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DIFFUSIONS WITH RANDOM COEFFICIENTS

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1. Introduction

Here we will be studying diffusion processes in R^d whose coefficients are random, stationary, stochastic processes. We will be mainly concerned with the behavior of such diffusion processes for large times for which we will establish a form of the ergodic theorem and deduce a central limit theorem from it.

Our diffusion processes will always start at $t=0$ from the point $x=0$ in R^d and will be governed by the diffusion generator:

$$L_\omega = \tfrac{1}{2}\sum a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Here $\omega \in \Omega$ where $(\Omega, \mathfrak{F}, \mu)$ is a probability space and $\{a_{ij}(x,\omega)\}$ constitutes a stationary stochastic process with $x \in R^d$ as the parameter set. For each x, the random variables $\{a_{ij}(x,\omega)\}$ constituting the diffusion coefficients at x, can be viewed as taking values in the space S_d of symmetric positive semi-definite matrices of size d. We will assume, throughout this paper, that the coefficients $\{a_{ij}(x,\omega)]$ are all almost surely continuous in x and that there exist constants $0 < C_1 \leqslant C_2 < \infty$ which are non-random such that,

$$C_1 \sum \xi_j^2 \leqslant \sum a_{ij}(x,\omega)\xi_i\xi_j \leqslant C_2 \sum \xi_j^2, \quad \forall x \in R^d, \quad \omega \in \Omega.$$

The last assumption we need is the ergodicity of the coefficient process $\{a(x,\omega)\}$ with respect to translations in space.

For each $\omega \in \Omega$, L_ω governs a diffusion and corresponding to the initial conditions of starting from $x=0$ at $t=0$ we have a measure P_ω on the space $C[[0,\infty); R^d]$ of continuous trajectories on $[0,\infty)$ with values in R^d. We want to look at the behavior of the distribution of $x_t/\sqrt{t}$ under P_ω as $t \to \infty$. Some form of the ergodic theorem should produce an averaging effect and the limiting distribution of $x_t/\sqrt{t}$ under P_ω should be independent of ω and should be a normal distribution with mean zero and covariance $\bar{a}$. The identification of $\bar{a}$ should be made as a precise form of averaging. This is the goal of this article.

Instead of random coefficients, one can take periodic coefficients and this case has been studied extensively in refs. [2, 1]. The case of random coefficients when L_ω is in divergence form has been studied in refs. [3, 4].

When the dimension, d, is one, we can use the method of random time change to study the problem explicitly and an easy computation establishes the result and calculates the limiting variance $\bar{a}$ to be the harmonic mean:

$$\bar{a} = \left[E \frac{1}{a(0;\omega)} \right]^{-1}.$$

2. An ergodic theorem

Although we formulated the theorem as a central limit theorem it is in reality an ergodic theorem that underlies the phenomenon. To see this, from the fact that P_ω is the diffusion corresponding to L_ω it follows that,

$$E^{P_\omega}\left[\exp\left\{ \mathrm{i}\langle\theta, x(t)\rangle + \tfrac{1}{2}\left\langle \theta, \int_0^t a[x(s),\omega]\,\mathrm{d}s\,\theta \right\rangle \right\}\right] = 1 \quad \text{a.e. } \omega.$$

By replacing θ by $\theta/\surd t$ we can rewrite this as:

$$E^{P_\omega}\left[\exp\left\{ \mathrm{i}\left\langle\theta, \frac{x(t)}{\surd t}\right\rangle + \tfrac{1}{2}\left\langle \theta, \frac{1}{t}\int_0^t a[x(s),\omega]\,\mathrm{d}s\,\theta \right\rangle \right\}\right] = 1.$$

If we had an ergodic theorem asserting,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t a(x(s),\omega)\,\mathrm{d}s = \bar{a}, \quad \text{a.e. } P_\omega, \text{a.e. } \mu, \tag{2.1}$$

it then follows that,

$$\lim_{t\to\infty} E^{P_\omega}\left[\exp\left\{ \mathrm{i}\left\langle\theta, \frac{x(t)}{\surd t}\right\rangle \right\}\right] = \exp\{-\tfrac{1}{2}\langle\theta, \bar{a}\theta\rangle\},$$

which is the central limit theorem that we are after.

Now we will set up the basic notation that we will be using. Let Ω denote the space of all continuous maps $\omega: R^d \to S_d$ satisfying

$$C_1 \sum \xi_j^2 \leqslant \sum \omega_{ij}(x)\xi_i\xi_j \leqslant C_2 \sum \xi_j^2,$$

for all $\xi \in R^d$ and $x \in R^d$. We will use also the customary notation $a_{ij}(x,\omega)$ to denote $\omega_{ij}(x)$ so we may conveniently view $a(x,\omega) = \{a_{ij}(x,\omega)\}$ either as a function of x for each fixed $\omega \in \Omega$ or as a function on Ω for each fixed x. We also have the translation maps $\tau_x: \Omega \to \Omega$ defined by:

$$a(y, \tau_x\omega) = a(x+y,\omega), \quad \forall x \in R^d, \quad y \in R^d \quad \text{and} \quad \omega \in \Omega.$$

There is a canonical Markov process with Ω as state space. Let us fix $\omega \in \Omega$. Then we have the diffusion operator,

$$L_\omega = \tfrac{1}{2}\sum a_{ij}(x,\omega)\frac{\partial^2}{\partial x_i \partial x_j},$$

on R^d and this generates a diffusion measure P_ω corresponding to these coefficients, with initial conditions corresponding to starting from the origin at time 0. P_ω is a measure on the space $C[[0,\infty); R^d]$ of trajectories starting from 0 at time 0.

Given a trajectory $x(t)$ with $x(0)=0$ we can map it onto a trajectory in the Ω space by setting

$$\omega_t = \tau_{x(t)}\omega.$$

This induces a measure Q_ω on the space of trajectories in Ω starting from ω at time 0 and a moment's reflection tells us that this is in fact a Markov process with Ω as state space. Let us denote by $q(t,\omega,d\omega')$ the transition probability of this Markov process. If we think of the topology of Ω as coming from uniform convergence on compact sets it is easy to see that $q(t,\omega,d\omega')$ has the Feller property. If we denote by $A(\omega)$ the function $a(0,\omega)$ then the property (2.1) that we want is really,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t A(\omega_s)\,ds = \bar{a} \quad \text{a.e. } Q_\omega \quad \text{a.e. } \mu, \tag{2.2}$$

where μ is a stationary ergodic measure on Ω. In other words, we need an ergodic theory for the canonical Markov process $\{Q_\omega\}$ on Ω. In the next section we will establish the following theorem.

Theorem 2.1. *Given any ergodic stationary measure μ on Ω, there exists a unique λ on Ω which is mutually absolutely continuous with respect to μ, and which is an ergodic invariant measure with respect to our canonical Markov process on Ω.*

As an immediate consequence we have:

Theorem 2.2. *For any ergodic stationary measure μ on Ω,*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t A(\omega_s)\,ds = \int_\Omega A(\omega)\lambda(d\omega), \quad a.e.\ Q_\omega, \quad a.e.\ \mu,$$

where λ is determined by μ according to Theorem 2.1.

3. Proof

In order to prove our main theorem we need two results from the theory of partial differential equations which we shall state as lemmas without proofs.

Lemma 3.1. *Let $a(x)$ be a set of coefficients on R^d which is periodic in each variable of period 1. We will think of*

$$L = \tfrac{1}{2}\sum a_{ij}\frac{\partial^2}{\partial x_i\,\partial x_j}$$

as a diffusion process on the d-dimensional torus, T_d. Then the resolvent,

$$R = (I-L)^{-1},$$

maps $L_d(T_d)$ into $L_\infty(T_d)$ with a bound K that depends only on the upper and lower bounds for eigenvalues of the coefficient matrix $\{a(x)\}$ and not on the smoothness of the coefficients:

Remark. This result can be found in ref. [6] in a slightly different form. The modifications needed to reduce it to Lemma 3.1 are routine.

Lemma 3.2. *Let $a(x)$ be a set of coefficients in our space Ω. Let $p(t, x, \mathrm{d}y)$ be the transition probability for the diffusion on R^d governed by a. Then for each $x \in R^d$ and $t>0$, the measure $p(t, x, \mathrm{d}y)$ is equivalent to the Lebesgue measure on R^d.*

Remark. The existence of a density for $p(t, x, \mathrm{d}y)$ can be found in ref. [7] and the fact that the density cannot vanish on a set of positive measure is essentially the strong maximum principle.

The next ingredient is the fact that any stationary process is a limit of periodic processes:

Lemma 3.3. *Let μ be a stationary process on Ω. Then we can find for a sequence $l_n \to \infty$, periodic functions $\omega_n \in \Omega$ of period l_n in each variable, such that, the measure μ_n on Ω obtained as the distribution of $\tau_x \omega_n$ where x is random and distributed uniformly on the cube of size l_n, converges weakly to μ as $n \to \infty$.*

Remark. This is a well-known fact from the theory of stationary process. See for instance reference [5] for a proof.

Proof of Theorem 2.1. Let us start with the coefficients $\omega_n \in \Omega$ given by Lemma 3.3 which is periodic of period l_n. By the map $x \to x/l_n$ we can shrink the torus of size l_n to one of unit size. If ω_n represents the coefficients $a^{(n)}(x)$ then we consider the coefficients $a^{(n)}(l_n x)$ on the unit torus. The diffusion generated by this on the torus will have an invariant density $\phi_n(x)$ representing an invariant probability distribution for this diffusion. Although the coefficients are rougher the upper and lower bounds C_1, C_2 stay the same, i.e.

$$C_1 \sum \xi_j^2 \leqslant a_{ij}^{(n)}(l_n x) \xi_i \xi_j \leqslant C_2 \sum \xi_j^2.$$

From Lemma 3.1 we know that the resolvent R maps L_d into L_∞ boundedly with a bound K independent of n. The adjoint R^* therefore maps L_1 boundedly into $L_{d/(d-1)}$ with the same bound K. Since $R^* \phi_n = \phi_n$ and ϕ_n has L_1 norm one we obtain,

$$\| \phi_n \|_{d/(d-1)} \leqslant K \quad \text{for all } n.$$

If we now go back to the torus of size l_n and denote by $\tilde{\phi}_n(x)$, the function $\phi_n(x/l_n)$ then,

$$\| \tilde{\phi}_n \|_{d/(d-1)} \leqslant K,$$

provided we compute the norms with respect to the normalized Lebesgue measure on the torus of size l_n. We denote the probability measure represented by $\tilde{\phi}_n$ by $\tilde{\lambda}_n$ and map it into Ω by the map $x \to \tau_x \omega_n$. The measure induced on Ω is denoted by λ_n. It is easily verified that λ_n is an invariant measure for the canonical Markov process, $\lambda_n \ll \mu_n$ and that

$$\int_\Omega \left| \frac{d\lambda_n}{d\mu_n} \right|^{d/(d-1)} d\mu_n \leqslant K^{d/(d-1)}.$$

Since μ_n is converging weakly to μ and K is independent of n, it is routine to check that λ_n has a weakly convergent subsequence, and the limit λ so obtained is again an invariant measure for our canonical Markov process. In addition, $\lambda \ll \mu$ and:

$$\int_\Omega \left| \frac{d\lambda}{d\mu} \right|^{d/(d-1)} d\mu \leqslant K^{d/(d-1)}.$$

We now assume that μ is ergodic and show that $\mu \ll \lambda$ and λ is ergodic for the canonical process. If μ is not absolutely continuous with respect to λ then since $\lambda \ll \mu$ we can use the Lebesgue decomposition to find a set $E \subset \Omega$ with the properties $\lambda(E)=0$, $0<\mu(E)<1$ and $\mu \ll \lambda$ on E^c. From $\lambda(E^c)=1$ we conclude,

$$q(t, \omega, E^c)=1 \quad \text{a.e. } \omega \quad \text{w.r.t. } \lambda.$$

Since $\mu \ll \lambda$ on E^c this implies

$$q(t, \omega, E^c)=1 \quad \text{a.e. } \omega \text{ on } E^c \quad \text{w.r.t. } \mu.$$

From the definition of q and Lemma 3.2, it now follows that

$$\tau_x \omega \in E^c \quad \text{a.e. } x \quad \text{a.e. } \omega \text{ on } E^c \quad \text{w.r.t. } \mu.$$

This in turn implies by Fubini's theorem that

$$\tau_x E^c = E^c \quad \text{for almost all } x,$$

as elements in the measure algebra of μ. Since $x \to \tau_x$ is continuous we conclude that E^c is an invariant set. By ergodicity of μ we must have $\mu(E^c)=0$ or 1 and we have a contradiction.

Let us now prove that as an invariant measure for our canonical process λ is ergodic; if not, we have a set $E \subset \Omega$ with $0<\lambda(E)<1$ such that

$$q(t, \omega, E)=1 \quad \text{a.e. } \omega \text{ in } E \quad \text{w.r.t. } \lambda.$$

Since $\mu \ll \lambda$, we conclude

$$q(t, \omega, E)=1 \quad \text{a.e. } \omega \text{ in } E \quad \text{w.r.t. } \mu.$$

Just as in the preceding argument we conclude now by Lemma 3.2 that E is invariant and by ergodicity of μ that $\mu(E)=0$ or 1. Since $\lambda \ll \mu$ we must also have $\lambda(E)=0$ or 1.

This completes the proof of Theorem 2.1. Theorem 2.2 of course follows immediately. Note that the ergodicity of any arbitrary λ which is absolutely continuous with respect to μ implies the uniqueness of λ.

4. Remarks

We remark that without any work we can obtain the weak convergence of the distribution of the process $(x(kt))/\sqrt{k}$ as $k \to \infty$ under Q_ω to a suitable Brownian motion.

If $d\lambda/d\mu = \psi(\omega)$ then $\psi(\tau_x \omega)$ defines a stochastic process $\psi(x, \omega)$ with the following properties: For a.e. ω $\psi(x, \omega)$ is a σ-finite non-negative invariant density for L_ω on R^d. It is a stationary stochastic process in x and is normalized to have $E\psi(x, \omega) = 1$.

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