A functional limit theorem for waves reflected by a random medium

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Abstract

We introduce a class of distribution-valued stochastic processes that arise in the study of pulse reflection from random media and we analyze their asymptotic properties when they are scaled in a natural way.

Keywords

Stochastic equations, limit theorems, random media, waves

AMS Classification

60H15, 35R60, 70B35

1 Introduction

Wave propagation in random media leads to many interesting and difficult problems in stochastic processes and differential equations. Several such problems arose in the study of pulse reflection from randomly layered media [4,2,15,10]. In this paper we give a more detailed mathematical analysis of the basic limit theorem used in [4] in the framework of reflected signal functionals introduced there. A comprehensive review of our work is given in [1].

In section 2 we formulate the acoustic pulse reflection problem for normally incident plane waves on a randomly layered half space and show how the study of the quantities of physical interest leads to the asymptotic analysis of a class of distribution-valued stochastic process, the reflection functionals of [4]. In section 3 these processes are considered in more detail and their asymptotic limit is analyzed in section 4 using a martingale formulation. The main technical issue in our analysis is the unique characterization and representation of the limit process. This is done partly by a duality argument that is useful in many other contexts in pulse reflection. The duality is defined and analyzed in section 6. The uniqueness of the limit process is given in section 5. The representation of the functional process is given in section 7.

The next step is to get limit theorems for a more general class of reflected signal functionals, the unsmoothed or localized functionals [4], which lead to more complex limit processes. They play an important role in the statistical inverse problems associated with pulse reflection [15].

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2 Formulation

We are interested in acoustic waves reflected by an one dimensional random medium. The physical problem and its scaling are discussed at length elsewhere [4]. We begin here with the mathematical problem in scaled and dimensionless form. The acoustic equations for the velocity $u(t,x)$ and pressure $p(t,x)$ with $x \in \mathbf{R}^1$, are

$$\rho(x)\frac{\partial u(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x} = 0$$

$$\frac{1}{\rho(x)c^2(x)}\frac{\partial p(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} = 0 \tag{2.1}$$

Here $\rho(x)$ and $c(x)$ are the density and local sound speed of the medium, respectively. We assume that the random medium occupies the negative half space $x < 0$ and that the medium is nonrandom and homogeneous in $x > 0$. Waves are incident on the random half space $x < 0$ from the homogeneous half space $x > 0$. As in [4], we model the random medium by

$$\rho(x) = \rho_0(x)\left[1 + \eta(\frac{x}{\varepsilon^2})\right] \tag{2.2}$$

$$\frac{1}{\rho(x)c^2(x)} = \frac{1}{\rho_0(x)c_0^2(x)}\left[1 + \nu(\frac{x}{\varepsilon^2})\right] \tag{2.3}$$

where $\rho_0(x)$ and $c_0(x)$ are deterministic functions of $x \in \mathbf{R}^1$ that are infinitely differentiable, uniformly positive, bounded and identically equal to constants ρ_0 and c_0 in $x > 0$. The fluctuations $\eta(y)$ and $\nu(y)$ are given stationary, mean zero stochastic processes with values in some interval $[-\beta, \beta]$ with $\beta < 1$ so that ρ and c in (2.2),(2.3) do not change sign. In the homogeneous half space $x > 0$ the fluctuations are zero. The parameter $\varepsilon > 0$ is the ratio of a typical microscopic to a macroscopic length scale and is assumed to be small so that the fluctuations are rapidly varying. Note that we do not assume that they are small. To simplify the analysis we will assume that $\eta(y)$ and $\nu(y)$ are mean zero, stationary Markov processes with strong ergodic properties although it is known [8] how to handle much more general, fluctuating coefficients. Define the travel time

$$\tau(x) = \int_0^x \frac{1}{c_0(s)}ds \tag{2.4}$$

for all x so that τ is negative for $x < 0$ and $\tau = x/c_0$ for $x > 0$. Let $\xi(\tau)$ be its inverse function that is zero at $\tau = 0$. Clearly

$$\frac{d\tau}{dx} = \frac{1}{c_0(x)}\ , \quad \frac{d\xi}{d\tau} = c_0(\xi(\tau)) \tag{2.5}$$

Equations (2.1) are provided with initial and boundary conditions by specifying that a pulse is incident from the positive half space

$$u = \frac{1}{(c_0\rho_0)^{1/2}}\frac{1}{\sqrt{\varepsilon}}f(\frac{t+\tau}{\varepsilon})$$

$$p = -(\rho_0 c_0)^{1/2}\frac{1}{\sqrt{\varepsilon}}f(\frac{t+\tau}{\varepsilon}) \tag{2.6}$$

for $t < 0$, with u and p continuous at $x = 0$. The pulse shape f is a smooth function of compact support in $(0, \infty)$. Note the scaling of the incident pulse: its width is of order ε and is thus intermediate to the microscopic scale ε^2 and the macroscopic scale 1. This is a good scaling for a pulse intended as a probe of the random medium [4]. The amplitude factor $\varepsilon^{-1/2}$ in (2.6) is not essential and serves to normalize the incident wave energy independently of ε.

We introduce the change of variables

$$\begin{aligned} u(t,x) &= (\rho_0 c_0)^{-1/2}\tilde{u}(t,\tau) \\ p(t,x) &= (\rho_0 c_0)^{+1/2}\tilde{p}(t,\tau) \end{aligned} \tag{2.7}$$

and let

$$\zeta = \frac{1}{\rho_0 c_0}\frac{d}{d\tau}(\rho_0 c_0) \tag{2.8}$$

where we may think of ρ_0 and c_0 as functions of x or of τ with $\rho_0(\tau)$ being $\rho_0(\xi(\tau))$ and similarly for $c_0(\tau)$. From (2.1) and (2.7) we derive equations for $\tilde{u}$ and $\tilde{p}$

$$\begin{aligned} (1+\eta)\tilde{u}_t + \tilde{p}_\tau + \frac{1}{2}\zeta\tilde{p} &= 0 \\ (1+\nu)\tilde{p}_t + \tilde{u}_\tau - \frac{1}{2}\zeta\tilde{u} &= 0 \end{aligned} \tag{2.9}$$

We now introduce right and left traveling waves A and B through

$$\tilde{u} = A + B, \tilde{p} = A - B \tag{2.10}$$

which satisfy the hyperbolic system

$$\begin{aligned} A_t + A_\tau + mA_t + nB_t - \frac{1}{2}\zeta B &= 0 \\ B_t - B_\tau + nA_t + mB_t + \frac{1}{2}\zeta A &= 0 \end{aligned} \tag{2.11}$$

with

$$A = 0, B = \frac{1}{\sqrt{\varepsilon}} f(\frac{t+\tau}{\varepsilon}) \tag{2.12}$$

for $t < 0$. This is equivalent to the initial conditions $A(0,\tau) = B(0,\tau) = 0$, $\tau < 0$ and the boundary condition $B(t,0) = \varepsilon^{-1/2} f(t/\varepsilon)$, $t > 0$. The quantity of principal interest in our asymptotic analysis is the reflected wave amplitude $A(t,\tau)$ evaluated at the interface $\tau = 0$ and for times $t > 0$. In (2.11) we have used the notation

$$\begin{aligned} m = m^\varepsilon(\tau) &= \frac{1}{2}\left[\eta(\xi(\tau)/\varepsilon^2) + \nu(\xi(\tau)/\varepsilon^2)\right] \\ n = n^\varepsilon(\tau) &= \frac{1}{2}\left[\eta(\xi(\tau)/\varepsilon^2) - \nu(\xi(\tau)/\varepsilon^2)\right] \end{aligned} \tag{2.13}$$

We can write the right going wave amplitude A as a linear functional of the left going wave amplitude B

$$A(t,\tau) = \int_0^t R(t-s,\tau)B(s,\tau)ds \ , \quad t \geq 0 \ , \quad \tau \leq 0$$

which we also write in short as

$$A = R * B \ .$$

The convolution operator with kernel $R(t,\tau)$ is the reflection operator. It is easily seen by direct computation and (2.11) that it satisfies the nonlinear hyperbolic equation

$$R_\tau + 2(1+m)R_t + n(\delta + R*R)_t - \frac{\zeta}{2}(\delta - R*R) = 0, \tag{2.14}$$

with $t \geq 0$, $\tau \leq 0$, $R(0,\tau) \equiv 0$ and with $\delta = \delta(t)$ the Dirac delta function. This equation is easily solved by Fourier transforms (cf. (2.17)). We are interested in the reflected amplitude at the interface $\tau = 0$

$$A(t,0) = \int_0^t R(t-s,0)\frac{1}{\sqrt{\varepsilon}}f(\frac{s}{\varepsilon})ds \tag{2.14a}$$

and so we must determine $R(s,0)$ for $0 \leq s \leq t$. The fact that the fluctuation processes η and ν are bounded by a constant β less than one and the relation (2.13) imply that $|m| < \beta$. Therefore the characterisics of (2.14) have always positive slope in the (τ, t) plane and for any fixed t the domain of dependence of $R(s,0)$, $0 \leq s \leq t$ is a subset of the region $\{-L \leq \tau \leq 0\} \cap \{0 \leq s \leq t\}$ for some L large that depends on t, $L > t/2(1-\beta)$. We may therefore assume that $R(s,-L) \equiv 0$ for all $s \geq 0$ if we are only interested in $R(s,0)$ for $0 \leq s \leq t$.

We will analyze the reflected signal $A(t,0), t > 0$, using Fourier transforms in time and an invariant imbedding representation of the time-harmonic reflection coefficient. Let

$$\hat{A}(\tau,\omega) = \int e^{i\omega t/\varepsilon}A(t,\tau)dt \ , \quad \hat{B}(\tau,\omega) = \int e^{i\omega t/\varepsilon}B(t,\tau)dt$$

be the Fourier transforms of the right and left going wave amplitudes with frequency ω scaled relative to the width of the incident pulse. Then $\hat{A}$ and $\hat{B}$ satisfy the ordinary differential equations

$$\frac{d}{d\tau}\begin{bmatrix}\hat{A}\\ \hat{B}\end{bmatrix} = \frac{i\omega}{\varepsilon}\begin{bmatrix}1+m & n\\ -n & -1-m\end{bmatrix}\begin{bmatrix}\hat{A}\\ \hat{B}\end{bmatrix} + \frac{\zeta}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\hat{A}\\ \hat{B}\end{bmatrix} \tag{2.15}$$

For any solution $\hat{A}(\tau,\omega), \hat{B}(\tau,\omega)$ of (2.15) let

$$\hat{R}(\tau,\omega) = \frac{\hat{A}(\tau,\omega)}{\hat{B}(\tau,\omega)}. \tag{2.16}$$

Then $\hat{R}$ satisfies the stochastic Riccati equation

$$\frac{d\hat{R}}{d\tau} = \frac{i\omega}{\varepsilon}\left\{2(1+m)\hat{R} + n(1+\hat{R}^2)\right\} + \frac{\zeta}{2}(1-\hat{R}^2) \tag{2.17}$$

which is the Fourier transform of (2.14). It is to be solved in some interval $-L < \tau \leq 0$ with $\hat{R}(-L,\omega)$ specified. Here L is a positive constant which represents the width of a layer of random medium in terms of travel time. The solution of (2.17) at $\tau = 0$, $\hat{R}(0,\omega)$, is the reflection coefficient at frequency ω of the layer $-L < \tau \leq 0$ when $\hat{R}$ is specified at $\tau = -L$.

The reflected amplitude $A(t,0), t > 0$ is given by (2.14a) or, using Fourier transforms and $\hat{R}$, by

$$A(t,0) = \frac{1}{2\pi\sqrt{\varepsilon}}\int_{-\infty}^{\infty} e^{-i\omega t/\varepsilon}\hat{f}(\omega)\hat{R}(0,\omega)d\omega. \tag{2.18}$$

Here $\hat{f}(\omega)$ is the Fourier transform of the incident pulse shape f in (2.12). Note that in (2.18), $\hat{R}(0,\omega)$ depends on the layer width[1] L and the value of $\hat{R}$ at $\tau = -L$. However, because of the hyperbolic nature of (2.11)-(2.12), for any t fixed, $0 < t < \infty$, the reflected process $A(t,0)$ does not depend on L or $\hat{R}(-L,\omega)$ if L is sufficiently large, for example $L > t/2(1-\beta)$, as we noted below (2.14a). In that case we can use (2.18) to represent the reflected process where $\hat{R}(0,\omega)$ is the solution of (2.17) with any given initial condition at $\tau = -L$, for example $\hat{R}(-L,\omega) = 0$. The hyperbolic nature of the initial-boundary value problem (2.11)-(2.12), and the finite propagation speed in particular, imply that the reflected signal observed at the interface $\tau = 0$ (or $x = 0$) at time

[1] In travel time units.

t does not depend on the properties of the medium in the region $\tau \le -L$ if $L > t/2(1-\beta)$. Thus, even though for each frequency ω the reflection coefficient at $\tau = 0$, $\hat{R}(0,\omega)$, contains information about material properties in all of the inhomogeneous half space $\tau < 0$, the reflected signal up to time t given by the Fourier integral (2.18) sees only a finite region $-L \le \tau \le 0$. This observation is important for the analysis that follows because it allows us to use Markovian ideas and techniques.

It is clear from (2.18) that in order to study $A(t,0)$ in the limit $\varepsilon \to 0$, it is necessary to know the process $\hat{R}(\tau,\omega)$ simultaneously for all $\omega \in \mathbf{R}$. While for each ω fixed the analysis of (2.17) as $\varepsilon \to 0$ is well known [9,14], the main purpose of this paper is to analyze the process $A(t,0), 0 < t < \infty$ in the limit $\varepsilon \to 0$ and this requires some new techniques.

3 The Functional Process

In [4] we noted that because $A(t,0)$ in (2.18) is not stationary and because of the nature of the scaling, it is necessary to introduce a windowed version of this process, namely the process $A(t+\varepsilon\sigma,0)$ with σ the window time parameter. We will introduce windowing in a somewhat different way in this paper. We will also denote $A(t,0)$ by $A(t)$ throughout.

Let $g(t)$ be a smooth function in $\mathbf{R}^1$ that decays rapidly at infinity and such that

$$\int_{\boldsymbol{R}^1} g^2(s)ds = 1 \tag{3.1}$$

Let $A_g(t,\omega)$ be defined by

$$A_g(t,\omega) = \int_{-\infty}^{\infty} A(s)e^{i\omega s/\varepsilon}\frac{1}{\varepsilon}g(\frac{t-s}{\varepsilon})ds \tag{3.2}$$

which is the windowed Fourier transform of A at t with g the window function. The reflected signal $A(t)$ can be recovered from $A_g(t,\omega)$ by the inversion formula [5]

$$A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_g(s,\omega)e^{-i\omega t/\varepsilon}\frac{1}{\varepsilon}g(\frac{s-t}{\varepsilon})d\omega ds \tag{3.3}$$

where we assume that $A(t)$ is smooth. Note that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_g(t,\omega)e^{-i\omega(t/\varepsilon-\tau)}d\omega = A(t-\varepsilon\tau)g(\tau) \tag{3.4}$$

Note also that if we use the representation (2.18) in (3.2) we get

$$A_g(t,\omega) = \frac{1}{2\pi\sqrt{\varepsilon}}\int e^{-i\omega_1 t/\varepsilon}\hat{g}(\omega_1)\hat{f}(\omega+\omega_1)\hat{R}(0,\omega+\omega_1)d\omega_1 \tag{3.5}$$

The quantity of principal interest to us is the *local power spectral density* of the signal $A(t)$

$$|A_g(t,\omega)|^2 = \frac{1}{(2\pi)^2\varepsilon}\int e^{-i\omega_1 t/\varepsilon}\hat{g}(\omega_1)\hat{R}_f(\omega+\omega_1)d\omega_1 \cdot \int e^{i\omega_2 t/\varepsilon}\hat{g}^*(\omega_2)\hat{R}_f^*(\omega+\omega_2)d\omega_2 \tag{3.6}$$

where

$$\hat{R}_f(\omega) = \hat{f}(\omega)\hat{R}(0,\omega) \tag{3.7}$$

If we let

$$\omega_1 = \tilde{\omega} - \frac{\varepsilon h}{2}, \omega_2 = \tilde{\omega} + \frac{\varepsilon h}{2} \tag{3.8}$$

then

$$|A_g(t,\omega)|^2 = \frac{1}{(2\pi)^2} \int\int e^{iht} \hat{g}(\tilde{\omega} - \frac{\varepsilon h}{2}) \hat{g}^*(\tilde{\omega} + \frac{\varepsilon h}{2})$$
$$\cdot \hat{R}_f(\omega + \tilde{\omega} - \frac{\varepsilon h}{2}) \hat{R}_f^*(\omega + \tilde{\omega} + \frac{\varepsilon h}{2}) d\tilde{\omega} dh \tag{3.9}$$

The local power spectral density is smoothed by both the pulse shape function f and by the window function g. Formally, as ε tends to zero we may write

$$|A_g(t,\omega)|^2 \approx$$
$$\frac{1}{(2\pi)^2} \int \left[|\hat{g}(\tilde{\omega})|^2 |\hat{f}(\omega + \tilde{\omega})^2 \cdot \int e^{iht} \hat{R}(0, \omega + \tilde{\omega} - \frac{\varepsilon h}{2}) \hat{R}^*(0, \omega + \tilde{\omega} + \frac{\varepsilon h}{2}) dh \right] d\tilde{\omega}$$

This expression is formal because the integral with respect to h does not converge in general. If we smooth $|A_g(t,\omega)|^2$ in t with a function $\phi \in C_0^\infty(\mathbf{R})$ then (3.9) simplifies correctly, as ε tends to zero, to

$$\int \phi(t) |A_g(t,\omega)|^2 dt \approx$$
$$\frac{1}{(2\pi)^2} \int \left[|\hat{g}(\tilde{\omega})|^2 |\hat{f}(\omega + \tilde{\omega})|^2 \int \hat{\phi}(h) \hat{R}(0, \omega + \tilde{\omega} - \frac{\varepsilon h}{2}) \hat{R}^*(0, \omega + \omega + \frac{\varepsilon h}{2}) dh \right] d\tilde{\omega} \tag{3.10}$$

If the smoothing in time is removed as $\varepsilon \to 0$ we have instead of (3.10)

$$\int \phi_\varepsilon(T - t) |A_g(t,\omega)|^2 dt \approx$$
$$\frac{1}{(2\pi)^2} \int \left[(\hat{g}(\tilde{\omega})|^2 |\hat{f}(\omega + \tilde{\omega})|^2 \int e^{ihT} \hat{\phi}_\varepsilon^*(h) \hat{R}(0, \omega + \tilde{\omega} - \frac{\varepsilon h}{2}) \hat{R}^*(0, \omega + \tilde{\omega} + \frac{\varepsilon h}{2}) dh \right] d\tilde{\omega}$$

where $\phi_\varepsilon(t)$ tends to the delta function $\delta(t)$ as $\varepsilon \to 0$. For example $\phi_\varepsilon(t) = \varepsilon^{-1}\phi(t\varepsilon^{-1})$ with $\phi(t) \in C_0^\infty$ and $\int \phi(t) dt = 1$.

From the above considerations we conclude that the quantity of interest to us is

$$\int e^{iht} \hat{R}(0, \omega - \frac{\varepsilon h}{2}) \hat{R}^*(0, \omega + \frac{\varepsilon h}{2}) dh \tag{3.11}$$

when suitably smoothed. We will actually study the more general quantity

$$W^{NM}(\tau, t, \omega) = \int e^{iht} \hat{R}^N(\tau, \omega - \frac{\varepsilon h}{2}) \hat{R}^{*M}(\tau, \omega + \frac{\varepsilon h}{2}) dh \tag{3.12}$$

which is only a formal expression in this unsmoothed form. To make sense of (3.12) we need to introduce an appropriate space of smoothing or test functions.

Let $\mathcal{S}$ denote the space of complex valued infinitely differentiable functions f in $\mathbf{R}^2$ that are rapidly decreasing[2] at infinity and let

$$\mathcal{S}_H = \Big\{ \lambda = (\lambda^{NM}), N \geq 0, M \geq 0 | (\lambda^{NM})^* = \lambda^{MN}, \lambda^{MN} \in \mathcal{S},$$

[2]This means that for every pair of integers p and q and every derivative D^q in t and ω of total order q, $|[(1 + t^2)(1+\omega^2)]^{p/2} D^q f|$ is uniformly bounded.

$$\text{and}(NM)^p\lambda^{NM} \in \mathcal{S} \text{ for each integer } p \geq 1, \text{ uniformly in } N, M\Big\} \tag{3.13}$$

which is the space of hermitian matrices whose entries are infinitely differentiable in t and ω and they and all their t and ω derivatives are rapidly decreasing as functions of t, ω, N and M. This space is endowed with the convergence

$$\lambda_n \to \lambda \text{ in } \mathcal{S}_H \text{ as } n \to \infty$$

if for each pair of integers p, q and each derivative D^q in t and ω of total order q, $[(1+t^2)(1+\omega^2)]^{p/2}(NM)^p D^q(\lambda_n^{NM} - \lambda^{NM})$ converges to 0 as $n \to \infty$, uniformly in t, ω, N and M. If on $\mathcal{S}_H$ we introduce the bilinear form

$$\langle \lambda_1, \lambda_2 \rangle = \int\int_{\boldsymbol{R}^2} dt d\omega \sum_{N,M=0}^{\infty} \lambda_1^{NM}\lambda_2^{MN} \tag{3.14}$$

then the space of distributions $\mathcal{S}'_H$ can be identified with the dual of $\mathcal{S}_H$, as usual.

From the formal expression (3.12) we see that if $\lambda \in \mathcal{S}_H$ then

$$W_\lambda = \langle W, \lambda \rangle = \int\int_{\boldsymbol{R}^2} dt d\omega \sum_{N,M=0}^{\infty} W^{NM}\lambda^{MN}$$

$$= \int\int_{\boldsymbol{R}^2} dt d\omega \sum_{N,M=0}^{\infty} \lambda^{MN}(t,\omega)\int e^{iht}\hat{R}^N(\tau, \omega - \frac{\varepsilon h}{2})\hat{R}^{*M}(\tau, \omega + \frac{\varepsilon h}{2})dh$$

$$= \sum_{N,M=0}^{\infty}\int\int_{\boldsymbol{R}^2} \hat{\lambda}^{MN}(h,\omega)\hat{R}^N(\tau, \omega - \frac{\varepsilon h}{2})\hat{R}^{*M}(\tau, \omega + \frac{\varepsilon h}{2})d\omega dh \tag{3.15}$$

is well defined. Therefore, starting with the solution $\hat{R}(\tau,\omega), -L \leq \tau \leq 0$ of (2.17) we can define a family of stochastic processes $W^\varepsilon(\tau)$ with values in $\mathcal{S}'_H$. That is, for each $\varepsilon > 0$, we have a probability measure P^ε on the space of continuous functions on $-L \leq \tau \leq 0$ with values in $\mathcal{S}'_H$. The continuity of $\langle W^\varepsilon(\tau), \lambda \rangle$ as a function of τ when ε is positive follows from (2.17). The scalar valued process $\langle W^\varepsilon(\tau), \lambda\rangle$ will also be denoted by $W^\varepsilon_\lambda(\tau)$.

The preceding process can be considered as a polynomial of degree 1 of W and we can define in the same way multinomial functionals of W of degree p. Let $\mathcal{S}^p$ denote the space of complex valued, rapidly decreasing, infinitely differentiable functions in $\mathbf{R}^{2p}$ and let

$$\mathcal{S}^p_H = \Big\{\lambda = (\lambda^{N_1M_1...N_pM_p}), N_i \geq 0, M_i \geq 0 \mid (\lambda^{N_1M_1...N_pM_p})^* = \lambda^{M_1N_1...M_pN_p},$$

$$(N_1M_1...N_pM_p)^q\lambda^{N_1M_1...N_pM_p} \in \mathcal{S}^p \text{ for each integer } q,$$

$$\text{uniformly in } N_1M_1...N_pM_p\} \tag{3.13'}$$

$\mathcal{S}^p_H$ is endowed with the same convergence as $\mathcal{S}_H$ and on $\mathcal{S}^p_H$ we can define the bilinear form.

$$\langle \lambda_1, \lambda_2 \rangle = \int ... \int_{\boldsymbol{R}^{2p}} dt_1...dt_p d\omega_1...d\omega_p$$

$$\cdot \sum_{N_1...M_p} \lambda_1^{N_1M_1...N_pM_p}(t_1,...,\omega_p)\lambda_2^{M_1N_1...M_pN_p}(t_1,...,\omega_p) \tag{3.14'}$$

We can now define a family of multinomial functionals of degree p, $F^{(p)}$ on $(\mathcal{S}^p_H)'$ by

$$F^{(p)} = \left\langle W^{(p)}, \lambda \right\rangle = \int ... \int_{\boldsymbol{R}^{2p}} dt_1,...,dt_p d\omega_1...d\omega_p$$

$$\cdot \sum_{N_1 \ldots M_p} W^{N_1 M_1}(t_1,\omega_1)\ldots W^{N_p M_p}(t_p,\omega_p)\lambda^{M_1 N_1 \ldots M_p N_p}(t_1,\ldots,\omega_p) \qquad (3.15')$$

where $W^{(p)}$ is the p-fold tensor product $W^{(p)} = W \otimes \ldots \otimes W$.

It is easy to see that the probability measure P^ε on $X = C([-L,0];\mathcal{S}'_H)$, the continuous functions on $[-L,0]$ with values in $\mathcal{S}'_H$, are tight. From the theorem of Mitoma and Fouque [12,6] it is enough to verify that for each $\lambda \in \mathcal{S}_H$ a Kolmogorov moment condition holds, such as

$$\limsup_{\varepsilon \downarrow 0} E^{P^\varepsilon}\left\{|W_\lambda(\tau_2) - W_\lambda(\tau_1)|^4\right\} \leq C|\tau_2-\tau_1|^2, -L \leq \tau_1 \leq \tau_2 \leq 0, \qquad (3.16)$$

where C is a constant that depends on λ and L but not on ε. This estimate follows easily from properties of (2.17) under the hypotheses on the random coefficients m and n introduced in section 2, and in more detail in the next section, but it necessary to first transform $\hat{R}$.

We introduce the *centered* reflection coefficient $\tilde{R}$ by

$$\hat{R}(\tau,\omega) = e^{2i\omega\tau/\varepsilon}\tilde{R}(\tau,\omega). \qquad (3.17)$$

It satisfies a modified form of (2.17)

$$\frac{d\tilde{R}}{d\tau} = \frac{i\omega}{\varepsilon}\left(2m\tilde{R} + ne^{-2i\omega\tau/\varepsilon} + ne^{2i\omega\tau/\varepsilon}\tilde{R}^2\right) + \frac{\zeta}{2}\left(e^{-2i\omega\tau/\varepsilon} + \tilde{R}^2 e^{2\omega\tau/\varepsilon}\right) \qquad (3.18)$$

with $\tilde{R}$ specified at $\tau = -L$. Since the quantity of interest $\tilde{R}(0,\omega)$ is not affected by this change of phase we will now assume that $\hat{R}$ in (3.15) is replaced by $\tilde{R}$. We will continue to denote the $\mathcal{S}'_H$ valued process by $W^\varepsilon(\tau)$.

From (2.17) or (3.18) we always have that

$$|\tilde{R}| \leq 1\ , \qquad (3.19)$$

as can be verified by direct calculation from (3.18) provided that $|\tilde{R}(-L,\omega)| \leq 1$, and that

$$\limsup_{\varepsilon \downarrow 0} E\left\{|\tilde{R}(\tau_2) - \tilde{R}(\tau_1)|^4\right\} \leq \tilde{C}(\tau_2-\tau_1)^2\ , \quad -L \leq \tau_1 \leq \tau_2 \leq 0. \qquad (3.20)$$

The proof of (3.20) is contained in the general results given in [3,13,11]. Properties (3.19) and (3.20) yield (3.16) easily. Note in particular that for $-L \leq \tau \leq 0$

$$|W^\varepsilon_\lambda(\tau)| \leq C(\lambda) \qquad (3.21)$$

uniformly in ε, where $C(\lambda)$ is a constant that depends on λ and L.

4 The Limit Process

The coefficient processes m and n that appear in (2.17) are defined by (2.13). We will denote $(m^\varepsilon(\tau), n^\varepsilon(\tau))$ by $q^\varepsilon(\tau)$ and assume, as noted below (2.3), that $q^\varepsilon(\tau)$ is a stationary Markov process with state space $[-\beta,\beta]^2$. We actually assume that the fluctuation processes η and ν in (2.2) and (2.3) are Markovian so that by (2.5) and (2.13) the generator of $q^\varepsilon(\tau)$ will have the form

$$\frac{1}{\varepsilon^2}c_0(\tau)Q \qquad (4.1)$$

where Q is a generator acting on bounded measurable functions on $[-\beta,\beta]^2$ and is assumed to be bounded for simplicity. We assume that Q is strongly ergodic which means that there is an invariant measure $\bar{p}$ on $[-\beta,\beta]^2$ for the process generated by Q and if g is any bounded measurable function on $[-\beta,\beta]^2$ the equation

$$Q\chi + g = 0 \tag{4.2}$$

has a bounded solution $\chi = -Q^{-1}g$ provided that

$$\int_{[-\beta,\beta]^2} g(q)\bar{p}(dq) = 0 \tag{4.3}$$

which is the Fredholm alternative for Q. The solution χ is made unique by assuming that

$$\int_{[-\beta,\beta]^2} \chi(q)\bar{p}(dq) = 0 \tag{4.4}$$

Condition (4.3) is the zero mean property of the fluctuations m and n which follows from the zero mean properties of η and ν in (2.2) and (2.3) via (2.13).

With this definition of m and n in (3.18) the process $\tilde{R}(\tau,\omega)$ is well defined and it is jointly Markovian along with $q^\varepsilon(\tau)$. From (3.15) it follows that $q^\varepsilon(\tau)$ and $W^\varepsilon_\lambda(\tau)$ are jointly Markovian with state space $[-\beta,\beta]^2 \times \mathcal{S}'_H$. When $\tau < 0$ this process does not have a physical interpretation and is not related directly to the original scattering problem (2.11), (2.12). However, at $\tau = 0$, if the test function λ has compact support in t and if L is large enough, then we recover a smoothed version of the quantity (3.12) which is related to the reflected amplitude (2.18). We use here the remarks following (2.18) regarding the hyperbolicity of (2.11), (2.12) in order to get this identification.

Let F be a smooth function from $[-\beta,\beta]^2 \times \mathbf{R} \to \mathbf{R}$. We will calculate the generator of $(q^\varepsilon(\tau), W^\varepsilon(\tau))$ on functions of the form $F(q, W_\lambda)$ with $\lambda \in \mathcal{S}_H$. Since the generator of the process q^ε is given by (4.1) with Q bounded, we see that

$$\frac{d}{dh}E_{W,q,\tau}\left\{F(q^\varepsilon(h), W^\varepsilon_\lambda(h))\right\}|_{h=0} =$$

$$\frac{c_0(\tau)}{\varepsilon^2}QF(q,W_\lambda) + F'(q,W_\lambda)\left\{\frac{1}{\varepsilon}\left\langle W, H_1(q,\tau,\frac{\tau}{\varepsilon})\lambda\right\rangle + \left\langle W, H_2(q,\tau,\frac{\tau}{\varepsilon})\lambda\right\rangle\right\} \tag{4.5}$$

Here F' denotes derivative of F with respect to its second argument and $E_{W,q,\tau}$ is conditional expectation given that $q^\varepsilon(\tau) = q, W^\varepsilon(\tau) = W \in \mathcal{S}'_H$. The explicit expression for the generators H_1 and H_2 in (4.5), which map $\mathcal{S}_H$ into $\mathcal{S}_H$, are obtained from (3.15) (with $\hat{R}$ replaced by $\tilde{R}$) and (4.6). In fact

$$\begin{aligned}\frac{d}{d\tau}\left\langle W^\varepsilon(\tau),\lambda\right\rangle = \frac{1}{\varepsilon}\left\langle W^\varepsilon(\tau), H_1(q^\varepsilon(\tau),\tau,\frac{\tau}{\varepsilon})\lambda\right\rangle \\ + \left\langle W^\varepsilon(\tau), H_2(q^\varepsilon(\tau),\tau,\frac{\tau}{\varepsilon})\lambda\right\rangle\end{aligned} \tag{4.6}$$

A long but straightforward calculation now gives

$$\begin{aligned}(H_1(q,\tau,l)\lambda)^{NM}(t,\omega) = -2i\omega m(N-M)\lambda^{NM}(t,\omega) \\ +i\omega n\left\{(M+1)e^{-2i\omega l}\lambda^{NM+1}(t-\tau,\omega) + (M-1)e^{2i\omega l}\lambda^{NM-1}(t+\tau,\omega)\right. \\ \left. -(N+1)e^{2i\omega l}\lambda^{N+1M}(t-\tau,\omega) - (N-1)e^{-2i\omega l}\lambda^{N-1M}(t+\tau,\omega)\right\}\end{aligned} \tag{4.7}$$

where $q = (m,n)$ and it is to be noted that H_1 does map $\mathcal{S}_H$ into $\mathcal{S}_H$. Similarly we have

$$(H_2(q,\tau,l)\lambda)^{NM}(t,\omega) = m(N+M)\frac{\partial}{\partial t}\lambda^{NM}(t,\omega)$$

$$+\frac{n}{2}\frac{\partial}{\partial t}\Big\{(M+1)e^{-2i\omega l}\lambda^{NM+1}(t-\tau,\omega)+(M-1)e^{2i\omega l}\lambda^{NM-1}(t+\tau,\omega)$$

$$+(N+1)e^{2i\omega l}\lambda^{N+1M}(t-\tau,\omega)+(N-1)e^{-2i\omega l}\lambda^{N-1M}(t+\tau,\omega)\Big\}$$

$$+\frac{\zeta(\tau)}{2}\Big\{(M+1)e^{-2i\omega l}\lambda^{NM+1}(t-\tau,\omega)+(M-1)e^{2i\omega l}\lambda^{NM-1}(t+\tau,\omega)$$

$$+(N+1)e^{2i\omega l}\lambda^{N+1M}(t-\tau,\omega)+(N-1)e^{-2i\omega l}\lambda^{N-1M}(t+\tau,\omega)\Big\} \tag{4.8}$$

Let $\mathcal{L}^{\varepsilon}_{\tau}$ denote the operator on the right side of (4.5), acting linearly on F. We can now summarize the definition of $(q^{\varepsilon}(\tau), W^{\varepsilon}(\tau))$ by saying that there is a unique probability measure on $\tilde{X} = C([-L,0];[-\beta,\beta]^2\times\mathcal{S}'_H)$, denoted by $\tilde{P}^{\varepsilon}, \varepsilon>0$, such that for each $\lambda\in\mathcal{S}_H$ and each smooth $F:[-\beta,\beta]^2\times\mathbf{R}\to\mathbf{R}$ the functional

$$F(q(s),W_\lambda(s))-F(q(\tau),W_\lambda(\tau))-\int_\tau^s \mathcal{L}^{\varepsilon}_{\gamma}F(q(\gamma),W_\lambda(\gamma))d\gamma \tag{4.9}$$

is a $\tilde{P}^{\varepsilon}$ martingale for all $s\geq\tau$, relative to the natural σ -algebras on $\tilde{X}$. If F depends on τ explicitly then the generator $\mathcal{L}^{\varepsilon}_{\gamma}$ must be replaced by $\frac{\partial}{\partial\gamma}+\mathcal{L}^{\varepsilon}_{\gamma}$ in (4.9).

The laws P^{ε} of $W^{\varepsilon}(\tau), -L\leq\tau\leq 0$, on $X=C([-L,0];\mathcal{S}'_H), \varepsilon>0$, are a tight family as noted at the end of section 3. The limit law can be identified using the martingales (4.9) with a suitable choice of function F.

The choice of the test function F for the asymptotic analysis is made in the usual way [13] based on perturbation theory. Given a smooth function $F:\mathbf{R}^+\times\mathbf{R}\to\mathbf{R}$ we shall construct a function $F^{\varepsilon}=F^{\varepsilon}(q,W_\lambda,\tau)$ which goes to F as $\varepsilon\to 0$ and such that

$$(\frac{\partial}{\partial\tau}+\mathcal{L}^{\varepsilon}_{\tau})F^{\varepsilon}\to(\frac{\partial}{\partial\tau}+\mathcal{L}_{\tau})F \tag{4.10}$$

This construction is done in several stages. First, based on our assumptions (4.1)-(4.4) regarding the coefficient process q, we introduce the bounded, mean zero solutions χ_m and χ_n of

$$Q\chi_m+m=0,\ Q\chi_n+n=0 \tag{4.11}$$

Set

$$\tilde{F}^{\varepsilon}=F+\frac{\varepsilon}{c_0(\tau)}F'\langle W,H_1^{\chi}\lambda\rangle+\frac{\varepsilon^2}{c_0(\tau)}F'\langle W,H_2^{\chi}\lambda\rangle \tag{4.12}$$

where F' denotes derivative of F with respect to the argument corresponding to W_λ and H_1^{χ} and H_2^{χ} denote the operators defined by (4.7) and (4.8), respectively, with m and n replaced by χ_m and χ_n everywhere. By direct calculation we find that

$$\mathcal{L}^{\varepsilon}_{\tau}\tilde{F}^{\varepsilon}=\tilde{\mathcal{L}}^{\varepsilon}_{\tau}F+\frac{\varepsilon}{c_0}\{F'\langle W,H_1H_2^{\chi}\lambda\rangle+F''\langle W,H_1\lambda\rangle\langle W,H_2^{\chi}\lambda\rangle$$

$$+F'\langle W,H_2H_1^{\chi}\lambda\rangle+F''\langle W,H_2\lambda\rangle\langle W,H_1^{\chi}\lambda\rangle\}$$

$$+\frac{\varepsilon^2}{c_0}\{F'\langle W,H_2H_2^{\chi}\lambda\rangle+F''\langle W,H_2\lambda\rangle\langle W,H_2^{\chi}\lambda\rangle\} \tag{4.13}$$

where

$$\tilde{\mathcal{L}}^{\varepsilon}_{\tau}F=\frac{1}{c_0}\{F'\langle W,H_1H_1^{\chi}\lambda\rangle+F''\langle W,H_1\lambda\rangle\langle W,H_1^{\chi}\lambda\rangle\} \tag{4.13$'$}$$

The coefficients of ε and ε^2 in (4.13) are bounded as ε tends to zero so we shall write $O(\varepsilon)$ after $\tilde{\mathcal{L}}_\tau^\varepsilon F$. Let $\bar{\mathcal{L}}_\tau^\varepsilon F$ denote the expectation of $\tilde{\mathcal{L}}_\tau^\varepsilon F$ with respect to the invariant measure $\bar{p}$ of the coefficient process $q(\tau)$, defined in (4.1)-(4.4). If we replace $\tilde{F}^\varepsilon$ by

$$\tilde{F}^\varepsilon - \frac{\varepsilon^2}{c_0} Q^{-1}[\tilde{\mathcal{L}}_\tau^\varepsilon F - \bar{\mathcal{L}}_\tau^\varepsilon F] \tag{4.14}$$

then for the new $\tilde{F}^\varepsilon$ we have

$$\mathcal{L}_\tau^\varepsilon \tilde{F}^\varepsilon = \bar{\mathcal{L}}_\tau^\varepsilon F + O(\varepsilon) \tag{4.15}$$

Finally let $\mathcal{L}_\tau F$ be the average of $\bar{\mathcal{L}}_\tau^\varepsilon F$ with respect to the rapid phase variations of the trigonometric functions in (4.7) and (4.8). In fact, to distinguish the τ dependence of $\bar{\mathcal{L}}_\tau^\varepsilon F$ due to the rapid variation of $l = \tau/\varepsilon$ and the slow variation τ we write it in the form $\bar{\mathcal{L}}_{\tau,\tau/\varepsilon}^\varepsilon F$. Then if we replace $\tilde{F}^\varepsilon$ by

$$\tilde{F}^\varepsilon + \varepsilon \int_0^\infty e^{-\delta s}(\bar{\mathcal{L}}_{\tau,s+\tau/\varepsilon}^\varepsilon F - \mathcal{L}_\tau F) ds \tag{4.16}$$

we find that

$$\begin{aligned} (\frac{\partial}{\partial \tau} + \mathcal{L}_\tau^\varepsilon)\tilde{F}^\varepsilon &= \mathcal{L}_\tau F + 0(\varepsilon) \\ +\delta \int_0^\infty e^{-\delta s}&(\bar{\mathcal{L}}_{\tau,s+\tau/\varepsilon}^\varepsilon F - \mathcal{L}_\tau F) ds \end{aligned} \tag{4.17}$$

The last term in (4.17) can be made small uniformly in ε by choosing δ small. This follows from the periodic (or almost periodic in general) dependence of H_1 and H_2 on $l = \tau/\varepsilon$.

In this way we have a sequence of test function $F^\varepsilon(q, W_\lambda, \tau)$ such that (4.10) holds and $F^\varepsilon \to F(W_\lambda)$ as $\varepsilon \to 0$. We can now pass to the limit in the martingale (4.9). A subsequence of the measures P^ε converges weakly to a measure P and the functionals converge strongly by the above construction so that in the limit $\varepsilon \to 0$, the functional

$$F(W_\lambda(s)) - F(W_\lambda(\tau)) - \int_\tau^s \mathcal{L}_\gamma F(W_\lambda(\gamma)) d\gamma \tag{4.18}$$

is a P martingale for every smooth and bounded function $F : \mathbf{R} \to \mathbf{R}$ and every test function $\lambda \in \mathcal{S}_H$.

The limit generator $\mathcal{L}_\tau$ can be computed explicitly from (4.13') by averaging with respect to the invariant measure $\bar{p}$ and with respect to the fast phase. It has a particularly simple form when $F(W_\lambda) = W_\lambda$. Then

$$\mathcal{L}_\tau W_\lambda = \left\langle W, \mathcal{L}_\tau^{(1)} \lambda \right\rangle \tag{4.19}$$

where

$$\begin{aligned} (\mathcal{L}_\tau^{(1)}\lambda)^{NM}(\tau, t, \omega) &= -4\omega^2 \alpha_{mm}(N-M)^2 \lambda^{NM}(t,\omega) \\ +2\omega^2 \alpha_{nn} \Big\{ &(M+1)(N+1)\lambda^{N+1,M+1}(t-2\tau,\omega) \\ &-(M^2+N^2)\lambda^{NM}(t,\omega) \\ &+(M-1)(N-1)\lambda^{N-1,M-1}(t+2\tau,\omega) \Big\} \end{aligned} \tag{4.20}$$

The coefficients α_{mm} and α_{nn} in (4.20) are given by

$$\alpha_{mm} = \alpha_{mm}(\tau) = \frac{1}{c_0(\tau)} \int_0^\infty E\left\{m(z)m(0)\right\} dz,$$

$$\alpha_{nn} = \alpha_{nn}(\tau) = \frac{1}{c_0(\tau)} \int_0^\infty E\{n(z)n(0)\}\, dz \tag{4.21}$$

Note that this notation differs from the one in [4] where the factor c_0^{-1} is not included in the definition of α_{mm} and α_{nn}.

Instead of working out the form of the generator $\mathcal{L}_\tau$ for general $F(W_\lambda)$, not necessarily the linear functions (4.19), we will give it for multilinear functions of arbitrary degree p. For $p > 1$ the space of test function $\mathcal{S}_H^p$, the inner product and the multinomial functions $F^{(p)}(W)$ are defined by (3.13'),(3.14') and (3.15') respectively. For such functions $F^{(p)}$ the generator $\mathcal{L}_\tau$ takes the form

$$\mathcal{L}_\tau F^{(p)}(W) = \left\langle W \otimes ... \otimes W, \mathcal{L}_\tau^{(p)} \lambda^{(p)} \right\rangle \tag{4.22}$$

where

$$(\mathcal{L}_\tau^{(p)}\lambda^{(p)})^{N_1M_1N_2M_2...N_pM_p}(t_1,\omega_1,t_2,\omega_2,...,t_p,\omega_p)$$
$$= \sum_{i=1}^{p} \mathcal{L}_\tau^{(1),i}\lambda^{(p)} + \sum_{1\le i<j\le p} B_\tau^i B_\tau^j \lambda^{(p)} \tag{4.23}$$

In (4.23) $\mathcal{L}_\tau^{(1),i}$ denotes the generator $\mathcal{L}_\tau^{(1)}$ of (4.20) acting on $\lambda^{(p)}$ as a function of its i-th indices N_i, M_i and its i-th arguments t_i, ω_i. The operators B_τ^i act multiplicatively and

$$B_\tau^i = 2\sqrt{-1}\sqrt{2\alpha_{mm}}\,\omega_i(M_i - N_i) \tag{4.24}$$

The imaginary unit is written as $\sqrt{-1}$ to avoid notational conflict with the indices.

5 Uniqueness

The laws P^ε of $W^\varepsilon(\tau), -L \le \tau \le 0$ on $X = C([-L,0];\mathcal{S}_H')$ have a weakly convergent subsequence which converges to a law P for which

$$\langle W(s),\lambda\rangle - \langle W(\tau),\lambda\rangle - \int_\tau^s \left\langle W(\gamma), \mathcal{L}_\gamma^{(1)}\lambda \right\rangle d\gamma \tag{5.1}$$

is a martingale for each $\lambda \in \mathcal{S}_H$ and with $\mathcal{L}_\tau^{(1)}$ defined by (4.20). This follows from (4.18) and (4.19). The same is true for multinomial functionals of degree p defined by (3.15'), in view of (4.22), (4.23), so that

$$\left\langle W^{(p)}(s), \lambda^{(p)} \right\rangle - \left\langle W^{(p)}(\tau), \lambda^{(p)} \right\rangle - \int_\tau^s \left\langle W^{(p)}(\gamma), \mathcal{L}_\gamma^{(p)}\lambda^{(p)} \right\rangle d\gamma \tag{5.2}$$

is also a P martingale. We will show that these properties characterize the measure P uniquely.

If in the definition (4.20) of $\mathcal{L}_\tau^{(1)}$ we replace $\lambda^{NM}(t,\omega)$ by $\lambda^{NM}(t+(N+M)\tau,\omega)$, which we will also denote by $\lambda^{NM}(t,\omega)$, and if λ^{NM} depends explicitly on τ as well then,

$$\left(\frac{\partial}{\partial\tau} + \mathcal{L}_\tau^{(1)}\right)\lambda^{NM}(\tau,t,\omega) = \frac{\partial}{\partial\tau}\lambda^{NM}(\tau,t,\omega) + (N+M)\frac{\partial}{\partial t}\lambda^{NM}(\tau,t,\omega)$$
$$-4\omega^2\alpha_{mm}(N-M)^2\lambda^{NM}(\tau,t,\omega) + 2\omega^2\alpha_{nn}\Big\{(M+1)(N+1)\lambda^{N+1M+1}(\tau,t,\omega)$$
$$-(M^2+N^2)\lambda^{NM}(\tau,t,\omega) + (M-1)(N-1)\lambda^{N-1M-1}(\tau,t,\omega)\Big\} \tag{5.3}$$

The process W is defined by (3.15) relative to test functions λ in $\mathcal{S}_H$ and since the quantity of physical interest is W at $\tau = 0$, it makes no difference if we shift the t argument of λ by $(N+M)\tau$ as above. We will assume in the sequel that W has been defined this way.

Suppose now that $\lambda \in \mathcal{S}_H$ and that we can find a $\lambda(\tau) \in \mathcal{S}_H, -L \leq \tau \leq 0$ such that

$$\left(\frac{\partial}{\partial\tau} + \mathcal{L}_\tau^{(1)}\right)\lambda = 0 \tag{5.4}$$

with $\lambda(0)$ equal to the given $\lambda \in \mathcal{S}_H$. Then since

$$\langle W(0), \lambda(0)\rangle - \langle W(-L), \lambda(-L)\rangle - \int_{-L}^{0} \left\langle W(s), (\frac{\partial}{\partial s} + \mathcal{L}_s^{(1)})\lambda(s)\right\rangle ds \tag{5.5}$$

is a bounded martingale, we conclude that

$$E^P\left\{\langle W(0), \lambda(0)\rangle\right\} = \langle W(-L), \lambda(-L)\rangle \tag{5.6}$$

This identifies the expectation of $W(0), E^P\{W(0)\} = \bar{W}(0)$, independently of the law P.

We see therefore that unique characterization of the expectation of the law P hinges on our ability to construct solutions $\lambda(\tau) \in \mathcal{S}_H$ of (5.4) with $\mathcal{L}_\tau^{(1)}$ given by (5.3). This is for the case $p = 1$ in (5.2) but the case $p > 1$ is relatively easy once the $p = 1$ is understood. By constructing a test function in $\mathcal{S}_H^p$ such that

$$\left(\frac{\partial}{\partial\tau} + \mathcal{L}_\tau^{(p)}\right)\lambda^{(p)} = 0 \tag{5.7}$$

for $-L \leq \tau \leq 0$ with $\lambda^{(p)}(0)$ equal to the p-th tensor product of $\lambda \in \mathcal{S}_H$ we conclude that

$$\begin{aligned} E^P\left\{\left\langle W^{(p)}(0), \lambda^{(p)}(0)\right\rangle\right\} &= E^P\left\{(\langle W(0), \lambda\rangle)^p\right\} \\ &= \left\langle W^{(p)}(-L), \lambda^{(p)}(-L)\right\rangle \end{aligned} \tag{5.8}$$

All moments of the bounded random variable $\langle W(0), \lambda\rangle$, $\lambda \in \mathcal{S}_H$, are identified this way and this identifies the law P. In the next section we will study in detail equations (5.4) and (5.7) and properties of their solutions.

6 Random walk representation and duality

The frequency ω is just a parameter in (4.20) or (5.3) so we will suppress dependence on it and let

$$2\omega^2\alpha_{nn} = \alpha > 0, 4\omega^2\alpha_{mm} = \beta > 0. \tag{6.1}$$

We will also assume that α and β are constants, shift the interval $(-L, 0)$ to $(0, T)$ and drop the subscript τ from $\mathcal{L}^{(1)}$ in this section. With this notation and after a slight rearrangement (5.4) becomes

$$\begin{aligned} &\frac{\partial}{\partial\tau}\lambda^{NM}(\tau,t) + (N+M)\frac{\partial}{\partial t}\lambda^{NM}(\tau,t) - (\beta+\alpha)(N-M)^2\lambda^{NM}(\tau,t) \\ &\quad +\alpha\left\{(M+1)(N+1)\lambda^{N+1M+1}(\tau,t) - 2NM\lambda^{NM}(\tau,t)\right. \\ &\quad \left. +(M-1)(N-1)\lambda^{N-1M-1}(\tau,t)\right\} = 0 \end{aligned} \tag{6.2}$$

for $0 \leq \tau \leq T$ with terminal conditions

$$\lambda^{NM}(T,t) = \lambda_0^{NM}(t) \tag{6.3}$$

Here λ_0 belongs to $\mathcal{S}_H$, with ω -dependence suppressed. In Appendix A we show that if the terminal data $\lambda_0 \in \mathcal{S}_H$ then $\lambda(0) \in \mathcal{S}_H$.

Let Δ_D be the generator

$$\Delta_D \lambda^{NM} = \alpha MN \left\{ \lambda^{N+1M+1} - 2\lambda^{NM} + \lambda^{N-1M-1} \right\} \tag{6.4}$$

and let V be the diagonal operator

$$V\lambda^{NM} = -(\beta+\alpha)(N-M)^2 \lambda^{NM} \tag{6.5}$$

If we denote by Δ_D^* the adjoint of Δ_D relative to the inner product (3.14) then (6.2) becomes

$$\frac{\partial}{\partial \tau} \lambda^{NM}(\tau,t) + (N+M)\frac{\partial}{\partial t}\lambda^{NM}(\tau,t) + (\Delta_D^* + V)\lambda^{NM}(\tau,t) = 0 \tag{6.6}$$

in $0 \leq \tau \leq T$ with the terminal condition (6.3). We note now that the generator (6.4) is the infinitesimal generator of a continuous time random walk on the nonnegative integers with the lines $N = 0$ and $M = 0$ absorbing sets and that $N(\tau) - M(\tau)$ is constant and equal to the initial value $N - M$ for all $\tau > 0$. Let $(N(\tau), M(\tau))$ be the process generated by Δ_D of (6.4) up to a blow up time T_∞. We will show that $T_\infty = \infty$ with probability one. Then we will analyze the existence, uniqueness and probabilistic representation of solutions of (6.2),(6.3). We prove that:

The blowup time T_∞ of the random walk $(N(\tau), M(\tau))$ generated by Δ_D in (6.4) is infinite with probability one.

There are many ways to prove this simple fact. We will give a proof based on a duality argument that we will introduce first. To motivate this duality recall that we are attempting to characterize the limit law of the process W^{NM} defined formally by (3.12) and rigorously by (3.15). Now the reflection coefficient $\hat{R}$ is a complex number of modulus less than or equal to one so it can be represented in the form

$$\hat{R} = \tanh\frac{\theta}{2} e^{i\phi} \tag{6.7}$$

with $\theta \geq 0$ and $0 \leq \phi \leq 2\pi$. Let

$$U^{NM}(\theta,\phi) = (\tanh\frac{\theta}{2}e^{i\phi})^N (\tanh\frac{\theta}{2}e^{-i\phi})^M$$

$$= (\tanh\frac{\theta}{2})^{N+M} e^{i(N-M)\phi} \tag{6.8}$$

and introduce the diffusion generator

$$\Delta_C = \alpha\left[\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}(\sinh\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}\right] + (\beta+\alpha)\frac{\partial^2}{\partial\phi^2} \tag{6.9}$$

acting on smooth functions of $\theta \geq 0$ and $\phi \in [0, 2\pi]$. If in the unit disc $|\hat{R}| \leq 1$ we introduce polar coordinates (6.7) with the metric

$$|dR|^2 = \sinh^2\theta (d\theta)^2 + (d\phi)^2 \tag{6.10}$$

and volume element

$$dv = \sinh\theta d\theta d\phi, \tag{6.11}$$

then we can write Δ_C in the form

$$\Delta_C = \alpha\Delta + (\beta+\alpha)\frac{\partial^2}{\partial\phi^2} \tag{6.12}$$

where Δ is the Laplace-Beltrami operator on the unit disc with metric (6.10), the hyperbolic disc.

The key observation in duality is the identity

$$\Delta_C U^{NM}(\theta,\phi) = (\Delta_D + V)U^{NM}(\theta,\phi) \tag{6.13}$$

where Δ_C acts on U^{NM} as a function of θ,ϕ and Δ_D+V as a function of the discrete indices N,M. This identity is easily verified by direct calculation and with be help of the identity

$$\frac{4}{\sinh^2\theta} = \tanh^2\frac{\theta}{2} - 2 + \frac{1}{\tanh^2\frac{\theta}{2}}$$

The subscripts C and D on Δ stand for discrete and continuous, respectively.

Now the diffusion generator (6.12) generates a process $(\theta(\tau),\phi(\tau))$ on the hyperbolic disc and it is easily shown that this process has two simple properties as $\tau\to\infty$. First, $\theta(\tau)\to\infty$ with probability one and this can be seen from the Ito stochastic equation for $\theta(\tau)$

$$d\theta(\tau) = \alpha\coth\theta d\tau + \sqrt{2\alpha}db(\tau)$$

that comes from (6.9). Here $b(\tau)$ is the standard one dimensional Brownian motion. The positive drift in this equation tends asymptotically to a constant and this gives the result $\tau^{-1}\theta(\tau)\to\alpha$ with probability one as $\tau\to\infty$. A general class of asymptotic results like this are given in [7]. Second, $\phi(\tau)$ is asymptotically independent from $\theta(\tau)$ and for any $n=0,\pm1,\pm2,\ldots$

$$E\left\{e^{in(\phi(\tau)-\phi(0))}\right\}e^{n^2(\alpha+\beta)\tau}\to 1 \tag{6.14}$$

as $\tau\to\infty$. This again follows from (6.9) and as a special case we have that $\phi(\tau)$ tends weakly to a uniform random variable as $\tau\to\infty$. We will use these properties in what follows.

Because of identity (6.13) the process

$$U^{N(\tau)M(\tau)}(\theta(T-\tau),\phi(T-\tau))e^{\int_0^\tau V(N(s),M(s))ds} \tag{6.15}$$

$0\le\tau\le T$, is a local martingale. Note that from the definition (6.5) V depends only on the difference $N-M$ and so it is a constant, independent of τ with probability one. We will however use the expression (6.15) without this simplification until it is needed. Let

$$T_K = \inf\{\tau>0 | N(\tau)+M(\tau)>K\} \tag{6.16}$$

which is a stopping time. Then the martingale property and the boundedness of $U^{NM}(\theta,\phi)$ gives

$$E_{NM}\left\{e^{\int_0^{T\wedge T_K} V(N(s),M(s))ds}U^{N(T\wedge T_K)M(T\wedge T_K)}(\theta(T-T\wedge T_K),\phi(T-T\wedge T_K))\right\}$$
$$= E_{\theta,\phi}\left\{U^{NM}(\theta(T),\phi(T))\right\}. \tag{6.17}$$

Since the integrands are bounded and T_K increases to T_∞ as $K\to\infty$ we can take the limit in (6.17) to get

$$E_{NM}\left\{e^{\int_0^{T\wedge T_\infty} V(N(s),M(s))ds}U^{N(T\wedge T_\infty)M(T\wedge T_\infty)}(\theta(T-T\wedge T_\infty),\phi(T-T\wedge T_\infty))\right\}$$
$$= E_{\theta,\phi}\left\{U^{NM}(\theta(T),\phi(T))\right\} \tag{6.18}$$

For $\theta > 0$ and fixed, $|U^{NM}(\theta,\phi)| < 1$ so (6.18) can be written in the form

$$E_{NM}\left\{e^{\int_0^T V(N(s),M(s))ds} U^{N(T)M(T)}(\theta,\phi), T \le T_\infty\right\}$$
$$= E_{\theta,\phi}\left\{U^{NM}(\theta(T),\phi(T))\right\} \tag{6.19}$$

As already noted, it is easily verified from the form (6.4) of the generator Δ_D that the lines $N - M = const$ are invariant sets for the random walk and hence (6.19) has also the form

$$E_{NM}\left\{(\tanh\frac{\theta}{2})^{N(T)+M(T)}, T \le T_\infty\right\}$$
$$= E_{\theta\phi}\left\{(\tanh\frac{\theta(T)}{2})^{N+M} e^{i(N-M)(\phi(T)-\phi(0))}\right\} e^{(\beta+\alpha)(N-M)^2 T} \tag{6.20}$$

Taking the limit $T \to \infty$ in (6.20) and recalling that $\theta(T) \to +\infty$ with probability one along with (6.14) we get

$$1 = \lim_{T\to\infty} E_{NM}\left\{(\tanh\frac{\theta}{2})^{N(T)+M(T)}, T \le T_\infty\right\}$$
$$\le \lim_{T\to\infty} P_{NM}\left\{T \le T_\infty\right\} \le 1.$$

This means that

$$\lim_{T\to\infty} P_{NM}\left\{T \le T_\infty\right\} = 1 \tag{6.21}$$

and hence $T_\infty = +\infty$ with probability one. This completes the proof of the statement above (6.7).

In the course of proving it we have introduced the duality given by (6.18) with $T_\infty = \infty$

$$E_{NM}\left\{e^{\int_0^T V(N(s),M(s))ds} U^{N(T)M(T)}(\theta,\phi)\right\}$$
$$= E_{\theta,\phi}\left\{U^{NM}(\theta(T),\phi(T))\right\} \tag{6.22}$$

Let $w^{NM}(T)$ denote the left member of this identity, with θ and ϕ suppressed in the notation. Then by the Feynman-Kac formula, w^{NM} satisfies the equation

$$\frac{\partial w^{NM}}{\partial \tau} = \Delta_D w^{NM} + V w^{NM}, \tau > 0,$$
$$w^{NM}(0) = U^{NM}(\theta,\phi) \tag{6.23}$$

We can also solve probabilistically the equation

$$\frac{\partial w^{NM}}{\partial \tau} + (N+M)\frac{\partial w^{NM}}{\partial t} = (\Delta_D + V)w^{NM}, \tau > 0, N, M \ge 0$$
$$w^{NM}(0,t) = w_0^{NM}(t) \tag{6.24}$$

provided $w_0^{NM}(t)$ is uniformly bounded and smooth. Let

$$t(\tau) = t - \int_0^\tau (N(s)+M(s))ds \tag{6.25}$$

Then

$$w^{NM}(\tau,t) = E_{NM}\left\{e^{\int_0^\tau V(N(s),M(s))ds} w_0^{N(\tau),M(\tau)}(t(\tau))\right\} \tag{6.26}$$

Similarly, the equation (6.24) with the boundary condition

$$w^{NM}(\tau,t) = 0, N + M \geq K \tag{6.24'}$$

can be solved using stopping times. Let T_K be defined by (6.16). Then (6.24-6.24') has the probabilistic representation

$$w^{NM}(\tau,t) = E_{NM}\left\{e^{\int_0^{\tau\wedge T_K} V(N(s),M(s))ds} \cdot w_0^{N(\tau\wedge T_K)M(\tau\wedge T_K)}(t(\tau\wedge T_K))\right\} \tag{6.27}$$

We are now ready to return to equations (6.2),(6.3). We can rewrite them in the form

$$\frac{\partial\lambda^{NM}(\tau,t)}{\partial\tau} + (N+M)\frac{\partial\lambda^{NM}(\tau,t)}{\partial t} + (\Delta_D^* + V)\lambda^{NM}(\tau,t) = 0$$

$$0 \leq \tau \leq T, N, M \geq 0, \tag{6.2'}$$

$$\lambda^{NM}(T,t) = \lambda_0^{NM}(t) \tag{6.3'}$$

It is easily verified that

$$\frac{\partial}{\partial\tau}\int dt \sum_{NM} \lambda^{NM}(\tau,t)w^{MN}(\tau,t) = 0$$

Therefore,

$$\sum_{NM}\int dt\lambda_0^{NM}(t)w^{MN}(T,t) = \sum_{NM}\int \lambda^{NM}(0,t)w^{MN}(0,t) \tag{6.28}$$

Existence of solutions for (6.2')-(6.3') is shown either directly as for (6.24) by writing Δ_D^* as a generator plus a potential or by using the duality (6.28). The fact that solutions are in $\mathcal{S}_H$ for $0 \leq \tau \leq T$ if the terminal data are in $\mathcal{S}_H$ is shown in the Appendix.

We can now return to (5.6) and complete the identification of $\bar{w}(0) = E^P\{W(0)\}$. If $\hat{R}(-L,\omega) =$, with $|,| \leq 1$ then from (3.12) we see that

$$\bar{w}^{NM}(-L,t,\omega) = 2\pi\delta(t), {}^N, {}^{*M} \tag{6.29}$$

Recalling the convention about translating $(-L,0)$ to $(0,T)$ stated below (6.1) we see that (5.6) is another form of (6.28) and therefore $w^{NM}(L,\tau) = w^{NM}(L,\tau,\omega)$ (when the ω -dependence of α and β in (6.1) is recalled) solves equation (6.24) i.e.,

$$\frac{\partial\bar{w}^{NM}}{\partial\tau} + (N+M)\frac{\partial\bar{w}^{NM}}{\partial t} - (\Delta_D + V)\bar{w}^{NM} = 0, -L < \tau \leq 0, N, M \geq 0,$$

$$\bar{w}^{NM}(-L,t,\omega) = 2\pi\delta(t), {}^N, {}^{*M} \tag{6.30}$$

This problem has a unique distribution solution (i.e. $\bar{w} \in \mathcal{S}'_H$) with the probabilistic representation (6.26) i.e.,

$$\bar{w}^{NM}(0,t,\omega) = 2\pi E_{NM}\left\{(\tanh\frac{\theta}{2})^{N(0)+M(0)}\delta(t(0))\right\} e^{i(N-M)\phi}e^{-(\alpha+\beta)(N-M)^2L} \tag{6.31}$$

The δ function should be replaced by a unit step function in (6.31) and then the distribution derivative with respect to t of this expression is the distribution solution of (6.30).

We will end this section with two important observations that follow easily from the probabilistic representation (6.31) and the duality (6.22). First, $\bar{w}^{NM}(0,t,\omega)$ tends to zero as $L \to \infty$ when

$N \neq M$. This is clear from (6.31). The δ -function in the initial conditions can be replaced by any smooth function $g_0(t)$ of compact support for this argument. Therefore

$$\bar{w}^{NM}(0,t,\omega) \to 0, L \to \infty, N \neq M. \tag{6.32}$$

Let us then concentrate on the diagonal part of $\bar{w}^{NM}$ which we denote by $\bar{w}^{N}$. It satisfies (6.30) which is now

$$\frac{\partial \bar{w}^N}{\partial \tau} + 2N\frac{\partial \bar{w}^N}{\partial t} - \alpha N^2 \left\{ \bar{w}^{N+1} - 2\bar{w}^N + \bar{w}^{N-1} \right\} = 0, -L < \tau \leq 0, N \geq 0,$$

$$\bar{w}^N(-L,t,\omega) = 2\pi\delta(t)|,|^{2N} \tag{6.33}$$

The probabilistic representation of the solution is

$$\bar{w}^N(0,t,\omega) = 2\pi E_N \left\{ (\tanh\frac{\theta}{2})^{2N(0)}\delta(t(0)) \right\} \tag{6.34}$$

which we write with the δ function as explained below (6.31). Recall that $N(\tau)$ is now the process generated by Δ_D in (6.4) in $-L \leq \tau \leq 0$ with $N = M, t(\tau)$ is given by (6.25) with $N = M$ (with the shifted process $N(\tau)$) and $|,| = \tanh\frac{\theta}{2}$ in polar coordinates (6.7).

The second observation we shall make is that if $L > t/2$ then $\bar{w}^N(0,t,\omega)$, the solution of (6.33), is independent of θ, or , , the initial reflection coefficient. To see this we recall that if $N = M$ initially then $N(\tau) = M(\tau)$ for all τ, so the line $N = M$ is invariant for the random walk. Next we show that when $N = M$ then $N(0) \to 0$ as $L \to \infty$. This follows from the special form of the duality identity (6.22) which we rewrite for the shifted process in $(-L, 0)$

$$E_N \left\{ (\tanh\frac{\theta}{2})^{2N(0)} \right\} = E_\theta \left\{ (\tanh\frac{\theta(L)}{2})^{2N} \right\}$$

Since $\theta(L) \to \infty$ as $L \to \infty$ with probability one we conclude that

$$\lim_{L\to\infty} E_N \left\{ (\tanh\frac{\theta}{2})^{2N(0)} \right\} = 1 \tag{6.35}$$

This implies that $N(0) \to 0$ with probability one as $L \to \infty$ because $\theta < \infty$ in (6.35).

We prove that $\bar{w}^N(0,t,\omega)$ does not depend on θ for $L > \frac{t}{2}$ as follows. In (6.34) $t(0)$ is given by

$$t(0) = t - 2\int_{-L}^{0} N(s)ds \tag{6.36}$$

and we may assume that $N = N(-L) \geq 1$ because $N = 0$ is an absorbing point. Suppose $L > t/2$. If $N(\tau) \geq 1$ in $-L \leq \tau \leq 0$ then $t(0)$ will be negative and the delta function in (6.34) will not play a role for larger L. So the paths with $N(\tau) \geq 1$ will not contribute to the expectation (6.34). If $N(\tau) = 0$ for some τ in $[-L, 0]$ then it will remain zero for all subsequent τ and hence $N(0) = 0$. Only such events contribute to (6.34) so for $L > t/2$ and $N \geq 1$

$$\bar{w}^N(0,t,\omega) = 2\pi E_N \left\{ \delta(t(0)) \right\}$$

which is manifestly independent of θ.

The two observations about $\bar{w}^{NM}$ that we made above (and proved using duality) translate to similar observations for the solution (6.2'),(6.3') of the adjoint equation via the identity (6.28).

At this point we may ask if it is possible to represent the solution of $\bar{w}^{NM}$ by a dual formula. We have of course the representation (6.31). Does there exist then some more complicated process similar to (θ,ϕ) for which some form of (6.22) holds for$\bar{w}^{NM}$? The answer is that there does indeed exist this possibility but it is not as useful as the representation (6.31) because one cannot readily see from it the $\theta-$independence property. Note that we can write (6.34) in the form

$$\bar{w}^N(0,t,\omega) = \int_{-\infty}^{\infty} d\xi e^{i\xi t} E_N \left\{ (\tanh\frac{\theta}{2})^{2N(0)} e^{-2i\xi\int_{-L}^{0} N(s)ds} \right\} \tag{6.37}$$

which follows from the Fourier integral representation of the δ function and The expectation is the joint generating function-characteristic function of $N(\tau)$ and its integral and there seems to be no nice duality for the pair.

To complete the uniqueness proof we must also show how to construct smooth solutions of

$$\left(\frac{\partial}{\partial\tau} + \mathcal{L}_\tau^{(p)}\right)\lambda^{(p)} = 0 \tag{6.38}$$

with $\lambda^{(p)}(0)$ given in $\mathcal{S}_H^p$ for $p>1$. The case $p=2$ is typical. From (4.23) we see that

$$\mathcal{L}_\tau^{(2)}\lambda^{(2)} = \mathcal{L}_\tau^{(1),1}\lambda^{(2)} + \mathcal{L}_\tau^{(1),2}\lambda^{(2)} + B_\tau^{(1)}B_\tau^{(2)}\lambda^{(2)} \tag{6.39}$$

where $\mathcal{L}_\tau^{(1),i}$ for $i=1,2$ is the operator $\mathcal{L}_\tau^{(1)}$ defined by (4.20) and acting only on the variables t_i,ω_i,N_i,M_i of $\lambda^{(2)}$. The multiplicative operators B_τ are defined by (4.24). It is enough to solve (6.38) with $\lambda^{(2)}(0)$ a product of matrices in $\mathcal{S}_H^{(1)}$ because linear combinations of such products are dense in $\mathcal{S}_H^{(2)}$. But the solution of (6.38) can be written explicitly as the product of two solutions of (6.2), (6.3) multiplied by an exponential factor

$$\lambda^{N_1M_1N_2M_2}(\tau,t_1,\omega_1,t_2,\omega_2) =$$

$$\lambda^{N_1M_1}(\tau,t_1,\omega_1)\lambda^{N_2M_2}(\tau,t_2,\omega_2)e^{8\alpha_{mm}\omega_1\omega_2(N_1-M_1)(N_2-M_2)\tau} \tag{6.40}$$

This completes the proof of uniqueness.

7 Representation of the W process

In the previous section we saw how the measure P for which (5.1) and (5.2) are martingales is uniquely determined and identified on linear and multilinear functionals of the process. In fact the expectation of W, $\bar{w} = E^P\{W\}$, is the unique solution of (6.30) in $\mathcal{S}'_H$, for any limit law P.

We will now use the martingale (5.2) with $p=2$ to show that (5.1) is a Brownian martingale of the form

$$\int_\tau^s \langle W(\gamma), B_\gamma\lambda\rangle\, db(\gamma) \tag{7.1}$$

where $b(\gamma)$ is the standard, one-dimensional Brownian motion and B_τ is the multiplicative generator defined by (4.24)

$$(B_\tau\lambda)^{NM}(t,\omega) = 2i\sqrt{2\alpha_{mm}}\omega(N-M)\lambda^{NM}(t,\omega) \tag{7.2}$$

with $\lambda\in\mathcal{S}_H$. To see that (5.1) has the form (7.1) we note that if

$$M_\lambda(s,\tau) = \langle W(s),\lambda\rangle - \langle W(\tau),\lambda\rangle - \int_\tau^s \left\langle W(\gamma), \mathcal{L}_\gamma^{(1)}\lambda\right\rangle d\gamma. \tag{7.3}$$

is the martingale (5.1) then

$$M_{\lambda_1}(s,\tau)M_{\lambda_2}(s,\tau) - \int_\tau^s \left\langle W(\gamma)\otimes W(\gamma), \mathcal{L}_\gamma^{(2)}\lambda_1\lambda_2 - \lambda_1\mathcal{L}_\gamma^{(1)}\lambda_2 - \lambda_2\mathcal{L}_\gamma^{(1)}\lambda_1\right\rangle d\gamma \tag{7.4}$$

is also a martingale [16], where $\mathcal{L}_\tau^{(2)}$ is defined by (4.23). But

$$\mathcal{L}_\gamma^{(2)}\lambda_1\lambda_2 - \lambda_1\mathcal{L}_\gamma^{(1)}\lambda_2 - \lambda_2\mathcal{L}_\gamma^{(1)}\lambda_1 = B_\gamma\lambda_1 B_\gamma\lambda_2 \tag{7.5}$$

and hence

$$E\left\{M_{\lambda_1}(s,\tau)M_{\lambda_2}(s,\tau)\right\} = E\left\{\int_\tau^s \langle W(\gamma), B_\gamma\lambda_1\rangle \cdot \langle W(\gamma), B_\gamma\lambda_2\rangle\, d\gamma\right\} \tag{7.6}$$

Since $M_\lambda(s,\tau)$ is P almost surely continuous as a function of s for each $\lambda \in \mathcal{S}_H$, it is bounded and (7.6) holds we conclude that it is indeed the Brownian martingale (7.1).

We can write (5.1) in the form

$$\langle W(s), \lambda\rangle = \langle W(\tau), \lambda\rangle + \int_\tau^s \left\langle W(\gamma), \mathcal{L}_\gamma^{(1)}\lambda\right\rangle d\gamma + \int_\tau^s \langle W(\gamma), B_\gamma\lambda\rangle\, db(\gamma) \tag{7.7}$$

for λ in $\mathcal{S}_H$. and the multinomial version in the form

$$\left\langle W^{(p)}(s), \lambda\right\rangle = \left\langle W^{(p)}(\tau), \lambda\right\rangle + \int_\tau^s \left\langle W^{(p)}(\gamma), \mathcal{L}_\gamma^{(p)}\lambda\right\rangle d\gamma + \int_\tau^s \left\langle W^{(p)}(\gamma), \sum_{i=1}^p B_\gamma^i\lambda\right\rangle db(\gamma).$$

We can also write the distribution solution $\bar{w}^{NM}(\tau,t,\omega)$ of (6.30) in the probabilistic form (6.31) or

$$\bar{w}^{NM}(\tau,t,\omega) = \tilde{w}^{NM}(\tau,t,\omega)e^{-\beta(N-M)^2(\tau+L)} \tag{7.8}$$

where β is defined by (6.1) and $\tilde{w}^{NM}$ is given by

$$\tilde{w}^{NM}(\tau,t,\omega) = 2\pi E_{NM}\left\{(\tanh\frac{\theta}{2})^{N(\tau)+M(\tau)}\delta(t(\tau))\right\} e^{i(N-M)\phi-\alpha(N-M)^2(\tau+L)} \tag{7.9}$$

and is the distribution solution of

$$\begin{aligned} &\frac{\partial\tilde{w}^{NM}}{\partial\tau} + (N+M)\frac{\partial\tilde{w}^{NM}}{\partial t} + \alpha(N-M)^2\tilde{w}^{NM} \\ &-\alpha NM\left\{\tilde{w}^{N+1M+1} - 2\tilde{w}^{NM} + \tilde{w}^{N-1M-1}\right\} = 0 \end{aligned} \tag{7.10}$$

for $-L < \tau \le 0$ with

$$\tilde{w}^{NM}(-L,t,\omega) = 2\pi\delta(t)(\tanh\frac{\theta}{2})^{N+M}e^{i(N-M)\phi} \tag{7.11}$$

Let

$$W^{NM}(\tau,t,\omega) = \tilde{w}^{NM}(\tau,t,\omega)e^{i\sqrt{2\beta}(N-M)(b(\tau)-b(-L))} \tag{7.12}$$

and recall that the martingale

$$\bar{M}(\tau) = e^{i\sqrt{2\beta}(N-M)(b(\tau)-b(-L))+\beta^2(N-M)^2(\tau+L)} \tag{7.13}$$

satisfies

$$d\bar{M}(\tau) = i\sqrt{2\beta}(N-M)\bar{M}(\tau)db(\tau)$$

Then we see that $W^{NM}(\tau,t,\omega)$ of (7.12) has the correct mean $\bar{w}^{NM}(\tau,t,\omega)$ given by (7.8) because the bounded martingale (7.13) has mean value equal to one. In fact all moments of $W^{NM}(\tau,t,\omega)$ in (7.12) agree with those of the W process so the two have the same law and this is the meaning of the equality in (7.12). There is no implication of strong uniqueness for (7.7), viewed as a stochastic differential equation, in our results.

A Appendix. Bounds for solutions of the transport equations

In this Appendix we will analyze solutions of the transport equations (6.2) which we write again here:

$$\frac{\partial}{\partial\tau}\lambda^{NM} + (N+M)\frac{\partial}{\partial t}\lambda^{NM} - (\beta+\alpha)(N-M)^2\lambda^{NM} \tag{A.1}$$
$$+\alpha\left\{(M+1)(N+1)\lambda^{N+1M+1} - 2NM\lambda^{NM} + (M-1)(N-1)\lambda^{N-1M-1}\right\} = 0,$$

for $0 \le \tau \le T$ with terminal conditions

$$\lambda^{NM}(T,t) = \lambda_0^{NM}(t)$$

and where α and β are positive constants. The expression in the braces in the second line of (A.1) can be written as a generator plus a potential and then the maximum principle for (A.1) gives the estimate

$$|\lambda^{NM}(0,t)| \le e^{2\alpha T}\sup_t \sup_{N,M\ge 0}|\lambda_0^{NM}(t)| \tag{A.2}$$

We want to show that if for any $p \ge 1$ and $q \ge 1$

$$\sup_t \sup_{N,M\ge 0}(1+t^2)^{p/2}(NM)^q|\lambda_0^{NM}(t)| < \infty, \tag{A.3}$$

then

$$\sup_t \sup_{N,M\ge 0}(1+t^2)^{p/2}(NM)^q|\lambda^{NM}(0,t)| \le C_{p,q}(T), \tag{A.4}$$

where $C_{p,q}(T)$ is a constant that depends on p and q, the bound (A.3) and T. Bounds like (A.4) for t derivatives of solutions of (A.1) follow immediately if (A.3) holds with t derivatives since equations (A.1) have constant in t coefficients.

We will first prove (A.4) with $p = 0$ and $q = 1$. Let $v^{NM} = NM\lambda^{NM}$. They satisfy the equations

$$\frac{\partial}{\partial\tau}v^{NM} + (N+M)\frac{\partial}{\partial t}v^{NM} - (\beta+\alpha)(N-M)^2 v^{NM}$$
$$+\alpha NM\left\{v^{N+1M+1} - 2v^{NM} + v^{N-1M-1}\right\} = 0 \tag{A.5}$$

If (A.3) holds with $p = 0$ and $q = 1$ then, since the expression in the braces in (A.5) is a generator, $|v^{NM}(0,t)|$ is bounded and hence (A.4) holds with $p = 0$ and $q = 1$.

For the more general case $p = 0$, $q > 1$ let

$$\Delta(\lambda^{NM}) = \lambda^{N+1M+1} - 2\lambda^{NM} + \lambda^{N-1M-1} \tag{A.6}$$

and

$$v_q^{NM} = (NM)^q\lambda^{NM}\ , \quad q \ge 1.$$

Note that $v_q^{NM} = 0$ if $N = 0$ or $M = 0$. Then for $N, M \ge 2$, v_q^{NM} satisfy the equations

$$\frac{\partial}{\partial\tau}v_q^{NM} + (N+M)\frac{\partial}{\partial t}v_q^{NM} - (\beta+\alpha)(N-M)^2 v_q^{NM}$$
$$+\alpha(NM)^q\Delta\left(\frac{1}{(NM)^{q-1}}v_q^{NM}\right) = 0. \tag{A.8}$$

We can now split the operator on the second line of (A.8) into a generator plus a potential:

$$(NM)^q \Delta \left(\frac{1}{(NM)^{q-1}} v_q^{NM} \right) = (NM)^q \Delta_G \left(\frac{1}{(NM)^{q-1}} v_q^{NM} \right) + V_q^{NM} v_q^{NM}. \qquad (A.9)$$

Here

$$V_q^{NM} = (NM)^q \Delta \left(\frac{1}{(NM)^{q-1}} \right) \ , \quad N, M \geq 2. \qquad (A.10)$$

and Δ_G is such that the first term on the right is a generator. It can be verified by a direct computation that there is a constant C_q, depending on q only, such that

$$-(N-M)^2 + V_q^{NM} \leq C_q < \infty, \qquad (A.11)$$

for all $N, M \geq 2$. Using the decomposition (A.9) in (A.8) and the bound (A.11), the maximum principle gives (A.4) with $p = 0$.

The bound (A.4) with $p \geq 1$ follows by induction on p. Let

$$v_{qp}^{NM} = (1+t^2)^{p/2} v_q^{NM}. \qquad (A.12)$$

From (A.1) we see that v_{qp}^{NM} satisfy the equations

$$\frac{\partial}{\partial \tau} v_{qp}^{NM} + (N+M) \frac{\partial}{\partial t} v_{qp}^{NM} - (\beta + \alpha)(N-M)^2 v_{qp}^{NM}$$

$$+\alpha (NM)^{q+1} \Delta \left(\frac{1}{(NM)^q} v_{qp}^{NM} \right) - (N+M) \frac{pt}{(1+t^2)^{1/2}} v_{qp-1}^{NM} = 0. \qquad (A.13)$$

If the bound (A.4) holds for v_{qp-1}^{NM} then the term

$$-(N+M) \frac{pt}{(1+t^2)^{1/2}} v_{qp-1}^{NM}$$

is uniformly bounded for $t \in \mathbf{R}$, $N, M \geq 0$ and $0 \leq \tau \leq T$. It follows then by the maximum principle for (A.13), as for (A.8), that the bound (A.4) holds for all $p \geq 1$.

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