SIAM J. APPL. MATH.
Vol. 49, No. 2, pp. 582-607, April 1989

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PROBING A RANDOM MEDIUM WITH A PULSE*

R. BURRIDGE†, G. PAPANICOLAOU‡, P. SHENG§, AND B. WHITE§

Abstract. This paper studies the reflection of pulses from a randomly layered half space. It characterizes the statistical properties of the reflected signals at the surface in a suitable asymptotic limit in which the pulse width is large compared to the typical size of the inhomogeneous layers but small compared to the macroscopic variations of the medium properties. It is shown how this characterization can be used to get the macroscopic properties of the medium from the reflected signals with optimal removal of the effects of the random inhomogeneities within the general framework of the model.

Key words. random media

AMS(MOS) subject classification. 82A42

1. Introduction. In [1] we have studied the reflection of pulses from a one-dimensional random medium using asymptotic analysis for stochastic equations. Our basic modeling hypothesis is that the pulse duration is short compared to macroscopic timescales but long compared to the time it takes to traverse a typical inhomogeneity of the medium. This is a reasonable model when pulses are used to probe a randomly inhomogeneous, stratified half-space. The pulse must be short enough to resolve the large-scale, deterministic structure of the medium but long enough compared to the size of the irregularities so that the statistics of the reflected signals stabilize and become independent of the detailed form of the irregularities.

In this paper we take up this subject using a different formulation than that of [1], although the main mathematical tools are the same. We obtain many new results, including the solution of an inverse problem for a random medium. We now briefly describe the contents of the paper.

In § 2 we formulate the problem, introduce a scaling that makes precise our basic modeling assumptions and define the quantities of interest. In [1] we have dealt only with a homogeneous, stratified random half-space. That is, the mean density and bulk modulus of the elastic (or other) medium were constant and their fluctuations were stationary processes. This is not a useful model when we want to do probing because the structure of the medium is trivial. Nevertheless it is important to see, as in [1], that the statistics of the reflected signal acquire a universal character in the limit and to describe them. In this paper we allow general inhomogeneous, stratified random media. As in [1] we introduce a new type of stochastic process we call a windowed process. It is well suited for the description of reflected signals. The limit law of these windowed processes is given in § 3.

In the case of a homogeneous random medium we recover the universal law we have obtained [1] and in addition we give the explicit form of the power spectral density (formulas (3.9) and (3.10)) that we had been unable to calculate in [1]. Extensive numerical simulations described in [2] agreed very well with the theory of [1]. The

* Received by the editors August 6, 1987; accepted for publication (in revised form) April 20, 1988.

† Schlumberger-Doll Research, Ridgefield, Connecticut 06877.

‡ Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012. The work of this author was supported by the National Science Foundation, The Air Force Office of Scientific Research, and the Office of Naval Research.

§ Exxon Research and Engineering Company, Route 22 East, Clinton Township, Annandale, New Jersey 08801.

localization length ideas that were introduced in [2] and proved very useful in the numerical calculations have not yet been fully understood analytically (cf. [3]).

A restriction that we introduce in this paper that we did not have in [1] is that the medium be (locally) totally reflecting. Roughly, this means that the medium is really random. To analyze reflected signals up to time t, under the present theory, the medium penetrated by the pulse up to this time must be random. This allows significant analytical simplifications but is not an essential restriction. The case of reflection and transmission of pulses by a slab of stratified random medium can be handled but the analysis is complicated. A different set of questions about the transmitted pulse, its form near the front, but not its coda or its reflection, were studied in [4].

In § 3 we also describe how an inverse problem can be solved using our theory. As we point out there, this is only a first step in understanding a whole collection of issues about the statistical estimation of medium properties from reflected signals.

In § 4 we give a formal derivation of the power spectral density of the windowed process. Although intuitive and elementary, this derivation is not satisfactory because the interchanges of limits involved are difficult to justify. Physically it is unsatisfactory because, being a frequency domain calculation, it obscures the support properties of the wave functions that come from the finite propagation speed of the signals. These properties are very important for a full understanding of the phenomena.

In § 5 we introduce a new way to analyze reflected signals, in the time domain, that makes use the finite propagation speed and its consequenses. This involves the study of infinite-dimensional processes (functional processes). We give only a brief introduction to the main ideas here. In [1] we have found that the law of the reflected signals (the windowed processes) is Gaussian in the limit. This was surprising to us because in the time-harmonic case the law of the reflection coefficient is not Gaussian at all [5], [6]. The argument we used in [1] was incomplete and not fully convincing. In the framework of § 5 the Gaussian property appears rather naturally but still in a surprising way. The usual thinking that gives the Gaussian law in the central limit theorem is not applicable here. This is one reason why the functional approach is useful.

2. Formulation and scaling. We consider a one-dimensional acoustic wave propagating in a random slab of material occupying the half-space $x<0$. We will analyze in detail the backscatter at $x=0$.

Let $p(t,x)$ be the pressure and $u(t,x)$ velocity. The linear conservation laws of momentum and mass governing acoustic wave propagation are

$$\begin{aligned}&\rho(x)\frac{\partial}{\partial t}u(t,x)+\frac{\partial}{\partial x}p(t,x)=0,\\ &\frac{1}{K(x)}\frac{\partial}{\partial t}p(t,x)+\frac{\partial}{\partial x}u(t,x)=0\end{aligned} \tag{2.1}$$

where ρ is the density and K the bulk modulus. We define means of ρ, $1/K$ as

$$\rho_0=E[\rho],\qquad \frac{1}{K_0}=E\left[\frac{1}{K}\right]. \tag{2.2}$$

In the special case that ρ and K are stationary random functions of position x, ρ_0, K_0 are the constant parameters of effective medium theory. That is, a pulse of long wavelength will propagate over distances that are not too large as if in a homogeneous medium with "effective" constant parameters ρ_0, K_0, and hence with propagation speed

$$c_0=\sqrt{K_0/\rho_0}. \tag{2.3}$$

Here we consider the case where ρ_0, K_0, c_0 are not constant, but vary slowly compared to the spatial scale l_0 of a typical inhomogeneity. We may take the "microscale" l_0 to be the correlation length of ρ and $1/K$. We introduce a "macroscale" l_0/ε^2 where $\varepsilon > 0$ is a small parameter. It is on this macroscale that ρ_0, K_0, and other statistics of ρ and K are allowed to vary. Thus we write the density and bulk modulus on the macroscale in the following scaled form:

$$\begin{aligned} \rho(x) &= \rho_0\left(\frac{x}{l_0}\right)\left[1+\eta\left(\frac{x}{l_0}, \frac{x}{\varepsilon^2 l_0}\right)\right], \\ \frac{1}{K(x)} &= \frac{1}{K_0(x/l_0)}\left[1+\nu\left(\frac{x}{l_0}, \frac{x}{\varepsilon^2 l_0}\right)\right] \end{aligned} \tag{2.4}$$

where the random fluctuations η and ν have mean zero and slowly varying statistics. The mean density ρ_0 and the mean bulk modulus K_0 are assumed to be differentiable functions of x.

Equations (2.1) are to be supplemented with boundary conditions at $x=0$ corresponding to different ways in which the pulse is generated at the interface. In the cases analyzed below the pulse width is assumed to be on a scale intermediate between the microscale and the macroscale. That is, the pulse is broad compared to the size of the random inhomogeneities, but short compared to the nonrandom variations. Thus the small scale structure will introduce only random effects in which the pulse is too broad to probe in detail. In contrast, the pulse is chosen to probe the nonrandom macroscale, from which it reflects and refracts in the manner of ray theory (geometrial optics). We will recover macroscopic variations of the medium by examination of reflections at $x=0$.

Let typical values of ρ_0, K_0 be $\bar{\rho}$, $\bar{K}$ with $\bar{c}=(\bar{K}/\bar{\rho})^{1/2}$. Then for $f(t)$ a smooth function of compact support in $[0,\infty)$ we define the incident pulse by

$$f^{\varepsilon}(t) = \frac{1}{\varepsilon^{1/2}} f\left(\frac{\bar{c}t}{\varepsilon l_0}\right). \tag{2.5}$$

This pulse, f^{ε}, will be convolved with the appropriate Green's function depending on how the wave is excited at the interface. The pre-factor $\varepsilon^{-1/2}$ is introduced to make the energy of the pulse independent of the small parameter ε.

We consider first the "matched medium" boundary condition (BCI). It is assumed that the wave is incident on the random medium occupying $x<0$ from a homogeneous medium occupying $x>0$ and characterized by the constant parameters $\rho_0(0)$, $K_0(0)$. We may similarly consider an unmatched medium where ρ_0, K_0 are discontinuous at $x=0$, but we do not carry this out here. To obtain the Green function for this problem we introduce the initial boundary condition for a left-traveling wave that strikes $x=0$ at time $t=0$:

$$\begin{aligned} \text{(BCI):}\quad u &= l_0\delta\left(t+\frac{x}{c_0(0)}\right), \\ p &= -l_0\rho_0(0)c_0(0)\delta\left(t+\frac{x}{c_0(0)}\right). \end{aligned} \tag{2.6}$$

The Green function G_1 will then be a right-going wave in $x>0$ and as $x\downarrow 0$:

$$\text{(GFI):}\quad G_1 = \frac{1}{2}\left[u(t,0)+\frac{p(t,0)}{(\rho_0(0)c_0(0))}\right]. \tag{2.7}$$

The "pressure release" boundary condition (BCII) corresponds to a pressure pulse imposed at $x=0$, following which $p=0$ there. Thus to obtain the Green function we impose the boundary condition

$$\text{(BCII):}\quad p(t,0)=l_0\rho_0(0)c_0(0)\delta(t). \tag{2.8}$$

The response is taken to be the velocity at the interface:

$$\text{(GFII):}\quad G_2=u(t,0). \tag{2.9}$$

We nondimensionalize by setting

$$\begin{aligned} x'&=x/l_0, & p'&=p/\bar\rho\bar c^2,\\ t'&=\bar c t/l_0, & u'&=u/\bar c. \end{aligned} \tag{2.10}$$

By inserting (2.10) into the above equations, and dropping primes, it can be shown that without loss of generality $\bar K$, $\bar\rho$, $\bar c$, l_0 may be taken equal to unity, after K, ρ, c are replaced by their normalized forms.

For either boundary condition we will determine the statistics of the Green function convolved with the pulse f^ε. Let

$$\begin{aligned} G^\varepsilon_{t,f}(\sigma)&=(G^*f^\varepsilon)(t+\varepsilon\sigma)\\ &=\int_0^{t+\varepsilon\sigma} G(t+\varepsilon\sigma-s)f^\varepsilon(s)\,ds. \end{aligned} \tag{2.11}$$

We consider the above expression as a stochastic process in σ, with t held fixed. That is, for each t we consider a "time window" centered at t, and of duration on the order of a pulse width, with the parameter σ measuring time within this window.

For the analysis of this problem, we Fourier transform in time, choosing a frequency scale appropriate to the pulse $f^\varepsilon(t)$. Thus, letting

$$\hat f(\omega)=\int_{-\infty}^{\infty} e^{i\omega t}f(t)\,dt \tag{2.12}$$

we transform (2.1) by

$$\hat u(\omega,x)=\int e^{i\omega t/\varepsilon}u(t,x)\,dt,\qquad \hat p(\omega,x)=\int e^{i\omega t/\varepsilon}p(t,x)\,dt \tag{2.13}$$

so that

$$G^\varepsilon_{t,f}(\sigma)=\frac{1}{2\pi\varepsilon^{1/2}}\int_{-\infty}^{\infty} e^{-i\omega[t+\varepsilon\sigma]/\varepsilon}\hat f(\omega)\hat G(\omega)\,d\omega. \tag{2.14}$$

In (2.14) $\hat G$ is the appropriate combination of $\hat u$, $\hat p$ obtained by Fourier transform of either (2.7) or (2.9).

From (2.1), (2.4), (2.13), $\hat u$, $\hat p$ satisfy

$$\begin{aligned} \frac{\partial}{\partial x}\hat p&=\frac{i\omega}{\varepsilon}\rho_0(x)\left[1+\eta\left(x,\frac{x}{\varepsilon^2}\right)\right]\hat u,\\ \frac{\partial}{\partial x}\hat u&=\frac{i\omega}{\varepsilon}\frac{1}{K_0(x)}\left[1+\nu\left(x,\frac{x}{\varepsilon^2}\right)\right]\hat p. \end{aligned} \tag{2.15}$$

In the frequency domain a radiation condition as $x\to-\infty$, is required for (2.15). One way to do this is to terminate the random slab at a finite point $x=-L$, and assume the medium is not random for $x<-L$. We can later let $L\to-\infty$ but in any case the reflected signal up to a time t is not affected by how we terminate the slab at a sufficiently distant point $-L$. This is a consequence of the hyperbolicity of (2.1).

We next introduce a right-going wave A and a left going wave B, with respect to the macroscopic medium. Let the travel time in the macroscopic medium be given by

$$\tau(x)=\int_x^0 \frac{ds}{c_0(s)}, \qquad x<0. \tag{2.16}$$

We define A, B by

$$\begin{aligned} \hat{u}&=\frac{1}{(K_0\rho_0)^{1/4}}[A\,e^{-i\omega\tau/\varepsilon}+B\,e^{i\omega\tau/\varepsilon}],\\ \hat{p}&=(K_0\rho_0)^{1/4}[A\,e^{-i\omega\tau/\varepsilon}-B\,e^{i\omega\tau/\varepsilon}]. \end{aligned} \tag{2.17}$$

Putting (2.16), (2.17) into (2.15) we obtain equations for A, B. Define the random functions $m^\varepsilon(x)$ and $n^\varepsilon(x)$ by

$$\begin{aligned} m^\varepsilon(x)&=m(x,x/\varepsilon^2)=\tfrac{1}{2}[\eta(x,x/\varepsilon^2)+\nu(x,x/\varepsilon^2)],\\ n^\varepsilon(x)&=n(x,x/\varepsilon^2)=\tfrac{1}{2}[\eta(x,x/\varepsilon^2)-\nu(x,x/\varepsilon^2)]. \end{aligned} \tag{2.18}$$

Then

$$\begin{aligned} \frac{d}{dx}\begin{bmatrix}A\\B\end{bmatrix}&=\frac{i\omega}{\varepsilon}\left(\frac{\rho_0}{K_0}\right)^{1/2}\begin{bmatrix} m^\varepsilon & n^\varepsilon e^{2i\omega\tau/\varepsilon}\\ -n^\varepsilon e^{-2i\omega\tau/\varepsilon} & -m^\varepsilon\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}\\ &\quad+\frac{1}{4}\frac{(K_0\rho_0)'}{(K_0\rho_0)}\begin{bmatrix}0 & e^{2i\omega\tau/\varepsilon}\\ e^{-2i\omega\tau/\varepsilon} & 0\end{bmatrix}\begin{bmatrix}A\\B\end{bmatrix}. \end{aligned} \tag{2.19}$$

We take as boundary conditions for (2.19) that there is no right-going wave at $x=-L$, and that there is a unit left-going wave at $x=0$;

$$\begin{aligned} A(-L)&=0, \qquad B(0)=1,\\ B(-L)&=T, \qquad A(0)=R=R^\varepsilon(-L,\omega). \end{aligned} \tag{2.20}$$

Here T is the transmission coefficient for the slab, and $R^\varepsilon(-L,\omega)$ is the reflection coefficient. Now (2.20) can be seen as corresponding to (BCI), equations (2.6), (2.7). There the left-going wave at $x=0$ has amplitude $[\rho_0(0)c_0(0)]^{1/2}$; however this factor has been taken out in the definition of the right-going Green's function G_1. Thus

$$\hat{G}_1=R^\varepsilon(-L,\omega). \tag{2.21}$$

We introduce the fundamental matrix solution of the linear system (2.19). That is, let $Y(x,-L)$ satisfy (2.19) with the initial condition that $Y(-L,-L)=I$ the 2×2 identity. From symmetries in (2.19) it is apparent that if $(a,\bar{b})^T$ is a vector solution (overbar denotes complex conjugate and T transpose), then so is $(b,\bar{a})^T$. Thus

$$Y=\begin{bmatrix}a & b\\ \bar{b} & \bar{a}\end{bmatrix}. \tag{2.22}$$

Furthermore, since the system has trace zero, Y has determinant one. Hence

$$|a|^2-|b|^2=1. \tag{2.23}$$

Now the reflection coefficient R^ε may be expressed in terms of a, b, by writing (2.20) in terms of propagators, i.e.,

$$\begin{bmatrix}a & b\\ \bar{b} & \bar{a}\end{bmatrix}\begin{bmatrix}0\\T\end{bmatrix}=\begin{bmatrix}R\\1\end{bmatrix},$$

and hence

$$R^\varepsilon=\frac{b}{\bar{a}}, \qquad T=\frac{1}{\bar{a}}. \tag{2.24}$$

We next express $\hat{G}_2$ in terms of R. Assuming (BCII) we again take $A(-L)=0$, but now (2.8), (2.17) imply

$$\hat{p}(0)=\rho_0(0)c_0(0)=[\rho_0(0)c_0(0)]^{1/2}[A(0)-B(0)]. \tag{2.25}$$

However, from (2.9), (2.17)

$$\hat{G}_2=\hat{u}(\omega,0)=[\rho_0(0)c_0(0)]^{-1/2}[A(0)+B(0)]. \tag{2.26}$$

Therefore

$$A(0)=\frac{[\rho_0(0)c_0(0)]^{1/2}}{2}[\hat{G}_2+1], \qquad B(0)=\frac{[\rho_0(0)c_0(0)]^{1/2}}{2}[\hat{G}_2-1]. \tag{2.27}$$

The propagator equation now becomes

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a}\end{bmatrix}\begin{bmatrix} 0 \\ T_{11}\end{bmatrix}=\frac{[\rho_2(0)c_0(0)]^{1/2}}{2}\begin{bmatrix}(\hat{G}_2+1)\\(\hat{G}_2-1)\end{bmatrix}. \tag{2.28}$$

Equations (2.24), (2.28) yield

$$\hat{G}_2=-\frac{1+R^\varepsilon}{1-R^\varepsilon}. \tag{2.29}$$

Now from (2.19), (2.22) we have that

$$\begin{aligned}\frac{da}{dx}&=\frac{i\omega}{\varepsilon}\left(\frac{\rho_0}{K_0}\right)^{1/2}[m^\varepsilon a+n^\varepsilon\bar{b}\,e^{2i\omega\tau/\varepsilon}]+\frac{1}{4}\frac{(\rho_0K_0)'}{\rho_0K_0}\bar{b}\,e^{2i\omega\tau/\varepsilon},\\ \frac{d\bar{b}}{dx}&=-\frac{i\omega}{\varepsilon}\left(\frac{\rho_0}{K_0}\right)^{1/2}[n^3a\,e^{-2i\omega\tau/\varepsilon}+m^\varepsilon\bar{b}]+\frac{1}{4}\frac{(\rho_0K_0)'}{\rho_0K_0}a\,e^{-2i\omega\tau/\varepsilon},\\ &a(-L)=1,\qquad b(-L)=0.\end{aligned} \tag{2.30}$$

Therefore, from (2.24), (2.30) we can derive the Riccati equation for R:

$$\begin{aligned}\frac{dR^\varepsilon}{dx}&=\frac{i\omega}{\varepsilon}\left(\frac{\rho_0}{K_0}\right)^{1/2}[n^\varepsilon e^{2i\omega\tau/\varepsilon}+2m^\varepsilon R^\varepsilon+n^\varepsilon(R^\varepsilon)^2e^{-2i\omega\tau/\varepsilon}]\\ &\quad+\frac{1}{4}\frac{(\rho_0K_0)'}{(\rho_0K_0)}[e^{2i\omega\tau/\varepsilon}-(R^\varepsilon)^2e^{-2i\omega\tau/\varepsilon}],\\ &R^\varepsilon(-L)=0.\end{aligned} \tag{2.31}$$

The boundary condition at $-L$ in (2.31) is for termination of the random slab by a uniform medium. If the medium is homogeneously random beyond $-L$ ($\rho_0(x)$, $K_0(x)$ constant) then we will have total reflection at $-L$ because the wave cannot penetrate the random medium to infinite depth. In fact in a statistically homogeneous random medium we have that

$$|T|\to 0 \quad \text{as } L\to-\infty \tag{2.32}$$

exponentially fast, which follows from Furstenberg's theorem [7]. Since (2.23), (2.24) imply that $|R|^2+|T|^2=1$ we have

$$|R|\to 1 \quad \text{as } L\to-\infty. \tag{2.33}$$

It is convenient to analyze (2.31) with a *totally reflecting termination*, so that

$$R^\varepsilon=e^{-i\psi^\varepsilon} \tag{2.34}$$

and the number of degrees of freedom is reduced by one. This simplification, not possible when we do have transmission, was not made in [1]. Putting (2.34) into (2.31), we obtain

$$\tag{2.35} \begin{aligned} \frac{d}{dx}\psi^\varepsilon = -\frac{\omega}{\varepsilon}\left(\frac{\rho_0(x)}{K_0(x)}\right)^{1/2}\left[2m^\varepsilon(x)+2n^\varepsilon(x)\cos\left(\psi^\varepsilon+\frac{2\omega\tau(x)}{\varepsilon}\right)\right] \\ +\frac{1}{2}\frac{(\rho_0K_0)'}{\rho_0K_0}\sin\left(\psi^\varepsilon+\frac{2\omega\tau(x)}{\varepsilon}\right) \end{aligned}$$

and we take ψ^ε to be asymptotically stationary as $x\to-\infty$.

To recapitulate, the asymptotically stationary solution of (2.35) evaluated at $x=0$ is put into (2.34) to yield the totally reflecting reflection coefficient R^ε at frequency ω. The frequency domain Green function is then given by either (2.21) if the matched media (BCI) (2.6) is assumed, or by (2.29) if the pressure release (BCII), (2.8) is assumed. The result is then transformed back to the time domain by (2.14).

3. Statement of the main results. Let $G^\varepsilon_{t,f}(\sigma)$ be the reflection process observed at $x=0$ within the time window centered at t. Then $G^\varepsilon_{t,f}(\cdot)$ converges weakly as $\varepsilon\downarrow 0$ to a stationary Gaussian process with mean zero and power spectral density

$$\tag{3.1} S_t(\omega)=|\hat{f}(\omega)|^2\mu(t,\omega),$$

where

$$\mu(t,\omega)=\mu_j(t,\omega),\qquad j=1,2$$

depending on whether ($j=1$) the matched medium boundary condition (BCI) or ($j=2$) the pressure release boundary condition (BCII) is assumed. The μ_j are computed as follows:

Let α_{nn} be the integral of the second moment of the medium properties defined by

$$\tag{3.2} \alpha_{nn}(x)=\int_0^\infty E[n(x,y)n(x,y+s)]\,ds.$$

Let $\tau(x)$ be travel time to depth x defined by (2.16), and let $\xi(\tau)$ be its inverse which is depth reached up to time t in the medium without fluctuations. Define

$$\tag{3.3} \gamma(\tau)=\frac{\alpha_{nn}(\xi(\tau))}{c_0(\xi(\tau))}.$$

Let $W^{(N)}(\tau,t,\omega)$, $N=0,1,2\cdots$ be the solution of

$$\tag{3.4} \begin{aligned} \frac{\partial W^{(N)}}{\partial\tau}+2N\frac{\partial W^{(N)}}{\partial t} \\ -2\omega^2\gamma(\tau)\{[N+1]^2W^{(N+1)}-2N^2W^{(N)}+[N-1]^2W^{(N-1)}\}=0 \end{aligned}$$

for $t,\tau>0$, $N=0,1,2,\cdots$ with $W^{(N)}\equiv 0$, for $t<0$, $N<0$. For $j=1$, (BCI) (3.4) is supplemented with the initial condition

$$\tag{3.5} \text{(BCI)}\; W^{(N)}(0,t,\omega)=\delta(t)\delta_{N,1}.$$

Then

$$\tag{3.6} \text{(BCI)}\,\mu_1(t,\omega)=\lim_{\tau\to\infty}W^{(0)}(\tau,t,\omega).$$

The system (3.4) is hyperbolic so it is not necessary to take a limit in (3.6) because $W^{(0)}$ is constant for $\tau > t/2$. Thus

$$\mu_1(t, \omega) = W^{(0)}\left(\frac{t}{2}, t, \omega\right). \tag{3.6a}$$

For $j = 2$, (BCII) (3.4) is supplemented with the initial condition

$$\text{(BCII)}\ W^{(N)}(0, t, \omega) = \begin{cases} \delta(t) & \text{for } N = 0, \\ 4\delta(t) & \text{for } N > 0. \end{cases} \tag{3.7}$$

Then

$$\text{(BCII):}\quad \mu_2(t, \omega) = \lim_{\tau\to\infty} W^{(0)}(\tau, t, \omega) \tag{3.8}$$

with the same remark about hyperbolicity holding here as in (3.6).

For the case of a homogeneous medium $[c_0, \gamma = \text{const.} = \tilde{\gamma}]$ the normalized power spectral density can be computed explicitly:

$$\text{(BCI):}\quad \mu_1(t, \omega) = \frac{\omega^2\tilde{\gamma}}{[1+\omega^2\tilde{\gamma}t]^2}, \tag{3.9}$$

$$\text{(BCII):}\quad \mu_2(t, \omega) = 4\tilde{\gamma}\omega^2 \tag{3.10}$$

Let us now consider inverse problems associated with the pulse reflection problem. Inverse problems associated with (2.1)-(2.7) (we consider BCI only) are of little interest before the limit $\varepsilon \to 0$ because the usual inverse scattering methods are overwhelmed by the fluctuations. So we want to pose inverse problems after the limit, i.e., for the reflected process $G_{t,f}(\sigma)$. Perhaps the simplest such problem is this: what can we say about the slowly varying properties of the medium if we know the power spectral density $\mu(t, \omega)$ of the windows given by (3.6) or (3.6a)?

We see that the limit

$$\lim_{\omega\to 0} \frac{1}{\omega^2}\mu(t, \omega) = \frac{2\alpha_{nn}(\xi(-t/2))}{c_0(\xi(-t/2))} \tag{3.11}$$

exists and is given by the right side of (3.11). This is an immediate and elementary consequence of (3.4)-(3.6). Let us consider a simple application of it. Let us assume that $\alpha_{nn} = \alpha = \text{const.}$ is known and let

$$\Theta(t) = \frac{1}{2\alpha}\lim_{\omega\to 0}\frac{1}{\omega^2}\mu(t, \omega) = \frac{1}{(\xi(-t/2))} \tag{3.12}$$

be the quantity that is measured. From (2.16) we see that

$$\frac{d\xi(t)}{dt} = c_0(\xi(t)) = \frac{1}{\Theta(-2t)}.$$

Thus, the distance traveled up to time t is obtained from the reflection statistics by

$$\xi(t) = \int_0^t \frac{1}{\Theta(-2s)}\, ds. \tag{3.13}$$

Since $\tau(x)$ and $\xi(\tau)$ are monotone inverse functions, (3.13) also determines the travel time $\tau(x)$, and hence the mean propagation speed $c_0(x)$ as a function of distance.

This result is in striking contrast with what the usual inverse methods give when applied to (2.1)–(2.7). It is known, for example, that only the impedance can be computed and only as a function of the (in general, unknown from normal incidence data) travel time. In the stochastic case, when fluctuations are statistically homogeneous, we are able to determine the mean speed c_0 as a function of distance into the medium from the power spectra via (3.11)–(3.13). This appears paradoxical at first because we get more out of (the limit of) the noisy problem. We get more because we have access to more, namely power spectra that are ensemble averages and contain therefore information from more than one realization. In addition we have used the fact that α is a constant. In many applied contexts we have only a single realization of the reflected signal available. Then $\mu(t, \omega)$ must be estimated statistically from this sample and so $\Theta(t)$ in (3.12) is known only approximately and depends on the realization, i.e., it is random. But we know in addition that the reflected signal is a Gaussian process (in the limit $\varepsilon \to 0$). This information can be used to get sharp estimates for the travel time statistics.

It is clear that there are many interesting problems, direct and inverse, that can be posed for the pulse reflection problem. Our framework and results provide a theoretical basis for solving some of them.

4. Calculation of power spectral density. We next calculate the power spectrum, as $\varepsilon \downarrow 0$, of the reflection process $G^\varepsilon_{t,f}(\sigma)$. From (2.11) we have the correlation function $C^\varepsilon_{t,f}$

$$\begin{aligned} C^\varepsilon_{t,f}(\sigma) &\equiv E[G^\varepsilon_{t,f}(\sigma) G^\varepsilon_{t,f}(0)], \\ &= \frac{1}{4\pi^2 \varepsilon} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, e^{-i\omega_1 t/\varepsilon} e^{-i\omega_1 \sigma} e^{i\omega_2 t/\varepsilon} \\ &\quad \cdot \hat{f}(\omega_1) \bar{\hat{f}}(\omega_2) E[\hat{G}(\omega_1) \bar{\hat{G}}(\omega_2)]. \end{aligned} \tag{4.1}$$

Let

$$u^\varepsilon(\omega, h) = E\left[\hat{G}\left(\omega - \frac{\varepsilon h}{2} \right) \bar{\hat{G}}\left(\omega + \frac{\varepsilon h}{2} \right) \right]. \tag{4.2}$$

We will show that the limit

$$u(\omega, h) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(\omega, h) \tag{4.3}$$

exists, and we will characterize it in this section. Then, after the change of variables $\omega = \frac{1}{2}(\omega_1 + \omega_2)$, $h = (\omega_2 - \omega_1)/\varepsilon$ in (4.1) we obtain in the limit $\varepsilon \to 0$

$$\begin{aligned} C_{t,f}(\sigma) &\equiv \lim_{\varepsilon \leftarrow 0} C^\varepsilon_{t,f}(\sigma) \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega\sigma} e^{iht} |\hat{f}(\omega)|^2 u(\omega, h)\, dh\, d\omega. \end{aligned} \tag{4.4}$$

Let

$$\mu(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iht} u(\omega, h)\, dh. \tag{4.5}$$

Then from (4.4), (4.5) the power spectral density, $S_t(\omega)$ is given by

$$\begin{aligned} S_t(\omega) &\equiv \int_{-\infty}^{\infty} e^{i\omega\sigma} C_{t,f}(\sigma)\, d\sigma \\ &= |\hat{f}(\omega)|^2 \mu(t, \omega). \end{aligned} \tag{4.6}$$

In the remainder of this section we characterize $u(\omega, h)$ and its transform $\mu(t, \omega)$. From (4.2), $u^\varepsilon(\omega, h)$ takes a different form, depending on which boundary condition is assumed. From (4.2) and from (2.21), (2.29), either case can be computed from knowledge of the joint statistics of R_1^ε, R_2^ε the reflection coefficients corresponding, respectively, to frequencies $\omega_1=\omega-\varepsilon h/2$, $\omega_2=\omega+\varepsilon h/2$. Since we are in the totally reflecting case (2.34)

$$R_1=e^{-i\psi_1^\varepsilon}, \qquad R_2^\varepsilon=e^{-i\psi_2^\varepsilon},$$

where ψ_1^ε, ψ_2^ε correspond, respectively, to frequencies ω_1 and ω_2 and each satisfies (2.35). We shall compute the joint distribution of ψ_1^ε, ψ_2^ε as ε tends to zero.

Let

$$\psi^\varepsilon=\begin{bmatrix}\psi_1^\varepsilon\\ \psi_2^\varepsilon\end{bmatrix}.$$

From (2.35) we see that ψ^ε satisfies the differential equation

$$\frac{d\psi^\varepsilon}{dx}=\frac{1}{\varepsilon}\mathbf{F}\left(x,\frac{x}{\varepsilon^2},\frac{\tau(x)}{\varepsilon},\psi^\varepsilon\right)+\mathbf{G}\left(x,\frac{x}{\varepsilon^2},\frac{\tau(x)}{\varepsilon},\psi^\varepsilon\right) \tag{4.7}$$

where

$$\mathbf{F}(x,y,\eta,\psi)=-2\omega\left(\frac{\rho_0(x)}{K_0(x)}\right)^{1/2}\begin{bmatrix}m(x,y)+n(x,y)\cos(\psi_1+2\omega\eta-h\tau(x))\\ m(x,y)+n(x,y)\cos(\psi_2+2\omega\eta+h\tau(x))\end{bmatrix} \tag{4.8}$$

and

$$\mathbf{G}(x,y,\eta,\psi)=\begin{bmatrix}G_1(x,y,\eta,\psi)\\ G_2(x,y,\eta,\psi)\end{bmatrix},$$

$$G_1(x,y,\eta,\psi)=h\left(\frac{\rho_0}{K_0}\right)^{1/2}(m(x,y)+n(x,y)\cos(\psi_1+2\omega\eta-h\tau(x))$$
$$+\frac{1}{2}\frac{(\rho_0(x)K_0(x))'}{\rho_0(x)K_0(x)}\sin(\psi_1+2\omega\eta-h\tau(x)),$$

$$G_2(x,y,\eta,\psi)=-h\left(\frac{\rho_0}{K_0}\right)^{1/2}(m(x,y)+n(x,y)\cos(\psi_2+2\omega\eta+h\tau(x)))$$
$$+\frac{1}{2}\frac{(\rho_0(x)K_0(x))'}{\rho_0(x)K_0(x)}\sin(\psi_2+2\omega\eta+h\tau(x)).$$

We assume, as in Appendix A, that the randomness in (4.7) is generated by an ergodic Markov process $q^\varepsilon(x)=q(x,x/\varepsilon^2)$ in Euclidean space R^d of arbitrary dimension d. It is assumed that $q(x,x/\varepsilon^2)$ is a random process on the fast, x/ε^2, spatial scale, but has slowly-varying statistics on the x scale. We express this mathematically by the assumption that $q(x,y)$ is, for fixed x, a stationary ergodic Markov process in y with infinitesimal generator Q_x, depending on x. We then write $m(x,x/\varepsilon^2)=\tilde{m}(x,q(x,x/\varepsilon^2))$, etc. (to simplify notation we will drop the tildes). A very wide class of processes with small scale randomness but slowly-varying statistics can be generated in this way.

The process $(q^\varepsilon,\psi^\varepsilon)\varepsilon R^{d+2}$, the solution of (4.7) together with its coefficients, is now jointly Markovian, with infinitesimal generator

$$\mathbf{L}_x^\varepsilon=\frac{1}{\varepsilon^2}Q_x+\frac{1}{\varepsilon}\mathbf{F}\left(x,\frac{x}{\varepsilon^2},\frac{\tau(x)}{\varepsilon},\psi\right)\cdot\nabla_\psi+\mathbf{G}\left(x,\frac{x}{\varepsilon^2},\frac{\tau(x)}{\varepsilon},\psi\right)\cdot\nabla_\psi. \tag{4.9}$$

From the results of Appendix A, we have that ψ^ε converges (weakly) to a process ψ that is Markovian by itself, without the necessity of including q. The limit process ψ has the x-dependent infinitesimal generator $\mathbf{L}_x$, where

$$
\begin{aligned}
\mathbf{L}_x = \frac{4\omega^2}{c_0^2(x)} \Bigg\{ & \alpha_{mm}(x)\left[\frac{\partial}{\partial\psi_1}+\frac{\partial}{\partial\psi_2}\right]^2 \\
& + \frac{1}{2}\alpha_{nn}(x)\left[\frac{\partial^2}{\partial\psi_1^2}+\frac{\partial^2}{\partial\psi_2^2}+2\cos(\psi_2-\psi_1+2h\tau(x))\frac{\partial^2}{\partial\psi_1\partial\psi_2}\right]\Bigg\}.
\end{aligned} \tag{4.10}
$$

In (4.10), the coefficients α_{mm}, α_{nn} are defined by the following averaged second moments:

$$
\begin{aligned}
\alpha_{mm}(x) &= \int_0^\infty E[m(x, q(x,y))m(x, q(x, y+r))]\,dr, \\
\alpha_{nn}(x) &= \int_0^\infty E[n(x, q(x,y))n(x, q(x, y+r))]\,dr.
\end{aligned} \tag{4.11}
$$

In Appendix A we show briefly how these results are obtained.

The generator (4.10) is better expressed in terms of the sum and difference variables

$$\psi = \psi_2 - \psi_1, \qquad \tilde\psi = \tfrac{1}{2}(\psi_2 + \psi_1).$$

Then

$$
\begin{aligned}
\mathbf{L}_x = \frac{4\omega^2}{c_0^2(x)} \Bigg\{ & \alpha_{mm}(x)\frac{\partial^2}{\partial\tilde\psi^2} + \frac{1}{4}\alpha_{nn}(x)[1+\cos(\psi+2h\tau(x))]\frac{\partial^2}{\partial\tilde\psi^2} \\
& + \alpha_{nn}(x)[1-\cos(\psi+2h\tau(x))]\frac{\partial^2}{\partial\psi^2}\Bigg\}.
\end{aligned} \tag{4.12}
$$

Using (4.12) we can now formulate the equations for $u(\omega, h)$ and its transform $\mu(t, \omega)$. For (BCI), we have from (2.21), (2.34) that $u_1(\omega, h)$ from (4.2), (4.3) is

$$u_1(\omega, h) = E[e^{i\psi}]. \tag{4.13}$$

Note that the coefficients in (4.12) do not depend on $\tilde\psi$, so that ψ is Markovian by itself. The function $u_1(\omega, h)$ can therefore be calculated from the solution V of the Kolmogorov backward equation

$$\frac{\partial V}{\partial x} + \frac{4\omega^2\alpha_{nn}(x)}{c_0^2(x)}[1-\cos(\psi+2h\tau(x))]\frac{\partial^2 V}{\partial\psi^2} = 0, \qquad x<0 \tag{4.14}$$

with the final condition

$$\text{(BCI):}\quad V|_{x=0} = e^{i\psi}. \tag{4.15}$$

The function u_1 is then

$$u_1(\omega, h) = \lim_{x\to-\infty} V(x, \psi, \omega, h), \tag{4.16}$$

where the limit in (4.16) exists and is independent of ψ. This follows easily assuming that α_{nn}, τ, and c_0 are constant for $-x$ large.

The derivation of the results for (BCII) is somewhat more involved. From (2.29), (2.34), and (4.2), (4.3), we have that

$$\begin{aligned} u_2(\omega, h) &= \lim_{r_1, r_2 \uparrow 1} E\left[\left(\frac{1+r_1\, e^{-i\psi_1}}{1-r_1\, e^{-i\psi_1}}\right)\left(\frac{1+r_2\, e^{i\psi_2}}{1-r_2\, e^{i\psi_2}}\right)\right] \\ &= \lim_{r_1, r_2 \uparrow 1} E\left[\frac{1+r_1 r_2\, e^{i\psi}+e^{i\psi/2}[r_1\, e^{-i\tilde{\psi}}+r_2\, e^{i\tilde{\psi}}]}{1+r_1 r_2\, e^{i\psi}-e^{i\psi/2}[r_1\, e^{-i\tilde{\psi}}+r_2\, e^{i\tilde{\psi}}]}\right]. \end{aligned} \tag{4.17}$$

In (4.17) we have replaced the unimodular coefficient $R = e^{-i\psi}$ of (2.34) by the limit $R = r\, e^{-i\psi}$ as $r \uparrow 1$ to resolve ambiguities resulting from the singularities in the denominator of (4.17). The choice $|R|$ approaching unity from below is a consequence of properties of the reflection coefficient of finite slab lengths.

Now we again note that $\tilde{\psi}$ does not appear in the coefficients of the generator $\mathbf{L}_x$ of (4.12). We conclude that $\tilde{\psi}$ may be taken to have a uniform distribution on $[-\pi, \pi)$, independent of the distribution of ψ. We may thus do the integration on $\tilde{\psi}$ in (4.17). The result from elementary contour integration is that

$$u_2(\omega, h) = \lim_{r \uparrow 1} E\left[\frac{(3r\, e^{i\psi}+1)}{(1-r\, e^{i\psi})}\right]. \tag{4.18}$$

Therefore u_2 may be obtained from the solution of V of (4.14) in $x<0$ that satisfies the alternative final condition

$$\text{(BCII):}\quad V|_{x=0} = \frac{(3r\, e^{i\psi}+1)}{(1-r\, e^{i\psi})}, \qquad 0<r<1. \tag{4.19}$$

As in (4.16)

$$u_2(\omega, h) = \lim_{\substack{x \to -\infty \\ r \uparrow 1}} V(x, \psi, \omega, h; r) \tag{4.20}$$

where the limit in (4.20) is independent of ψ.

Equation (4.14) may be simplified somewhat by the change of variables

$$\hat{\psi} = \psi + 2h(\tau(x)), \qquad \hat{x} = x \tag{4.21}$$

Then upon dropping hats, it becomes

$$\frac{\partial V}{\partial x} - \frac{2h}{c_0(x)}\frac{\partial V}{\partial \psi} + \frac{4\omega^2 \alpha_{nn}(x)}{c_0^2(x)}[1-\cos\psi]\frac{\partial^2 V}{\partial \psi^2} = 0 \quad \text{for } x<0. \tag{4.22}$$

To summarize, (4.22) is to be solved for V subject to the final condition (4.15) corresponding to (BCI), or (4.19) corresponding to (BCII). Then u_1, u_2 are obtained from (4.16) or (4.20), respectively.

Equation (4.22) can be solved by Fourier series in ψ. Let

$$V = \sum_{N=-\infty}^{\infty} V^{(N)}\, e^{iN\psi}, \quad \psi \text{ in } [-\pi, \pi). \tag{4.23}$$

Then (4.22) becomes the infinite-dimensional system

(4.24)

$$\begin{aligned} &\frac{\partial V^{(N)}}{\partial x} - \frac{2ihN}{c_0(x)} V^{(N)} \\ &\quad + \frac{2\omega^2 \alpha_{nn}(x)}{c_0^2(x)}\left\{[N+1]^2 V^{(N+1)} - 2N^2 V^{(N)} + [N-1]^2 V^{(N-1)}\right\} = 0 \quad \text{for } x<0 \end{aligned}$$

with boundary condition for u_1

$$\text{(BCI):}\quad V^{(N)}|_{x=0}=\delta_{N,1} \tag{4.25}$$

and boundary condition for u_2

$$\text{(BCII):}\quad V^{(N)}|_{x=0}=\begin{cases}0 & \text{for } N<0,\\ 1 & \text{for } N=0,\\ 4 & \text{for } N>0.\end{cases} \tag{4.26}$$

In either case, $V^{(N)}\to 0$ as $x\to-\infty$ for $N\neq 0$, and $u(\omega, h)$ is given by $\lim_{x\to-\infty} V^{(0)}$.

Equivalently, we may now formulate an infinite set of coupled linear first-order partial differential equations for $\mu(t, \omega)$ the Fourier transform in h of $u(\omega, h)$, equation (4.5). Let $W^{(N)}$ be the Fourier transform in h, of $V^{(N)}$. Then we obtain easily from (4.24)-(4.26) that

$$\begin{aligned}\frac{\partial W^{(N)}}{\partial x}-\frac{2N}{c_0(x)}\frac{\partial W^{(N)}}{\partial t}&\\ +\frac{2\omega^2\alpha_{nn}(x)}{c_0^2(x)}&\{[N+1]^2W^{(N+1)}-2N^2W^{(N)}+[N-1]^2W^{(N-1)}\}=0.\end{aligned} \tag{4.27}$$

For (BCI) we use

$$\text{(BCI):}\quad W^{(N)}|_{x=0}=\delta(t)\delta_{N,1} \tag{4.28}$$

for (BCII) we use

$$\text{(BCII):}\quad W^{(N)}|_{x=0}=\begin{cases}0 & \text{for } N<0,\\ \delta(t) & \text{for } N=0,\\ 4\delta(t) & \text{for } N>0.\end{cases} \tag{4.29}$$

The normalized power spectral density $\mu(t, \omega)$ is then given by the limit of $W^{(0)}$ as $x\to-\infty$. The formulation given in § 3 follows from making the change of variables in (4.27) from depth, x, to traveltime $\tau(x)$, equation (2.16).

We will next calculate μ explicitly for the case of a statistically homogeneous medium. That is, we assume that c_0 and

$$\tilde{\gamma}=\frac{\alpha_{nn}(x)}{c_0(x)} \tag{4.30}$$

do not depend on x. We will use a forward Kolmogorov equation formulation based on (4.22) and (4.15) or (4.19). For the case of c_0 and $\tilde{\gamma}$ constant, great simplification is achieved by first making the transformation

$$z=\cot\frac{\psi}{2}. \tag{4.31}$$

Then $z\in[-\infty,\infty)$ when $\psi\in[-\pi,\pi)$. The Kolmogorov backward equation for z is obtained by change of variables in (4.22)

$$\frac{\partial V}{\partial x}+h\frac{(1+z^2)}{c_0}\frac{\partial V}{\partial z}+2\frac{\omega^2\tilde{\gamma}}{c_0}\frac{\partial}{\partial z}\left\{(1+z^2)\frac{\partial}{\partial z}\right\}=0. \tag{4.32}$$

The probability density associated with (4.32), $P(x, z)$, satisfies the Kolmogorov forward equation

$$c_0\frac{\partial P}{\partial x}=-h\frac{\partial}{\partial z}\{(1+z^2)P\}+2\omega^2\tilde{\gamma}\frac{\partial}{\partial z}\left\{(1+z^2)\frac{\partial P}{\partial z}\right\}. \tag{4.33}$$

The invariant density $\bar{P}_h(z)$ is obtained by setting $\partial P/\partial x = 0$ in (4.33). For $h > 0$ we have

$$\bar{P}_h(z) = \frac{h}{2\pi\omega^2\tilde{\gamma}} \int_0^\infty \frac{e^{-h\xi/2\omega^2\tilde{\gamma}}}{[1+(\xi+z)^2]}\, d\xi. \tag{4.34}$$

For $h < 0$, symmetries in (4.33) imply that

$$\bar{P}_{-h}(-z) = \bar{P}_h(z). \tag{4.35}$$

Now from (4.31) we have that

$$e^{i\psi} = \left(\frac{z+i}{z-i}\right). \tag{4.36}$$

Therefore

$$\begin{aligned} u_1(\omega, h) = E[e^{i\psi}] &= \int_{-\infty}^{\infty} \bar{P}_h(z)\left(\frac{z+i}{z-i}\right) dz \\ &= \frac{h}{2\omega^2\tilde{\gamma}} \int_0^\infty e^{-h\xi/2\omega^2\tilde{\gamma}}\left(\frac{\xi}{\xi+2i}\right) d\xi \quad \text{for } h > 0. \end{aligned} \tag{4.37}$$

From (4.35), (4.37) it follows that

$$u_1(\omega, -h) = \overline{u_1(\omega, h)}. \tag{4.38}$$

Therefore

$$\mu(t, \omega) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{iht} u_1(\omega, h)\, dh. \tag{4.39}$$

Substitution of (4.37) into (4.39) then gives, after some elementary integrations

$$\mu_1(t, \omega) = \frac{\omega^2\tilde{\gamma}}{[1+\omega^2\tilde{\gamma}t]^2}, \tag{4.40}$$

which is the result (3.9).

To analyze (BCII) for $\tilde{\gamma}$ constant, we may again use the invariant density $\bar{P}_h(z)$ from (4.34). Now however, we have from (4.19) that

$$\begin{aligned} u_2(\omega, h) &= \lim_{r\uparrow 1} E\left[\frac{(3r\, e^{i\psi}+1)}{(1-r\, e^{i\psi})}\right] \\ &= \lim_{r\to 1} \int_{-\infty}^{\infty} P_h(z)\left[\frac{[(3r+1)z+i(3r-1)]}{[(1-r)z-i(1+r)]}\right] dz. \end{aligned} \tag{4.41}$$

For $h > 0$ we have from (4.41), (4.34)

$$u_2(\omega, h) = \lim_{r\uparrow 1} \int_0^\infty d\xi \frac{e^{-h\xi}h}{2\pi\omega^2\tilde{\gamma}} \int_{-\infty}^\infty dz \frac{[(3r+1)z+i(3r-1)]}{[(1-r)z-i(1+r)][1+(\xi+z)^2]}. \tag{4.42}$$

The inner integral in (4.42) is equal to the residue at the pole $z = -\xi - i$, after closing the contour in the lower halfplane. The assumption that $0 < r < 1$ is necessary here so that the pole at $z = i(1+r)/(1-r)$ does not contribute. After doing the z integral we can then let $r\uparrow 1$ to obtain

$$u_2(\omega, h) = 1 - i\frac{4\omega^2\tilde{\gamma}}{h-i0} \quad \text{for } h > 0. \tag{4.43}$$

Again (4.35) implies that $u_2(\omega, -h) = \overline{u_2(\omega, h)}$. Fourier transform of (4.43) now gives that

$$(\text{BCII})\mu_2(t, \omega) = \delta(t) + 4\omega^2\tilde{\gamma}. \tag{4.44}$$

For $t>0$ the delta function in (4.44) does not contribute, so it is simply $\mu_2 = 4\omega^2\bar{\alpha}$, independent of t, that appears in (3.10).

5. The method of functionals.

5.1. The generalized reflection functional. In this section we shall consider only the case (BCI) (2.6) and we shall show that the windowed reflection process $G^{\varepsilon}_{t,f}(\sigma)$ defined by (2.11) tends to a Gaussian process. The power spectral density of this process is given by (3.1) with $\mu = \mu_1$ defined by (3.6) or (3.6a). The calculations in § 4 are formal because a number of interchanges of limits and integration have not been justified. In this section the calculations are carried out in a different framework that involves functional processes and avoids these difficulties. For this reason we reformulate briefly the problem and reintroduce the quantities of interest in a more general form.

The equations of motion in scaled form are

$$\rho_0\left[1+\eta\left(\frac{x}{\varepsilon^2}\right)\right]u_t + p_x = 0,$$
$$\frac{1}{\rho_0(x)c_0^2(x)}\left[1+\nu\left(\frac{x}{\varepsilon^2}\right)\right]p_t + u_x = 0, \tag{5.1}$$

for $x<0$ and $t>0$. We assume that $\rho_0(x)$ and $c_0(x)$ are identically constant in $x>0$ and that they are differentiable functions in all of R^1, bounded and positive. The random fluctuations η and ν are taken to be stationary here, to simplify the writing. They have mean zero, take values in the interval $[-\frac{1}{2}, \frac{1}{2}]$, say, and are Markovian. The last hypothesis is unnecessary again but simplifies the analysis. The methods of [1] can be used here also in the general mixing case.

Define

$$\tau(x) = \int_0^x \frac{1}{c_0(s)}\,ds, \tag{5.2}$$

the travel time, for all x and note that it is as in (2.16) except for signs and it is increasing. Let $\xi(\tau)$ be its inverse function that is zero at τ equal to zero. Clearly,

$$\dot{\tau} = \frac{1}{c_0}, \qquad \dot{\xi} = c_0. \tag{5.3}$$

Equations (5.1) are provided by initial and boundary conditions by specifying that a pulse is incident from the right:

$$u = \frac{1}{(c_0\rho_0)^{1/2}}\frac{1}{\sqrt{\varepsilon}}f\left(\frac{t+\tau}{\varepsilon}\right), \qquad p = -(c_0\rho_0)^{1/2}\frac{1}{\sqrt{\varepsilon}}f\left(\frac{t+\tau}{\varepsilon}\right) \tag{5.4}$$

for $t<0$ with u and p continuous at $x=0$. The pulse shape is a smooth function, rapidly decreasing at infinity. Recall that c_0 and ρ_0 are identically constant in $x>0$.

We introduce the change of variables

$$u(t,x) = \frac{1}{(\rho_0 c_0)^{1/2}}\tilde{u}(t,\tau), \qquad p(t,x) = (\rho_0 c_0)^{1/2}\tilde{p}(t,\tau) \tag{5.5}$$

and let

$$\eta^{\varepsilon}(\tau) = \eta(\xi(\tau)/\varepsilon^2), \qquad \nu^{\varepsilon}(\tau) = \nu(\xi(\tau)/\varepsilon^2), \tag{5.6}$$

$$c_0(\tau) = c_0(\xi(\tau)), \qquad \zeta(\tau) = \frac{1}{\rho_0 c_0}\frac{d}{d\tau}(\rho_0 c_0). \tag{5.7}$$

From (5.1) we obtain the following equations for $\tilde{u}$ and $\tilde{p}$, with the tilde dropped from now on:

$$(5.8) \qquad (1+\eta^{\varepsilon})u_t + p_\tau + \tfrac{1}{2}\zeta p = 0, \qquad (1+\nu^{\varepsilon})p_t + u_\tau - \tfrac{1}{2}\zeta u = 0$$

for $\tau < 0$ and $t > 0$ with

$$(5.9) \qquad u = \frac{1}{\sqrt{\varepsilon}} f\left(\frac{t+\tau}{\varepsilon}\right), \qquad p = \frac{-1}{\sqrt{\varepsilon}} f\left(\frac{t+\tau}{\varepsilon}\right)$$

for $t < 0$. Equations (5.8) and (5.9) along with continuity of u and p provide a well-defined initial boundary value problem.

Because of the form (5.9) of the excitation by a wave traveling to the left and striking the interface $\tau = 0$ at time $t = 0$, it is convenient to introduce right and left traveling wave amplitudes A and B by

$$(5.10) \qquad u = A + B, \qquad p = A - B.$$

We also let

$$(5.11) \qquad m^{\varepsilon}(\tau) = \frac{\eta^{\varepsilon}(\tau) + \nu^{\varepsilon}(\tau)}{2}, \qquad n^{\varepsilon}(\tau) = \frac{\eta^{\varepsilon}(\tau) - \nu^{\varepsilon}(\tau)}{2}.$$

Then $A(\tau, t)$ and $B(\tau, t)$ satisfy the system

$$(5.12) \qquad A_t + A_\tau + m^{\varepsilon} A_t + n^{\varepsilon} B_t - \frac{\zeta}{2} B = 0, \qquad B_t - B_\tau + n^{\varepsilon} A_t + m^{\varepsilon} B_t + \frac{\zeta}{2} A = 0$$

for $\tau < 0$ and $t > 0$, with

$$(5.13) \qquad A = 0, \qquad B = \frac{1}{\sqrt{\varepsilon}} f\left(\frac{t+\tau}{\varepsilon}\right)$$

for $t < 0$.

We now proceed as in § 2 via Fourier transforms to obtain an expression for the reflected signal $A(0, t)$ for $t > 0$. Let $\hat{f}$ be defined by (2.12) and let

$$(5.14) \qquad \hat{A}(\tau, \omega) = \int e^{i\omega t/\varepsilon} A(\tau, t)\, dt, \qquad \hat{B}(\tau, \omega) = \int e^{i\omega t/\varepsilon} B(\tau, t)\, dt.$$

Then $\hat{A}$ and $\hat{B}$ satisfy the ordinary differential equations

$$(5.15) \qquad \frac{d}{d\tau}\begin{bmatrix}\hat{A}\\ \hat{B}\end{bmatrix} = \frac{i\omega}{\varepsilon}\begin{bmatrix}1+m^{\varepsilon} & n^{\varepsilon}\\ -n^{\varepsilon} & -(1+m^{\varepsilon})\end{bmatrix}\begin{bmatrix}\hat{A}\\ \hat{B}\end{bmatrix} + \frac{1}{2}\zeta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\hat{A}\\ \hat{B}\end{bmatrix}.$$

Consider now the fundamental solution matrix Y of system (5.15). It has the form (2.22), (2.23). If, as in (2.24), we let

$$(5.16) \qquad R(\tau, \omega) = \frac{b(\tau, \omega)}{\bar{a}(\tau, \omega)}$$

then R satisfies the Riccati equation

$$(5.17) \qquad \frac{dR}{d\tau} = \frac{i\omega}{\varepsilon}[n^{\varepsilon} + 2(1+m^{\varepsilon})R + n^{\varepsilon}R^2] + \tfrac{1}{2}\zeta[1 - R^2].$$

The meaning of this equation is as follows. Suppose that for some $\tau = -L < 0$ the reflection coefficient at frequency ω for the region $(-\infty, -L]$ is known. Then to obtain the reflection coefficient at frequency ω for the region $(-\infty, 0]$ we solve (5.17) in

$-L < \tau \leqq 0$ with initial condition $R(-L, \omega)$ equal to the known value and then evaluate the result at $\tau = 0$. Assuming as in § 2 that we are in the totally reflecting case we may look for R in the form

$$R = e^{-i\psi}. \tag{5.18}$$

Then the phase $\psi(\tau, \omega)$ satisfies the equation

$$\frac{d\psi}{d\tau} = \frac{-2\omega}{\varepsilon}[1 + m^\varepsilon(\tau) + n^\varepsilon(\tau) \cos \psi] - \zeta(\tau) \sin \psi, \tag{5.19}$$

in $-L < \tau \leqq 0$, with an initial condition at $\tau = -L$. We also introduce the *centered phase*.

$$\tilde{\psi} = \psi + \frac{2\omega\tau}{\varepsilon}, \tag{5.20}$$

which satisfies

$$\frac{d\tilde{\psi}}{d\tau} = \frac{-2\omega}{\varepsilon}\left[m^\varepsilon + n^\varepsilon \cos\left(\tilde{\psi} - \frac{2\omega\tau}{\varepsilon}\right)\right] - \zeta \sin\left(\tilde{\psi} - \frac{2\omega\tau}{\varepsilon}\right). \tag{5.21}$$

The functional of interest is

$$R^\varepsilon_{f,t}(\tau, \sigma) = \frac{1}{2\pi\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{-i\omega(t+\varepsilon\sigma)/\varepsilon} \hat{f}(\omega)\, e^{-i\psi(\tau,\omega)}\, d\omega. \tag{5.22}$$

When this is evaluated at $\tau = 0$ it is identical to (2.14) (for (BCI)) and it is the windowed reflected signal. Instead of (5.22) it is convenient to study another more general quantity, the *generalized reflection functional* defined as follows. Let $\lambda^N(s, \omega)$ be real test functions (C^∞ rapidly decreasing), $N = 0, \pm 1, \pm 2, \cdots$ with $\lambda^N(s, \omega) = \lambda^{*N}(s, -\omega)$. Here and in the sequel star denotes complex conjugate. We define the distribution-valued process $R^\varepsilon(\tau)$ by

$$\langle R^\varepsilon(\tau), \lambda\rangle = \frac{1}{\sqrt{\varepsilon}} \sum_{N=-\infty}^{\infty} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\omega\, e^{-i\omega s/\varepsilon} \lambda^N(s, \omega)\, e^{-iN\psi(\tau,\omega)}. \tag{5.23}$$

Clearly when $\lambda^N(s, \omega) = \delta_{N,1}\delta(t-s)(1/2\pi)\, e^{-i\omega\sigma}\hat{f}(\omega)$, the generalized reflection functional (5.23) is the same as (5.22).

We will think of the generalized reflection functional as a distribution valued stochastic process with $\tau \leqq 0$ being the "time" parameter. The objective here is to study the limit law of the process defined by the stochastic equation (5.21) as ε tends to zero, *simultaneously* for all ω so that the law of the functional (5.22) can be analyzed. From (5.19) or (5.21) we see that for each set of rest functions $\{\lambda^N\}$, the law of the process $\langle R^\varepsilon(\tau), \lambda\rangle$ is known. We shall study the limit of this law as ε tends to zero.

5.2 The limit theorem. We shall assume that the fluctuations $(\eta(x), \nu(x))$ in (5.1) or $(m(x), n(x))$ in (5.11) are Markov processes (or projections of a higher-dimensional Markov process) in addition to being stationary. We denote this coefficient process by $q(x)$ and let S be its state space, a compact subset of R^2 for example. We denote by Q the infinitesimal generator of this process and by $\bar{P}$ its invariant measure. The coefficients m and n have mean zero with respect to $\bar{P}$.

Let $q^\varepsilon(\tau)$ denote the process $(m^\varepsilon(\tau), n^\varepsilon(\tau))$ defined by (5.6). Let $R^\varepsilon(\tau)$ (we omit dependence on t, N and λ) be the generalized reflection functional introduced by (5.22). The pair $(q^\varepsilon(\tau), R^\varepsilon(\tau))$ is a Markov process with the latter component distribution-valued and with τ the time parameter. The Markov property is a consequence of the corresponding one for q^ε and ψ^ε, the solution of (5.19).

We will write the infinitesimal generator of the $(q^\varepsilon, R^\varepsilon)$ process acting on suitable test functions. Such functions can be taken in the following form. Let F be a smooth function from $S\times R$ to R and let $\{\lambda^N\}$ be smooth test functions on R vanishing at infinity. We shall write the infinitesimal generator acting on functions of the form

$$F=F(q, R)=F(q, \langle R, \lambda\rangle). \tag{5.24}$$

Now from the definition (5.23) and the differential equation (5.19) we see that the infinitesimal generator has the form

$$\mathbf{L}_\tau^\varepsilon F=\frac{c_0(\tau)}{\varepsilon^2} QF+F'\left[\frac{1}{\varepsilon}\langle R, G_1\lambda\rangle+\langle R, G_2\lambda\rangle\right] \tag{5.25}$$

where $F'(q, \xi)=(\partial F/\partial\xi)(q, \xi)$ and

$$\begin{aligned}(G_1\lambda)^N(s, \omega)&=2i\omega mN\lambda^N(s, \omega)\\&\quad+i\omega n[(N+1)\lambda^{N+1}(s, \omega)-(N-1)\lambda^{N-1}(s, \omega)],\end{aligned} \tag{5.26}$$

$$\begin{aligned}(G_2(\tau)\lambda)^N(s, \omega)&=2N\frac{\partial\lambda^N(s, \omega)}{\partial s}\\&\quad+\frac{\zeta(\tau)}{2}[(N+1)\lambda^{N+1}(s, \omega)-(N-1)\lambda^{N-1}(s, \omega)].\end{aligned} \tag{5.27}$$

Note that the infinitesimal generator $\mathbf{L}_\tau^\varepsilon$ depends on τ so the process is inhomogeneous in the τ parameter.

To analyze the limit of the process $R^\varepsilon(\tau)$ we shall calculate the limit form of its generator defined on various classes of test functions. Test functions of the form (5.24) are not enough however when we want to analyze $R_{t,f}^\varepsilon$ of (5.22) and its moments at a single fixed t. We begin with the limit of (5.25).

This is obtained in much the same way as in the elementary case described in the Appendix (and in the references cited therein). Let $F=F(\langle R, \lambda\rangle)$ be a fixed test function independent of q, because we are interested only in the limit of the R process, and let

$$F^\varepsilon=F+\frac{\varepsilon F'}{c_0(\tau)}\langle R, G_1^\chi\lambda\rangle. \tag{5.28}$$

Here

$$(G_1^\chi\lambda)^N=2i\omega\chi_m N\lambda^N+i\omega\chi_n[(N+1)\lambda^{N+1}+(N-1)\lambda^{N-1}] \tag{5.29}$$

with $\chi_m(q)$ and $\chi_n(q)$ the unique, zero mean solutions of the equation

$$Q\chi_m+m=0, \qquad Q\chi_n+n=0. \tag{5.30}$$

Note that in the Appendix we use the same construction to solve (A11) and we have expressed the inverse of Q in the form (A15). A direct calculation now yields

$$\begin{aligned}\mathbf{L}_\tau^\varepsilon F^\varepsilon&=\frac{1}{c_0(\tau)}F''\{\langle R, G_1\lambda\rangle\langle R, G_1^\chi\lambda\rangle\}\\&\quad+F'\left\{\frac{1}{c_0(\tau)}\langle R, G_1G_1^\chi\lambda\rangle+\langle R, G_2\lambda\rangle\right\}+O(\varepsilon).\end{aligned} \tag{5.31}$$

Let E denote expectation with respect to the invariant measure $\bar{P}$ of the q process. Let

$$\begin{aligned}\mathbf{L}_\tau F&=\frac{1}{c_0(\tau)}F''\{\langle R\times R, E[G_1\times G_1^\chi\rangle]\lambda\times\lambda\}\\&\quad+F'\left\{\frac{1}{c_0(\tau)}\langle R, E[G_1G_1^\chi]\lambda\rangle+\langle R, G_2\lambda\rangle\right\}\end{aligned} \tag{5.32}$$

where $\times$ denotes tensor product. This is then the form of the limit generator for functionals of the form (5.23). If F depends explicitly on τ then the generator is $(\partial_\tau + \mathbf{L}_\tau)F$ with $\mathbf{L}_\tau$ defined by (5.32).

It is useful to write the form of the generator for the functional (5.23) when ψ is replaced by $\tilde{\psi}$, the centered phase given by (5.20). Let

$$\tilde{\lambda}^N(s) = \lambda^N(s - 2N\tau). \tag{5.33}$$

If $\tilde{R}^\varepsilon(\tau)$ is defined by (5.23) with ψ replaced by $\tilde{\psi}$, then we see that

$$\langle \tilde{R}^\varepsilon(\tau), \lambda \rangle\rangle = \langle R^\varepsilon(\tau), \tilde{\lambda}\rangle. \tag{5.34}$$

Also, with this change we have

$$\frac{\partial F}{\partial \tau} \to \frac{\partial F}{\partial \tau} + F'\langle R, -2N(\partial\lambda/\partial s)\rangle.$$

Thus,

$$\begin{aligned}(\partial_\tau + \mathbf{L}_\tau)F(\tau, \langle \tilde{R}, \lambda\rangle) = \frac{\partial F}{\partial \tau} &+ F''\{\langle \tilde{R}\times\tilde{R}, E(G_1 \times G_1^\chi)\lambda\times\lambda\rangle\} \\ &+ F'\{\langle \tilde{R}, E(G_1 G_1^\chi)\lambda\rangle + \langle \tilde{R}, G_3\lambda\rangle\}.\end{aligned} \tag{5.35}$$

Here G_3 is defined by the right side of (5.27) without the term with the s derivative.

For λ^N of the form $\delta(t-s)\delta_{N,N'}\lambda(\omega)$ the generalized reflection functional has the form

$$\langle R_{N,t}(\tau), \lambda\rangle = \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{\infty} e^{-i\omega t/\varepsilon}\lambda(\omega)\, e^{-iN\psi(\tau,\omega)}\, d\omega \tag{5.36}$$

and similarly for the centered functional $\tilde{R}_{N,t}$ where ψ is replaced by the centered phase $\tilde{\psi}$. The explicit form of the generator (5.35) in this case is as follows:

(5.37)

$$\begin{aligned}(\partial_\tau + \mathbf{L}_\tau)F = \frac{\partial F}{\partial \tau} + \frac{N^2}{c_0(\tau)} & F''\{4\alpha_{mm}\langle \tilde{R}_{N,t}, i\omega\lambda\rangle\langle \tilde{R}_{N,t}, i\omega\lambda\rangle \\ &+ 2\alpha_{mn}\langle \tilde{R}_{N,t}, i\omega\lambda\rangle\langle \tilde{R}_{N-1,t+2\tau}, i\omega\lambda\rangle + 2\alpha_{mn}\langle \tilde{R}_{N,t}, i\omega\lambda\rangle \\ &\cdot \langle \tilde{R}_{N+1,t-2\tau}, i\omega\lambda\rangle \\ &+ 2\alpha_{nm}\langle \tilde{R}_{N-1,t+2\tau}, i\omega\lambda\rangle\langle \tilde{R}_{N,t}, i\omega\lambda\rangle + \alpha_{nn}\langle \tilde{R}_{N-1,t+2\tau}, i\omega\lambda\rangle \\ &\cdot \langle \tilde{R}_{N-1,t+2\tau}, i\omega\lambda\rangle \\ &+ \alpha_{nn}\langle \tilde{R}_{N-1,t+2\tau}, i\omega\lambda\rangle\langle \tilde{R}_{N+1,t-2\tau}, i\omega\lambda\rangle + 2\alpha_{nm}\langle \tilde{R}_{N+1,t-2\tau}, i\omega\lambda\rangle \\ &\cdot \langle \tilde{R}_{N,t}, i\omega\lambda\rangle \\ &+ \alpha_{nn}\langle \tilde{R}_{N+1,t-2\tau}, i\omega\lambda\rangle\langle \tilde{R}_{N-1,t+2\tau}, i\omega\lambda\rangle + \alpha_{nn}\langle \tilde{R}_{N+1,t-2\tau}, i\omega\lambda\rangle \\ &\cdot \langle \tilde{R}_{N+1,t-2\tau}, i\omega\lambda\rangle\} \\ + \frac{N}{c_0(\tau)} & F'\{4N\alpha_{nm}\langle \tilde{R}_{N,t}, -\omega^2\lambda\rangle \\ &+ 2N\alpha_{mn}\langle \tilde{R}_{N-1,t+2\tau}, -\omega^2\lambda\rangle \\ &+ 2N\alpha_{mn}\langle \tilde{R}_{N+1,t+2\tau}, -\omega^2\lambda\rangle \\ &+ 2(N-1)\alpha_{nm}\langle \tilde{R}_{N-1,t+2\tau}, -\omega^2\lambda\rangle \\ &+ (N-1)\alpha_{nn}\langle \tilde{R}_{N-2,t+4\tau}, -\omega^2\lambda\rangle \\ &+ (N-1)\alpha_{nn}\langle \tilde{R}_{N,t}, -\omega^2\lambda\rangle \\ &+ 2(N+1)\alpha_{nm}\langle \tilde{R}_{N+1,t-2\tau}, -\omega^2\lambda\rangle\end{aligned}$$

$$+(N+1)\alpha_{nn}\langle \tilde{R}_{N,t}, -\omega^2\lambda\rangle$$
$$+(N+1)\alpha_{nn}\langle \tilde{R}_{N+2,t-4\tau}, -\omega^2\lambda\rangle\}$$
$$+N\frac{\zeta(\tau)}{2}F'\{\langle \tilde{R}_{N-1,t+2\tau}, \lambda\rangle - \langle \tilde{R}_{N+1,t-2\tau}, \lambda\rangle\}.$$

Here the coefficients α_{mm}, α_{nm}, and α_{nn} are defined by

$$\alpha_{mm} = \int_0^\infty E\{m(x)m(0)\}\,dx, \qquad \alpha_{mn} = \int_0^\infty E\{m(x)n(0)\}\,dx,$$
$$\alpha_{nn} = \int_0^\infty E\{n(x)n(0)\}\,dx \tag{5.38}$$

as in (3.2). Note that the components of $\tilde{R}_{N,t}(\tau)$ of the generalized, centered, reflection functional are coupled for different N's and t's.

The question of existence and uniqueness of a distribution-valued process $R(\tau)$ with generator $\mathbf{L}_\tau$ as defined above and the detailed proof of the weak convergence as ε tends to zero will not be discussed here. We note only that the existence of the limit process and the compactness of the family R^ε follow from elementary estimates derived from (5.21) or (5.19). The uniqueness of the limit law is obtained by exploiting the "linearity" of the generalized reflection process that allows us to obtain closed equations for its moments of each order. The second-order moments are considered in the next section.

In the context of the physical problem under consideration with the time t fixed, unless $\tau \leqq -t/2$ we must give the (right) reflection coefficient of the region $(-\infty, \tau]$ as initial condition for (5.19) or (5.21) or for the full process R. We want to study the generalized reflection functional only in the case $\tau < -t/2$ here. Then the reflection functional for the region $(-\infty, 0]$, which is the physically interesting quantity, does not depend on the subregion $(-\infty, \tau]$ at all. This is because signals propagate at a finite speed and (in the present scaled variables) the reflected signal observed up to time t could not have come from scattering in the region $\tau < -t/2$. This is seen more analytically in the next section.

5.3. The Wigner functional. Let $\lambda^{NM}(s, \omega)$ be a doubly indexed array of test functions each of which has the same properties as the test functions λ^N of the previous section. We define the generalized Wigner functional of the reflected process by

$$\langle W^\varepsilon(\tau), \lambda\rangle = \sum_{N,M} \int dt \int d\omega \lambda^{NM}(t,\omega) \int dh\, e^{iht} \exp(-iN\psi(\tau, \omega - \varepsilon h/2) + iM\psi(\tau, \omega + \varepsilon h/2)). \tag{5.39}$$

Formally the Wigner functional is defined by

$$W^{NM}(\tau, t, \omega) = \frac{1}{2\pi}\int e^{iht} \exp(-iN\psi(\tau, \omega - \varepsilon h/2) + iM\psi(\tau, \omega + \varepsilon h/2)\,dh, \tag{5.40}$$

but this expression makes sense only through (5.39). The reason we introduce the Wigner functional is as follows.

To study moments of $R^\varepsilon_{f,t}(\tau, \sigma)$ at a fixed t we must go beyond test functions of the form (5.24) which, in view of (5.23), contain integration over t. Formally, we could take $\lambda^N(s, \omega)$ to be a delta function in s but then the powerful calculus of generators is lost. Consider the expression

$$\begin{aligned} &R^\varepsilon_{f,t}(\tau, \sigma)R^\varepsilon_{f,t}(\tau, 0) \\ &= \frac{1}{(2\pi)^2}\frac{1}{\varepsilon}\iint d\omega_1\, \omega_2\, e^{-i\omega_1(t+\varepsilon\sigma)/\varepsilon}\hat{f}(\omega_1)\, e^{-i\psi(\tau,\omega_1)}\, e^{i\omega_2 t/\varepsilon}\hat{f}^*(\omega_2)\, e^{i\psi(\tau,\omega_2)} \end{aligned} \tag{5.41}$$

obtained from (5.22). Let

$$\omega = \frac{\omega_1+\omega_2}{2}, \qquad h = \frac{\omega_2-\omega_1}{\varepsilon}.$$

Then, after dropping some formally small terms as $\varepsilon \to 0$, we see that

$$R^\varepsilon_{f,t}(\tau, \sigma)R^\varepsilon_{f,t}(\tau, 0) \approx \frac{1}{2\pi}\int e^{-i\omega\sigma}|\hat{f}(\omega)|^2 W^{11}(\tau, t\omega)\, d\omega \tag{5.42}$$

with W^{NM} defined by (5.40). This explains how the Wigner functional enters and why its expectation is interesting. In fact, the power spectral density $\mu(t, \omega)$ in § 3 is simply the expectation of W^{11} in the limit $\varepsilon \to 0$.

In the same way that we calculated the generator of the Markov process $(q^\varepsilon(\tau), R^\varepsilon(\tau))$ in section (5.2) we can now calculate the generator of the Markov process $(q^\varepsilon(\tau), W^\varepsilon(\tau))$ on functions of the form $F(q, \langle W, \lambda\rangle)$. We have

$$\mathbf{M}^\varepsilon_\tau F = \frac{c_0(\tau)}{\varepsilon^2}QF + F'\left\{\frac{1}{\varepsilon}\langle W, H_1\lambda\rangle + \langle W, H_2\lambda\rangle\right\} \tag{5.43}$$

where

$$\begin{aligned} (H_1\lambda)^{NM}(s, \omega) = {}& 2i(N-M)\omega\lambda^{NM}(s, \omega) + m2i(N-M)\omega\lambda^{NM}(s, \omega) \\ &+ n\{i\omega(N+1)\lambda^{N+1,M}(s, \omega) + i\omega(N-1)\lambda^{N-1,M}(s, \omega) \\ &- i\omega(M-1)\lambda^{N,M-1}(s, \omega) \\ &- i\omega(M+1)\lambda^{N,M+1}(s, \omega)\} \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} (H_2\lambda)^{NM}(s, \omega) = {}& (N+M)\frac{\partial\lambda^{NM}}{\partial s} + m(N+M)\frac{\partial\lambda^{NM}}{\partial s} \\ &+ \frac{\zeta(\tau)}{2}\{(N+1)\lambda^{N+1,M} - (N-1)\lambda^{N-1,M} \\ &- (M-1)\lambda^{N,M-1} + (M+1)\lambda^{N,M+1}\} \\ &- \frac{n}{2}\left\{(N+1)\frac{\partial}{\partial s}\lambda^{N+1,M} + (N-1)\frac{\partial}{\partial s}\lambda^{N-1,M}\right. \\ &\left. + (M-1)\frac{\partial}{\partial s}\lambda^{N,M-1} + (M+1)\frac{\partial}{\partial s}\lambda^{N,M+1}\right\}. \end{aligned} \tag{5.45}$$

The generator on functions that depend on τ explicitly is $\partial_\tau + \mathbf{M}^\varepsilon_\tau$.

To calculate the limit generator we fix a function $F(\langle W, \lambda\rangle)$ and let

$$F^\varepsilon = F + \frac{\varepsilon}{c_0(\tau)}F'\langle W, H_1^\chi\lambda\rangle. \tag{5.46}$$

Here H_1^χ is identical to H_1 of (5.44) except that m and n are replaced by χ_m and χ_n that are defined by (5.30). We also restrict attention to test functions λ^{NM} such that

$$\lambda^{NM}=0 \quad \text{for } N\neq M. \tag{5.47}$$

We call these *diagonal* test functions. We now calculate

$$\mathbf{M}_\tau^\varepsilon F^\varepsilon=\frac{1}{c_0(\tau)}F''\{\langle W, H_1\lambda\rangle\langle W, H_1^\chi\lambda\rangle\}+F'\left\{\frac{1}{c_0(\tau)}\langle W, H_1H_1^\chi\lambda\rangle+\langle W, H_2\lambda\rangle\right\}+O(\varepsilon).$$

Taking expectation with respect to $\bar{P}$, we define the limit generator $\mathbf{M}_\tau$:

$$\begin{aligned}\mathbf{M}_\tau F=&\frac{1}{c_0(\tau)}F''\langle W\times W, E[H_1\times H_1^\chi]\lambda\times\lambda\rangle\\&+F'\left\{\frac{1}{c_0(\tau)}\langle W, E[H_1H_1^\chi]\lambda\rangle+\langle W, E[H_2]\lambda\rangle\right\}\end{aligned} \tag{5.48}$$

with the test functions λ^{NM} diagonal (5.47).

The interesting thing about the generator (5.48) is that it preserves homogeneity (is "linear") as before but in addition if $F=\langle W, \lambda\rangle$ with λ diagonal, then $\mathbf{M}_\tau F$ is linear in W with a diagonal test function. This follows from the fact that if λ is diagonal with $\lambda^{NN}=\lambda^N$, then

$$(E[H_1H_1^\chi]\lambda)^N=\frac{4\omega^2\alpha_{nn}}{c_0(\tau)}\left\{\frac{(N+1)^2}{2}\lambda^{N+1}-N^2\lambda^N+\frac{(N-1)^2}{2}\lambda^{N-1}\right\}.$$

Here α_{nn} is defined by (3.2) or (5.38).

Let us now use this important property of the limit generator to calculate the conditional expectation of the Wigner functional at $\tau=0$, given its value W, say, at a fixed negative τ. Let $\tilde{\lambda}^N=\tilde{\lambda}^N(\tau, t, \omega)$ be the solution of the system

$$\frac{\partial\tilde{\lambda}^N}{\partial t}+2N\frac{\partial\tilde{\lambda}^N}{\partial s}+\frac{4\omega^2\alpha_{nn}}{c_0(\tau)}\left\{\frac{(N+1)^2}{2}\tilde{\lambda}^{N+1}-N^2\tilde{\lambda}^N+\frac{(N-1)^2}{2}\tilde{\lambda}^{N-1}\right\}=0 \tag{5.49}$$

for $\tau<0$ and $s<t$ with the terminal condition

$$\tilde{\lambda}^N(0, s, \omega)=\frac{1}{2\pi}\delta_{N1}\delta(t-s)$$

and let $\lambda^N(\tau, t, \omega)=\tilde{\lambda}^N(\tau, t, \omega)\delta(\omega)$. Then

$$E_{W,\tau}\{\langle W(0), \lambda(0)\rangle\}=\langle W, \lambda(\tau)\rangle \tag{5.50}$$

because for test functions that satisfy (5.49) the scalar process $\langle W(\tau), \lambda(\tau)\rangle$ is a martingale. The subscripts on the expectation on the left side of (5.50) denote conditioning.

The terminal conditions in (5.49) were chosen so that

$$E_{W,\tau}\{\langle W(0), \lambda(0)\rangle\}=E_{W,\tau}\{W^{11}(0, t, \omega)\}. \tag{5.51}$$

From (5.42) we see that this must be the power spectral density $\mu(t, \omega)$ of the limit reflection process as described in § 2. For this to be true, the right-hand side of (5.50) must not depend on W and τ if $\tau\leqq -t/2$. But this is clearly true because of the hyperbolic nature of the system (5.49). First, $\lambda^N\equiv 0$ for $N<0$. Second, $\lambda^N\equiv 0$ for $N\geqq 1$ when $\tau<-t/2$ by the support properties in the variables t and τ of the solution λ^N of (5.49) (dependence on ω is parametric and the $\delta(\omega)$ is a multiplicative factor).

Thus, only λ^0 is different from zero when $\tau < -t/2$ and it is in fact constant in this range. Since $W^{00} = \delta(t)$ in all cases, we see that

$$\langle W, \lambda(\tau)\rangle = \tilde{\lambda}^0\left(\frac{-t}{2}, t, \omega\right) \tag{5.52}$$

for all $\tau < -t/2$. Except for changes in notation, $\tilde{\lambda}^0(-t/2, t, \omega)$ is exactly the same as $W^{(0)}(t/2, t, \omega)$ of (3.6a).

The above calculation shows that the finite speed of propagation, which makes (5.49) hyperbolic, has very significant implications for the generalized reflection process. In the previous paragraph we saw how the calculation of the mean Wigner distribution is done directly in the right framework and this should be compared with the calculation of § 4 where the support properties of the quantities of interest are hidden and the interchanges of limits are unjustified (are impossible to justify in that setting).

5.4. The Gaussian property. If the generator of the generalized reflection functional $R^\varepsilon(\tau)$ of (5.23) has the form (5.32) (or (5.35) in the centered case) then it cannot possibly be a Gaussian process, which would make it a Gauss-Markov or Ornstein-Uhlenbeck process. So in what sense then do we have the Gaussian property for the reflected signals as we have claimed in § 3? The answer to this question rests entirely on the finite propagation speed of the signals because then the law of the reflection process in the time interval $[0, t]$ at $\tau = 0$ becomes constant, indendent of τ for $\tau < -t/2$ and independent of the conditioning at τ. In other words the generalized reflection process at $\tau = 0$ becomes ergodic after a finite, negative starting value of τ that depends on the support of the test functions as functions of time. It is this ergodic law that is Gaussian.

The ergodicity of the law of the reflection functional is shown through its moments, using the finite propagation speed as with the Wigner functional. We will not give the details here because they are lengthy. Now consider the generalized reflection functional $\tilde{R}_{N,t}(\tau)$ defined by (5.36) with ψ replaced by $\tilde{\psi}$, the centered phase. We want to show that for each sequence of test functions $\{\lambda^N(\omega)\}$

$$E_{W,\tau}\left\{\exp\left[i\sum_N \langle \tilde{R}_{N,t}(0), \lambda^N\rangle\right]\right\} = \exp\left[-\frac{1}{2}\sum_N E_{W,\tau}\{\langle \tilde{W}_t^{NN}(0), |\lambda^N|^2\rangle\}\right] \tag{5.53}$$

with the expectations being taken relative to the limit process and with the conditioning at $\tau < -t/2$ indicated with subscripts. The expectations do not depend on the conditioning in this case as we saw above. On the right side of (5.53), $\tilde{W}^{NN}$ denotes the shifted Wigner functional at time t

$$\tilde{W}_t^{NN}(\tau, \omega) = W^{NN}(\tau, t - 2N\tau, \omega). \tag{5.54}$$

The expectation of the Wigner functional on the right side of (5.53) can be computed in a manner similar to the one we used in (5.49)-(5.52). We describe briefly the procedure. The expectation $E_{W,\tau}\{W^{N'N'}(0, t, \omega)\}$ for $\tau < -t/2$ is equal to $\tilde{\lambda}^{0N'}(-t/2t, \omega)$ where $\tilde{\lambda}^{NN'}(\tau, t, \omega)$ satisfies (5.49) for $\tau < 0$, $s < t$ with terminal condition

$$\tilde{\lambda}^{NN'}(0, s, \omega) = \frac{1}{2\pi}\delta_{NN'}\delta(t - s). \tag{5.55}$$

Then

$$E_{W,\tau}\{\langle W^{N'N'}(0, t, \cdot), |\lambda^{N'}|^2\rangle\} = \int d\omega \tilde{\lambda}^{0N'}\left(\frac{-t}{2}, t, \omega\right)|\lambda^{N'}(\omega)|^2. \tag{5.56}$$

For the shifted Wigner functional appearing on the right side of (5.53) we use (5.56) with the second argument t on the right side replaced by $t-2N\tau$.

Now suppose that we fix $t>0$ and $\tau<-t/2$ and let (5.53) define a Gaussian law for $\tilde{R}_{N,t}(0)$. We want to show that this law is invariant with respect to the Markovian evolution that generates the process. We must then show that

$$E\left\{\mathbf{L}_\tau \exp\left[i\sum_N \langle \tilde{R}_{N,t}, \lambda^N\rangle\right]\right\}=0 \tag{5.57}$$

where E stands for expectation with respect to the Gaussian law (5.53) and $\mathbf{L}_\tau$ is given by (5.37). In addition, we will use the independence of $R_{N,t}$ (when $\tau<-t/2$) for different t's. As we noted earlier we omit the proof of this fact in the brief description given here. To prove (5.57) under the independence condition we first calculate the expectation of each term in (5.57) when (5.37) is used for $\mathbf{L}_\tau$. This is a standard calculation with characteristic functions of Gaussian integrals. An identity then results involving the expectation of the Wigner functionals and it must be true for (5.57) to hold. This identity is none other than (5.49) with the terminal condition (5.55), the time t shifted to $t-2N\tau$ and with $\tau<-t/2$ so that the $\partial/\partial\tau$ term drops. This then shows that the law (5.53) is invariant. A uniqueness argument finally tells us that (5.53) is the ergodic limit law.

There are many details that we have omitted here that are needed to make the above a complete proof. The brief description we have given introduces the main ideas and the framework of the functional processes.

Appendix A. Limit theorem for a stochastic differential equation. References to work on the asymptotics of stochastic equations are cited in [1].

We consider here the behavior, as $\varepsilon\downarrow 0$, of ψ^ε given by (4.7). As discussed in § 4, we assume that the equation is driven by a Markov process with slowly varying parameters. Thus, we let $q(x, y)$ with values in R^d be, for each fixed x, a stationary ergodic Markov process in y, with infinitesimal generator Q_x. Equation (4.7) is then of the form

$$\frac{d}{dx}\psi^\varepsilon=\frac{1}{\varepsilon}\mathbf{F}\left(x, q\left(x, \frac{x}{\varepsilon^2}\right), \frac{\tau(x)}{\varepsilon}, \psi^\varepsilon\right)+\mathbf{G}\left(x, q\left(x, \frac{x}{\varepsilon^2}\right), \frac{\tau(x)}{\varepsilon}, \psi^\varepsilon\right). \tag{A1}$$

By ergodicity $q(x, \cdot\,)$ has an invariant measure $\bar{P}_x(dq)$ that satisfies

$$\int Q_x f(q)\bar{P}_x(dq)=0 \tag{A2}$$

for any test function f. We define expectation with respect to $\bar{P}_x$ by

$$E\{\cdot\}\equiv\int \cdot\, \bar{P}_x(dq). \tag{A3}$$

Since m, n have mean zero, it is apparent from (4.8) that

$$E\{\mathbf{F}\}=0. \tag{A4}$$

Now the infinitesimal generator of the Markov process q^ε, ψ^ε with $q^\varepsilon(x)=q(x, x/\varepsilon^2)$ is given by (4.9), so that the Kolmogorov backward equation for this process may be written as follows:

$$\frac{\partial V^\varepsilon}{\partial x}+\frac{1}{\varepsilon^2}Q_x V^\varepsilon+\frac{1}{\varepsilon}\mathbf{F}\cdot\nabla_\psi V^\varepsilon+\mathbf{G}\cdot\nabla_\psi V^\varepsilon=0, \qquad x<0. \tag{A5}$$

We consider final conditions at $x=0$ that do not depend on q, i.e.,

(A6) $$V^\varepsilon(x, q, \psi)_{x=0} = H(\psi).$$

Let

(A7) $$\eta = \tau/\varepsilon$$

so that $\mathbf{F} = \mathbf{F}(x, q, \eta, \psi)$, etc. in (A5). We will solve (A5), (A6) asymptotically as $\varepsilon \downarrow 0$ by the multiple scale expansion

(A8) $$V^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n V^n(x, q, \eta, \psi)|_{\eta=\tau(x)/\varepsilon}.$$

To expand (A5) in multiple scales x, η, we replace $\partial/\partial x$ by $\partial/\partial x + \tau'(x)/\varepsilon\, \partial/\partial\eta$, and note that $\tau'(x) = -1/c_0(x) < 0$. Thus (A5) becomes

(A9) $$\frac{1}{\varepsilon} Q_x V^\varepsilon + \frac{1}{\varepsilon}\left\{\mathbf{F}\cdot\nabla_\psi V^\varepsilon - \frac{1}{c_0(x)}\frac{\partial}{\partial\eta} V^\varepsilon\right\} + \left\{\mathbf{G}\cdot\nabla_\psi V^\varepsilon + \frac{\partial}{\partial x} V^\varepsilon\right\} = 0.$$

Now substitution of (A8) into (A9) yields a hierachy of equations for $V^{(n)}$ of which the first three are

(A10) $$Q_x V^0 = 0,$$

(A11) $$Q_x V^1 + \mathbf{F}\cdot\nabla_\psi V^0 - \frac{1}{c_0(x)}\frac{\partial}{\partial\eta} V^0 = 0,$$

(A12) $$Q_x V^2 + \mathbf{F}\cdot\nabla_\psi V^1 - \frac{1}{c_0(x)}\frac{\partial}{\partial\eta} V^1 + \mathbf{G}\cdot\nabla_\psi V^0 + \frac{\partial}{\partial x} V^0 = 0.$$

From (A10) and ergodicity of $q(x, \cdot)$ we conclude that V^0 does not depend on q:

(A13) $$V^0 = V^0(x, \eta, \psi).$$

We next take the expectation of (A11). Since $\mathbf{F}$ has mean zero as noted in (A4), using (A2) we see that (A11) implies that

$$-\frac{1}{c_0(x)}\frac{\partial}{\partial\eta} V^0 = 0,$$

whence V^0 does not depend on η:

(A14) $$V^0 = V^0(x, \psi).$$

Now by ergodicity, Q_x has the one-dimensional null space consisting of functions that do not depend on q. Thus Q_x does not have an inverse. However, by the Fredholm alternative, which we assume to hold for the process q, Q_x has an inverse on the subspace of functions that have mean zero with respect to $\bar{P}_x$. We define a particular inverse Q_x^{-1} such that its range consists of functions with vanishing mean

(A15) $$-Q_x^{-1} = \int_0^\infty e^{Q_x r}\, dr.$$

In terms of this Q_x^{-1} we can solve (A11) for V^1

(A16) $$V^1 = -Q_x^{-1}\{\mathbf{F}\cdot\nabla_\psi V^0\} + V^{1,0}$$

where $V^{1,0}$ does not depend on q.

We now substitute (A16) into (A12) and take expectations. We also average this equation with respect to η:

$$\text{(A17)} \qquad \langle\cdot\rangle_\eta = \lim_{\eta_0\to\infty} \frac{1}{\eta_0}\int_0^{\eta_0} \cdot\, d\eta.$$

Since $E\{\mathbf{F}\}=0$ and $\langle \mathbf{G}\rangle_\eta = 0$ (see (4.8)) we see that V^0 must satisfy

$$\text{(A18)} \qquad \frac{\partial}{\partial x} V^0 + \langle E\{\mathbf{F}\cdot\nabla_\psi(-Q_x^{-1})\mathbf{F}\cdot\nabla_\psi\}\rangle_\eta V^0 = 0.$$

This is the solvability condition for (A12).

Equation (A18) is the limiting backward Kolmogorov equation for ψ. It has the form

$$\text{(A19)} \qquad \frac{\partial}{\partial x} V^0 + \mathbf{L}_x V^0 = 0, \qquad x<0,$$

$$V^0|_{x=0} = H(\psi).$$

From (A18) the limit infinitesimal generator $\mathbf{L}_x$ is given by

$$\text{(A20)} \qquad \mathbf{L}_x = \int_0^\infty dr \langle E\{\mathbf{F}\cdot\nabla_\psi e^{rQ_x}\mathbf{F}\cdot\nabla_\psi\}\rangle_\eta.$$

Using the probabilistic interpretation of the semigroup e^{rQx} and expectation $E\{\cdot\}$ with respect to the invariant measure $P_x(dq)$ and the averaging $\langle\cdot\rangle$, we can write (A20) in the form

$$\text{(A21)} \qquad \mathbf{L}_x = \int_0^\infty dr E\{\langle \mathbf{F}(x, q(x,y), \eta, \psi)\cdot\nabla_\psi(\mathbf{F}(x, q(x, y+r), \eta, \psi)\cdot\nabla_\psi\cdot\,)\rangle_\eta\}.$$

In the application of this result in § 4, the explicit form of **F** in (4.7) is used in (A21) to obtain (4.11).

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