, and $\hat{\gamma}^2$ is the integral of the correlation time (or power spectral density) for q^2:

$$\hat{\gamma}^2 = 2\int_0^\infty E\{(q^2(s) - \overline{q^2})(q^2(0) - \overline{q^2})\}ds.$$

The formula (3.34) is calculated using the Green's function for the classical Black-Scholes PDE (see Appendix C). A plot of the pricing bands around the Black-Scholes average when $q(\cdot)$ is a two-state Markov chain is shown in Figure 3. The bandwidth is widest in the neighbourhood of the strike price $x \approx K$ and tends to zero as $x \to 0, \infty$. Thus one can be confident of the Black-Scholes price (with the correct averaged volatility) deep in- and out-of-the-money, but the greatest uncertainty about the option price arising from uncertainty about the volatility is at-the-money.

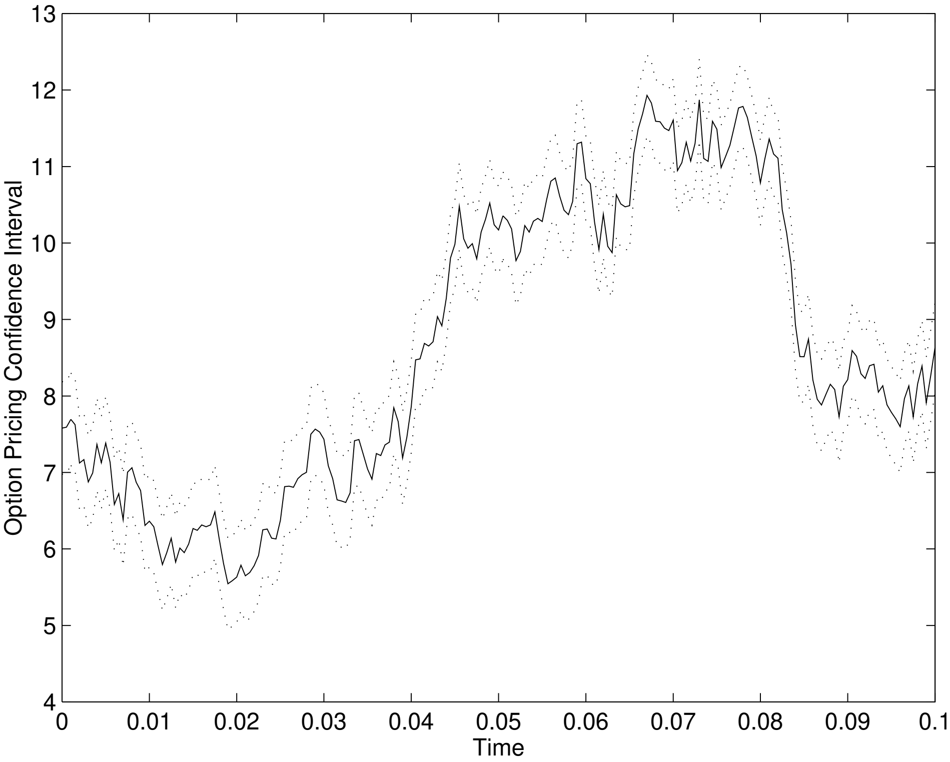

Figure 3: Pricing bands for separable $\sigma(\tau, x) = xq(\tau)$ with $q(\cdot)$ a two-state Markov chain taking values 0.2 or 0.3. The option strike price is $K = 100$, expiration $T = 0.5$ and $\varepsilon = 0.05$. The option price is shown along a particular realisation of the asset price.

3.2.2 Hedging

We now address the problem of hedging the risk of having written an option on the stock, which is an important issue for large trading institutions. In the classical Black-Scholes case, a perfect hedge can be achieved by holding $\Delta^{BS} = \partial C^{BS}/\partial x$ shares of the underlying to eliminate the risk from the Brownian fluctuations. The feedback effect of a significant amount of such hedging strategies on market volatility was studied in [36].

In the traditional stochastic volatility case, the procedure is less clear. For a perfect hedge, another derivative security on the same stock is needed and hedging ratios are given by formulas like (1.11,1.12) when, for example, an option with a later expiration date is used. This is known as Δ-Σ hedging (because the original notation had Δ of the underlying and Σ of the other derivative defining the strategy), but is often regarded as unsatisfactory because of the transaction costs involved in purchasing the other derivative. Romano & Touzi [29] looked for an optimal second option strike price to hedge with to lower such costs, and Renault & Touzi [28] studied the effects of just hedging with the underlying stock.

Another line of research by Föllmer & Sondermann [13] and Schweizer [32] tackles the implicit uncertainty of how to hedge represented by the dependence of the pricing measure $Q(\lambda^\sigma)$ on the volatility risk parameter λ^σ (see expression (1.2)) by choosing the measure that minimises the expected variance of the cost of hedging with just the underlying. Solving this stochastic least-squares problem defines a so-called minimal equivalent martingale measure that characterises the risk preference of the investor, and yields a hedging strategy that is self-financing on the average. The analysis for a more general market model (in which, for example, the volatility process can also depend on the stock price as we have allowed in this section) is presented in Hofmann *et al.*[18].

In this vein, we are interested in hedging strategies that can be adapted to the amount of volatility risk each investor will permit him or herself to be exposed to. This is characterised by hedging confidence intervals. In the generalised Black-Scholes case, the hedging ratio $\overline{\Delta} = \partial\overline{C}/\partial x$ gives the number of shares of the underlying stock that an option writer should buy to entirely hedge away the risk from the randomness in the stock price. It satisfies the PDE

$$\overline{\Delta}_\tau = \frac{1}{2}\Sigma^2(\tau,x)\overline{\Delta}_{xx} + \left(rx + \frac{1}{2}\left[\Sigma^2(\tau,x)\right]_x\right)\overline{\Delta}_x,$$

with initial condition $\overline{\Delta}(0,x) = \mathcal{H}(x-K)$ for a call option[11]. When the volatility is stochastic, we define the *stochastic* hedging ratio $\Delta^\varepsilon(\tau,x)$ as the solution of the corresponding stochastic PDE

$$\Delta^\varepsilon_\tau = \frac{1}{2}\sigma^2(\frac{\tau}{\varepsilon},x)\Delta^\varepsilon_{xx} + \left(rx + \frac{1}{2}\left[\sigma^2(\frac{\tau}{\varepsilon},x)\right]_x\right)\Delta^\varepsilon_x, \tag{3.35}$$

where we have assumed that $\sigma(\tau,x)$ is restricted to a space of admissible processes that are smooth enough that $\sigma_x(\tau,x)$ exists almost surely. As $\varepsilon \downarrow 0$,

$$\mathcal{P}\left(\|\Delta^\varepsilon - \overline{\Delta}\| > \delta\right) \to 0,$$

for any $\delta > 0$ and each $0 \le \tau \le T$, where $\overline{\Delta}(\tau,x)$ satisfies the averaged hedging PDE

$$\overline{\Delta}_\tau = \frac{1}{2}\overline{\sigma^2}(x)\overline{\Delta}_{xx} + \left(rx + \frac{1}{2}\overline{a}(x)\right)\overline{\Delta}_x,$$

with $a(\tau,x) := \left[\sigma^2(\tau,x)\right]_x$ and $\overline{a}(x) := E\{a(\tau,x)\}$. We assume the technical restrictions of Appendix B on the process $a(\tau,\cdot)$.

Thus the fluctuation $Y^\varepsilon(\tau,x)$ defined by

$$\Delta^\varepsilon(\tau,x) = \overline{\Delta}(\tau,x) + \sqrt{\varepsilon}Y^\varepsilon(\tau,x),$$

[11] Here, $\mathcal{H}(z)$ denotes the Heaviside function ($\mathcal{H}(z) = 0$ if $z < 0$ and $\mathcal{H}(z) = 1$ if $z > 0$)

satisfies

$$\begin{aligned} Y_\tau^\varepsilon &= \frac{1}{2}\sigma^2\left(\frac{\tau}{\varepsilon},x\right)Y_{xx}^\varepsilon + \left(rx + \frac{1}{2}a\left(\frac{\tau}{\varepsilon},x\right)\right)Y_x^\varepsilon \\ &\quad + \frac{1}{2}\left[\frac{\sigma^2\left(\frac{\tau}{\varepsilon},x\right) - \overline{\sigma^2}(x)}{\sqrt{\varepsilon}}\right]\overline{\Delta}_{xx} + \frac{1}{2}\left[\frac{a\left(\frac{\tau}{\varepsilon},x\right) - \overline{a}(x)}{\sqrt{\varepsilon}}\right]\overline{\Delta}_x. \end{aligned}$$

Under our regularity assumptions, a central limit theorem of the form (3.29) holds for $a(\tau,x)$

$$\frac{1}{\sqrt{\varepsilon}}\int_0^\tau \left\langle a(\frac{s}{\varepsilon},\cdot) - \overline{a}(\cdot), g(s,\cdot)\right\rangle ds \to \int_0^\tau \left\langle g(s,\cdot), d\tilde{B}(s,\cdot)\right\rangle, \tag{3.36}$$

where $\tilde{B}(\tau,x)$ is a Brownian motion with covariance $\Gamma^2(x,y)$ defined by

$$\Gamma^2(x,y) = 2\int_0^\infty \hat{R}(s,x,y)ds, \tag{3.37}$$

$$\hat{R}(|s_1-s_2|,x,y) = E\{(a(s_1,x)-\overline{a}(x))(a(s_2,y)-\overline{a}(y))\}. \tag{3.38}$$

The Brownian motion $\tilde{B}(\tau,x)$ is related to $\tilde{W}(\tau,x)$, the driving Brownian in the limit process for $\sigma^2(\tau,x)$ in (3.29), by

$$E\left\{\int_0^\tau \langle\phi(s,\cdot), d\tilde{W}(s,\cdot)\rangle \int_0^\tau \langle\psi(s,\cdot), d\tilde{B}(s,\cdot)\rangle\right\} = \int_0^\tau \iint \phi(s,x)\gamma(x,y)\Gamma(x,y)\psi(s,y)dxdyds.$$

Hence Y^ε converges weakly to a process $\{Y(\tau,x)\}$ satisfying the linear stochastic PDE

$$dY = \left[\frac{1}{2}\overline{\sigma^2}(x)Y_{xx} + \left(rx + \frac{1}{2}\overline{a}(x)\right)Y_x\right]d\tau + \frac{1}{2}\left(\overline{\Delta}_{xx}d\tilde{W} + \overline{\Delta}_x d\tilde{B}\right). \tag{3.39}$$

As we did with the stochastic price, we characterise the confidence bands for the hedging strategy taking into account the randomness of the volatility by the variance of the limiting fluctuation process, $\Upsilon(\tau,x) := E\{Y^2(\tau,x)\}$. In general, the covariance function $V(\tau,x,y) := E\{Y(\tau,x)Y(\tau,y)\}$ will satisfy the following (Lyapunov) PDE,

$$V_\tau = \frac{1}{2}\overline{\sigma^2}(x)V_{xx} + \frac{1}{2}\overline{\sigma^2}(y)V_{yy} + [rx + \frac{1}{2}\overline{a}(x)]V_x + [ry + \frac{1}{2}\overline{a}(y)]V_y + M(\tau,x,y),$$

where

$$\begin{aligned} M(\tau,x,y) &:= \frac{1}{4}\Big[\overline{\Delta}_{xx}(\tau,x)\gamma^2(x,y)\overline{\Delta}_{xx}(\tau,y) \\ &\quad + \overline{\Delta}_{xx}(\tau,x)\gamma(x,y)\Gamma(x,y)\overline{\Delta}_x(\tau,y) + \overline{\Delta}_x(\tau,x)\Gamma^2(x,y)\overline{\Delta}_x(\tau,y)\Big], \end{aligned}$$

with $V(0,x,y) \equiv 0$.

In the 'traditional' separable case, we find, using the Green's function for the classical Black-Scholes hedging PDE (see Appendix C), that

$$\sqrt{\Upsilon(\tau,x)} = \frac{\hat{\gamma}}{2\sqrt{6\pi\overline{q^2}}}\left(\frac{K}{x}\right)^{a+1}\left(\left[\log\left(\frac{x}{K}\right)\right]^2 + \left(2r + 3\overline{q^2}\right)\tau\right)\exp\left(-\nu\tau - \frac{[\log(x/K)]^2}{2\overline{q^2}\tau}\right), \tag{3.40}$$

where a and ν are as defined after (3.34). Typical risk bands

$$\overline{\Delta}(\tau,x) \pm 2\sqrt{\varepsilon\Upsilon(\tau,x)}$$

for the two-state Markov chain example are shown in Figure 4.

The results of this section depend only on the statistics $\overline{\sigma^2}(x), \overline{a}(x), \gamma(x,y), \Gamma(x,y)$ of the process $\sigma(\tau,x)$ in the most general case, and so are not sensitive to specific modelling of the volatility. This will be important for estimation.

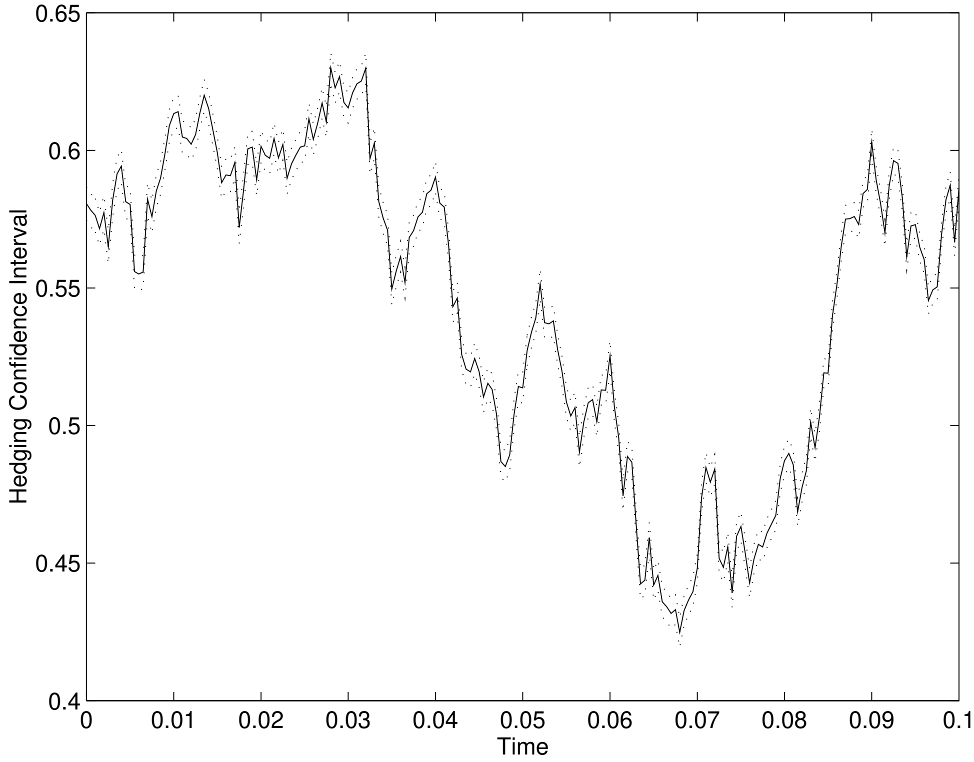

Figure 4: Hedging bands for the example described in Figure 3.

3.2.3 Numerical Simulations

We simulate a stock price trajectory that is conditionally lognormal, driven by a rapidly fluctuating two-state Markov chain volatility process, and compare two hedging strategies, one with the averaged volatility, corresponding approximately to the traditional approach (through the ergodic theorem (3.25)), and one at the edge of the hedging band described above which ought to provide similar protection against catastrophic losses, but for a lower cost.

Thus the asset price follows

$$dX^\varepsilon = \mu X^\varepsilon dt + q\left(\frac{t}{\varepsilon}\right) X^\varepsilon dW_t,$$

and our hedging strategies against the risk of having sold a call option are

$$\Delta^{SV} = \frac{\partial C^{BS}}{\partial x}(\sqrt{\overline{q^2}}), \tag{3.41}$$

$$\Delta^\varepsilon = \frac{\partial C^{BS}}{\partial x}(\sqrt{\overline{q^2}}) - 2\sqrt{\varepsilon}\Upsilon. \tag{3.42}$$

The second stategy costs less since it involves buying less of the underlying, but it should provide the same protection. The writer using the pure Black-Scholes strategy sells the option for $C^{BS}(0, X(0); \sqrt{\overline{q^2}})$, and the writer employing the edge-of-band hedge sells it for $C^{BS}(0, X(0); \sqrt{\overline{q^2}}) + 2\sqrt{\varepsilon\zeta(T, X(0))}$.

In Figure 5 we show the stock, volatility and hedging processes along a typical realisation, and in Figure 6 we show histograms of 1000 runs implementing the strategies over the length of a twelve month contract ($T = 1$) with 200 equally spaced re-hedgings and the asset price beginning at 90% of the strike price. Clearly greater profits are obtained with the second strategy.

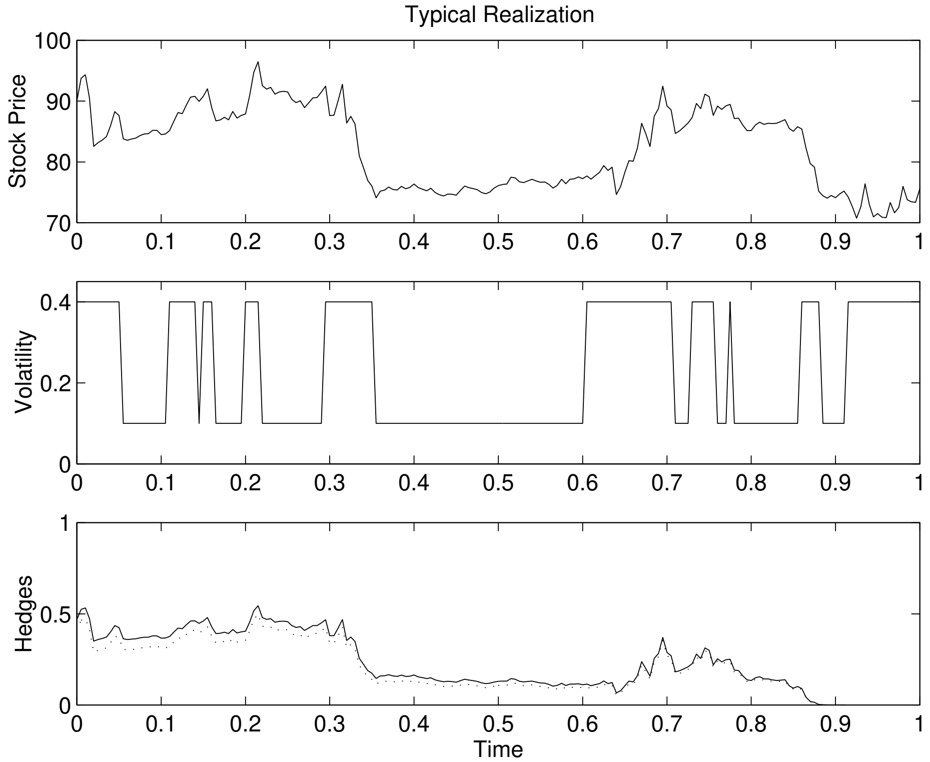

Figure 5: Stock price, Volatility and hedging strategies when $\varepsilon = 0.0005, K = 100, T = 1$. In the last, the solid curve is Δ^{SV}, the dotted curve is Δ^{ε} up to first-order, and ε is exaggerated in the plot to show the difference.

3.3 Connection with Smile Curve

We detail here the connection between rapidly fluctuating volatility processes, traditional stochastic volatility models and the smile curve. In particular, this will show that the stochastic option price C^{ε} for the special separable volatility case gives the smile on average, and that we can obtain expressions for the geometry of the implied volatility curve by asymptotic methods combining the averaging techniques of section (3.2.1) and the calculations of section (2.3).

Thus we look at $\sigma(\tau, x) = xq(\tau)$ where $q^2(\tau)$ is a stationary ergodic stochastic process. The stock price $X^{\varepsilon}(\cdot)$ satisfies

$$dX^{\varepsilon}(t) = \mu X^{\varepsilon}(t)dt + q\left(\frac{T-t}{\varepsilon}\right) X^{\varepsilon}(t)d\hat{W}(t).$$

Then recall that the 'traditional' stochastic volatility price $C^{SV}(X^{\varepsilon}(\tau), q(\tau), \tau)$ of a European call option is defined to be

$$C^{SV} = E_t\{C^{BS}(\sqrt{\overline{V}^{\varepsilon}(\tau)})\},$$

where we assume the market price of volatility $\lambda \equiv 0$, and $\overline{V}^{\varepsilon}(\tau)$ is defined in (3.25). We now use the following well-known result that is the averaging and central limit theorem for finite-dimensional processes to construct an asymptotic expansion for C^{SV} (see, for example, [27]).

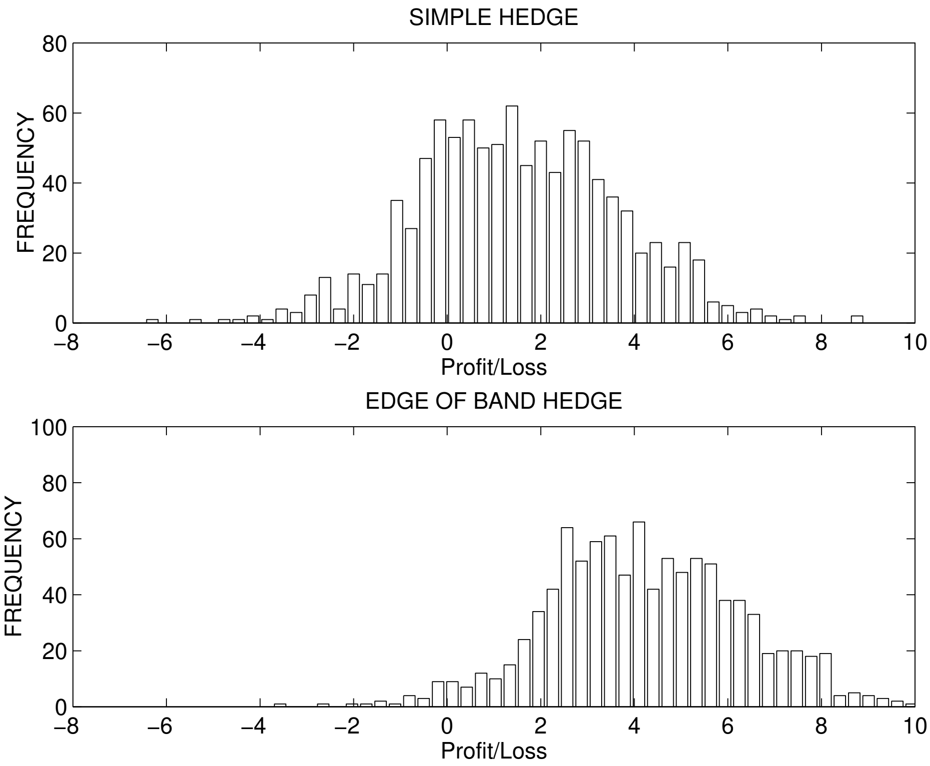

Figure 6: Profit-loss statistics after 1000 runs of hedging against having sold a call option struck at $K = 100$ with $T = 1$.

Theorem 2 Finite-dimensional averaging *Under the conditions on $q(\cdot)$ stated above, $\overline{V}^{\varepsilon}(\tau)$ converges strongly to the average $\overline{q^2}$ in the sense that*

$$\lim_{\varepsilon \downarrow 0} \mathcal{P}\left\{ \sup_{0 \le \tau \le \hat{T}} |\overline{V}^{\varepsilon}(\tau) - \overline{q^2}| > \delta \right\} = 0,$$

for any $\delta > 0$ and $\hat{T} < \infty$. In addition, the appropriately scaled error process in the approximation of $\overline{V}^{\varepsilon}(\tau)$ by $\overline{q^2}$, also referred to as the fluctuation and defined by $R^{\varepsilon}(\tau) := (\overline{V}^{\varepsilon}(\tau) - \overline{q^2})/\sqrt{\varepsilon}$, converges weakly to a Gaussian random variable $R_{t,T}$ with mean zero[12].

To derive the asymptotic smile arising from the rapidly fluctuating volatility process, we write $\overline{V}^{\varepsilon}(\tau) = \overline{q^2} + \sqrt{\varepsilon} R^{\varepsilon}(\tau)$, and expand

$$C^{SV} = C^{BS}(\sqrt{\overline{q^2}}) + \sqrt{\varepsilon} C^{BS}_{\sigma}(\sqrt{\overline{q^2}}) \frac{E_t\{R^{\varepsilon}\}}{2\sqrt{\overline{q^2}}} + \frac{\varepsilon}{8}\left(\frac{C^{BS}_{\sigma\sigma}(\sqrt{\overline{q^2}})}{\overline{q^2}} - \frac{C^{BS}_{\sigma}(\sqrt{\overline{q^2}})}{\left(\overline{q^2}\right)^{\frac{3}{2}}} \right) E_t\{(R^{\varepsilon})^2\}.$$

Equating this with $C^{BS}(I^{\varepsilon})$ where $I^{\varepsilon} = \sqrt{\overline{q^2}} + \sqrt{\varepsilon} I_1 + \varepsilon I_2 + \mathcal{O}(\varepsilon^{3/2})$, we find

$$I_1 \quad = \quad \frac{E_t\left[R_{t,T}\right]}{2\sqrt{\overline{q^2}}} = 0, \tag{3.43}$$

[12]Specifically, we could write $R_{t,T} = -\frac{\hat{\gamma}}{\tau}\int_t^T dB(s)$, where $\{B(\cdot)\}$ is a standard Brownian motion adapted to the filtration $\mathcal{F}_t$ from $\hat{q}(t)$.

$$
\begin{aligned}
I_2 &= \frac{1}{8\overline{q^2}}\mathrm{var}_t[R_{t,T}]\frac{C^{BS}_{\sigma\sigma}(\sqrt{\overline{q^2}})}{C^{BS}_{\sigma}(\sqrt{\overline{q^2}})} - \frac{1}{8\left(\overline{q^2}\right)^{\frac{3}{2}}}E_t[R^2_{t,T}] \\
&= \frac{\mathrm{var}_t\left[R_{t,T}\right]}{8\left(\overline{q^2}\right)^{3/2}}\left(\frac{\tilde{L}^2}{\overline{q^2}\tau} - \frac{1}{4}\overline{q^2}\tau - 1\right),
\end{aligned} \tag{3.44}
$$

where we have replaced $Z^\varepsilon(\tau)$ by its weak Gaussian mean zero limit in the expectations and variance. As in section (2.3), these expressions give the asymptotic geometry of the smile curve arising from a rapidly fluctuating volatility process.

The connection with the stochastic volatility model for derivative prices introduced in section (3.1) is clear. When the process $\sigma(\tau, x)$ is separable as $q(\tau)x$, the stochastic option price is

$$
C^\varepsilon(\tau, x) = C^{BS}(\sqrt{\overline{V}^\varepsilon(\tau)}),
$$

as can be shown by the usual change of time variable in the stochastic PDE (3.24). Thus $C^{SV}(x, q, \tau) = E_t\{C^\varepsilon(\tau, x) | q(T - \tau) = q\}$ and so, the averaged stochastic option price gives the asymptotic smile curve.

3.4 Estimation issues

Traditionally, the approach to the estimation aspect of implementing stochastic volatility models has been to write down a specific model of the volatility process (as in the table at the end of section (1.3)), and then use historical stock price data to identify the relevant parameters using the method of moments. Furthermore, it is necessary to assume *a priori* a specific functional form for the volatility risk premium as explained in Wiggins [40], who derives his λ using a market equilibrium argument for investors with log utility functions.

Scott [34] assumes λ is a constant and estimates it and today's value of volatility as minimizers of the squared discrepancy between observed and model options prices. Renault & Touzi [28] suggest using implied volatilities to obtain the stochastic volatility parameters from a maximum likelihood estimator. In this manner, we plan to use observed smile curves to estimate the parameters needed to calculate the pricing and hedging bands from our rapidly fluctuating volatility theory. Since traditional models coupled with estimation themselves lead to confidence intervals for predicted prices (because of the uncertainty in the parameter identification process), our seeking of model-independent bands in the first place might not be too great a weakening of the goals of stochastic volatility theory in practice.

Use of implied volatility or option price data from European options for market analysis purposes has been reported recently by Shimko [35], Dupire [12], Derman & Kani [10] and Rubinstein [31], who all sought a deterministic term-structure of volatility (analogous to $\Sigma(\tau, x)$ in (3.23)) and a corresponding risk-neutral probability density for the asset price. The first three exploit the Breeden-Litzenberger formula relating the density function to the option price convexity in strike price, C^{obs}_{KK}, using interpolation or binomial trees. Rubinstein uses an optimization scheme to find the probability density values (at the nodes of a binomial tree) that are least-squares closest to lognormal risk-neutral probabilities, and allow expected stock and option prices to lie in the observed bid-ask spreads. His results indicate greater skew and kurtosis than lognormal distributions from data after the 1987 crash. However, the extraction of such detailed information about the market's view of volatility risk from today's data has poor predictive ability - tomorrow's estimate of the term-structure will likely be different from today's. This problem of consistency over time is further discussed in Wilmott *et al.* [42]. Of course, since we take the view that volatility is genuinely stochastic (rather than just as a function of the stock price $\Sigma(t, X(t))$), we do not seek the whole term-structure, but a few (model-independent) statistics.

We outline a simple procedure for the separable volatility case for which we have explicit (asymptotic) formulas. Suppose that at discrete times t_n we observe the stock price x_n and a set of implied volatilities $I^{obs}(x_n, t_n; K_{ij}, T_i)$ from options with expiration dates T_i and strike prices K_{ij}, and that for each T_i, the interpolated points seem to form a shallow smile as a function of K. Then, for our theory, we need to estimate $\varepsilon, \overline{q^2}$ and $\hat{\gamma}$ for use in (3.34) or (3.40).

We could view each observed smile curve as an approximation to the average smile curve I^ε coming from $E\{C^\varepsilon\} = C^{BS}(I^\varepsilon)$. By a suitable best-fit of the implied volatility data to curves described explicitly by $I^\varepsilon = \sqrt{\overline{q^2} + \varepsilon I_2}$ and (3.44)[13], we can obtain estimates for $\varepsilon, \overline{q^2}$ and $\mathrm{var}_{t_n}[R_{t_n,T_i}]$. It remains to relate $\mathrm{var}_{t_n}[R_{t_n,T_i}]$ to $\hat{\gamma}$. This is trivial in the separable case by the representation $R_{t,T} = -\frac{\hat{\gamma}}{T-t}\int_t^T dB(s)$, and $\mathrm{var}_{t_n}[R_{t_n,T_i}] = \hat{\gamma}^2/(T_i - t_n)$. Thus in fact we shall end up estimating the product $\varepsilon\hat{\gamma}^2$, but it is exactly the square root of this that is needed in the confidence band formulas. What we have assumed is that the market through its trading of options comes to some 'average' view of volatility risk, and that this is reflected in market prices. Thus we want to infer the market's quantification of this risk contained in the observed smile curves, and from this obtain a first-pass approximation to our parameters. More precise information will come from spectral analysis of filtered stock price data. Our taking actual market prices to produce bands is consistent with the philosophy given by Wilmott [41] who writes: "*If we drop the assumption that markets are always correct and that financial modelling is at best an approximate science, then there is no such thing as risk-free hedging $\cdots$. [An] alternative approach assumes uncertain volatility: if volatility is so difficult to measure, why not accept this situation and instead work on volatility ranges? A consequence is that there is no such thing as a single fair value for an option, all prices within a given range are possible.*"

3.5 Numerical Simulation of Smile Fitting

To demonstrate how smile data can be used to infer $\overline{q^2}$ and $\varepsilon\hat{\gamma}^2$, we ran the following experiment using generated stock and options prices. We assumed that prices for options with 10 evenly distributed strike prices between \$90 and \$110 all with the same expiration date $T = 1$ were observed at times $0 \le t_n \le t_f = 0.025 < T$, where $t_n = n\Delta t$, $n = 0, \cdots, N$ and $\Delta t = 0.005$.

1. Simulate a realisation of the stock price $X^\varepsilon(t)$ at discrete times t_n, $n = 0, \cdots, N$, using a realisation of $q(t/\varepsilon)$, a rapidly fluctuating two-state Markov chain. We chose $\varepsilon = 0.0005$ and a volatility $q(\cdot)$ taking values 0.1 or 0.4 with power spectral density $\hat{\gamma} = 1$.

2. At each time-step n, generate $C^{obs}_{n,i}$ corresponding to strike price K_i $(i = 1, \cdots, I)$ by running 5000 realizations of $\{q^\varepsilon(t_k), t_n \le t_k \le T\}$, calculating $\overline{V}^\varepsilon = \sum_k [q^\varepsilon(t_k)]^2$ and then averaging $C^{BS}(\sqrt{\overline{V}^\varepsilon}; K_i)$ over the 5000 paths. The C^{obs} produced this way display a shallow smile as required for the experiment, because they are an approximation to $E\{C^\varepsilon\}$ to $\mathcal{O}(1/\sqrt{5000} \approx 10^{-2})$.

3. Calculate the observed implied volatility curves $I^{obs}(x_n, t_n; K_i, T)$ by numerically inverting the Black-Scholes formula.

4. Set-up $I \times (N+1)$ least-squares equations to fit I^{obs} to the formula (3.44) for I^ε. In our case, $I = 10$ and $N = 5$. This is effectively an interpolation of the smile data simultaneously in $K - t$ space to functions that are quadratic in $\log K$ and linear in t. Such basis functions

[13] The formula tells us we should fit each strike time's data to a quadratic curve in $\tilde{L}$.

arise naturally from the theory and suggest an improvement to the aribitrary quadratic-in-K basis used by Shimko [35] to find the term-structure of volatility.

5. The first-order equations for the least-squares problem give a 2×2 nonlinear system of equations for $\sqrt{\overline{q^2}}$ and $\varepsilon\hat{\gamma}^2$ that is easily solved by Newton-Raphson since the Jacobian can be calculated explicitly.

We ran this simulation and inversion algorithm 100 times (so that each time, the realised asset price path is different). Most of the time (94%), we obtained a good estimate of $\sqrt{\overline{q^2}}$ within 3% of the theoretical value[14] of 0.2915 and an estimate of $\varepsilon\hat{\gamma}^2$ that was of the right order of magnitude (10^{-4}), but inaccurate by up to 26%. On other runs, the estimate of $\varepsilon\hat{\gamma}^2$ was negative (4%), or the algorithm did not converge (2%). These preliminary findings from a simple fitting procedure indicate that a first-pass extraction of the necessary parameters can be obtained from smile data alone, before even looking at historical stock price data for further accuracy. In particular, the potential problem of $\varepsilon\hat{\gamma}^2$ being estimated as negative can be fixed by constraining it to be positive. We defer further improvements of the algorithm until we use real market data, which will be tackled in future work. Furthermore, empirical work [8] suggests that implied volatility from options prices is not a good forecaster of future realised volatility, and that historical volatility estimates from stock price data are superior in this sense. Thus it is crucial to employ a detailed analysis of past stock prices to extract a good estimate of $\varepsilon\hat{\gamma}^2$ and $\overline{q^2}$.

4 Conclusions

We have studied the pricing and hedging problem for options (and by extension, other European-style derivative securities) on stocks whose price volatility is stochastic, motivated by the success of such models in accounting for the frequently observed smile curve. New asymptotic formulas, valid when volatility has a small stochastic component or fluctuates on a distinct time scale give detailed description of the geometry of implied volatility curves and are effective for interpolation of observed implied volatilities, or as a tool for parameter identification.

The usefulness of separating the time scales on which the separate price and volatility processes fluctuate enables us to present a new approach to stochastic volatility modelling starting with the Black-Scholes option pricing PDE with a random volatility coefficient. This leads to an approximation that is the Black-Scholes price plus a Gaussian random variable quantifying the risk from the uncertain volatility. In particular, we characterise the risk by pricing and hedging bands which satisfy a PDE and depend only on the average and power spectral density of the volatility process. This robustness to different models of the volatility is in the spirit of the global smile result.

In a special (separable) case, we have explicit formulas for the bands that can be combined with the Black-Scholes formula. The explicit representation of the average asymptotic smile curve provides a simple method for obtaining a first-pass approximation to the statistics of the volatility process that are needed.

We plan to address more accurate estimation procedures using stock price data in future work, as well as extension of this approach to American securities for which there is little stochastic volatility literature at present.

In broad terms, we have presented a set of tools that can be used to quantify risk in an incomplete markets setting where the modelling identifies distinct time-scales of fluctuation between stochastic state variables. The methodology, which is easily extendable to derivative securities on $n > 2$

[14] This is the theoretical value if C^{obs} were computed by averaging over all volatility paths, whereas we generate our data from an average over only 5000 paths

underlying assets and factors (such as volatility), is an alternative to the general pricing PDE of Garman [15] which needs estimation of an unseen market-price-of-risk-type function for each untradeable state factor, and whose solution will be specific to models of these factors.

The separation-of-scales technique values the contribution to the derivative price from the hedgeable factors classically (no arbitrage) and quantifies the risk from the untradeable factors as pricing bands. Thus there is no longer one number known as the price, but there is no dependence on unobservable market risk premium factors or specific models of the non-traded factors. In addition, this approach evaluates the risk for the investor of hedging with just the tradeable underlyings, which is preferable to hedging with other derivatives in view of transaction costs. The Garman approach would give a theoretically perfect hedge, but with hedging strategies involving other derivatives.

The new methodology presents a wide variety of challenging modelling and estimation problems for future work on pricing and hedging in incomplete markets.

A Induction step

Suppose $\overline{V}$ is an n-state random variable taking values $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ with probabilities $\alpha_1, \cdots, \alpha_n$ respectively. Analogous to the case $n = 2$, we are interested in $\operatorname{sign}(f(\boldsymbol{\alpha}^{(n)}))$ where

$$f(\boldsymbol{\alpha}^{(n)}) := \sum_{i=1}^{n} \alpha_i C^{BS}(\sigma_i) - C_K^{BS}(I(\boldsymbol{\alpha}^{(n)}, K)),$$

$\boldsymbol{\alpha}^{(n)}$ is the vector of probabilities and $I(\boldsymbol{\alpha}^{(n)}, K)$ is defined by

$$C^{BS}(I(\boldsymbol{\alpha}^{(n)}, K)) = \sum_{i=1}^{n} \alpha_i C^{BS}(\sigma_i).$$

Differentiating these with respect to α_j for $j = 1, \cdots, n-1$ and using $\alpha_n = 1 - (\alpha_1 + \cdots + \alpha_{n-1})$, we find

$$I_{\alpha_j} C_\sigma^{BS}(I(\boldsymbol{\alpha}^{(n)}, K)) = C^{BS}(\sigma_j) - C^{BS}(\sigma_n) < 0,$$

and

$$f_{\alpha_j \alpha_k} = 2[C^{BS}(\sigma_j) - C^{BS}(\sigma_n)] \frac{\tilde{L} I_{\alpha_k}}{I^3 \tau},$$

so that $I_{\alpha_j} < 0$ and $\operatorname{sign}(f_{\alpha_j \alpha_k}) = \operatorname{sign}(\tilde{L})$ for $j = 1, \cdots, n-1$. Finally, we write $f(\boldsymbol{\alpha}^{(n)}) = f(\boldsymbol{\alpha}^{(n-1)}, \alpha_n)$ with $\boldsymbol{\alpha}^{(n-1)}$ denoting the vector of the first $n-1$ probabilities, and suppose (for induction) that $\operatorname{sign}(f(\boldsymbol{\alpha}^{(n-1)}, 0) = \operatorname{sign}(\tilde{L})$ on the hyperplane $\alpha_1 + \cdots + \alpha_{n-1} = 1$. This is the $n-1$ result. This implies that $f(\mathbf{1}, 0) = 0$, and we also know that $f(\mathbf{0}, 1) = 0$. Since f can only achieve one type of extremum given the sign of $\tilde{L}$, it follows that f is of one sign in the region $0 < \alpha_1 + \cdots + \alpha_{n-1} < 1$, provided $\alpha_n = 1 - (\alpha_1 + \cdots + \alpha_{n-1})$, which is the desired result for n-states.

B Proofs of stochastic averaging and CLT

Averaging

The averaging problem for deterministic parabolic PDEs with rapidly varying coefficients is considered in Bensoussan *et al.*[5], and Watanabe [39] presents results for the case when the coefficients are stochastic, under assumptions of stationarity and strong regularity of coefficients. Similar results are given by Dawson and Kouritzin [9] under less stringent regularity. The coefficients also need not be stationary, but must satisfy a weak law of large numbers and a weak invariance principle pointwise in x and t. The basic method of the proofs, in which an 'extra' term of the asymptotic series representation of the series is taken and the error shown to be small, is due to Khas'minskii [21].

We first use a multiple scales analysis to motivate the approximation of $C^\varepsilon(\tau,x)$ in (3.24) by $\overline{C}(\tau,x;\sqrt{\overline{\sigma^2}}(x))$. Considering the fast time scale as an independent variable $s := \tau/\varepsilon$, we look for a solution of the form

$$C^\varepsilon(\tau,x) = C^{(0)}(\tau,s,x) + \varepsilon C^{(1)}(\tau,s,x) + \cdots.$$

Substituting into (3.24) with the formal replacement

$$\frac{\partial}{\partial\tau} \longmapsto \frac{\partial}{\partial\tau} + \frac{1}{\varepsilon}\frac{\partial}{\partial s},$$

and comparing powers of ε gives $C^{(0)}_s = 0$, from the $\mathcal{O}(1/\varepsilon)$ term, so that $C^{(0)} = C^{(0)}(\tau,x)$. The $\mathcal{O}(1)$ terms then give

$$C^{(1)} = -s\left[C^{(0)}_\tau - r\left(xC^{(0)}_x - C^{(0)}\right)\right] + \frac{1}{2}C^{(0)}_{xx}\int_0^s \sigma^2(s',x)ds'. \tag{B.45}$$

Now as $\varepsilon \downarrow 0$, $s \to \infty$, and the integral in the second term behaves like $s\overline{\sigma^2}(x)$ by the ergodic theorem. Hence this expression for $C^{(1)}$ is non-asymptotic on average unless the right-hand side has mean zero:

$$-C^{(0)}_\tau + \frac{1}{2}\overline{\sigma^2}(x)C^{(0)}_{xx} + r\left(xC^{(0)}_x - C^{(0)}\right) = 0. \tag{B.46}$$

Thus $C^{(0)}(\tau,x) = \overline{C}(\tau,x;\sqrt{\overline{\sigma^2}}(x))$. We now make this approximation concrete by showing the error is small. We shall need the following result and then explain how it can be used in the problem at hand.

Theorem 3 Infinite-dimensional averaging *Consider the PDE for $u^\varepsilon(\tau,y) : [0,T]\times\mathcal{R}^n \to \mathcal{R}$,*

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial\tau} &= \sum_{i,j=1}^n a^\varepsilon_{ij}(\tau,y)\frac{\partial^2 u^\varepsilon}{\partial y_i\partial y_j} + \sum_{j=1}^n b^\varepsilon_j(\tau,y)\frac{\partial u^\varepsilon}{\partial y_j}, \\ u^\varepsilon(0,\tau) &= h(y), \end{aligned} \tag{B.47}$$

where the coefficients $a^\varepsilon_{ij} = a_{ij}(\frac{\tau}{\varepsilon},y)$, $b^\varepsilon_j = b_j(\frac{\tau}{\varepsilon},y)$ are random on a probability space $(\Omega,\mathcal{F},\mathcal{P})$, $\varepsilon > 0$ and $T < \infty$. Assume the following conditions hold

1. *$a_{ij}(\tau,y,\omega)$, $b_j(\tau,y,\omega)$ are measurable functions of $(\tau,y,\omega)\in[0,T]\times\mathcal{R}^n\times\Omega$.*
2. *The coefficients $\{a_{ij}\}$ are uniformly elliptic:*

$$c_1\sum_{i=1}^n \xi_i^2 \le \sum_{i,j=1}^n a_{ij}(\tau,y,\omega)\xi_i\xi_j \le c_2\sum_{i=1}^n \xi_i^2,$$

   for some $0 < c_1 \le c_2$ with probability 1.
3. *The $\{a_{ij}(\tau,\cdot)\}$ and $\{b_j(\tau,\cdot)\}$ are stationary ergodic processes with at least four uniformly bounded and continuous spatial derivatives:*

$$\max\left(\left\|D^\beta_y a_{ij}\right\|, \left\|D^\beta_y b_j\right\|\right) \le c_3 < \infty,$$

where

$$D_y^\beta := \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1}\cdots\partial y_n^{\beta_n}},\ |\beta| = \beta_1 + \cdots \beta_n,$$

for $|\beta| \le 4$, *with probability* 1. *The norm is defined by*

$$\|f\| := \sup_{0\le\tau\le T}\sup_{y\in\mathcal{R}^n} f(\tau, y). \tag{B.48}$$

4. *The initial data has at least four bounded and continuous derivatives:* $\sup|D_y^\beta h(y)| \le c_4$ *for* $|\beta| \le 4$.

Then

$$\mathcal{P}\left(\|u^\varepsilon - \overline{u}\| > \delta\right) \to 0,$$

as $\varepsilon \downarrow 0$ *for each* $\delta > 0$, *where* $\overline{u}(\tau, y)$ *satisfies*

$$\begin{aligned} \frac{\partial\overline{u}}{\partial\tau} &= \sum_{i,j=1}^n \overline{a}_{ij}(y)\frac{\partial^2\overline{u}}{\partial y_i\partial y_j} + \sum_{j=1}^n \overline{b}_j(y)\frac{\partial\overline{u}}{\partial y_j}, \\ \overline{u}(0,y) &= h(y), \end{aligned} \tag{B.49}$$

and $\overline{a}_{ij}(y) = E\{a_{ij}(\tau,y)\}$, $\overline{b}_j(y) = E\{b_j(\tau,y)\}$.

Proof We present the calculation for $n = 1$ for simplicity of notation; the higher dimensional case goes through analogously. Let

$$\begin{aligned} \chi_1(s,y) &= \int_0^s [a(s',y) - \overline{a}(y)]ds', \\ \chi_2(s,y) &= \int_0^s [b(s',y) - \overline{b}(y)]ds', \end{aligned}$$

and define $H^\varepsilon(\tau, y)$ by

$$u^\varepsilon(\tau,y) = \overline{u}(\tau,y) + \varepsilon\chi_1(\frac{\tau}{\varepsilon},y)\overline{u}_{yy} + \varepsilon\chi_2(\frac{\tau}{\varepsilon},y)\overline{u}_y + H^\varepsilon(\tau,y). \tag{B.50}$$

Then from (B.47) and (B.49), H^ε satisfies

$$H_\tau^\varepsilon = a^\varepsilon(\tau,y)H_{yy}^\varepsilon + b^\varepsilon(\tau,y)H_y^\varepsilon + R^\varepsilon(\tau,y), \tag{B.51}$$

with zero initial data where

$$R^\varepsilon = \varepsilon\left(a^\varepsilon\frac{\partial^2}{\partial y^2} + b^\varepsilon\frac{\partial}{\partial y}\right)\left(\chi_1\left(\frac{\tau}{\varepsilon},y\right)\overline{u}_{yy} + \chi_2\left(\frac{\tau}{\varepsilon},y\right)\overline{u}_y\right).$$

We have used the assumed smoothness of the coefficients and initial data which guarantees four derivatives (in space) on the solution $\overline{u}(\tau, y)$ of (B.49), so that R^ε is well-defined. Now, by the boundedness of the coefficients and the ergodic theorem:

$$\varepsilon\left\|D_y^\beta\chi_{1,2}\left(\frac{\tau}{\varepsilon},y\right)\right\| \to 0,\ \text{w.p.1 as } \varepsilon\downarrow 0,$$

for $|\beta| \le 2$, we have that $\|R^\varepsilon\| \to 0$ almost surely. Thus by the maximum principle for (B.51), $\|H^\varepsilon\| \to 0$ almost surely, and the result follows from (B.50).

Remarks

1. For our application, we shall assume that we can write $\sigma(\tau, x) \equiv \tilde{\sigma}(\tau, x)x$ with $\tilde{\sigma}(\tau, \cdot)$ having four continuous bounded derivatives almost surely. A logarithmic transformation ($y = \log x$) reduces (3.24) to the case above.

2. The payoff function must be four times continuously differentiable. For call and put options, this means that some slight artificial smoothing will be required at the kink at $x = K$, but this can be designed to have relatively small consequences for the pricing and hedging bands some time away from expiration ($\tau >> 0$).

CLT for Fluctuation

The statistics of the fluctuation of the stochastic option price and hedging ratio about the Black-Scholes average are deduced from the following result, similar versions of which can be found in [9, 39].

Theorem 4 Infinite-dimensional fluctuation theory *We assume that the coefficients satisfy a weak invariance principle*

$$\left| \frac{1}{\sqrt{\varepsilon}} \int_0^\tau \left\langle g(s,\cdot), a_{ij}^\varepsilon(s,\cdot) - \overline{a}_{ij}(\cdot) \right\rangle ds - \int_0^\tau \langle g(s,\cdot), dW_{ij}(s,\cdot) \rangle \right| \to 0 \ \text{as} \ \varepsilon \downarrow 0,$$

with convergence in the sense of distributions on the probability space. Here the Brownian motions $\{W_{ij}(s,\cdot)\}$ *have covariance functions*

$$\left[\gamma_{ij}^a(x,y)\right]^2 := 2\int_0^\infty E\{[a_{ij}(s,x) - \overline{a}_{ij}(x)][a_{ij}(0,y) - \overline{a}_{ij}(y)]\} ds,$$

with $x, y \in \mathcal{R}^n$, $g(s,\cdot) \in \mathcal{S}(\mathcal{R}^n)$, *the space of Schwarz distributions, and* $\langle \cdot, \cdot \rangle$ *denotes integration over* $\mathcal{R}^n$. *Similar is assumed to hold for the* b_j*'s with corresponding* $\gamma_j^b(x,y)$ *and* $\{W_j(s,\cdot)\}$. *Then the fluctuation*

$$S^\varepsilon(\tau, y) := (u^\varepsilon(\tau,y) - \overline{u}(\tau,y))/\sqrt{\varepsilon}$$

converges weakly to a process $S(\tau, y)$ *satisfying*

$$\begin{aligned} dS &= \left[\sum_{i,j=1}^n \overline{a}_{ij}(y) \frac{\partial^2 S}{\partial y_i \partial y_j} + \sum_{j=1}^n \overline{b}_j(y) \frac{\partial S}{\partial y_j} \right] d\tau + \sum_{i,j=1}^n \frac{\partial^2 \overline{u}}{\partial y_i \partial y_j} dW_{ij}(\tau, y) \\ &\quad + \sum_{j=1}^n \frac{\partial \overline{u}}{\partial y_j} dW_j(\tau, y). \end{aligned} \tag{B.52}$$

Proof Again we work with $n = 1$ for simplicity. By (B.47) and (B.49), $S^\varepsilon(\tau, y)$ satisfies

$$S_\tau^\varepsilon = a^\varepsilon S_{yy}^\varepsilon + b^\varepsilon S_y^\varepsilon + \left(\frac{a^\varepsilon - \overline{a}}{\sqrt{\varepsilon}} \right) \overline{u}_{yy} + \left(\frac{b^\varepsilon - \overline{b}}{\sqrt{\varepsilon}} \right) \overline{u}_y.$$

Let $\tilde{S}^\varepsilon(\tau, y)$ satisfy

$$\tilde{S}_\tau^\varepsilon = \overline{a}(y) \tilde{S}_{yy}^\varepsilon + \overline{b}(y) \tilde{S}_y^\varepsilon + \left(\frac{a^\varepsilon - \overline{a}}{\sqrt{\varepsilon}} \right) \overline{u}_{yy} + \left(\frac{b^\varepsilon - \overline{b}}{\sqrt{\varepsilon}} \right) \overline{u}_y.$$

Then we can write $S^\varepsilon(\tau, y) = I_1(\tau, y) + I_2(\tau, y)$ where

$$I_1 = \int_0^\tau \int_{-\infty}^\infty G_{\overline{a},\overline{b}}(y, \eta, \tau, s) \left(\frac{a^\varepsilon(s,\eta) - \overline{a}(\eta)}{\sqrt{\varepsilon}} \right) \overline{u}_{yy}(s,\eta) d\eta ds,$$

and similarly I_2. Here $G_{\overline{a},\overline{b}}$ is the Green's function associated with the averaged coefficients. Under the prevailing assumptions, the invariance principles of the coefficients apply and we see that $\tilde{S}^{\varepsilon}(\tau, y)$ converges weakly to $S(\tau, y)$ defined by (B.52). Finally, we can write the equation satisfied by $(S^{\varepsilon} - \tilde{S}^{\varepsilon})$ and conclude

$$\mathcal{P}\left(\|S^{\varepsilon} - \tilde{S}^{\varepsilon}\| > \delta\right) \to 0 \text{ as } \varepsilon \downarrow 0,$$

by the maximum principle.

C Classical Black-Scholes with stochastic volatility coefficient

Here, we briefly give details of the special case at the end of section (3.2.1) where the covariance $\zeta(\tau, x)$ is calculated for $\sigma(\tau, x) = xq(\tau)$. The Green's function for the classical Black-Scholes PDE with constant volatility σ_0 is given by

$$G(x, \xi, \tau, s) = -\frac{x^{\theta-1}}{\xi^{\theta}\sigma_0\sqrt{2\pi(\tau - s)}} \exp\left(-\nu(\tau - s) - \frac{[\log(x/\xi)]^2}{2\sigma_0^2(\tau - s)}\right), \tag{C.53}$$

where

$$\begin{aligned} \theta &= \frac{3}{2} - \frac{r}{\sigma_0^2}, \\ \nu &= \frac{r}{2} + \frac{1}{8}\sigma_0^2 + \frac{r^2}{2\sigma_0^2}. \end{aligned}$$

Thus the price of a put option $P(\tau, x)$, for example, can be written

$$P(\tau, x) = \int_0^{\infty} G(x, \xi, \tau, 0)(K - \xi)^+ d\xi.$$

Then the solution $Z(\tau, x)$ of (3.32) is

$$Z(\tau, x) = \frac{1}{2}\int_0^{\tau} \left\langle G(x, \xi, \tau, s)\xi^2 \overline{C}_{xx}(s, \xi), d\tilde{W}(s, \xi)\right\rangle.$$

Finally, $\zeta(\tau, x)$ is calculated from the expression

$$\zeta(\tau, x) = \frac{1}{4}\hat{\gamma}^2 \int_0^{\tau}\left[\int_0^{\infty} G(x, \xi, \tau, s)\xi^2 \overline{C}_{xx}(s, \xi) d\xi\right]^2 ds,$$

because the covariance function is $\gamma^2(x, y) = \hat{\gamma}^2 x^2 y^2$, which gives (3.34).

The Green's function for the classical Black-Scholes hedging PDE is identical to (C.53) with θ replaced by $\theta - 1$. This is used in the calculation in section (3.2.2).

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