Reflection of Wavefronts by Randomly Layered Media

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Abstract

We consider reflection of a pulse in a random medium with a strong reflector. We show that the wavefront of the reflected wave observed in the frame moving with the random propagation speed stabilizes to the deterministic waveform. The problem is studied using invariant imbedding. The results of numerical experiments illustrating the theory are presented.

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1 Introduction

We study the evolution of the front of the wave in a layered random medium that has variations on two scales. On the micro-scale comparable with the width of the incident pulse the properties of the medium change rapidly with depth i.e. the direction of layering, but the amplitudes of the fluctuations in the medium-related parameters are small. On the macro-scale comparable with the distances of propagation, the average properties of the medium are piece-wise constant functions. The rapid fluctuations on the micro-scale produce backscattering. Although weak, it accumulates over long distances of propagation and causes significant changes in the shape of the front of the wave. However, each of the abrupt changes in the average properties of the medium on the macro-scale initiates a strong reflected signal at the moment the transmitted pulse reaches it. While the evolution of the transmitted pulses in the undisturbed medium is fairly well understood by now [5],[2],[3],[4] the evolution of the reflected pulse was not investigated before.

The theory that originated with the work of O'Doherty and Anstey (OD-A) in [7] suggests that if the front of a pulse traveling in a random medium is observed relative to a frame moving with the random propagation speed, it stabilizes to a deterministic wave form, which is the convolution of the initial pulse with a Gaussian whose variance is determined by the statistics of the fluctuating properties of the medium. This is based on the analysis of the evolution equation for the front of the transmitted wave in the undisturbed medium. However, a similar evolution equation for the front of the reflected pulse is not closed, as it involves the interactions between the reflected pulse and the backscattering of the transmitted one. Consequently, the averaging principle used in the papers cited above does not seem to apply.

We use here the approach based on invariant imbedding that was used successfully in [1]. One expects that the interactions between the reflected pulse and the backscattering of the transmitted one do not change the evolution of the reflected wavefront. Consequently, this evolution is like the one for the transmitted pulse in the undisturbed medium that originates at the reflector and is traveling towards the surface. We prove that this is in fact true in the same limit in which the transmitted pulse stabilizes: the size of the fluctuations decreases to 0 while the distance traveled increases to infinity.

In section 2 we formulate the problem for the acoustic pulse in a one-dimensional random medium with a single strong reflector at the end. The variance of the reflected pulse is expressed in terms of the time harmonic reflection coefficient that satisfies a stochastic Riccati equation in the depth variable. The transport equations for the moments of the time harmonic coefficients and their limit are analyzed in section 3. In section 4 we carry out the asymptotic evaluation of the mean and variance of the reflected pulse. Equation (48) giving the mean amplitude of the reflected pulse, along with showing that its variance is zero asymptotically (equation (24)), is the main result of this paper. A comparison between the coordinate systems moving with the mean velocity and with the random velocity is given in section 5. In section 6 we present a numerical experiment illustrating the convergence of the observed reflected pulse to its OD-A limit waveform.

2 Formulation of the problem

We are interested in the reflection of acoustic waves by an one-dimensional random medium. The momentum and mass conservation equations for the velocity $u(z,t)$ and the pressure $p(z,t)$ are

$$\begin{aligned} &\rho u_t + p_z = 0 \\ &\frac{1}{K} p_t + u_z = 0 \end{aligned} \qquad (1)$$

where $\rho = \rho(z)$ is the density, $K = K(z)$ is the bulk modulus. The local sound speed is given by

$$c(z) = \sqrt{\frac{K(z)}{\rho(z)}}. \tag{2}$$

We assume that a slab of thickness L, $z \in [-L, 0]$, contains the random medium, while the medium above and below it is non-random and homogeneous. The constant acoustic parameters in the half space $z > 0$ are denoted by ρ_1, K_1, c_1. In the random regime the density is for simplicity constant $\rho(z) = \rho_1$ but the bulk modulus has the form

$$\frac{1}{K(z)} = \frac{1}{K_1}\left(1 + \epsilon\nu(\frac{z}{\epsilon^2})\right) \tag{3}$$

where $\nu(\cdot)$ is a zero-mean, bounded stationary random process with strong ergodic properties. The parameter ϵ^2 is the ratio of a typical microscopic to a macroscopic length scales and is assumed to be small. The random fluctuations are rapidly varying but their amplitude is small. Note that the mean acoustic parameters are the same for the homogeneous half space $z > 0$ and the random slab $[-L, 0]$. Below the slab the constant parameters are ρ_2, K_2, c_2. Summarizing:

$$\rho(z) = \begin{cases} \rho_1 & z > 0 \\ \rho_1 & -L < z < 0 \\ \rho_2 & z < -L \end{cases} \tag{4}$$

$$K^{-1}(z) = \begin{cases} K_1^{-1} & z > 0 \\ K_1^{-1}\left(1 + \epsilon\nu(\frac{z}{\epsilon^2})\right) & -L < z < 0 \\ K_2^{-1} & z < -L \end{cases} \tag{5}$$

The initial and boundary conditions for equation (1) are provided by specifying the incident pulse in the positive half space to be

$$u(z,t) = -(c_1\rho_1)^{-1/2}\frac{1}{\epsilon}f\left(\frac{t + z/c_1}{\epsilon^2}\right)$$
$$p(z,t) = (c_1\rho_1)^{1/2}\frac{1}{\epsilon}f\left(\frac{t + z/c_1}{\epsilon^2}\right) \tag{6}$$

where f is a smooth function with compact support in $(0, +\infty)$. Note that the incident pulse is scaled so that the total energy released is independent of ϵ.

We analyze the pulse in the frequency domain. Let

$$\hat{f}(\omega) = \int e^{i\omega t}\, f(t)\, dz \tag{7}$$

be the standard Fourier transform. Let

$$\hat{p}(z,\omega) = \int e^{i\omega t/\epsilon^2}\, p(z,t)\, dz$$
$$\hat{u}(z,\omega) = \int e^{i\omega t/\epsilon^2}\, u(z,t)\, dz \tag{8}$$

be the Fourier transforms scaled relative to the width of the incident pulse (6).
Let $\tau(z)$ be the random travel time defined by

$$\tau(z) = \int_0^z \frac{ds}{c(s)} \tag{9}$$

and let $\chi(\tau)$ be its inverse. Note that $\tau < 0$ for $z < 0$. Let

$$\zeta(z) = (\rho(z)K(z))^{1/2} \tag{10}$$

be the acoustic impedance.
We define the up- and down-going wave amplitudes A, B by

$$\begin{aligned} \hat{p} &= \zeta^{1/2}\left(Ae^{i\omega\tau/\epsilon^2} - Be^{-i\omega\tau/\epsilon^2}\right) \\ \hat{u} &= \zeta^{-1/2}\left(Ae^{i\omega\tau/\epsilon^2} + Be^{-i\omega\tau/\epsilon^2}\right). \end{aligned} \tag{11}$$

Substituting (11) into the Fourier transformed equations (1) we find that $A(z,\omega)$ and $B(z,\omega)$ satisfy a system of stochastic differential equations

$$\begin{aligned} \frac{dA}{dz} &= \frac{d}{dz}\left(\ln \zeta^{1/2}(z)\right) e^{-2i\omega\tau/\epsilon^2} B \\ \frac{dB}{dz} &= \frac{d}{dz}\left(\ln \zeta^{1/2}(z)\right) e^{2i\omega\tau/\epsilon^2} A \end{aligned} \tag{12}$$

Our analysis is based on the invariant imbedding representation of the time harmonic reflection coefficient

$$\Gamma(z,\omega) = \frac{A(z,\omega)}{B(z,\omega)}. \tag{13}$$

It satisfies the stochastic Riccati equation

$$\frac{d\Gamma}{dz} = \frac{d}{dz}\left(\ln \zeta^{1/2}(z)\right)\left[e^{-2i\omega\tau(z)/\epsilon^2} - \Gamma^2 e^{2i\omega\tau(z)/\epsilon^2}\right] \tag{14}$$

in $-L < z < 0$. The initial condition for (14) i.e. $\Gamma(-L)$ is found from the continuity of $\hat{p}, \hat{u}$ across the interface $z = -L$. In fact

$$\Gamma|_{z=-L} = e^{-2i\omega\tau(-L)/\epsilon^2}\,\Gamma_I^\epsilon(-L) \tag{15}$$

where

$$\Gamma_I^\epsilon(-L) = \frac{\zeta(-L^+) - \zeta(-L^-)}{\zeta(-L^+) + \zeta(-L^-)} = \frac{\zeta(-L^+) - \zeta_2}{\zeta(-L^+) + \zeta_2}. \tag{16}$$

Note that when $\epsilon \to 0$ the interface reflection coefficient $\Gamma_I^\epsilon(-L)$ converges to a ϵ-independent constant value

$$\Gamma_I = \frac{\zeta_1 - \zeta_2}{\zeta_1 + \zeta_2}. \tag{17}$$

The fact that $\lim_{\epsilon\to 0}\Gamma_I^\epsilon(z)$ is not zero at $z = -L$ distinguishes the interface $z = -L$ from all other interfaces $z = z_0, z_0 \neq -L$ where this limit is zero. This is why we call $z = -L$ a strong reflector.

The quantity of interest is the reflected pressure at $z = 0$. According to (8),(11) the total pressure is given by

$$p(z,t) = \frac{1}{2\pi\epsilon^2}\int \zeta^{1/2}(z)\left(A(z,\omega)e^{i\omega(\tau-t)/\epsilon^2} - B(z,\omega)e^{i\omega(-\tau-t)/\epsilon^2}\right) d\omega. \tag{18}$$

Therefore, the reflected pressure is represented in (18) by the component A i.e.

$$p_{\text{refl}}(0,t) = \frac{1}{2\pi\epsilon^2}\zeta_1^{1/2}\int A(0,\omega)e^{-i\omega t/\epsilon^2}\, d\omega. \tag{19}$$

As we want to base our analysis of p_{refl} on the initial value problem (14),(15) for , , we use (13) and (6),(11) to express A in terms of , and $\hat{f}$. In fact, we have

$$A(0,\omega) = -\epsilon \hat{f}(\omega),\ (0,\omega). \tag{20}$$

Substituting (20) into (19) we obtain

$$p_{\text{refl}}(0,t) = \frac{-1}{2\pi\epsilon}\zeta_1^{1/2} \int e^{-i\omega t/\epsilon^2},\ (0,\omega)\hat{f}(\omega)\, d\omega. \tag{21}$$

The OD-A theory predicts that the pulse observed in the frame moving with the random velocity stabilizes with probability one to a deterministic shape. We prove this claim by calculating asymptotically as $\epsilon \to 0$ the variance of the reflected pressure at the random time when the pulse reaches the surface after being reflected by the strong reflector at $z = -L$. Therefore, we are interested in the coherently reflected field

$$\langle p_{\text{refl}}(0,t)\rangle = \mathrm{E}\{p_{\text{refl}}(0,t)\} \tag{22}$$

and in the intensity function

$$\langle p_{\text{refl}}(0,t)^2\rangle = \mathrm{E}\{p_{\text{refl}}(0,t)^2\} \tag{23}$$

Note that the time $-2\tau(-L)$ is the time it takes the pulse to reach the reflector at $z = -L$ and come back to the surface. Therefore, the OD-A theory is equivalent to

$$\lim_{\epsilon\to 0} \langle p_{\text{refl}}(0,-2\tau(-L))^2\rangle - \langle p_{\text{refl}}(0,-2\tau(-L))\rangle^2 = 0. \tag{24}$$

which says that the fluctuations in the reflected pressure are negligible at the random arrival time. In the analysis of the above expression, a generalization of (23), the two-point intensity function

$$I(t,\bar{t}) = \lim_{\epsilon\to 0} \left\langle p_{\text{refl}}\left(0, t+\frac{\epsilon^2\bar{t}}{2}\right) p_{\text{refl}}\left(0, t-\frac{\epsilon^2\bar{t}}{2}\right)\right\rangle. \tag{25}$$

is very useful. Note that the offset in time in (25) is of the order ϵ^2 which is the correlation range of the random process $\nu(\frac{\cdot}{\epsilon^2})$ of (3).

The asymptotic expression for the two-point intensity function I is found by multiplying two expressions of the form (21) with integration variables ω_1, ω_2 and changing variables in the double integral to

$$\omega_1 = \omega - \frac{\epsilon^2 h}{2} \quad \omega_2 = \omega + \frac{\epsilon^2 h}{2}. \tag{26}$$

We have

$$I(t,\bar{t}) = \frac{1}{2\pi}\zeta_1 \int e^{-i\omega\bar{t}} |\hat{f}(\omega)|^2 \tilde{W}^{11}(0,t,\omega)\, d\omega \tag{27}$$

where

$$\tilde{W}^{NM}(z,t,\omega) = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int e^{iht} \left\langle ,^{N}\left(z, \omega - \frac{\epsilon^2 h}{2}\right) ,^{-M}\left(z, \omega + \frac{\epsilon^2 h}{2}\right)\right\rangle dh. \tag{28}$$

The intensity function I is thus given in terms of (28). However, the formulas (21) and (22) imply that the asymptotic behavior of the coherently reflected pressure is found from the knowledge of $\lim_{\epsilon\to 0}\langle, (0,\omega)\rangle$, which is also a special case $(N = 1, M = 0)$ of $\tilde{W}^{NM}$. In the next section we obtain the equations that govern the behavior of $\tilde{W}^{NM}$.

3 Transport equations

We are interested in the reflected pressure at the surface $z = 0$ for the time when the pulse emerges from the random medium after being reflected by the interface $z = -L$. For this purpose it is convenient to consider the reflected pressure as a function of τ, t i.e. $p(\chi(\tau), t)$. Let $\zeta(\tau)$, $\Gamma(\tau, \omega)$ stand for $\zeta(\chi(\tau))$, $\Gamma(\chi(\tau), \omega)$. Then, the formula (21) expressing $p_{\mathrm{refl}}(0, t)$ in terms of Γ does not change (as $\chi(0) = 0$) but Γ satisfies now

$$\frac{d\Gamma}{d\sigma} = \frac{d}{d\sigma}\left(\ln \zeta^{1/2}(\sigma)\right)\left[e^{-2i\omega\sigma/\epsilon^2} - \Gamma^2 e^{2i\omega\sigma/\epsilon^2}\right] \tag{29}$$

in $-T < z < 0$ where T is the random initial time defined by $-T = \tau(-L)$. The initial condition for (29) is found from (15) to be

$$\Gamma|_{\sigma=-T} = e^{-2i\omega(-T)/\epsilon^2}\,\Gamma_I^{\epsilon}(-L) \tag{30}$$

In the sequel, we denote the random coefficient in the equation (29) by n i.e.

$$n \equiv \frac{d}{d\sigma}\left(\ln \zeta^{1/2}(\sigma)\right) \tag{31}$$

As a consequence of the change of variables in ζ from material coordinate z to travel time τ the dependence of n on the stochastic process $\nu(\cdot)$(3) is no longer through $\nu(\frac{\sigma}{\epsilon^2})$, but through $\nu(\frac{\chi(\sigma)}{\epsilon^2})$ where $\chi(\cdot)$ is the inverse of the travel time.

We now return to the quantities of interest $\tilde{W}^{NM}$. We want to find their behavior as functions of σ. For this purpose, we calculate the equations for the moments of Γ at two ω-s using the Riccati equation (29). Let

$$\Gamma^{NM}(\sigma, \omega, h) = \Gamma^N\left(\sigma, \omega - \frac{\epsilon^2 h}{2}\right)\bar{\Gamma}^M\left(\sigma, \omega + \frac{\epsilon^2 h}{2}\right). \tag{32}$$

The Γ^{NM} satisfy the following infinite-dimensional system of linear equations:

$$\begin{aligned}\frac{d\Gamma^{NM}}{d\sigma} &= n\Big[N e^{-2i\omega\sigma/\epsilon^2 + ih\sigma}\Gamma^{N-1,M} - N e^{2i\omega\sigma/\epsilon^2 - ih\sigma}\Gamma^{N+1,M} \\ &\qquad + M e^{2i\omega\sigma/\epsilon^2 + ih\sigma}\Gamma^{N,M-1} - M e^{-2i\omega\sigma/\epsilon^2 - ih\sigma}\Gamma^{N,M+1}\Big] \\ \Gamma^{NM}|_{\sigma=-T} &= e^{-2i\omega(N-M)(-T)/\epsilon^2} e^{-ih(-T)(N+M)}\,\Gamma_I^N \bar{\Gamma}_I^M.\end{aligned} \tag{33}$$

The initial condition in (33) takes into account that $\lim_{\epsilon\to 0}\Gamma_I^{\epsilon}(-L) = \Gamma_I$.

The expressions for $\tilde{W}^{NM}$ involve the Fourier transforms of Γ^{NM} (cf.(28)). Therefore, we define

$$\tilde{\Gamma}^{NM}(\sigma, t + (N+M)\sigma, \omega) = \frac{1}{2\pi}\int e^{iht}\,\Gamma^{NM}(\sigma, \omega, h)\,dh. \tag{34}$$

The offset in the t variable enables us to get rid of the factors $e^{\pm ih\sigma}$ appearing in the equations (33) for Γ^{NM}. In fact, by differentiating (34) with respect to σ and using (33) we find that $\tilde{\Gamma}^{NM}$ satisfies

$$\begin{aligned}\frac{\partial\tilde{\Gamma}^{NM}}{\partial\sigma} + (N+M)\frac{\partial\tilde{\Gamma}^{NM}}{\partial t} &= n\Big[N e^{-2i\omega\sigma/\epsilon^2}\tilde{\Gamma}^{N-1,M} - N e^{2i\omega\sigma/\epsilon^2}\tilde{\Gamma}^{N+1,M} \\ &\qquad + M e^{2i\omega\sigma/\epsilon^2}\tilde{\Gamma}^{N,M-1} - M e^{-2i\omega\sigma/\epsilon^2}\tilde{\Gamma}^{N,M+1}\Big] \\ \tilde{\Gamma}^{NM}|_{\sigma=-T} &= e^{-2i\omega(N-M)(-T)/\epsilon^2}\,\Gamma_I^N \bar{\Gamma}_I^M\,\delta(t).\end{aligned} \tag{35}$$

Note that at $\sigma = 0$ the shift in the t argument in the definition of $\tilde{,}^{NM}$ is 0 and therefore, the asymptotic behavior of $\mathrm{E}\tilde{,}^{NM}(0,t,\omega)$ as $\epsilon \to 0$ will give $\tilde{W}^{NM}(0,t,\omega)$ of (28).

The asymptotic behavior of systems like (35) is mathematically well understood. In fact, we show in the Appendix that

$$\tilde{W}^{NM}(0,t,\omega) = e^{-2i\omega(N-M)(-T)/\epsilon^2} W^{NM}(0,t,\omega) \tag{36}$$

where W^{NM} satisfies an infinite-dimensional system of equations

$$\begin{aligned} \frac{\partial W^{NM}}{\partial \sigma} &+ (N+M)\frac{\partial W^{NM}}{\partial t} \\ &= 2a_R NM \left[W^{N-1,M-1} - 2W^{NM} + W^{N+1,M+1}\right] \\ &\quad -2\left((N-M)^2 a_R + i(N-M)a_I\right) W^{NM} \\ W^{NM}|_{\sigma=-T} &= ,_I^N ,_I^{-M} \delta(t) \end{aligned} \tag{37}$$

in $N, M \geq 0$, $-T < \sigma < 0$, $t \in R$. The constants a_R, a_I are given by

$$a_R = \frac{c_1}{16}\int_0^\infty r(s)\cos(\frac{2\omega s}{c_1})\,ds \quad a_I = \frac{c_1}{16}\int_0^\infty r(s)\sin(\frac{2\omega s}{c_1})\,ds \tag{38}$$

where

$$r(z) = \mathrm{E}\{\nu'(\cdot + z)\nu'(\cdot)\}. \tag{39}$$

We observe that the diagonal part of W^{NM} decouples and $W^N \equiv W^{NN}$ satisfies

$$\begin{aligned} \frac{\partial W^N}{\partial \sigma} + 2N\frac{\partial W^N}{\partial t} &= 2a_R N^2\left[W^{N-1} - 2W^N + W^{N+1}\right] \\ W^N|_{\sigma=-T} &= ,_I^{2N}\delta(t) \end{aligned} \tag{40}$$

in $N \geq 0$, $-T < \sigma < 0$, $t \in R$.

4 Reflected pressure near the coherent arrival

The reflected pressure is given by (21). Therefore, the mean reflected pressure is

$$\langle p_{\mathrm{refl}}(0,t)\rangle = \frac{-1}{2\pi\epsilon}\zeta_1^{1/2}\int e^{-i\omega t/\epsilon^2}\langle ,\,(0,\omega)\rangle \hat{f}(\omega)\,d\omega. \tag{41}$$

According to (28) and (36)

$$\langle ,\,(0,\omega)\rangle \simeq \int dt\,\tilde{W}^{10}(0,t,\omega) = e^{-2i\omega(-T)/\epsilon^2}\int dt\,W^{10}(0,t,\omega). \tag{42}$$

We take $N=1, M=0$ and integrate (37) over t. The equation reduces to the following equation for $w(\sigma) = \int dt\,W^{10}(\sigma,t,\omega)$:

$$\begin{aligned} \frac{\partial w}{\partial \sigma} &= -2(a_R + i a_I)w \\ w|_{\sigma=-T} &= ,_I. \end{aligned} \tag{43}$$

Combining (41),(42),(43) we have

$$\langle p_{\text{refl}}(0,t)\rangle \simeq \frac{-1}{2\pi\epsilon}\zeta_1^{1/2},{}_I \int e^{-i\omega(t-2T)/\epsilon^2} e^{-2(a_R+ia_I)T} \hat{f}(\omega)\, d\omega. \tag{44}$$

It is natural to express the coefficients a_R, a_I in terms of the Fourier transform of the correlation function of $\nu(\cdot)$:

$$R(z) = \mathrm{E}\{\nu(\cdot+z)\nu(\cdot)\}. \tag{45}$$

In fact, since $r(z) = -R''(z)$,

$$a_R = \omega^2\alpha_R; \quad a_I = \omega^2\alpha_I - \frac{\omega}{8}R(0) \tag{46}$$

where

$$\alpha_R = \frac{1}{4c_1}\int_0^\infty R(s)\cos(\frac{2\omega s}{c_1})\, ds; \quad \alpha_I = \frac{1}{4c_1}\int_0^\infty R(s)\sin(\frac{2\omega s}{c_1})\, ds. \tag{47}$$

Then, formula (44) for the coherent reflected pressure is

$$\langle p_{\text{refl}}(0,t)\rangle \simeq \frac{-1}{2\pi\epsilon}\zeta_1^{1/2},{}_I \int e^{-i\omega(t-2T)/\epsilon^2} e^{i\omega R(0)2T/8-\omega^2(\alpha_R+i\alpha_I)2T} \hat{f}(\omega)\, d\omega. \tag{48}$$

We now interpret formula (48), which is the main result of the OD-A theory. The mean pulse emerges from the random medium convoluted with the complex Gaussian whose Fourier transform is $exp(-\omega^2(\alpha_R+i\alpha_I)2T)$ and retarded by $\epsilon^2 R(0)2T/8$. As $2T$ is the travel time from the surface to the reflector and back to the surface, we conclude from (48) that the pulse travels in the medium with retardation increasing linearly as a function of time. The shape of the pulse is convoluted by a complex Gaussian whose spreading depends also linearly on the travel time. This description is valid only in the immediate vicinity of the front i.e. for $\tau - t$ of the order ϵ^2 and when α_R, α_I are approximately constant over support of $\hat{f}$.

The OD-A theory, however, claims that the description (48) is true not only for the mean reflected pulse $\langle p_{\text{refl}}(0,t)\rangle$ but for the reflected pulse $p_{\text{refl}}(0,t)$ itself as well. This is seen by proving (24). Using (48) we calculate asymptotically $\langle p_{\text{refl}}(0,t)\rangle^2$ as follows. Multiply two expressions (48) with integration variables ω_1, ω_2 and change the variables in the double integral according to

$$\omega_1 = \omega - \frac{\epsilon^2 h}{2} \quad \omega_2 = \omega + \frac{\epsilon^2 h}{2}.$$

We get

$$\langle p_{\text{refl}}(0,t)\rangle^2 \simeq \frac{1}{2\pi}\zeta_1,{}^2_I \int |\hat{f}(\omega)|^2 e^{-4\omega^2\alpha_R T}\, d\omega\, \delta(t-2T). \tag{49}$$

Now, we calculate asymptotically $\langle p^2_{\text{refl}}(0,t)\rangle$. According to (25) this is by definition $I(t,0)$ which, by (28) and (36) is equal to

$$\begin{aligned} I(t,0) &= \frac{1}{2\pi}\zeta_1 \int |\hat{f}(\omega)|^2 \tilde{W}^{11}(0,t,\omega)\, d\omega \\ &= \frac{1}{2\pi}\zeta_1 \int |\hat{f}(\omega)|^2 W^1(0,t,\omega)\, d\omega \end{aligned} \tag{50}$$

where W^N is the solution of the equation (40).

We need a closed formula for the solution of equation (40) if we want to apply (50) successfully in our asymptotic analysis. However, as it was already noted in [1], the W-equation (40) allows for the probabilistic representation of its solution. This is because the operator in the right hand side of W-equation (40)

$$(\Delta W)^N = 2a_R N^2 \left[W^{N-1} - 2W^N + W^{N+1}\right] \tag{51}$$

is a infinitesimal generator of a Markov chain $N(\sigma)$ with the state space consisting of all non-negative integers $N \geq 0$. This chain is uniquely determined by the infinitesimal generator Δ and it is defined for all times σ. (see section (3.5) of [1]). In terms of this canonical chain, the solution of (40) is given by the Faynman-Kac formula:

$$W^N(\sigma, t) = \overline{\mathrm{E}}_N\{, {}^{2N(\sigma)}_{I}\delta(t - 2\int_{-T}^{\sigma} N(s)\, ds)\} \tag{52}$$

where $\overline{\mathrm{E}}_N$ is the expectation over all trajectories of the Markov chain that start from N at the moment $\sigma = -T$.

We are interested in

$$W^1(0, t) = \overline{\mathrm{E}}_1\{, {}^{2N(0)}_{I}\delta(t - 2\int_{-T}^{0} N(s)\, ds)\} \tag{53}$$

for times t s.t. $t - 2T \to 0$. There are two kinds of paths of the random chain $N(\sigma)$ that contribute to (53): those that start form $N = 1$ and switch to 0 and those that start from $N = 1$ and do not switch to 0. If we look for $t - 2T$ small then only the path $N(\cdot) \equiv 1$ contributes in the second case. For if the path switches to $N > 1$ it must return to $N = 1$ quickly or else the delta function will be 0, but the probability of such paths tends to 0 as $t - 2T \to 0$. Therefore

$$\begin{aligned}\overline{\mathrm{E}}_1\{, {}^{2N(0)}_{I}\delta(t - 2\int_{-T}^{0} N(s)\, ds)\} &\simeq \overline{\mathrm{P}}_1\{N(\sigma) \equiv 1; -T \leq \sigma \leq 0\}, {}^{2}_{I}\delta(t - 2T) \\ &+ \overline{\mathrm{E}}_1\{, {}^{2N(0)}_{I}\delta(t - 2\int_{-T}^{0} N(s)\, ds), z_T^\star \leq 0\}\end{aligned} \tag{54}$$

where $z_T^\star = \inf\{\sigma \geq -T : N(\sigma) = 0\}$. It is shown in [1] that the conditional law of the second line in (54) has continuous density. The probability that $N(\sigma) \equiv 1$ for $-T \leq \sigma \leq 0$ given $N(-T) = 1$ is found from (51) to be

$$e^{-2a_R 2T} = e^{-4\omega^2 \alpha_R T} \tag{55}$$

Combining (48),(54),(55) we find that the intensity function for $t - 2T$ small is given by

$$I(t, 0) = \frac{1}{2\pi}\zeta_1 \int |\hat{f}(\omega)|^2, {}^{2}_{I}e^{-4\omega^2\alpha_R T}\, d\omega\delta(t - 2T). \tag{56}$$

Note now that (56) and (49) are the same. It proves the asymptotic equivalence of $\langle p_{\text{refl}}(0, t)\rangle^2$ and $\langle p_{\text{refl}}^2(0, t)\rangle$ for $t = -2\tau(-L)$ and therefore it concludes the proof of the OD-A theory.

5 Pulse in the mean velocity frame

The stabilization of the pulse predicted by the OD-A theory is specific to the frame moving with the random velocity. In the frame moving with the mean velocity, however, both the time of the arrival of the pulse at the surface and the shape of the pulse fluctuate randomly. In this section we analyze the mean shape of the pulse and its variance for times near the mean arrival time. We find that the fluctuations in the shape of the pulse *do not* die out asymptotically as it was the case for the pulse observed in the random velocity frame. We show however that these fluctuations are solely produced by the fluctuations of the two-way travel time, reconfirming therefore the OD-A theory.

The analysis of the pulse in the mean velocity frame is very similar to that of pulse in the random velocity frame, so we will only summarize it here. The scaled Fourier transforms $\hat{p}(z,\omega), \hat{u}(z,\omega)$ are defined again by (8) but the up- and down-going wave amplitudes A and B are defined in relation to the mean travel time

$$\tau = \frac{z}{c_1} \tag{57}$$

and the mean impedance

$$\zeta_1 = (\rho_1 K_1)^{1/2} \tag{58}$$

by

$$\begin{aligned} \hat{p} &= \zeta_1^{1/2}\left(Ae^{i\omega\tau/\epsilon^2} - Be^{-i\omega\tau/\epsilon^2}\right) \\ \hat{u} &= \zeta_1^{-1/2}\left(Ae^{i\omega\tau/\epsilon^2} + Be^{-i\omega\tau/\epsilon^2}\right). \end{aligned} \tag{59}$$

The time harmonic reflection coefficient $\Gamma = A/B$ satisfies the stochastic Riccati equation

$$\frac{d\Gamma}{dz} = -\frac{i\omega n}{\epsilon}\left[e^{-2i\omega\tau(z)/\epsilon^2} - 2\Gamma + \Gamma^2 e^{2i\omega\tau(z)/\epsilon^2}\right] \tag{60}$$

in $-L < z < 0$ where

$$n(\frac{z}{\epsilon^2}) \equiv \frac{1}{2c_1}\nu(\frac{z}{\epsilon^2}). \tag{61}$$

The initial condition for (60) is found from the continuity of $\hat{p}, \hat{u}$ across the interface $z = -L$:

$$\Gamma|_{z=-L} = e^{-2i\omega\tau(-L)/\epsilon^2}\Gamma_I \tag{62}$$

where Γ_I is defined by (17). The reflected pressure at $z = 0$ which is the quantity of interest has the integral representation analogous to (21):

$$p^{\star}_{\text{refl}}(0,t) = \frac{-1}{2\pi\epsilon}\zeta_1^{1/2}\int e^{-i\omega t/\epsilon^2}\Gamma(0,\omega)\hat{f}(\omega)\,d\omega. \tag{63}$$

The superscript $\star$ will serve to distinguish the formulas of this section from the ones of sections 2-4. Note however that the reflected pressure defined by formulas (63) and (21) is the same quantity.

The two-point intensity function I is defined again by (25). While its relation (27) with the quantity $\tilde{W}^{11}$ of (28) holds in the same form, the asymptotic behavior of $\tilde{W}^{NM}$ is found now from the Riccati equation (60). In fact, we find that the analog of (34) i.e.

$$\tilde{\Gamma}^{NM}(z, t + (N+M)\tau(z), \omega) = \frac{1}{2\pi}\int e^{iht}\Gamma^N(z, \omega - \frac{\epsilon^2 h}{2}, h)\bar{\Gamma}^M(z, \omega + \frac{\epsilon^2 h}{2}, h)\,dh \tag{64}$$

satisfies the equation

$$\begin{aligned} &\frac{\partial\tilde{\Gamma}^{NM}}{\partial z} + \frac{(N+M)}{c_1}\frac{\partial\tilde{\Gamma}^{NM}}{\partial t} = \\ &\qquad \frac{i\omega n}{\epsilon}\Big[-Ne^{-2i\omega\tau/\epsilon^2}\tilde{\Gamma}^{N-1,M} - Ne^{2i\omega\tau/\epsilon^2}\tilde{\Gamma}^{N+1,M} \\ &\qquad\quad + Me^{2i\omega\tau/\epsilon^2}\tilde{\Gamma}^{N,M-1} + Me^{-2i\omega\tau/\epsilon^2}\tilde{\Gamma}^{N,M+1} \\ &\qquad\quad + 2(N-M)\tilde{\Gamma}^{NM}\Big] \\ &\tilde{\Gamma}^{NM}|_{z=-L} = e^{-2i\omega(N-M)\tau(-L)/\epsilon^2}\,\Gamma_I^N\bar{\Gamma}_I^M\delta(t). \end{aligned} \tag{65}$$

The asymptotic behavior of $\mathrm{E}\tilde{,}^{NM}(0,t,\omega)$ as $\epsilon \to 0$ will give the quantity of interest $\tilde{W}^{NM}(0,t,\omega)$.

The calculation similar to the one in the Appendix yields that

$$\tilde{W}^{NM}(0,t,\omega) = e^{-2i\omega(N-M)\tau(-L)/\epsilon^2} W^{NM}(0,t,\omega) \tag{66}$$

where W^{NM} satisfies an infinite-dimensional system of equations

$$\begin{aligned} \frac{\partial W^{NM}}{\partial z} &+ \frac{(N+M)}{c_1}\frac{\partial W^{NM}}{\partial t} \\ &= \frac{2\omega^2\alpha_R}{c_1} NM \left[W^{N-1,M-1} - 2W^{NM} + W^{N+1,M+1}\right] \\ &- \frac{2\omega^2}{c_1}\Big((N-M)^2(\alpha_R + 2\alpha_0) + i(N-M)\alpha_I\Big) W^{NM} \\ W^{NM}|_{z=-L} &= ,_I^N ,_I^M \delta(t) \end{aligned} \tag{67}$$

in $N, M \geq 0$, $-L < z < 0$, $t \in R$. The constants α_R, α_I are given by (47) and

$$\alpha_0 = \frac{1}{4c_1}\int_0^\infty R(s)\, ds \tag{68}$$

where $R(\cdot)$ is the correlation function (45).

The mean reflected pressure is found in the same way as (48) in section 4. We have

$$\langle p^\star_{\text{refl}}(0,t)\rangle \simeq \frac{-1}{2\pi\epsilon}\zeta_1^{1/2},_I \int e^{-i\omega(t-\frac{2L}{c_1})/\epsilon^2} e^{-\omega^2(\alpha_R + i\alpha_I + 2\alpha_0)\frac{2L}{c_1}} \hat{f}(\omega)\, d\omega. \tag{69}$$

Let us compare the formulas (48) and (69) for the mean reflected pressure. Note that the two-way travel time $2T$ of (48) is random and consequently it is the mean of (48) with respect to the distribution of T that must be equal to (69). The distribution of T is easy to find. In fact, it is

$$T \sim \frac{L}{c_1} + \epsilon^2\mathcal{N}(-\frac{L}{c_1}R(0)/8, \frac{2L}{c_1}\alpha_0) \tag{70}$$

where $\mathcal{N}(m,\sigma^2)$ is a Gaussian random variable with mean m and variance σ^2. Let us denote the expectation with respect to the distribution of T by E_T. Then a standard formula $\mathrm{E}e^{i\omega\mathcal{N}} = e^{i\omega m - \omega^2\sigma^2/2}$ yields

$$\mathrm{E}_T\langle p_{\text{refl}}(0,t)\rangle \simeq \frac{-1}{2\pi\epsilon}\zeta_1^{1/2},_I \int e^{-i\omega(t-\frac{2L}{c_1})/\epsilon^2} e^{-\omega^2(\alpha_R + i\alpha_I + 2\alpha_0)\frac{2L}{c_1}} \hat{f}(\omega)\, d\omega \tag{71}$$

which is exactly (69). Thus, both the random velocity frame approach of sections 2-4 and the mean velocity frame approach of this section yield the same formula for the mean pulse. The above calculation shows also that $2\alpha_0\frac{2L}{c_1}$ part of the variance of the Gaussian convoluted with the initial pulse in (69) comes form the variance of the random travel time T.

finally, we find the variance of the reflected pulse in the mean velocity frame. ¿From (69) we obtain, similarly to (49), that

$$\langle p^\star_{\text{refl}}(0,t)\rangle^2 \simeq \frac{1}{2\pi}\zeta_1,_I^2 \int |\hat{f}(\omega)|^2 e^{-\omega^2 2[\alpha_R + 2\alpha_0]\frac{2L}{c_1}}\, d\omega\, \delta(t - \frac{2L}{c_1}). \tag{72}$$

Using the W-equation (67) for $N = M$, as in (56), we find that

$$\langle p^\star_{\text{refl}}(0,t)^2\rangle \simeq \frac{1}{2\pi}\zeta_1,_I^2 \int |\hat{f}(\omega)|^2 e^{-\omega^2 2[\alpha_R]\frac{2L}{c_1}}\, d\omega\, \delta(t - \frac{2L}{c_1}). \tag{73}$$

Therefore, because the exponentials in (72) and (73) are different, the factors standing by the delta functions are different. As a result, unlike the OD-A theory situation, and in agreement with our expectations, the fluctuations in the shape of the pulse observed in the mean velocity frame *do not* die out. Note however, that the discrepancy between (72) and (73) is in the term $e^{-\omega^2 2[2\alpha_0]\frac{2L}{c_1}}$ which we found above to be solely related to the variance of the random travel time T. This fact confirms again that the shape of the pulse stabilizes when observed in the proper i.e. random velocity frame.

6 Numerical experiments

We conducted a series of numerical simulations to illustrate the accuracy of the approximation given by the OD-A limit shape formula (48). We considered a Goupillaud medium that consists of a stack of layers with the same travel time across each one and with impedance ζ constant within each one. The number of layers in the slab, which is now parameterized by the travel time $\tau \in [-T, 0]$, is assumed to be $N = \epsilon^{-2}$. We denote the constant value of the impedance within the k-th layer by ζ_k. In the Goupillaud medium equations (1) become a difference equation for the amplitudes of down- and up-going waves at mesh points midway between interfaces. A detailed description of the difference equation is found in [5]. The coefficients of this equation are expressed in terms of a sequence of characteristic impedances $\{\zeta_k\}_{k=0}^{\infty}$ defining a particular realization of the Goupillaud medium. We used for $\{\zeta_k\}_{k=0}^{\infty}$ a single realization of a certain Markov chain described below.

In the experiment we observe the shape of the pulse not only when it strikes the surface but in its whole passage through the random medium. The passage has two phases: in the first the pulse is traveling in the direction of the reflector and in the second it is returning back to surface. The first phase i.e. the behavior of the transmitted pulse was investigated fully in [5]. In the figures below, we include the pictures of the transmitted pulse and its OD-A limit shape to illustrate that the rates of convergence in both phases of the passage through the medium are similar.

In the transmission phase the wavefront travels along the curve $\{(\chi(-t), t) : t \in [0, T]\}$ where T is the time at which the pulse reaches the reflector i.e. $T = -\tau(-L)$. In the reflection phase the wavefront travels along the curve $\{(\chi(-2T + t), t) : t \in [T, 2T]\}$. In the pictures we rescale the time axis so that $T = 1$.

For travel time t up to 1.0 (the transmission) we plot the values of

$$\{p(\chi(-t), t + \epsilon^2 j) : j = 0, \dots, w\}$$

as a histogram curve, and the corresponding values of the OD-A limit shape as a continuous curve. Similarly, for travel time t after to 1.0 (the reflection) we plot the values of

$$\{p_{\text{refl}}(\chi(-2T + t), t + \epsilon^2 j) : j = 0, \dots, w\}$$

as a histogram curve, and the corresponding values of the OD-A limit shape as a continuous curve. The number w describes how far away after the first arrival time we observe the pulse. It is of order $O(1)$ compared to the number of layers N of order $O(\epsilon^{-2})$.

The Markov chain generating the sequence of impedances $\{\zeta_k\}_{k=0}^{\infty}$ is defined as follows. Let $\alpha(x)dx$ be a fixed probability distribution. Let p, q be fixed positive numbers s.t. $p + q = 1$. The initial value ζ_0 is drawn form the distribution $\alpha(x)dx$. Assume that ζ_n is defined. Then, ζ_{n+1} is equal to ζ_n with probability p (no real interface between the layers) and it is drawn from the distribution $\alpha(x)dx$ with probability q. This type of random Goupillaud medium was first studied in [8]. In our experiment we took the $\alpha(x)\,dx$ distribution to be Gaussian with mean 0 and variance 4.0. The parameter p, which is the probability of change in the characteristic impedance on the interface, is 1 for Figures 1 and 2 and it is 0.5 for Figures 3 and 4. The width w is usually 50. The number of layers, N, is given at each plot separately. Each plot consist of 5 pairs of functions. Each pair is indexed by the time of arrival (marked on the vertical axis): 0.0, 0.3, ..., 2.0. The pairs are positioned in the picture so to make them more readable: it is only the position of one function in a pair with respect to the other function in the same pair which matters. We shifted the pairs to the right, but in fact all pairs have the first non-zero value at the first observed position.

The number in parentheses below time of the arrival is the relative error between functions in each pair i.e. the L_2 norm of the difference divided by the L_2 norm of the OD-A limit waveform. As we see, the errors for the reflected pulse are bigger than the errors for the transmitted pulse. This is due to the interactions between the reflected pulse and the backscatters produced by the transmitted pulse in its passage to the reflector. Note however that while N increases, the error decreases and eventually the errors for times up to 1.0 are not much smaller than the ones for times after 1.0.

While we present here only plots for $p = 1, 0.5$ and $\alpha(x)\,dx$ Gaussian, we conducted the experiment for many different values of p and different densities $\alpha(x)\,dx$. All the results share the same properties as the ones presented above.

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A Appendix

We analyze here the asymptotic behavior of the solution of the $\tilde{,}$-equation (35) when $\epsilon \to 0$. The random coefficient n in (35) allows for the ϵ-expansion of the form

$$n = \frac{1}{\epsilon}\mu_0(\frac{\chi(\sigma)}{\epsilon^2}) + \mu_1(\frac{\chi(\sigma)}{\epsilon^2}) + \epsilon e_\mu(\nu(\frac{\chi(\sigma)}{\epsilon^2}), \epsilon) \tag{74}$$

where

$$\mu_0(z) = -\frac{c_1}{4}\frac{d\nu(z)}{dz} \quad \mu_1(z) = \frac{3c_1}{8}\nu(z)\frac{d\nu(z)}{dz} \tag{75}$$

and $e_\mu(y, \epsilon)$ is the 2-nd order error term from the Taylor expansion of the function $f(y) = (1 + \epsilon y)^{-3/2}$.

Let $q(z)$ be an R^3-valued random process defined by

$$q(\sigma) = (\mu_0(\sigma), \mu_1(\sigma), \nu(\sigma)), \qquad \sigma < 0. \tag{76}$$

The randomness enters the equation (35) only through process q. We assume that it is a stationary ergodic Markov process with infinitesimal generator Q and an invariant measure $P^\star(dq)$ defined on R^3 that satisfies

$$\int (Q\phi)(q)\, P^\star(dq) = 0 \tag{77}$$

for any test function ϕ. We define the expectation with respect to $P^\star$ by

$$\mathrm{E}^\star \cdot = \int \cdot P^\star(dq). \tag{78}$$

Now, using the expansion (74) of n, we can rewrite equation (35) in the concise form, exhibiting the influence of ϵ openly. Let X be the space of double index sequences taking values in $S(R)$, the space of rapidly decreasing functions, i.e.

$$X = \{, \; = [, \; ^{NM}]_{N,M=0}^{\infty}, \; , \; ^{N,M} = , \; ^{N,M}(t) \in S(R)\}.$$

Let $A(l), l \in R$; D be linear operators defined on X by

$$\begin{aligned}(A(l), \;)^{N,M} &= \Big[Ne^{-2i\omega l}, \; ^{N-1,M} - Ne^{2i\omega l}, \; ^{N+1,M} \\ &\quad + Me^{2i\omega l}, \; ^{N,M-1} - Me^{-2i\omega l}, \; ^{N,M+1}\Big] \\ (D, \;)^{N,M} &= (N+M)\frac{\partial}{\partial t}, \; ^{N,M}. \end{aligned} \tag{79}$$

Note that the dependence of A on l is periodic, with the period π/ω. Let $F_i, i = 0,1,2$ be X-valued functions defined on on X by

$$\begin{aligned} F_0(l,q,, \;) &= \mu_0 A(l), \\ F_1(l,q,, \;) &= \mu_1 A(l), \; + D, \\ F_2^{\epsilon}(l,q,, \;) &= e_{\mu}(\nu,\epsilon)A(l), \end{aligned} \tag{80}$$

where $l \in R,\; q = (\mu_0, \mu_1, \nu) \in R^3$. We suppress the superindex ϵ in F_2 in what follows.
Then the equation (35) is given by

$$\frac{d}{d\sigma}, \; ^{\epsilon} = \frac{1}{\epsilon}\sum_{i=0}^{2} \epsilon^i F_i\left(\frac{\sigma}{\epsilon^2}, q(\frac{\chi(\sigma)}{\epsilon^2}),, \; ^{\epsilon}\right) \tag{81}$$

where we explicate mark the ϵ-dependence of the solution by the superindex.

The first coordinate μ_0 of the stationary process q has mean 0 with respect to the invariant measure $P^{\star}$, because it is a derivative of the stationary process $\nu(\cdot)$. As a result, for any $l \in R, \sigma < 0,, \; \in X$

$$\mathrm{E}^{\star} F_0(l, q(\sigma),, \;) = 0. \tag{82}$$

We study the behavior of equation (81) by investigating the augmented Markov process $(q^{\epsilon}(\cdot), l^{\epsilon}(\cdot),, \; ^{\epsilon}(\cdot))$ where

$$q^{\epsilon}(\sigma) = q(\frac{\chi(\sigma)}{\epsilon^2}) \quad l^{\epsilon}(\sigma) = \frac{\sigma}{\epsilon^2} \bmod \pi/\omega \tag{83}$$

for $\sigma < 0$. The state space of the augmented process is $R^3 \times S_{\omega} \times X$ where S_{ω} is the circle identified with the interval $[0, \pi/\omega]$.

To find the infinitesimal generator of the augmented process $(q^{\epsilon}, l^{\epsilon},, \; ^{\epsilon})$ we have to analyze the equation satisfied by the inverse of the travel time $\chi(\tau)$. In fact, (cf (9))

$$\frac{d}{d\sigma}\chi(\sigma) = c(\chi(\sigma)) = c_1(1 - \epsilon\frac{\nu}{2} + \epsilon^2\frac{3}{8}\nu^2 + \epsilon^3 e_{\chi}(\nu,\epsilon))|_{\nu=\nu(\chi(\sigma)/\epsilon^2)} \tag{84}$$

where $e_{\chi}(y,\epsilon)$ is the 3-rd order error term from the Taylor expansion of the function $f(y) = (1+\epsilon y)^{-1/2}$.

The generator of the augmented process $(q^{\epsilon}, l^{\epsilon},, \; ^{\epsilon})$ is given by

$$\mathcal{L}^{\epsilon} = \frac{1}{\epsilon^2}c_1\left(1 - \epsilon\frac{\nu}{2} + \epsilon^2\frac{3}{8}\nu^2 + \epsilon^3 e_{\chi}(\nu,\epsilon)\right)Q + \frac{1}{\epsilon}\sum_{i=0}^{2}\epsilon^i F_i \cdot \nabla_{\Gamma} + \frac{1}{\epsilon^2}\partial_l. \tag{85}$$

and the backward Kolmogorov equation for the augmented process is

$$(\partial_\sigma + \mathcal{L}^\epsilon)V^\epsilon = 0, \quad \sigma < 0. \tag{86}$$

As we are interested in the slowly changing ,i.e. l, q-independent, part of the solution V^ϵ, we impose the final condition for (86) in the form

$$Y^\epsilon(\sigma, q, l, ,)|_{\sigma=0} = v(,).$$

We solve the equation (86) asymptotically, as $\epsilon \to 0$ by the multiply scale expansion:

$$V^\epsilon = \sum_{i=0}^{\infty} \epsilon^i V^i(\sigma, q, l, ,). \tag{87}$$

We are interested in the solution V^ϵ with the main term V^0 that is slowly changing i.e. we assume that

$$V^0 = V^0(\sigma, ,). \tag{88}$$

Substituting (87) into (86) yields a hierarchy of equations for V^i. We write the first three:

$$(c_1 Q + \partial_l)V^0 = 0 \tag{89}$$

$$(c_1 Q + \partial_l)V^1 + (F_0 \cdot \nabla_\Gamma - c_1 \frac{\nu}{2} Q)V^0 = 0 \tag{90}$$

$$(c_1 Q + \partial_l)V^2 + (F_0 \cdot \nabla_\Gamma - c_1 \frac{\nu}{2} Q)V^1 + (F_1 \cdot \nabla_\Gamma + c_1 \frac{3}{8}\nu^2 Q + \partial_\sigma)V^0 = 0 \tag{91}$$

Note that the operator $\tilde{Q} = (c_1 Q + \partial_l)$ is an infinitesimal generator of a process $(q(c_1\sigma), \sigma), \sigma < 0$ with the state space $R^3 \times S_\omega$. The invariant measure of this process is $P^\star \otimes \bar{dl}$ where $\bar{dl}$ is the Lebesgue measure on S_ω normalized to mass 1. By ergodicity of the process $q(\cdot)$, the kernel of Q consists of functions that do not depend on q. This fact and the l-independence of V^0 implies that the equation (89) is clearly satisfied. Moreover, we see that the terms $-c_1 \frac{\nu}{2} Q V^0$ and $c_1 \frac{3}{8}\nu^2 Q V^0$ in equations (90) and (91), respectively, vanish.

By the Fredholm alternative, which we assume to hold for the process $(q(c_1\sigma), \sigma)$, the operator $\tilde{Q}$ has an inverse on the space of functions with mean 0 with respect to the measure $P^\star \otimes \bar{dl}$. The particular inverse with the range consisting of functions with vanishing mean is given by

$$-\tilde{Q}^{-1} = \int_0^\infty e^{\sigma\tilde{Q}}\, d\sigma \tag{92}$$

where $e^{\sigma\tilde{Q}} = e^{c_1\sigma Q} \circ Tr(\sigma)$ and $e^{c_1\sigma Q}$ is the semigroup of the process $q(c_1\sigma)$ acting on a trial function $\phi(q)$ by

$$(e^{c_1\sigma Q}\phi)(q) = \mathrm{E}\{\phi(q(c_1\sigma))|q(0) = q\} \tag{93}$$

while $Tr(\sigma)$ is the translation semigroup acting on the trial function $\psi(l)$ by

$$(Tr(\sigma)\psi)(l) = \psi(l + \sigma). \tag{94}$$

Now, we can solve (90) for V^1:

$$V^1 = -\tilde{Q}^{-1} F_0 \cdot \nabla_\Gamma V^0. \tag{95}$$

We use this formula to obtain the limit equation for V^0 as follows. We take the expectation of (91) with respect to the measure $P^\star \otimes \bar{dl}$. The first term of (91) vanishes, due to (77) and the

l-periodicity of V^2. However, using (95) we see that the contribution from Q in the second term of (91) involving V^1 also disappears:

$$
\begin{aligned}
&\int \oint P^\star(dq) \otimes \bar{d}l\, \nu c_1 Q(-\tilde{Q}^{-1} F_0) \\
&= \int \oint P^\star(dq) \otimes \bar{d}l\, \nu c_1 Q \int_0^\infty d\sigma\, e^{\sigma c_1 Q} Tr(\sigma) F_0 \\
&= -\int P^\star(dq)\, \nu \int_0^\infty d\sigma\, e^{\sigma Q} (Tr(\sigma) F_0)|_{l=0}^{l=\pi/\omega} + \oint \bar{d}l \int P^\star(dq)\, \nu F_0 \\
&= 0.
\end{aligned} \tag{96}
$$

The passage from the 2-nd line to the 3-rd line in (96) is due to the integration by parts formula. The first integral is 0 because F_0 is a periodic function in l (cf (80)). The second integral is 0, too , because $\nu\mu_0 = -\frac{c_1}{8}\frac{d(\nu(z)^2)}{dz}$ is a derivative of a stationary quantity, and such expressions have always mean 0.

Therefore, the limit equation for V^0 is

$$
\partial_\sigma V^0 + \mathcal{L} V^0 = 0 \tag{97}
$$

where

$$
\begin{aligned}
\mathcal{L} = \int \oint P^\star(dq) \otimes \bar{d}l \Big\{ & F_0(l,q,,) \cdot \nabla_\Gamma [\int_0^\infty F_0(l+\sigma, q(c_1\sigma),,) \cdot \nabla_\Gamma\, d\sigma] \\
& + F_1(l,q,,) \cdot \nabla_\Gamma \}
\end{aligned} \tag{98}
$$

The equation (97) is the backward Kolmogorov equation. The limit of the mean of $,^\epsilon$ i.e. W, which is the main object of our interest in section 3, satisfies the equation (cf [6])

$$
\partial_\sigma W = \overline{FF}_0 W \tag{99}
$$

where

$$
\overline{FF} = \int \oint P^\star(dq) \otimes \bar{d}l \Big\{ \int_0^\infty G_0(l+\sigma, q(c_1\sigma)) G_0(l,q)\, d\sigma + G_1(l,q) \Big\} \tag{100}
$$

and G_i is the linear operator on X defined by $F_i(l,q,,) = G_i,$.

The application of (100) for the specific F_i defined by (80) yields the W-equation (37) of section 3.

We remark finally that the above calculation was based solely on the mean zero properties of $F_0(q,l)$ and $\nu F_0(q,l)$ with respect to $P^\star(dq)$ and the l-periodicity of $F_0(q,l)$.

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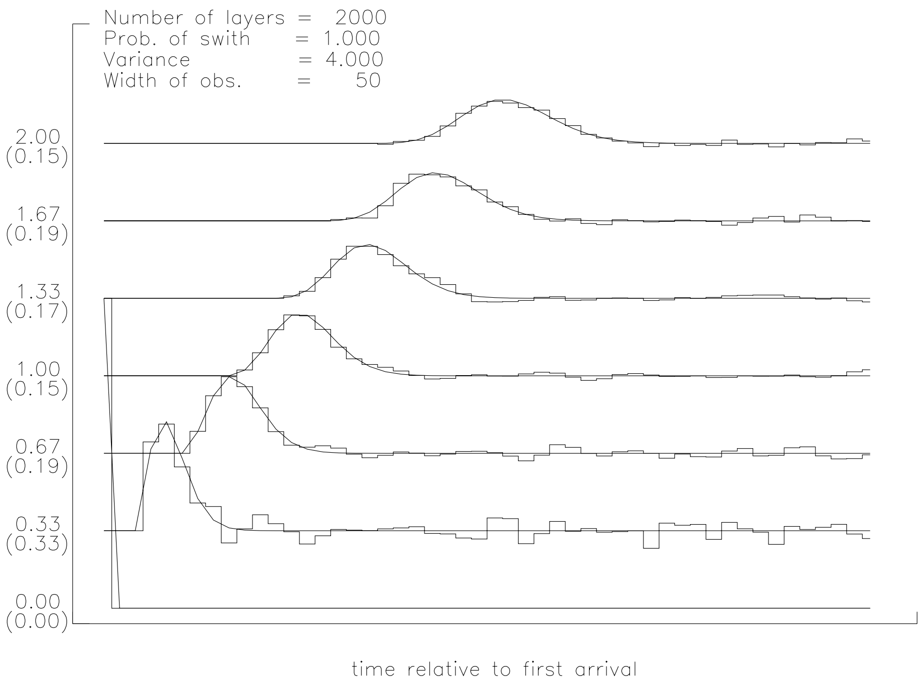
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Variance = 4.000
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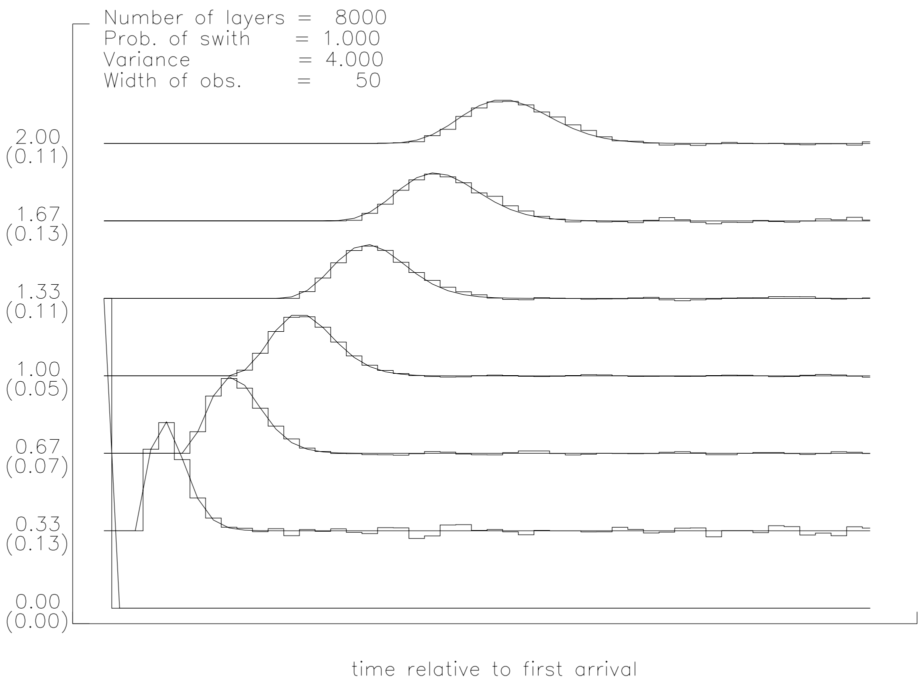
(0.15)
1.67
(0.19)
1.33
(0.17)
1.00
(0.15)
0.67
(0.19)
0.33
(0.33)
0.00
(0.00)
time relative to first arrival

Figure 1: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Variance = 4.000
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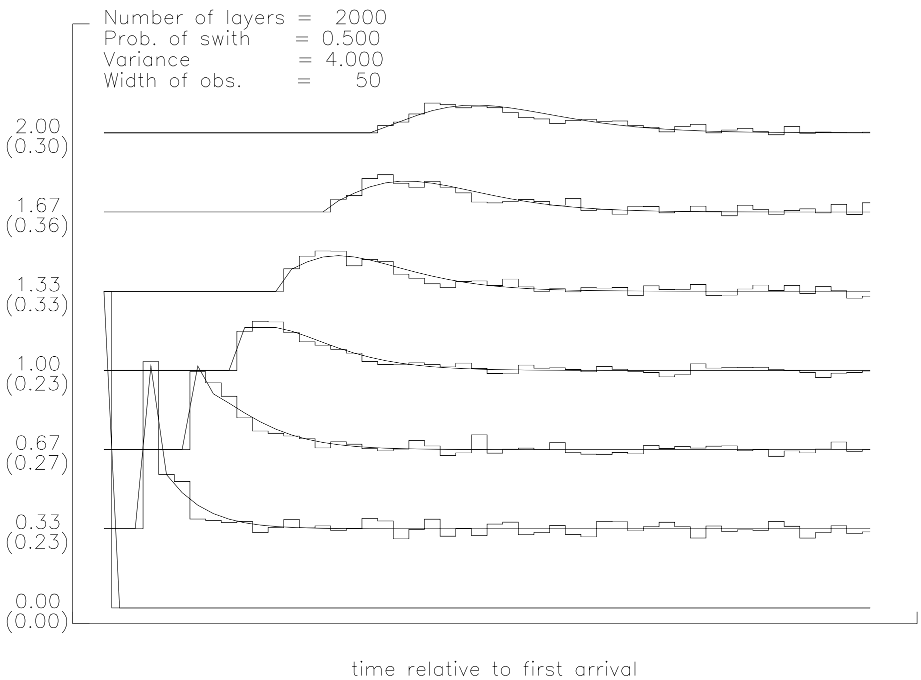
(0.11)
1.67
(0.13)
1.33
(0.11)
1.00
(0.05)
0.67
(0.07)
0.33
(0.13)
0.00
(0.00)
time relative to first arrival

Figure 2:



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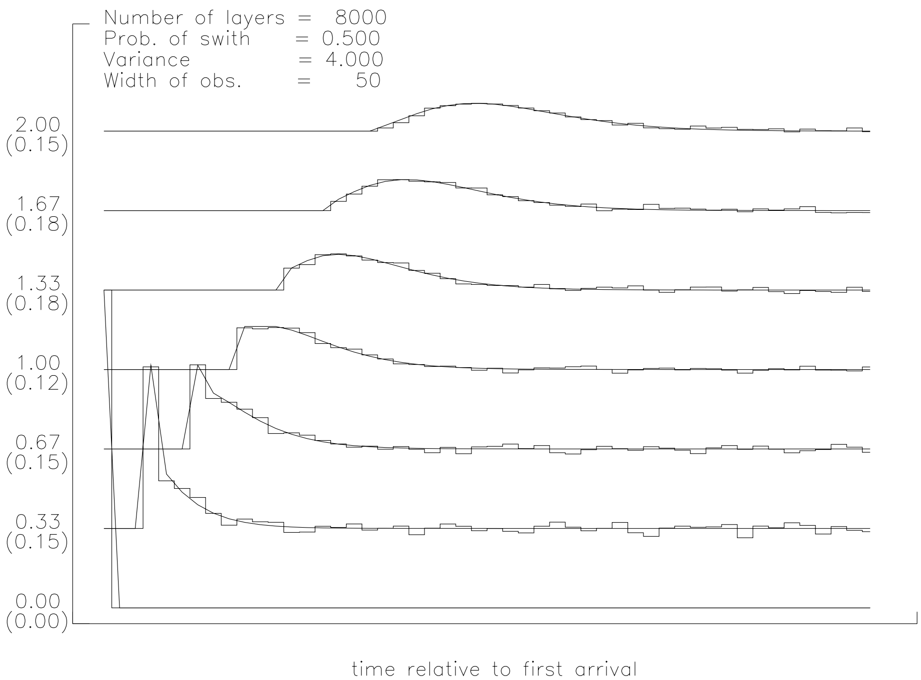
(0.30)
1.67
(0.36)
1.33
(0.33)
1.00
(0.23)
0.67
(0.27)
0.33
(0.23)
0.00
(0.00)
time relative to first arrival

Figure 3:



Number of layers = 8000
Prob. of swith = 0.500
Variance = 4.000
Width of obs. = 50
2.00
(0.15)
1.67
(0.18)
1.33
(0.18)
1.00
(0.12)
0.67
(0.15)
0.33
(0.15)
0.00
(0.00)
time relative to first arrival

Figure 4: