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BOUNDARY VALUE PROBLEMS WITH RAPIDLY OSCILLATING RANDOM COEFFICIENTS

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

We shall introduce the problem under consideration and its analysis in an informal way at first. The detailed mathematical formulation begins in the next section and the main theorem is stated in Section 3. Another version of this theorem is given in Section 5 and a probabilistic one in Section 8.

Consider a conductor occupying a region $\mathcal{O}$ in R^3 and suppose that the conducting material is inhomogeneous as for example in the case of a composite or a mixture of several materials with different conductivities. We shall model the material with a conductivity that is a random function of position changing rapidly as x varies over lengths comparable to the size of the region $\mathcal{O}$. To articulate this last feature we introduce a parameter $\epsilon > 0$ which is the ratio of a typical length scale associated

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with the region $\mathcal{O}$ to a typical length scale associated with the variations in conductivity. We take then the conductivity to be a random function of the form $a\left(\frac{x}{\epsilon}\right)$ where $a(x)$ is a given stationary random function that is strictly positive and bounded. The temperature $u^{\epsilon}(x)$ satisfies the stochastic heat equation

$$(1.1) \qquad -\nabla \cdot \left(a\left(\frac{x}{\epsilon}\right)\nabla u^{\epsilon}(x)\right) = f(x) \qquad (x \in \mathcal{O}),$$

and we assume for simplicity here that $u^{\epsilon}(x) = 0$ for $x \in \partial\mathcal{O}$, the boundary of $\mathcal{O}$. The function $f(x)$ is a given deterministic (for simplicity) function which represents the density of heat sources in $\mathcal{O}$.

The problem now is to analyze the behaviour of the random temperature distribution $u^{\epsilon}(x)$ as $\epsilon \to 0$. It is found that when the random conductivity is strictly stationary and ergodic (cf. Section 2), there exist constants q_{ij} $(i, j = 1, 2, 3)$ such that if $u(x)$ is the solution of the deterministic heat equation

$$(1.2) \qquad -\sum_{i,j=1}^{3} q_{ij}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) \qquad (x \in \mathcal{O},\ u(x) = 0,\ x \in \partial\mathcal{O}),$$

then

$$(1.3) \qquad \int_{\mathcal{O}} E\{|u^{\epsilon}(x) - u(x)|^2\}\, dx \to 0 \quad \text{as} \quad \epsilon \to 0,$$

where $E\{\cdot\}$ denotes expectation value. The tensor $q = (q_{ij})$ is called the effective conductivity tensor and can be computed by solving certain auxiliary problems (cf. Section 3).

This result shows that the randomly inhomogeneous conducting medium will behave like a homogeneous deterministic medium with conductivity tensor q when ϵ is small. When the conductivity $a(x)$ is a periodic function the above convergence result, considered from a variety of viewpoints and in several more general or refined forms, has been obtained and discussed by several authors [1]-[10]. When $a(x)$ is almost periodic the result was obtained by Kozlov [11]. In the stochastic case (1.3) was obtained by Jurinskii [12] assuming that the random conductivity $a(x)$ satisfies a mixing condition, and in general by Kozlov [13].*

Our purpose here is to review the problem and formulate it in a suitable analytical framework, which is described in Section 2, in which one can obtain (1.3) and related facts with just stationarity, ergodicity and uniform ellipticity assumptions on $a(x)$. Thus the periodic, almost periodic and random cases come out of the same theorem in a unified way.

In the remainder of this section we shall describe briefly the calculations that enter in obtaining (1.3) and in determining the effective conductivity tensor (q_{ij}).

Let $(\Omega, \mathscr{F}, P)$ be a probability space with $\omega \in \Omega$ labeling the realization of the medium $(a(x) = a(x, \omega))$. It is natural to attempt to expand the temperature $u^\epsilon(x) = u^\epsilon(x, \omega)$ in the form

$$u^\epsilon(x, \omega) = u(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}, \omega\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}, \omega\right) + \dots \tag{1.4}$$

which is the usual formalism of multiple scales used in [10] and in many other situations. Nothing that if $v^\epsilon(x) = v\left(x, \frac{x}{\epsilon}\right)$, then

$$\nabla v^\epsilon(x) = \left[\nabla_x v(x, y) + \frac{1}{\epsilon} \nabla_y v(x, y)\right]_{y = \frac{x}{\epsilon}}, \tag{1.5}$$

we insert (1.4) into (1.1), use (1.5) and equate coefficients of powers of ϵ. This gives the following sequence of problems in which $y = \frac{x}{\epsilon}$.

$$-\nabla_y \cdot (a(y, \omega)\nabla_y u(x)) = 0 \tag{1.6}$$

$$\begin{aligned} &-\nabla_y \cdot (a(y, \omega)\nabla_y u_1(x, y, \omega)) - \nabla_y \cdot (a(y, \omega)\nabla_x u(x)) - \\ &- \nabla_x \cdot (a(y, \omega)\nabla_y u(x)) = 0 \end{aligned} \tag{1.7}$$

*The authors obtained the results given here after seeing that Jurinskii's analysis did not require mixing. Kozlov's work than came to our attention wherein similar results and methods of analysis as here are given.

$$-\nabla_y \cdot (a(y,\omega)\nabla_y u_2(x,y,\omega)) - \nabla_y \cdot (a(y,\omega)\nabla_x u_1(x,y,\omega)) -$$

$$(1.8) \qquad - \nabla_x \cdot (a(y,\omega)\nabla_y u_1(x,y,\omega)) - \nabla_x \cdot (a(y,\omega)\nabla_x u(x)) =$$

$$= f(x),$$

and similarly for the higher order terms.

Clearly (1.6) is satisfied since u was taken as a function of x only. The function $u(x)$ is usually determined from solvability conditions in the equations for u_1 or u_2, etc. We consider (1.7) and note that if we let

$$(1.9) \qquad u_1(x,y,\omega) = \sum_{k=1}^{d} \chi^k(y,\omega)\,\frac{\partial u(x)}{\partial x_k},$$

then (1.7) will be satisfied if $\chi^k(y,\omega)$ solves the problem

$$(1.10) \qquad -\nabla_y \cdot (a(y,\omega)\nabla_y \chi^k(y,\omega)) = \nabla_y \cdot (a(y,\omega)e_k) \qquad (k = 1, 2, 3),$$

where e_k is the column vector with components (δ_{ik}). Once (1.10) has been solved in a suitable sense, (1.9) is inserted into (1.8) which becomes another equation of the form (1.10) with a different right-hand side. The condition that this form of (1.8) have a suitable solution leads to the equation (1.2) that determines $u(x)$.

The problem then reduces effectively to the analysis of the stochastic partial differential equation (1.10) which we rewrite as

$$(1.11) \qquad -\nabla_y \cdot (a(y,\omega)\nabla_y \chi(y,\omega)) = h(y,\omega) \qquad (y \in R^3),$$

with $a(y,\omega)$ and $h(y,\omega)$ given stationary stochastic processes. In the periodic case, that is when for each $\omega \in \Omega$, $a(y,\omega)$ and $h(y,\omega)$ are periodic of period one in y_1, y_2 and y_3 with $y = (y_1, y_2, y_3)$, problem (1.11) is elementary. If $a(y,\omega)$ is strictly positive and bounded, uniformly in y and ω, (1.11) interpreted in a weak or variational form has a unique periodic solution $\chi(y,\omega)$ if and only if the integral of h over the period cell vanishes. For uniqueness we require that the integral of χ over the period cell also vanishes. In the almost periodic and stationary random case it is not possible in general to find almost periodic or stationary solutions $\chi(y,\omega)$ even when h has mean zero. We show in Theorem 2 below that

(1.11) always has a solution $\chi(y, \omega)$ that is not stationary but its gradient $\nabla\chi(y, \omega)$ is stationary. Furthermore when a and h are ergodic and h has mean zero then $\chi(y, \omega)$ increases slower than $|y|$ in mean square as $|y| \to \infty$.

Returning to (1.10) and (1.9) we see that this is sufficient to make some sense out of (1.4) because then $\epsilon u_1\left(x, \frac{x}{\epsilon}, \omega\right)$ is truly small in mean square as $\epsilon \to 0$. We also see that the solvability condition for (1.8) leads to (1.2) where, if we let $a_{ij}(x, \omega) = a(x, \omega)\delta_{ij}$, we have

$$(1.12) \qquad q_{ij} = E\left\{\sum_{k=1}^{3} a_{ik}(x)\left(\delta_{kj} + \frac{\partial \chi^j(x)}{\partial x_k}\right)\right\} \qquad (i, j = 1, 2, 3).$$

Note that the effective conductivity $q = (q_{ij})$ is indeed constant, independent of x, since $\nabla\chi^k(x, \omega)$ is stationary. Formula (1.12) is exactly the one that holds in the periodic case and had been obtained previously. We show here that it holds also in the almost periodic and stationary random case when the solution χ^k of (1.10) is interpreted suitably.

In the method of averaging for ordinary differential equations one encounters a similar but simpler problem with the one encountered here namely that integrals of almost periodic functions with mean zero are not in general almost periodic but grow more slowly than linearly as their argument tends to infinity. It is also encountered in the analysis of stochastic ordinary differential equations by multiscaling which is frequently similar to averaging (cf. [14] and references therein).

Let us give a physical interpretation for the formula (1.12) defining the effective conductivity (q_{ij}). Conductivity is by Fourier's law heat flow per unit temperature gradient. This is, of course, already a macroscopic concept but we take it to hold microscopically here while allowing for very rapid irregular fluctuations. If the temperature distribution $u^\epsilon(x)$ stabilizes when ϵ, the scale of fluctuations, is small, then locally the conductor sees a uniform temperature gradient impressed on it. Consider now the conductor in this local picture and suppose unit temperature gradient is impressed in the y_j direction. Compute the resulting temperature distribution $y_j + \chi^j(y, \omega)$ (this is the meaning of (1.10)) and the heat flux

$J_{ij}(y, \omega)$ in the y_i direction due to the impressed temperature gradient in the y_j direction

$$(1.13) \qquad J_{ij}(y, \omega) = \sum_{k=1}^{3} a_{ik}(y, \omega)\left(\delta_{kj} + \frac{\partial \chi^j(y, \omega)}{\partial y_k}\right).$$

Then average the result over realizations. This gives the effective conductivity (q_{ij}) in (1.12) as average local heat flux per unit local temperature gradient. Note finally that if $w^k(y, \omega) = y_k + \chi^k(y, \omega)$ is the temperature distribution in the local picture so that $\nabla \cdot (a \nabla w^k) = 0$, which is (1.10), then the statement that unit temperature gradient is impressed in the y_k direction is interpreted properly by the statement that $w^k(y, \omega) - y_k = \chi^k(y, \omega) = o(|y|)$ (in mean square) as $|y| \to \infty$ which is what is shown below.

2. THE ABSTRACT FRAMEWORK

We begin with a precise formulation of (1.1). Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $(a_{ij}(y, \omega))$ $(i, j = 1, 2, \ldots, d)$ be a strictly stationary matrix-valued random field with $y \in R^d$. It is assumed throughout that there is a positive constant a_0 such that

$$(2.1) \qquad a_0 |\xi|^2 \leqslant \sum_{i,j=1}^{d} a_{ij}(y, \omega)\xi_i \xi_j \leqslant a_0^{-1} |\xi|^2$$

for all $y \in R^d$, $\omega \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_d)$. Strict stationarity means that for any $h \in R^d$, any integer $n = 1, 2, \ldots$ and any points $y_1, y_2, \ldots, y_n$ in R^d the joint distribution of $a_{ij}(y_1, \omega), a_{ij}(y_2, \omega), \ldots \ldots, a_{ij}(y_n, \omega)$ is the same as that of $a_{ij}(y_1 + h, \omega), \ldots, a_{ij}(y_n + h, \omega)$ $(i, j = 1, 2, \ldots, d)$. The processes $a_{ij}(y, \omega)$ are assumed to be stochastically continuous.

$$(2.2) \qquad \lim_{|h| \downarrow 0} P\Big\{\sum_{i,j=1}^{d} |a_{ij}(y + h, \omega) - a_{ij}(y, \omega)| > \delta\Big\} = 0,$$

for all $\delta > 0$ and $y \in R^d$. On account of this assumption the process can be taken to be separable (with separating set the rationals in R^d) and jointly measurable in $(y, \omega) \in R^d \times \Omega$ (R^d taken with its Lebesgue measurable sets and the Lebesgue measure). We take $\mathscr{F}$ to be countably generated.

We shall now introduce some notation that will be used throughout. Let $\mathscr{H} = L^2(\Omega, \mathscr{F}, P)$ be the Hilbert space of square integrable functions on Ω with inner product

$$E\{gh\} = \int_\Omega P(d\omega) g(\omega) h(\omega) \tag{2.3}$$

and norm $E^{\frac{1}{2}}\{|g|^2\}$. $\mathscr{H}$ is separable since $\mathscr{F}$ is countably generated. Let $\mathcal{O} \subset R^d$ be an open set. Denote by $H = L^2(\mathcal{O}; \mathscr{H}) = L^2(\mathcal{O} \times \Omega)$ the Hilbert space of square integrable functions on $\mathcal{O}$ with values in $\mathscr{H}$ and with inner product

$$(u, v) = \int_{\mathcal{O}} dx E\{u(x)v(x)\} = \int_{\mathcal{O}} dx \int_\Omega P(d\omega) u(x, \omega) v(x, \omega). \tag{2.4}$$

We denote by $H^1 = H^1(\mathcal{O}; \mathscr{H})$ the Hilbert space of $\mathscr{H}$ valued functions whose distribution derivatives are square integrable over $\mathcal{O}$ and with inner product

$$\begin{aligned}(u, v)_1 &= (u, v) + \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right) = \\ &= \int_{\mathcal{O}} dx \int_\Omega P(d\omega) + \sum_{i=1}^d \int_{\mathcal{O}} dx \int_\Omega P(d\omega) \frac{\partial u(x, \omega)}{\partial x_i} \frac{\partial v(x, \omega)}{\partial x_i}.\end{aligned} \tag{2.5}$$

We denote by $H_0^1 = H_0^1(\mathcal{O}; \mathscr{H})$ the Hilbert space of $\mathscr{H}$ valued functions u with square integrable distribution derivatives and such that $u = 0$ on $\partial\mathcal{O}$.

Let $\{e_n(\omega)\}$ be an orthonormal basis in $\mathscr{H}$. Then $u \in H^1(\mathcal{O}; \mathscr{H})$ if and only if $u_n(x) = E\{u(x)e_n\}$ is in $H^1(\mathcal{O}; R^1)$ and

$$\sum_{n=1}^\infty |u_n|^2_{H^1(\mathcal{O}; R^1)} < \infty.$$

We shall denote by $\mathscr{D}(\mathcal{O}; \mathscr{H})$ or $C_0^\infty(\mathcal{O}; \mathscr{H})$ the space of infinitely differentiable $\mathscr{H}$-valued functions that vanish outside a compact subset of $\mathcal{O}$.

If $u_n(x)$ are the coordinates of u relative to the basis $\{e_n\}$, then $u \in \mathscr{D}(\mathcal{O}; \mathscr{H})$ if and only if the u_n are infinitely differentiable, vanish outside a fixed compact subset of $\mathcal{O}$ and

$$\sum_{n=1}^{\infty} \int_{\mathcal{O}} dx \left| \left(\frac{\partial}{\partial x_1}\right)^{k_1} \ldots \left(\frac{\partial}{\partial x_d}\right)^{k_d} u_n(x) \right|^2 < \infty$$

for all multiindices $k = (k_1, k_2, \ldots, k_d)$ with $|k| = 1, 2, \ldots$.

Let V be a closed subset of $H^1(\mathcal{O}; \mathscr{H})$ containing $H_0^1(\mathcal{O}; \mathscr{H})$ i.e., $H_0^1 \subseteq V \subseteq H^1$. The precise meaning of the problem under considerating (1.1) with general homogeneous variational boundary conditions is this. Find $u^\epsilon(x, \omega) \in V$ such that

$$\begin{aligned} &\int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial u^\epsilon(x, \omega)}{\partial x_j} \frac{\partial \varphi(x, \omega)}{\partial x_i} + \\ (2.6) \qquad &+ \alpha \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) u^\epsilon(x, \omega)\varphi(x, \omega) = \\ &= \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) f(x)\varphi(x, \omega), \end{aligned}$$

for all $\varphi(x, \omega) \in V$. Here $\alpha > 0$ is a fixed constant and $f(x) \in L^2(\mathcal{O}; R^1)$ is given. In view of (2.1) this problem has a unique solution for each $\epsilon > 0$ by the Lax – Milgram lemma [15]. By letting $\varphi = u^\epsilon$ in (2.6) and using (2.1) we obtain the bound

$$(2.7) \qquad (u^\epsilon, u^\epsilon)_1 \leqslant C \int_{\mathcal{O}} dx \, |f(x)|^2$$

which C a constant independent of ϵ. The case $V = H_0^1$ corresponds to the Dirichlet problem while $V = H^1$ to the generalized Neumann problem. For additional information regarding boundary value problems in variational form we refer to [15].

Before going to the theorem regarding the behavior of u^ϵ as $\epsilon \to 0$ in the following sections we shall consider more closely the implications of stationarity and stochastic continuity of the coefficients (a_{ij}).

We may take the probability space $(\Omega, \mathscr{F}, P)$ to be as follows. The set Ω is the set of Lebesgue measurable $d \times d$ matrix-valued functions on R^d. The value of $\omega \in \Omega$ at $y \in R^d$ is defined almost everywhere and is denoted by $\omega_{ij}(y, \omega)$. Thus Ω is the set of all coefficients for (2.6). We take for the σ-algebra $\mathscr{F}$ the one generated by cylinder sets with base points that have rational coordinated in R^d and range sets that are spheres in R^{d^2} with rational centers and rational radii so that $\mathscr{F}$ is countably generated. The probability measure P is defined on $(\Omega, \mathscr{F})$ and invariant with respect to the translation group τ_x: $\Omega \to \Omega$ defined by

$$(2.8) \qquad (\tau_x \omega)(y) = \omega(y - x) \qquad (x, y \in R^d).$$

The translation group τ_x will also be assumed to be ergodic: the only sets A in $\mathscr{F}$ that are invariant, i.e. $\tau_x A \subset A$, have $P(A) = 0$ or 1. We note that the mapping $(x, \omega) \to \tau_x \omega$ is jointly measurable in (x, ω) with respect to $\mathscr{L}(R^d) \times \mathscr{F}$ with $\mathscr{L}(R^d)$ the σ-algebra of Lebesgue measurable sets in R^d.

Let $\tilde{f} \in \mathscr{H} = L^2(\Omega, \mathscr{F}, P)$. For almost all ω we let

$$(2.9) \qquad (T_x \tilde{f})(\omega) = \tilde{f}(\tau_{-x} \omega) \qquad (x \in R^d)$$

and note that the operators T_x form a unitary group on $\mathscr{H}$. This group is strongly continuous. To see this we note that any $\tilde{f}$ in $\mathscr{H}$ can be approximated by a function in $\mathscr{H}$ that is measurable with respect to the algebra of cylinder sets and therefore depends on the value of ω at a finite number of points $y \in R^d$. But for such functions the strong continuity of T_x follows from the stochastic continuity (2.2) (the $a_{ij}(x, \omega)$ is identified with $\omega_{ij}(x, \omega)$ below).

With any $\tilde{f}$ in $\mathscr{H}$ we may associate the stationary process

$$(2.10) \qquad f(x, \omega) = (T_x \tilde{f})(\omega) = f(\tau_{-x} \omega).$$

We shall use the tilde notation exclusively for associating a function with its translates that form the stationary process. In particular we define

$$(2.11) \qquad \tilde{a}_{ij}(\omega) = \omega_{ij}(0, \omega) \qquad (i, j = 1, 2, \ldots, d)$$

for P almost all ω so that

$$(2.12) \qquad a_{ij}(x,\omega) = (T_x \tilde{a}_{ij})(\omega) = \tilde{a}_{ij}(\tau_{-x}\omega) = \omega_{ij}(x,\omega).$$

Thus the stationary random coefficients in (2.6) are the sample points themselves. Note that the functions $\tilde{a}_{ij}(\omega)$ satisfy (2.1) which can also be expressed by saying that the support of P is all $d \times d$ matrix functions that satisfy (2.1).

When one coordinate variable of $x = (x_1, \ldots, x_d)$ varies at a time in the group T_x, we obtain d one-parameter strongly continuous unitary groups in $\mathscr{H}$ that commute with each other. Let $D_1, D_2, \ldots, D_d$ denote the infinitesimal generators of these groups. They are closed and densely defined linear operators with domains $\mathscr{D}(D_i)$ in $\mathscr{H}$. For $f \in \mathscr{D}(D_i)$ we have

$$(2.13) \qquad (D_i\tilde{f})(\omega) = \frac{\partial}{\partial x_i}(T_x\tilde{f})(\omega)\Big|_{x=0}$$

where the differentiation is in the $\mathscr{H} = L^2(\Omega, \mathscr{F}, P)$ sense. Since T_x is a unitary group, the generators are skew adjoint

$$(2.14) \qquad E\{\tilde{g}D_i\tilde{f}\} = -E\{\tilde{f}D_i\tilde{g}\}$$

for all $\tilde{f}, \tilde{g}$ in $\mathscr{D}(D_i)$.

The hypothesis that the action of the translation group τ_x is ergodic on Ω takes the following form in $\mathscr{H}$: the only functions in $\mathscr{H}$ that are invariant under T_x are the constant functions.

The linear subspace

$$(2.15) \qquad \mathscr{H}^1 = \bigcap_{i=1}^{d} \mathscr{D}(D_i)$$

is a dense subset of $\mathscr{H}$. Equipped with the inner product

$$E\{\tilde{f}\tilde{g}\} + \sum_{i=1}^{d} E\{D_i\tilde{f}D_i\tilde{g}\},$$

$\mathscr{H}^1$ is a Hilbert space since the D_i are closed. Let $H_{\mathrm{loc}}(R^d; \mathscr{H})$ be the space of functions from R^d to $\mathscr{H}$ with inner product

$$\int_{\mathscr{O}} dx E\{f(x)g(x)\} = \mathrm{vol}\,(\mathscr{O})E\{\tilde{f}\tilde{g}\}$$

and finite norm for every bounded open set $\mathcal{O}$. Here $f(x, \omega) = \tilde{f}(\tau_{-x}\omega)$ and $g(x, \omega) = \tilde{g}(\tau_{-x}\omega)$ and we have used the translation invariance of P. Let $H_S(R^d; \mathcal{H})$ be the space of all stationary random processes on R^d Clearly H_S is in one-to-one correspondence with $\mathcal{H}$ since it is simply the space of all translates of $\mathcal{H}$. H_S is also a closed subset of $H_{\text{loc}}(R^d; \mathcal{H})$ that is invariant under T_x. The group T_x acts on $H_{\text{loc}}(R^d; \mathcal{H})$ in the manner $f(x, \omega) \to f(x, \tau_{-x}\omega)$.

We may similarly identify $\mathcal{H}^1$ with the set of mean square differentiable, stationary processes $H_S^1(R^d; \mathcal{H})$. Clearly $H_S^1(R^d; \mathcal{H})$ is a closed subset of $H_{\text{loc}}^1(R^d; \mathcal{H})$ i.e. the set of all $\mathcal{H}$ valued functions on R^d that have square integrable distribution derivatives over every bounded open set $\mathcal{O}$. We note that if $f \in H_S^1$, then its x derivatives form stationary processes and

$$\frac{\partial f(x, \omega)}{\partial x_i} = D_i f(x, \omega)$$

with equality holding $\mu \times P$ almost everywhere (μ = Lebesgue measure on R^d).

The unitary group of operators T_x is ergodic on $\mathcal{H}$ since the action τ_x is ergodic on Ω. Let B_N be the cube of side $2N$ in R^d centered at the origin. By the mean ergodic theorem, for any $\tilde{f} \in \mathcal{H}$,

$$\text{(2.16)} \qquad \frac{1}{(2N)^d} \int_{B_N} (T_x \tilde{f})(\omega) dx \to E\{\tilde{f}\}$$

in $\mathcal{H}$ as $N \to \infty$. We also have the individual ergodic theorem [16] which says that (2.16) is valid for P almost all ω when $\tilde{f} \in L^1(\Omega, \mathcal{F}, P)$. We note finally thet the spectral resolution of T_x is

$$\text{(2.17)} \qquad T_x = \int_{R^d} e^{i\lambda x} U(d\lambda)$$

where $U(d\lambda)$ is the associated projection valued measure.

Periodic and almost periodic coefficients (a_{ij}) can be considered within the above framework in the following manner.

Consider the periodic case first. Let $(\tilde{a}_{ij}(x))$ be the given Lebesgue measurable periodic functions of period one in each coordinate and satisfying (2.1). Let T^d be the unit d-dimensional torus. We take $\Omega = \{\tilde{a}_{ij}(\cdot + \omega)$ with $\omega \in T^d\}$ so that in fact Ω is identical with T^d in this case and is a much smaller space than it can be in general. Let $\mathscr{F}$ be the σ-algebra of Lebesgue measurable sets and P Lebesgue measure on T^d which is invariant under translation $\tau_x \omega = \omega - x \pmod 1$. The action of τ_x on Ω is ergodic and the infinitesimal generators of the unitary group T_x are now the usual partial derivatives

$$D_i = \frac{\partial}{\partial \omega_i} \qquad (\omega = (\omega_1, \ldots, \omega_d)).$$

The coefficients a_{ij} are given by $a_{ij}(x, \omega) = \tilde{a}_{ij}(x + \omega)$.

The periodic case is thus put into the present abstract framework by the essentially trivial process of letting the center of the period cell be a random variable that is uniformly distributed over the unit torus.

In the almost periodic case we shall only consider coefficients (a_{ij}) that are continuous as follows.

Consider R^d as a locally compact abelian group. There exists [17] a compact abelian group G, the Bohr compactification of R^d, containing R^d as a dense subgroup and such that the continuous almost periodic functions on R^d are precisely the continuous functions on G, $C(G)$, restricted to R^d. Let $(\tilde{a}_{ij}(x))$ be the given continuous almost periodic coefficients satisfying (2.1). These functions are restrictions to R^d of functions $(\tilde{a}_{ij}(g))$ in $C(G)$ which are uniquely defined by continuity and density of R^d in G. Let τ_g be the group action on G which on R^d reduces to the usual translation. We take for Ω the subset of all continuous functions on G with values $d \times d$ matrices that consists of the given functions $(\tilde{a}_{ij}(x))$ and all their translates by τ_g, that is

$$\Omega = \{\tilde{a}_{ij}(\tau_{-g} x),\ g \in G\}.$$

Thus the set Ω may be identified with G. For $\mathscr{F}$ we take the σ-algebra generated by "rational" cylinder sets in Ω as above. This $\mathscr{F}$ is countably generated and it is a sub σ-algebra of the one relative to which all continuous $d \times d$ matrix functions on G are measurable (this one is not countably generated). We take for P the Haar measure which is invariant under τ_g. Moreover, the action τ_g is ergodic on Ω.

3. WEAK CONVERGENCE THEOREM

Let the coefficients $a_{ij}(x, \omega) = \tilde{a}_{ij}(\tau_{-x}\omega)$ be given satisfying (2.1) with $x \in R^d$, $\omega \in \Omega$, $(\Omega, \mathscr{F}, P)$ the probability space introduced in the previous section with $\tau_x: \Omega \to \Omega$ the measure preserving, ergodic translation group. Let $u^\epsilon \in V$ be the solution of (2.6) for fixed $f \in L^2(\mathcal{O}; R^1)$, $\alpha > 0$ and each $\epsilon > 0$.

Let $\{e_n(\omega)\}$ be an orthonormal basis in $\mathscr{H} = L^2(\Omega, \mathscr{F}, P)$ with $e_1(\omega) \equiv 1$ and let $V_1 = E\{Ve_1\}$ by which we mean the set of all first coordinates of elements in V. V_1 is a closed subspace of $H^1(\mathcal{O}; R^1)$ and contains $H_0^1(\mathcal{O}; R^1)$. V_1 is also a closed subspace of V.

Theorem 1. *The solution $u^\epsilon \in V$ of (2.6) converges weakly in V to the solution $u(x) \in V_1$ of the deterministic variational problem*

$$(3.1) \qquad \int_{\mathcal{O}} dx \sum_{i,j=1}^{d} q_{ij} \frac{\partial u(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} + \alpha \int_{\mathcal{O}} dx u(x)\varphi(x) = \int_{\mathcal{O}} dx f(x)\varphi(x),$$

for all $\varphi(x) \in V_1$. Here (q_{ij}) is a constant matrix that satisfies (2.1) and is defined in Theorem 2 below. It is symmetric when (a_{ij}) is symmetric.

Remark 1. By weak convergence we mean that for every $\varphi(x, \omega) \in$ $\in V$,

$$(3.2) \qquad (u^\epsilon, \varphi)_1 \to (u, \varphi)_1 \quad \text{as} \quad \epsilon \to 0$$

where the inner product is defined by (2.5).

Remark 2. Another convergence result is given in Section 5 and still another in Section 8.

Theorem 2. *There exist uniquely defined functions $\tilde{\psi}_i^k(\omega)$ in $\mathscr{H}$, $(i, k = 1, 2, \dots, d)$, such that*

$$(3.3) \qquad \sum_{i,j=1}^{d} E\{\tilde{a}_{ij}(\delta_{jk} + \tilde{\psi}_j^k) D_i \tilde{\varphi}\} = 0, \quad \forall \tilde{\varphi} \in \mathscr{H}^1 \quad (k = 1, 2, \dots, d),$$

$$(3.4) \qquad E\{\tilde{\psi}_i^k\} = 0,$$

$$(3.5) \qquad E\{\tilde{\psi}_i^k D_i \tilde{\varphi}\} = E\{\tilde{\psi}_i^k D_j \tilde{\varphi}\}, \quad \forall \tilde{\varphi} \in \mathscr{H}^1 \qquad (i, j, k = 1, 2, \ldots, d).$$

The coefficients (q_{ij}) *in* (3.1) *are defined by*

$$(3.6) \qquad q_{ij} = E\Big\{\sum_{k=1}^{d} \tilde{a}_{ik}(\delta_{kj} + \tilde{\psi}_k^j)\Big\} \qquad (i, j = 1, 2, \ldots, d).$$

There exist furthermore uniquely defined processes $\chi^k(x, \omega)$ $(k = 1, 2, \ldots, d)$ *which are in* $H^1_{\text{loc}}(R^d; \mathscr{H})$, *they are not stationary,* $\chi^k(0, \omega) = 0$ *and*

$$(3.7) \qquad \frac{\partial \chi^k(x, \omega)}{\partial x_i} = \psi_i^k(x, \omega) = \tilde{\psi}_i^k(\tau_{-x}\omega) \qquad (i, k = 1, 2, \ldots, d)$$

so that their gradients are stationary. For any compact subset K *of* R^d *they satisfy the estimate*

$$(3.8) \qquad \lim_{e \downarrow 0} \sup_{x \in K} E\Big\{\Big(\epsilon\chi^k\Big(\frac{x}{\epsilon}\Big)\Big)^2\Big\} = 0 \qquad (k = 1, 2, \ldots, d).$$

Remark 1. In the periodic case there exist functions $\tilde{\chi}^k(\omega)$ in $\mathscr{H}^1$ which satisfy

$$(3.9) \qquad \int_{T^d} d\omega \sum_{i,j=1}^{d} \tilde{a}_{ij}(\omega)\Big(\delta_{jk} + \frac{\partial \tilde{\chi}^k(\omega)}{\partial \omega_j}\Big) \frac{\partial \tilde{\varphi}(\omega)}{\partial \omega_i} = 0,$$

for all $\tilde{\varphi} \in \mathscr{H}^1$ $(k = 1, 2, \ldots, d)$. This is the usual cell problem in homogenization (cf. for example [10], Chapter 1, Sections 2 and 3). In the general case one cannot obtain stationary $\chi^k(x, \omega)$ $(= \tilde{\chi}^k(\tau_{-x}\omega))$ but it turns out that Theorem 2 is sufficient.

Remark 2. Theorem 2 also be stated in the following way. There are functions $\chi^k(x, \omega)$ in $H^1_{\text{loc}}(R^d; \mathscr{H})$ whose gradients are in $H_S(R^d; \mathscr{H})$ (i.e. stationary) such that if B_N is as in (2.16),

$$(3.10) \qquad \lim_{N \uparrow \infty} \frac{1}{(2N)^d} \int_{B_N} dx \sum_{i,j=1}^{d} a_{ij}(a, \omega) \times \\ \times \Big(\delta_{jk} + \frac{\partial \chi^k(x, \omega)}{\partial x_j}\Big) \frac{\partial \varphi(x, \omega)}{\partial x_i} = 0$$

for all $\varphi \in H_S^1(R^d; \mathscr{H})$ $(k = 1, 2, \ldots, d)$ and with the limit in (3.10) being in $\mathscr{H}$ or P almost everywhere. The gradients of χ^k have mean zero and

$$(3.11) \qquad q_{ij} = \lim_{N \uparrow \infty} \frac{1}{(2N)^d} \int_{B_N} dx \sum_{k=1}^{d} a_{ik}(x, \omega) \left(\delta_{kj} + \frac{\partial \chi^k(x, \omega)}{\partial x_j} \right),$$

with the limit again in $\mathscr{H}$ or P almost everywhere.

The translation of Theorem 2 to the above statement is immediate in view of the ergodic theorem (2.16). The statement (3.10), (3.11) is perhaps a bit more physical and in line with the discussion in Section 1. However, we have found the abstract formulation very convenient analytically.

Remark 3. In the next section we show that formula (3.6) for q_{ij} can also be written in the form

$$(3.12) \qquad q_{ij} = E\left\{ \sum_{k,l=1}^{d} \widetilde{a}_{lk} (\delta_{kj} + \widetilde{\psi}_k^j)(\delta_{li} + \widetilde{\psi}_l^i) \right\}.$$

From this formula we see that (q_{ij}) is symmetric when a_{ik} is symmetric. Using (2.1) and (3.4) we also see that (2.1) holds for the (q_{ij}). For example

$$\sum_{i,j=1} q_{ij} \xi_i \xi_j \geqslant a_0 E\left\{ \sum_{i,i,j=1}^{d} (\delta_{kj} + \widetilde{\psi}_k^j) \xi_j (\delta_{ki} + \widetilde{\psi}_k^i) \xi_i \right\} =$$

$$= a_0 |\xi|^2 + a_0 E\left\{ \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \widetilde{\psi}_k^j \xi_j \right)^2 \right\} \geqslant a_0 |\xi|^2 .$$

4. PROOF OF WEAK CONVERGENCE THEOREM

The proof of Theorem 1 is a modification of Tartar's proof for the periodic case (cf. [10], Chapter 1, Section 3). Of course we use in it Theorem 2 so we shall prove it first.

Proof of Theorem 2. For each $\beta > 0$ consider the problem: Find $\widetilde{\chi}^{k,\beta}$ in $\mathscr{H}^1$ such that

$$\int_\Omega P(d\omega) \sum_{i,j=1}^{d} \widetilde{a}_{ij}(\omega)(\delta_{jk} + D_j\widetilde{\chi}^{k,\beta}(\omega))D_i\widetilde{\varphi}(\omega) + \tag{4.1}$$
$$+ \beta \int_\Omega P(d\omega)\widetilde{\chi}^{k,\beta}(\omega)\widetilde{\varphi}(\omega) = 0$$

for all $\widetilde{\varphi} \in \mathscr{H}^1$ $(k = 1, 2, \dots, d)$. This problem has a unique solution by the Lax – Milgram lemma [15]. Letting $\widetilde{\varphi} = \widetilde{\chi}^{k,\beta}$ in (4.1) and unisg (2.1) yields

$$\int_\Omega P(d\omega) \sum_{j=1}^{d} (D_j\widetilde{\chi}^{k,\beta}(\omega))^2 \leqslant C_1 \tag{4.2}$$

$$\beta \int_\Omega P(d\omega)(\widetilde{\chi}^{k,\beta}(\omega))^2 \leqslant C_2 \tag{4.3}$$

for $k = 1, 2, \dots, d$ with C_1 and C_2 constants independent of β.

Because of (4.2) there is a subsequence $\beta' \to 0$ such that $D_j\widetilde{\chi}^{k,\beta'} \to$ $\to \widetilde{\psi}_j^k$ (some limit) in $\mathscr{H}$ weakly. Using (4.3) we can pass to the limit in (4.1) along this subsequence to obtain

$$\int_\Omega P(d\omega) \sum_{i,j=1}^{d} \widetilde{a}_{ij}(\omega)(\delta_{jk} + \widetilde{\psi}_j^k(\omega))D_i\widetilde{\varphi}(\omega) = 0, \quad \forall \widetilde{\varphi} \in \mathscr{H}^1 \quad (k = 1, 2, \dots, d). \tag{4.4}$$

By passage to the limit we obtain in addition the following:

$$\int_\Omega P(d\omega)\widetilde{\psi}_j^k(\omega)D_i\widetilde{\varphi}(\omega) = \int_\Omega P(d\omega)\widetilde{\psi}_i^k(\omega)D_j\widetilde{\varphi}(\omega), \quad \forall \widetilde{\varphi} \in \mathscr{H}^1 \quad (i, j, k = 1, 2, \dots, d) \tag{4.5}$$

and

$$\int_\Omega P(d\omega)\widetilde{\psi}_j^k(\omega) = 0. \tag{4.6}$$

We shall show next that (4.4)-(4.6) has a unique solution which proves the first part of Theorem 2. For this purpose we use the spectral resolution of the unitary group T_x,

$$(4.7) \qquad T_x = \int_{R^d} e^{ix\lambda} U(d\lambda) \qquad (x \in R^d),$$

with $U(d\lambda)$ the spectral projections.

For each $\gamma > 0$ we define

$$(4.8) \qquad \widetilde{g}^{k,\beta}(\omega) = \int_{R^d} \sum_{j=1}^{d} \frac{(-i\lambda_j - \gamma)}{|i\lambda - \gamma|^2} U(d\lambda)\widetilde{\psi}_j^k(\omega) \qquad (k = 1, 2, \ldots, d)$$

where $|i\lambda - \gamma|^2 = \sum_{j=1}^{d} (i\lambda_j - \gamma)(-i\lambda_j - \gamma)$. On using (4.5) we conclude that $\widetilde{g}^{k,\gamma} \in \mathscr{H}^1$ and

$$(4.9) \qquad (D_j - \gamma)\widetilde{g}^{k,\gamma} = \widetilde{\psi}_i^k \qquad (k, j = 1, 2, \ldots, d).$$

Moreover,

$$|\gamma\widetilde{g}^{k,\gamma}|^2 = \int_{\Omega} P(d\omega)(\gamma\widetilde{g}^{k,\gamma}(\omega))^2 =$$

$$(4.10) \qquad = \int_{R^d} \gamma^2 \sum_{j,l=1}^{d} \frac{(-i\lambda_j - \gamma)(i\lambda_l - \gamma)}{|i\lambda - \gamma|^4} (U(d\lambda)\widetilde{\psi}_j^k, \widetilde{\psi}_l^k) \to$$

$$\to (U(\{0\})\widetilde{\psi}_j^k, \widetilde{\psi}_l^k) \quad \text{as} \quad \gamma \to 0,$$

by the Lebesgue dominated convergence theorem. But $U(\{0\})$ is the projection operator into the functions invariant by T_x. The only such functions are the constants, by ergodicity, and because of (4.6) we conclude that

$$(4.11) \qquad \lim_{\gamma \downarrow 0} |\gamma\widetilde{g}^{k,\gamma}|^2 = 0 \qquad (k = 1, 2, \ldots, d).$$

Now suppose $\widetilde{\psi}_j^k$ is a solution of the homogeneous version of (4.4)-(4.6). From (4.4) we have

$$\int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} \widetilde{a}_{ij}(\omega)\widetilde{\psi}_j^k(\omega)D_i\widetilde{\varphi}(\omega) \qquad (k = 1, 2, \ldots, d).$$

Substitute for $\widetilde{\varphi}$ the function $\widetilde{g}^{k,\gamma}$. Using (4.9), passing to the limit $\gamma \to 0$ and using (4.11) we find that

$$\int_{\Omega} dP(\omega) \sum_{i,j=1}^{d} \tilde{a}_{ij}(\omega) \tilde{\psi}_j^k(\omega) \tilde{\psi}_i^k(\omega) = 0 \qquad (k = 1, 2, \ldots, d).$$

From the uniform ellipticity we conclude that $\tilde{\psi}_j^k = 0$ and hence uniqueness has been shown.

We continue with the construction of $\chi^k(x, \omega)$ satisfying (3.7) and (3.8).

We define $\chi^k(x, \omega)$ by

$$\chi^k(x, \omega) = \int_{R^d} (e^{i\lambda x} - 1) \frac{1}{|\lambda|^2} \sum_{j=1}^{d} (-i\lambda_j) U(d\lambda) \tilde{\psi}_j^k(\omega) \tag{4.12}$$

$$(k = 1, 2, \ldots, d),$$

which is nonstationary because it is not of the form $\chi^k = T_x \tilde{\chi}^k$. However, $\chi^k(x, \omega)$ is in $H^1_{\text{loc}}(R^d; \mathscr{H})$ as can be verified directly, $\chi^k(0, \omega) = 0$ and (3.7) holds.

It remains to show (3.8). We have

$$\int_{\Omega} P(d\omega) \left(\epsilon \chi^k \left(\frac{x}{\epsilon}, \omega \right) \right)^2 = \int_{R^d} \left| \frac{e^{ix\frac{\lambda}{\epsilon}} - 1}{\left| \frac{\lambda}{\epsilon} \right|} \right|^2 \sum_{j,l=1}^{d} \frac{\lambda_j \lambda_l}{|\lambda|^2} \hat{R}_{jl}^k(d\lambda) \tag{4.13}$$

where

$$\hat{R}_{jl}^k(d\lambda) = \int_{\Omega} P(d\omega) U(d\lambda) \tilde{\psi}_j^k(\omega) \tilde{\psi}_l^k(\omega), \tag{4.14}$$

is the power spectral matrix measure of the stationary process $\psi_j^k(x, \omega) = \tilde{\psi}_j^k(\tau_{-x}\omega)$. From the estimate

$$\frac{1}{|\lambda|^2} \sum_{j,l=1}^{d} \lambda_j \lambda_l \hat{R}_{jl}^k(d\lambda) \leq \sum_{j=1}^{d} \hat{R}_{jj}^k(d\lambda),$$

which follows from Schwartz's inequality, we obtain

$$\int_{\Omega} P(d\omega) \left(\epsilon \chi^k \left(\frac{x}{\epsilon}, \omega \right) \right)^2 \leq \int_{R^d} \left| \frac{e^{ix\frac{\lambda}{\epsilon}} - 1}{\left| \frac{\lambda}{\epsilon} \right|} \right|^2 \sum_{j=1}^{d} \hat{R}_{jj}(d\lambda). \tag{4.15}$$

By the ergodicity hypothesis and (4.6) it follows that $\hat{R}_{jj}(\{0\}) = 0$. Application of the Lebesgue convergence theorem to (4.15) yields the result (3.8). The proof of Theorem 2 is complete.

Before continuing with the proof of Theorem 1 we observe that formula (3.12) for (q_{ij}) is obtained by replacing $D_i\widetilde{\varphi}$ in (3.3) by $\widetilde{\psi}_i^l$ and adding the result to (3.6). That this substitution is permissible follows from the uniqueness argument for $\widetilde{\psi}_j^k$ given above.

Proof of Theorem 1. We follow Tartar's elegant argument ([10], Chapter 1, Section 3). Define

$$\xi_i^\epsilon(x, \omega) = \sum_{j=1}^{d} a_{ij}\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial u^\epsilon(x, \omega)}{\partial x_j} \qquad (i = 1, 2, \ldots, d), \tag{4.16}$$

which is in $L^2(\mathcal{O}; \mathcal{H})$. In view of (2.1) and (2.7) there is a subsequence, denoted again by $\xi_i^\epsilon(x, \omega), u^\epsilon(x, \omega)$ which converges weakly in $(L^2(\mathcal{O}; \mathcal{H}))^{d+1}$ to some limit $\xi_i(x, \omega), u(x, \omega)$. Passing to the limit in (2.6) we obtain

$$\begin{aligned} &\int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{i=1}^{d} \xi_i(x, \omega) \frac{\partial \varphi(x, \omega)}{\partial x_i} + \\ &+ \int_{\mathcal{O}} dx \int P(d\omega) u(x, \omega)\varphi(x, \omega) = \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) f(x)\varphi(x, \omega) \end{aligned} \tag{4.17}$$

for all $\varphi(x, \omega) \in V$.

Let $\chi^k(x, \omega)$ be the function defined in Theorem 2 with $(\widetilde{a}_{ij})$ replaced by $(\widetilde{a}_{ji})$. Define $w_k(x, \omega)$ by

$$w_k(x, \omega) = x_k + \chi^k(x, \omega).$$

We may rewrite (3.3) in the form

$$\int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} a_{ij}(x, \omega) \frac{\partial w_k(x, \omega)}{\partial x_i} \frac{\partial \varphi(x, \omega)}{\partial x_j} = 0 \tag{4.18}$$

for all $\varphi(x, \omega) = \widetilde{\varphi}(\tau_{-x}\omega)$ with $\widetilde{\varphi} \in \mathcal{H}^1$ and $k = 1, 2, \ldots, d$. Since $D_j\varphi = \dfrac{\partial \varphi}{\partial x_j}$ we may rewrite (4.18) in the form

$$(4.19) \qquad \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x,\omega)\frac{\partial w_k(x,\omega)}{\partial x_j}\right) = 0$$

which makes sense as an equality of distribution valued stationary processes.

Define

$$(4.20) \qquad w_k^\epsilon(x,\omega) = \epsilon w_k^\epsilon\left(\frac{x}{\epsilon},\omega\right) = x_k + \epsilon\chi^k\left(\frac{x}{\epsilon},\omega\right).$$

Clearly (4.19) may also be written in scaled form

$$(4.21) \qquad \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon},\omega\right)\frac{\partial w_k^\epsilon(x,\omega)}{\partial x_i}\right) = 0.$$

Let $\theta(x,\omega)$ be any function in $\mathscr{D}(\mathscr{O};L^\infty(\Omega,\mathscr{F},P))$. Since $u^\epsilon(x,\omega)\in$ $\in V\subset H^1(\mathscr{O};\mathscr{H})$ it follows that $\theta(x,\omega)u^\epsilon(x,\omega)\in H_0^1(\mathscr{O};\mathscr{H})$. Multiplying (4.21) by θu^ϵ, integrating and integrating by parts gives the identity

$$(4.22) \qquad \begin{aligned} &\int_{\mathscr{O}} dx \int P(d\omega) \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon},\omega\right)\frac{\partial}{\partial x_j}(\theta(x,\omega)u^\epsilon(x,\omega)) \times \\ &\times \frac{\partial w_k^\epsilon(x,\omega)}{\partial x_i} = 0. \end{aligned}$$

The functions $w_k^\epsilon(x,\omega)$ are in $H^1_{\mathrm{loc}}(R^d;\mathscr{H})$ and hence $\theta(x,\omega)w_k^\epsilon(x,\omega)$ is in $H_0^1(\mathscr{O};\mathscr{H})\subseteq V$. Thus, replacing φ by θw_k^ϵ in (2.6) gives

$$(4.23) \qquad \begin{aligned} &\int_{\mathscr{O}} dx \int_\Omega P(d\omega)\sum_{i=1}^{d} \xi_i^\epsilon(x,\omega)\frac{\partial}{\partial x_i}(\theta(x,\omega)w_k^\epsilon(x,\omega)) + \\ &+ \int_{\mathscr{O}} dx \int_\Omega P(d\omega)u^\epsilon(x,\omega)\theta(x,\omega)w_k^\epsilon(x,\omega) = \\ &= \int_{\mathscr{O}} dx \int_\Omega P(d\omega)f(x)\theta(x,\omega)w_k^\epsilon(x,\omega). \end{aligned}$$

Subtracting (4.22) from (4.23) and cancelling some terms yields the identity

$$\int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{i=1}^{d} \xi_i^{\epsilon}(x, \omega) w_k^{\epsilon}(x, \omega) \frac{\partial \theta(x, \omega)}{\partial x_i} -$$

$$(4.24) \quad - \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial w_k^{\epsilon}(x, \omega)}{\partial x_i} u^{\epsilon}(x, \omega) \times$$

$$\times \frac{\partial \theta(x, \omega)}{\partial x_j} + \alpha \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) u^{\epsilon}(x, \omega) \theta(x, \omega) w_k^{\epsilon}(x, \omega) =$$

$$= \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) f(x) \theta(x, \omega) w_k^{\epsilon}(x, \omega).$$

We may now pass to the limit $\epsilon \to 0$ in (4.24) along the subsequence. From (4.20) and (3.8) we conclude that

$$(4.25) \quad \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) (w_k^{\epsilon}(x, \omega) - x_k)^2 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Using (4.25) to pass to the limit in (4.24) and then using (4.17) gives us the relation

$$\int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \xi_k(x, \omega) \theta(x, \omega) =$$

$$(4.26) \quad = - \lim_{\epsilon \downarrow 0} \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial w_k^{\epsilon}(x, \omega)}{\partial x_i} \times$$

$$\times u^{\epsilon}(x, \omega) \frac{\partial \theta(x, \omega)}{\partial x_j}$$

for $k = 1, 2, \ldots, d$. We shall show that the right side in (4.26) equals

$$(4.27) \quad \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{j=1}^{d} q_{kj} u(x, \omega) \frac{\partial \theta(x, \omega)}{\partial x_j}$$

with (q_{ij}) defined by (3.6) (after noting that $(\widetilde{a}_{ij})$ has been replaced by $(\widetilde{a}_{ji})$). Assuming this fact we shall complete the proof.

Since $\theta(x, \omega) \in \mathcal{D}(\mathcal{O}; L^{\infty}(\Omega, \mathcal{F}, P))$ is arbitrary and $\mathcal{D}(\mathcal{O}; L^{\infty}(\Omega, \mathcal{F}, P))$ is dense in $L^2(\mathcal{O}; \mathcal{H})$ we conclude that

$$(4.28) \quad \xi_k(x, \omega) = \sum_{j=1}^{d} q_{kj} \frac{\partial u(x, \omega)}{\partial x_j} \qquad (k = 1, 2, \ldots, d).$$

Inserting this expression into (4.17) gives

$$(4.29)\quad \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} q_{ij} \frac{\partial u(x,\omega)}{\partial x_j} \frac{\partial \varphi(x,\omega)}{\partial x_i} + \alpha \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) u(x,\omega)\varphi(x,\omega) = \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) f(x)\varphi(x,\omega)$$

for all $\varphi(x,\omega) \in V$. This problem has a unique solution in V since (q_{ij}) is positive definite and α is positive. But the solution $u(x) \in V_1$ of (3.1) satisfies (4.29) since the (q_{ij}) are constants. Thus $u(x) = u(x,\omega)$, i.e. the limit is deterministic and we have the result stated in Theorem 1.

It remains to show that the right side of (4.26) equals (4.27). Define g^{kj} by

$$(4.30)\quad g^{kj}(x,\omega) = \tilde{g}^{kj}(\tau_{-x}\omega) = \sum_{i=1}^{d} a_{ij}(x,\omega) \frac{\partial w_k(x,\omega)}{\partial x_i} - q_{kj}.$$

In terms of g^{kj} what we must show is that

$$(4.31)\quad \lim_{\epsilon \downarrow 0} \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{j=1}^{d} g^{kj}\left(\frac{x}{\epsilon}, \omega\right) u^{\epsilon}(x,\omega) \frac{\partial \theta(x,\omega)}{\partial x_j} = 0 \qquad (k = 1, 2, \ldots, d).$$

Define $h_l^{kj}(x,\omega)$ by

$$(4.32)\quad h_l^{kj}(x,\omega) = \int_{R^d} (e^{ix\lambda} - 1) \frac{(-i\lambda_l)}{|\lambda|^2} U(d\lambda)\tilde{g}^{kj}(\omega).$$

Since $E\{\tilde{g}^{kj}\} = 0$, a calculation similar to the one used for χ^k below (4.12) gives

$$(4.33)\quad \lim_{\epsilon \downarrow 0} \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \left(\epsilon h_l^{kj}\left(\frac{x}{\epsilon}, \omega\right)\right)^2 = 0 \qquad (k, j, l = 1, 2, \ldots, d).$$

We also have that

$$(4.34)\quad \sum_{l=1}^{d} \frac{\partial h_l^{kj}(x,\omega)}{\partial x_l} = \int_{R^d} e^{ix\lambda} U(d\lambda)\tilde{g}^{kj} = g^{kj}(x,\omega).$$

Now we use (4.34) suitably scaled in (4.31) and integrate by parts.

$$\int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{j,l=1}^{d} \Big(\epsilon \frac{\partial}{\partial x_l} h_l^{kj}\big(\frac{x}{\epsilon}, \omega\big)\Big) u^{\epsilon}(x, \omega) \frac{\partial \theta(x, \omega)}{\partial x_j} =$$

$$(4.35) \qquad = - \int_{\mathcal{O}} dx \int_{\Omega} P(d\omega) \sum_{j,l=1}^{d} \epsilon h_l^{kj}\big(\frac{x}{\epsilon}, \omega\big) \times$$

$$\times \Big[\frac{\partial u^{\epsilon}(x, \omega)}{\partial x_l} \frac{\partial \theta(x, \omega)}{\partial x_j} + u^{\epsilon}(x, \omega) \frac{\partial^2 \theta(x, \omega)}{\partial x_j \partial x_l}\Big].$$

The result (4.31) follows from this by using (4.34) and the estimate (2.7) which is uniform in ϵ. The proof of Theorem 1 is complete.

5. STRONG CONVERGENCE THEOREM

We shall state and prove the strong convergence theorem for the whole of R^d since the result is local in character. The cutoff arguments of [10], Chapter 1, Section 5 apply here also so that the same problem over a subset $\mathcal{O}$ with Dirichlet conditions can be analyzed without difficulty.

We assume that all hypotheses about (a_{ij}) stated in Section 2 and at the beginning of Section 3 hold here also. With $f \in L^2(R^d; R^1)$ and $\alpha > 0$ fixed we consider the problem: find $u^{\epsilon}(x, \omega)$ in $H^1(R^d; \mathcal{H})$ such that for each $\epsilon > 0$

$$\int_{R^d} dx \int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} a_{ij}\big(\frac{x}{\epsilon}, \omega\big) \frac{\partial u^{\epsilon}(x, \omega)}{\partial x_j} \frac{\partial \varphi(x, \omega)}{\partial x_i} +$$

$$(5.1) \qquad + \alpha \int_{R^d} dx \int_{\Omega} P(d\omega) u^{\epsilon}(x, \omega) \varphi(x, \omega) =$$

$$= \int_{R^d} dx \int_{\Omega} P(d\omega) f(x) \varphi(x, \omega)$$

for all $\varphi(x, \omega)$ in $H^1(R^d; \mathcal{H})$. This problem has a unique solution satisfying (2.7) with $\mathcal{O}$ replaced by R^d.

Theorem 3. *Let* $u^{\epsilon}(x, \omega) \in H^1(R^d; \mathcal{H})$ *be the solution of* (5.1) *and let* $u(x) \in H^1(R^d; R^1)$ *be the solution of*

$$(5.2) \qquad \int_{R^d} dx \sum_{i,j=1}^{d} q_{ij} \frac{\partial u(x)}{\partial x_j} \frac{\partial \varphi(x)}{\partial x_i} + \alpha \int_{R^d} dx u(x)\varphi(x) = \int_{R^d} dx f(x)\varphi(x)$$

for all $\varphi \in H^1(R^d; R^1)$ *and with* (q_{ij}) *given by* (3.6). *Then*

$$(5.3) \qquad \lim_{\epsilon \downarrow 0} \int_{R^d} dx \int_{\Omega} P(d\omega) | u^\epsilon(x, \omega) - u(x)|^2 = 0$$

and

$$(5.4) \qquad \lim_{\epsilon \downarrow 0} \int_{R^d} dx \int_{\Omega} P(d\omega) \sum_{i=1}^{d} \left| \frac{\partial u^\epsilon(x, \omega)}{\partial x_i} - \frac{\partial u(x)}{\partial x_i} - \sum_{k=1}^{d} \psi_i^k \left(\frac{x}{\epsilon}, \omega\right) \frac{\partial u(x)}{\partial x_k} \right|^2 = 0$$

where $\psi_i^k(x, \omega) = \widetilde{\psi}_i^k(\tau_{-x}\omega)$ *and* $\widetilde{\psi}_i^k(\omega)$ *is defined in Theorem* 2.

Remark 1. This theorem complements Theorem 1 since it gives stronger convergence. Note in particular that u^ϵ does not tend to u in $H^1(R^d; \mathscr{H})$ strongly as can be seen from (5.4). The principal advantage of Theorem 1 is that it works for arbitrary homogeneous boundary value problems in variational form while the present theorem is of a local character.

Remark 2. We shall give two proofs of Theorem 3 both of which are more direct and hence more intuitive than Tartar's proof of Theorem 1. The first proof is exactly analogous to the one given in the periodic case and uses Theorem 2. The second uses Theorem 2 without taking advantage of the nonstationary χ^k and (3.8). As a result more information is needed about problem (4.1) that approximates (3.3)-(3.5). This being of independent interest it is stated as Theorem 4 in the next section. In the second proof and Theorem 4 we require that the (a_{ij}) be symmetric since we use the spectral theorem.

6. PROOF OF THE STRONG CONVERGENCE THEOREM

In view of (2.7) it suffices to prove (5.3) and (5.4) for each $f(x) \in C_0^\infty(R^d)$. In that case (5.2) has a C^∞ solution $u(x)$ that goes to zero exponentially fast $(\alpha > 0)$ as $|x| \to \infty$.

Let

$$(6.1)\qquad z^{\epsilon}(x,\omega) = u^{\epsilon}(x,\omega) - u(x) - \sum_{k=1}^{d} \epsilon \chi^{k}\left(\frac{x}{\epsilon},\omega\right)\frac{\partial u(x)}{\partial x_k}$$

where $\chi^k(x, \omega)$ is the process defined in Theorem 2. We note first that because of the estimate

$$(6.2)\qquad \int_{\Omega} P(d\omega)(\chi^k(x,\omega))^2 \leqslant C_2(1 + |x|^2) \quad \text{for all} \quad x,$$

which is a cruder version of (3.8) and follows from (4.13), and because of (2.7) there is a constant C_1 independent of ϵ such that

$$(6.3)\qquad \int_{R^d} dx \int_{\Omega} P(d\omega) \sum_{i=1}^{d} \left|\frac{\partial z^{\epsilon}(x,\omega)}{\partial x_i}\right|^2 \leqslant C_1 < \infty.$$

We note further that

$$(6.4)\qquad \lim_{\epsilon \downarrow 0} \int_{R^d} dx \int_{\Omega} P(d\omega)\left(\sum_{k=1}^{d} \epsilon\chi^k\left(\frac{x}{\epsilon},\omega\right)\frac{\partial u(x)}{\partial x_k}\right)^2 = 0$$

which follows from (3.8), the rapid decay of $u(x)$ and its derivatives as $|x| \to \infty$ and from (6.2).

We write (5.1) in the form

$$(6.5)\qquad \mathscr{L}_{\epsilon} u^{\epsilon} = f$$

with

$$(6.6)\qquad \mathscr{L}_{\epsilon} = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\epsilon},\omega\right)\frac{\partial}{\partial x_j}\right) + \alpha.$$

By direct computation and by using Theorem 2 we find that

$$(6.7)\qquad \mathscr{L}_{\epsilon} z^{\epsilon} = \sum_{i,j,k=1}^{d}\left[a_{ik}\left(\frac{x}{\epsilon},\omega\right)\left(\delta_{kj} + \frac{\partial\chi^j\left(\frac{x}{\epsilon},\omega\right)}{\partial x_u}\right) - q_{ij}\right] \times$$

$$\times \frac{\partial^2 u(x)}{\partial x_i \partial x_j} - \epsilon \sum_{i,j,k=1}^{d} \frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\epsilon},\omega\right)\chi^k\left(\frac{x}{\epsilon},\omega\right)\right]\frac{\partial^2 u(x)}{\partial x_k \partial x_j} +$$

$$+ \epsilon \sum_{i,j,k=1}^{d} a_{ij}\left(\frac{x}{\epsilon}, \omega\right) \chi^k\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial^3 u(x)}{\partial x_k \partial x_j \partial x_i} -$$

$$- \alpha \sum_{k=1}^{d} \chi^k\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial u(x)}{\partial x_k}.$$

The precise meaning of this identity is that if both sides are multiplied by $\varphi(x, \omega) \in H^1(R^d; \mathscr{H})$, integrated with respect to x and Ω and $\mathscr{L}^\epsilon$ is written in variational form, then a correct equation results. In this equation we may replace φ by z^ϵ in view of (6.3). Using then the uniform ellipticity (2.1) and (6.4) (along with some obvious generalizations of it) we obtain the estimate

$$\begin{aligned} &\overline{\lim_{\epsilon \downarrow 0}}\, a_0 \int_{R^d} dx \int_\Omega P(d\omega) \sum_{i=1}^{d} \left| \frac{\partial z^\epsilon(x, \omega)}{\partial x_i} \right|^2 + \\ &+ \overline{\lim_{\epsilon \downarrow 0}}\, \alpha \int_{R^d} dx \int_\Omega P(d\omega) |z^\epsilon(x, \omega)|^2 \leqslant \\ &\leqslant \lim_{\epsilon \downarrow 0} \left| \int_{R^d} dx \int_\Omega P(d\omega) \sum_{i,j,k=1}^{d} \left[a_{ik}\left(\frac{x}{\epsilon}, \omega\right) \times \right.\right. \\ &\left.\left. \times \left(\delta_{kj} + \psi_k^j\left(\frac{x}{\epsilon}, \omega\right)\right) - q_{ij}\right] \frac{\partial^2 u(x)}{\partial x_j \partial x_i} z^\epsilon(x, \omega) \right|. \end{aligned} \tag{6.8}$$

We observe now that the right side of (6.8) is precisely of the same form as the limit in (4.27) i.e. just like (4.32). Since the uniform in ϵ estimate (6.3) holds the arguments of Section 4 proving (4.31) applies again and the proof of Theorem 3 is complete.

The second proof of Theorem 3 goes as follows. Instead of defining the error z^ϵ with the term

$$\sum_{k=1}^{d} \epsilon \chi^k\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial u(x)}{\partial x_k}$$

we define it with $\chi^k\left(\frac{x}{\epsilon}, \omega\right)$ replaced by $\chi^{k,\epsilon^2}\left(\frac{x}{\epsilon}, \omega\right)$ where $\chi^{k,\beta}(x, \omega) = \tilde{\chi}^{k,\beta}(\tau_{-x}\omega)$ is stationary and $\tilde{\chi}^{k,\beta}(\omega)$ solves (4.1). A computation nearly identical to the one that gave (6.7) gives an equation for the new error z^ϵ. Inspection of this equation reveals that Theorem 3 will follow if we can prove that

$$\lim_{\beta \downarrow 0} \beta \int_{\Omega} P(d\omega)(\tilde{\chi}^{k,\beta}(\omega))^2 = 0 \tag{6.9}$$

and

$$\lim_{\beta \downarrow 0} \int_{\Omega} P(d\omega) | D_j \tilde{\chi}^{k,\beta}(\omega) - \tilde{\psi}_j^k(\omega)|^2 = 0 \qquad (j, k = 1, 2, \ldots, d). \tag{6.10}$$

The two estimates (6.9) and (6.10) say that the approximating solutions $\tilde{\chi}^{k,\beta}$ of (3.3)-(3.5), which satisfy (4.1), not only converge weakly in $\mathscr{H}^1$ but also strongly. We state this as a separate theorem.

Theorem 4. *Estimates* (6.9) *and* (6.10) *hold for* $\tilde{\chi}^{k,\beta}$, *the solution of* (4.1), *provided the* $(\tilde{a}_{ij})$ *are symmetric.*

Proof of Theorem 4. Suppose first that (6.9) holds. Subtracting (3.3) from (4.1), replacing $\tilde{\varphi}$ by $\tilde{\chi}^{k,\beta}$ and using (6.9) we obtain

$$\lim_{\beta \downarrow 0} \int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} \tilde{a}_{ij}(\omega)(D_j \tilde{\chi}^{k,\beta}(\omega) - \tilde{\psi}_j^k(\omega)) D_i \tilde{\chi}^{k,\beta}(\omega) = 0. \tag{6.11}$$

By the replacement used in Section 4 to prove uniqueness of the solution of (3.3)-(3.5) we conclude that

$$\int_{\Omega} P(d\omega) \sum_{i,j=1}^{d} (D_j \tilde{\chi}^{k,\beta}(\omega) - \tilde{\psi}_j^k(\omega)) \tilde{\psi}_j^k(\omega) = 0. \tag{6.12}$$

Subtracting (6.12) from (6.11) and using the uniform ellipticity gives (6.10).

So it remains to prove (6.9). Let A be the operator

$$A = - \sum_{i,j=1}^{d} D_i(\tilde{a}_{ij} D_j \cdot) \tag{6.13}$$

defined on $\mathscr{H}^1$ as a quadratic form. We denote by A also its Friedrichs extension [19], p. 372 in $\mathscr{H}$. This operator is a nonnegative selfadjoint operator (using symmetry of $(\tilde{a}_{ij})$) with spectral resolution

$$A = \int_0^\infty \lambda G(d\lambda). \tag{6.14}$$

Write

(6.15) $\quad \tilde{f}^k(\omega) = \sum_{i=1}^{d} D_i(\tilde{a}_{ij}(\omega)) \qquad (k = 1, 2, \ldots, d),$

with the right side interpreted in the weak sense

(6.16) $\quad \int_\Omega P(d\omega)\tilde{g}D_i\tilde{a}_{ik} = -\int_\Omega P(d\omega)\tilde{a}_{ik}D_i\tilde{g}, \quad \forall \tilde{g} \in \mathscr{H}^1.$

We shall show below that $\tilde{f}^k$ is in the range of $A^{\frac{1}{2}}$. Assuming this we shall complete the proof of (6.9).

In terms of the spectral resolution we may write $\tilde{\chi}^{k,\beta}$ in the form

$$\sqrt{\beta}\tilde{\chi}^{k,\beta} = \sqrt{\beta}(\beta + A)^{-1}\tilde{f}^k = \int_0^\infty \frac{\sqrt{\beta}}{\beta + \lambda} G(d\lambda)\tilde{f}^k(\omega)$$

so that

(6.17) $\quad \beta \int_\Omega P(d\omega)(\tilde{\chi}^{k,\beta}(\omega))^2 = \int_0^\infty \frac{\beta}{(\beta + \lambda)^2} E\{\tilde{f}^k G(d\lambda)\tilde{f}^k\}.$

Since we have assumed that $\tilde{f}^k$ is in the range of $A^{\frac{1}{2}}$, there exists an $\tilde{h}^k$ in $\mathscr{D}(A^{\frac{1}{2}})$ such that $\tilde{f}^k = A^{\frac{1}{2}}h^{\frac{1}{2}}$. But then

(6.18) $\quad E\{\tilde{f}^k G(d\lambda)\tilde{f}^k\} = E\{A^{\frac{1}{2}}\tilde{h}^k G(d\lambda)A^{\frac{1}{2}}\tilde{h}^k\} = Eh^k G(d\lambda)h^k$

and (6.17) becomes

(6.19) $\quad \beta \int_\Omega P(d\omega)(\tilde{\chi}^{k,\beta}(\omega))^2 = \int_0^\infty \frac{\lambda\beta}{(\lambda + \beta)^2} E\{h^k G(d\lambda)\tilde{h}^k\}.$

Since $\lambda\beta(\lambda + \beta)^{-2}$ is bounded by $\frac{1}{4}$ independently of $\beta > 0$ the Lebesgue convergence theorem applied to (6.19) gives the desired result (6.9).

We must now show that $\tilde{f}^k$ of (6.15) is in the range of $A^{\frac{1}{2}}$.

Lemma. *If there is a constant* $C < \infty$ *such that*

(6.20) $\quad |E\{\tilde{f}\tilde{g}\}|^2 \leqslant CE\{\tilde{g}A\tilde{g}\} \quad \forall \tilde{g} \in \mathscr{D}(A)$

then f *is in the range of* $A^{\frac{1}{2}}$ *and* $E\{|A^{-\frac{1}{2}}\tilde{f}|^2\} \leqslant C.$

We prove the lemma after verifying that the f^k of (6.15) satisfy (6.20). Since $\mathscr{D}(A) \subset \mathscr{H}^1$; for all $\tilde{g} \in \mathscr{D}(A)$ we have

$$|E\{\tilde{f}^k \tilde{g}\}|^2 = \left| E\left\{\bar{\tilde{g}} \sum_{i=1}^{d} D_i \tilde{a}_{ik}\right\}\right|^2 =$$

$$= \left| \sum_{i=1}^{d} E\{\tilde{a}_{ik} D_i \tilde{g}\}\right|^2 \leqslant$$

$$\leqslant \left(\frac{1}{a_0}\right)^2 \sum_{i=1}^{d} E\{|D_i \tilde{g}|^2\} \leqslant \qquad \text{(using (2.1))}$$

$$\leqslant \left(\frac{1}{a_0}\right)^3 \sum_{i,j=1}^{d} E\{D_j \tilde{g}\, \tilde{a}_{ij} D_j \tilde{g}\} = \qquad \text{(using (2.1))}$$

$$= \left(\frac{1}{a_0}\right)^3 E\{\tilde{g} A \tilde{g}\}.$$

Then (6.2) holds and the proof of Theorem 4 is complete.

Proof of Lemma. Put

(6.21) $$\tilde{h}_\beta = (\beta + A)^{-\frac{1}{2}} \tilde{f} \qquad (\beta > 0).$$

Then for all $\tilde{g} \in \mathscr{D}(A)$

$$|E\{\tilde{g}\, \tilde{h}_\beta\}| = |E\{\tilde{g}(\beta + A)^{-\frac{1}{2}} \tilde{f}\}|^2 =$$

$$= |E\{\tilde{f}(\beta + A^{-\frac{1}{2}} \tilde{g}\}|^2 \leqslant$$

$$\leqslant CE\{(\beta + A)^{-\frac{1}{2}} \tilde{g} A (\beta + A)^{-\frac{1}{2}} \tilde{g}\} = \qquad \text{(by (6.20))}$$

$$= CE\{A(\beta + A)^{-1} \tilde{g}\, \tilde{g}\} \leqslant$$

$$\leqslant CE\{|\tilde{g}|^2\}.$$

Since $\mathscr{D}(A)$ is dense in $\mathscr{H}$ this means that $E\{|h_\beta|^2\} \leqslant C$ and hence there exists a subsequence, also denoted by $\tilde{h}_\beta$, that converges weakly in $\mathscr{H}$ to some limit $\tilde{h}$ as $\beta \to 0$.

By (6.21), $(\beta + A)^{\frac{1}{2}} \tilde{h}_\beta = \tilde{f}$ and hence

$$E\{\tilde{h}_\beta (\beta + A)^{\frac{1}{2}} \tilde{g}\} = E\{\tilde{f}\tilde{g}\}, \quad \forall \tilde{g} \in \mathscr{D}(A^{\frac{1}{2}}).$$

Passing to the limit $\beta \to 0$ gives

$$E\{\tilde{h} A^{\frac{1}{2}} \tilde{g}\} = E\{\tilde{f}\tilde{g}\}, \quad \forall \tilde{g} \in \mathscr{D}(A^{\frac{1}{2}}).$$

This implies that $\tilde{h}$ is in the domain of $A^{\frac{1}{2}}$ and that $A^{\frac{1}{2}} \tilde{h} = \tilde{f}$, (cf. [19], p. 322) as was to be shown.

7. THE BLOCH REPRESENTATION

Problem (5.1) can be solved explicitly in terms of Fourier transforms and the solution of a problem in the abstract space $(\Omega, \mathscr{F}, P)$. We call this the Bloch representation, or the Bloch expansion, because in the periodic case the problem in the abstract space is a boundary value problem over a period cell that can be solved by Bloch waves (cf. [10], Chapter 4, Section 3). Although we do not make use of this representation in our asymptotic analysis we give it here since it may be useful in other contexts.

We write (5.1) with $\epsilon = 1$ in the form

$$-\sum_{j,l=1}^{d} \frac{\partial}{\partial x_j} \left[a_{jl}(x, \omega) \frac{\partial u(x, \omega)}{\partial x_l} \right] + \alpha u(x, \omega) = f(x) \qquad (x \in R^d). \tag{7.1}$$

Since the coefficients $a_{jl}(x, \omega) = \tilde{a}_{jl}(\tau_{-x}\omega)$ depend on ω through $\tau_{-x}\omega$ we may look for u in the form

$$u(x, \omega) = v(x, \tau_{-x}\omega). \tag{7.2}$$

We then have $\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i} + D_i v$, and hence (7.1) becomes

$$-\sum_{j,l=1}^{d} \left(D_j + \frac{\partial}{\partial x_j}\right) \left[\tilde{a}_{jl}(\omega) \left(D_l + \frac{\partial}{\partial x_l}\right) v(x, \omega) \right] + \alpha v(x, \omega) = f(x) \qquad (x \in R^d). \tag{7.3}$$

Suppose $f(x)$ is a smooth function decaying rapidly as $|x| \to \infty$ and let $\tilde{f}(k)$ be its Fourier transform

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{R^d} dx e^{-ikx} f(x).$$

We write formally at the moment

$$\nu(x, \omega) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{R^d} dk e^{ikx} \hat{\nu}(k, \omega). \tag{7.4}$$

Then $\hat{\nu}(k,\omega)$ satisfies

$$-\sum_{j,l=1}^{d} (D_j + ik_j)[\tilde{a}_{jl}(\omega)(D_l + ik_l)\hat{\nu}(k, \omega)] + \alpha\hat{\nu}(k, \omega) = \hat{f}(k) \tag{7.3'}$$

which is an equation in the abstract space $(\Omega, \mathscr{F}, P)$ with $k \in R^d$ a parameter.

Define formally for each $k \in R^d$

$$A(k) = -\sum_{j,l=1}^{d} (D_j + ik_j)[\tilde{a}_{jl}(\omega)(D_l + ik_l)]. \tag{7.5}$$

The precise definition of $A(k)$ is as a quadratic form over $\mathscr{H}^1$ and then by the Friedrichs extension of this form [19]. It is a nonnegative selfadjoint operator with domain $\mathscr{D}(A(k))$ dense in $\mathscr{H}^1$ (we always assume uniform ellipticity (2.1)). It can be verified easily that for each $\alpha > 0$ the operator $(A(k) + \alpha)^{-1}$ is bounded in $\mathscr{H}$ uniformly in $k \in R^d$ and is strongly continuous as a function of k.

Let

$$\hat{\nu}(k, \omega) = (A(k) + \alpha)^{-1} \hat{f}(k)(\omega) \tag{7.6}$$

with $\hat{f}(k)$ smooth and rapidly decaying as $|k| \to \infty$. Then the integral (7.4) makes sense for almost all ω and we have Parseval's identity,

$$\int_{R^d} dx \int_{\Omega} P(d\omega) |\nu(x, \omega)|^2 = \int_{R^d} dx \int_{\Omega} P(d\omega) |\hat{\nu}(k, \omega)|^2 .$$

This and the boundedness of $(A(k) + \alpha)^{-1}$ uniformly in k imply that the mapping $f(x) \to v(x, \omega)$ is bounded from $L^2(R^d)$ to $L^2(R^d; \mathscr{H})$ and the formula

$$u(x, \omega) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{R^d} dk e^{ikx} (A(k) + \alpha)^{-1} \hat{f}(k)(\tau_{-x}\omega) \tag{7.7}$$

extends by continuity to all $f \in L^2(R^d)$.

We summarize the discussion of this section as follows: The solution $u(x, \omega) \in H^1(R^d; \mathscr{H})$ of (5.1) with $\epsilon = 1$ (and with $\alpha > 0$ and (2.1)) for $f \in L^2(R^d)$, admits the representation (7.7) with the integral understood in the sense of $L^2(R^d; \mathscr{H})$. The operator $A(k)$ is the Friedrichs extension of (7.5) defined as a quadratic form on $\mathscr{H}^1$ for each $k \in R^d$.

Thus the problem of solving (7.1) is reduced to solving the abstract cell problems (7.3′) (the shifted cell problems in the terminology of [10]) for each $k \in R^d$ and then constructing the Fourier integral (7.7).

8. PROBABILISTIC CONVERGENCE THEOREM

We shall assume that the coefficients $a_{ij}(x, \omega) = \tilde{a}_{ij}(\tau_{-x}\omega)$ are symmetric and satisfy (2.1) on $(\Omega, \mathscr{F}, P)$ with the translation group τ_x ergodic on M as described in Section 2. We shall assume in addition that

(8.1) for almost all ω the coefficients $(a_{ij}(x, \omega)$ are continuous functions of $x \in R^d$,

(8.2) the functions $b_j(x, \omega) = \frac{1}{2} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(x, \omega))$ are bounded and measurable in (x, ω).

For each $\epsilon > 0$ define the operator

$$\mathscr{L}_\omega^\epsilon = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\epsilon} \sum_{j=1}^{d} b_j\left(\frac{x}{\epsilon}, \omega\right) \frac{\partial}{\partial x_j}. \tag{8.3}$$

Let $X = \mathscr{C}([0, \infty); R^d)$ be the space of continuous trajectories in R^d with the topology of uniform convergence on compact sets. Let Σ denote the σ-algebra of Borel sets and let Σ_t $(t \geqslant 0)$, denote the σ-algebra generated by events depending on paths up to time t. A point in X is denoted by ζ and the point in R^d at time t through which ζ passes is denoted by $x(t) = x(t, \zeta)$.

Under assumptions (8.1), (8.2) and (2.1) there is a uniquely defined diffusion process on R^d associated with $\mathscr{L}^\epsilon_\omega$. That is, for each $\epsilon > 0$, $x \in R^d$ and almost all $\omega \in \Omega$ there is a probability measure $Q_x(\cdot, \omega)$ on (X, Σ) which solves the martingale problem [20] associated with $\mathscr{L}^\epsilon_\omega$:

(8.4)

(i) $Q^\epsilon_x(x(0) = x, \omega) = 1$,

(ii) for each $f(x)$ in $C^2(R^d)$ that is bounded and has bounded derivatives the expression

$$f(x(t)) - \int_0^t (\mathscr{L}^\epsilon_\omega f)(x(s))\, ds$$

is a $(\Sigma_t, Q^\epsilon_x(\cdot, \omega))$ martingale.

Theorem 5. *Let* Q_x *be the diffusion process (Brownian motion) associated with*

$$\mathscr{L} = \frac{1}{2} \sum_{i,j=1}^{d} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{8.5}$$

with the constant coefficients (q_{ij}) *defined in Theorem* 2. *Let* $\varphi(x)$ *be any nonnegative square integrable function on* R^d *and let* $F(\zeta)$ *be any bounded continuous function on* X. *Then*

$$\lim_{\epsilon \downarrow 0} \int_\Omega dP(\omega) \Big| \int_{R^d} dx \varphi(x) \int_X F(\zeta) Q^\epsilon_x(d\zeta, \omega) - \int_{R^d} dx \varphi(x) \int_X F(\zeta) Q_x(d\zeta) \Big|^2 = 0. \tag{8.6}$$

Remark 1. In the periodic case it is possible to analyze processes with generators of the form (8.3) that do not satisfy (8.2) i.e. are not in divergence form. However, the large drift b_j must satisfy certain other conditions (cf. [10], Chapter 3 and [9]).

Remark 2. The asymptotic analysis of the process corresponding to (8.3) with $b_j \equiv 0$ can also be carried out but requires additional consideration. It is given in [22].

Remark 3. The main advantage of the probabilistic convergence Theorem 5 is that it can be used to obtain convergence results for functionals F that are associated with solutions of boundary value problems. Thus, it provides a convenient tool for localization i.e. for separating what happens away from boundaries from what happens near them.

Remark 4. Let us assume that the matrix (a_{ij}) has a symmetric square root

$$a_{ij}(x, \omega) = \sum_{k=1}^{d} \sigma_{ik}(x, \omega)\sigma_{jk}(x, \omega) \qquad (i, j = 1, 2, \ldots, d), \tag{8.7}$$

where $\sigma_{ij}(x, \omega) = \tilde{\sigma}_{ij}(\tau_{-x}\omega)$. Let us also assume that b_j and σ_{ij} satisfy a Lipschitz condition with Lipschitz constant independent of ω. Let $\beta(t) = (\beta_j(t))$ denote the standard Brownian motion processs on R^d. Then the Itǒ stochastic differential equations

$$\begin{gathered} dx_j(t) = b_j(x(t), \omega)\, dt + \sum_{k=1}^{d} \sigma_{jk}(x(t), \omega) d\beta_j(t) \\ (x(0) = x \in R^d,\ j = 1, 2, \ldots, d), \end{gathered} \tag{8.8}$$

have for almost all $\omega \in \Omega$ a solution which is a diffusion process (with path measure $Q_x^\epsilon(\cdot, \omega)$ as above) expressed explicitly as a functional of a standard Brownian motion. We have what may be called a diffusion process in a random medium or a random environment with the realization of the medium being labeled by $\omega \in \Omega$. A natural question to ask is how does $x(t) - x$ behave for t large? Does $t^{-\frac{1}{2}}(x(t) - x)$ tend in some sense to a Gaussian as $t \uparrow \infty$? In fact if we let

$$x^\epsilon(t) = \epsilon x\left(\frac{t}{\epsilon^2}\right) \qquad (x^\epsilon(0) = x), \tag{8.9}$$

we find that (8.8) becomes

$$\begin{gathered} dx_j^\epsilon(t) = \frac{1}{\epsilon} b_j\left(\frac{x^\epsilon(t)}{\epsilon}, \omega\right) dt + \sum_{k=1}^{d} \sigma_{jk}\left(\frac{x^\epsilon(t)}{\epsilon}, \omega\right) d\beta_j(t) \\ (x^\epsilon(0) = x,\ j = 1, 2, \ldots, d), \end{gathered} \tag{8.10}$$

which is precisely the Ǐto equation that goes with (8.3). Thus, Theorem 5 says that indeed $x^\epsilon(t)$ tends to Brownian motion (generator (8.5)) as $\epsilon \to 0$ weakly (as a measure in X) in $L^2(\Omega, \mathscr{F}, P)$, i.e. in the sense (8.6).

We note again that Theorem 5 has been shown only under condition (8.2) which says that (8.3) is in divergence form (cf. Remark 2 above).

9. PROOF OF PROBABILISTIC CONVERGENCE THEOREM

We shall first show that the measures $Q_x^\epsilon(\cdot, \omega)$ $(\epsilon > 0,\ \omega \in \Omega,\ x \in R^d)$ are relatively weakly compact. For this purpose it suffices to show that

$$E^{Q_x^\epsilon}\{|x(t) - x(s)|^4\} = \tag{9.1}$$
$$= \int_X |x(t,\zeta) - x(s,\zeta)|^4 Q_x^\epsilon(d\zeta, \omega) \leqslant C|t-s|^2,$$

where C is a constant independent of t, s, ϵ, ω and x. To prove (9.1) we use the divergence form of (8.3) and Nash's estimates [21].

Specifically, suppose for the moment that the coefficients (a_{ij}) are smooth and satisfy (2.1). Then the diffusion equation

$$\frac{\partial u}{\partial t} = \mathscr{L}_\omega^\epsilon u$$

has a fundamental solution $\varphi^\epsilon(t, x, y; \omega)$. The estimate of Nash says that there is a constant $\gamma > 0$ that depends only on the uniform ellipticity constants in (2.1) such that

$$\varphi^\epsilon(t, x, y; \omega) \leqslant \frac{\gamma}{t^{\frac{d}{2}}} e^{-\gamma \frac{|x-y|}{\sqrt{t}} \log \gamma \frac{|x-y|}{\sqrt{t}}}, \tag{9.2}$$

for all $t \geqslant 0$, $x, y \in R^d$ and $\omega \in \Omega$. Estimate (9.2) immediately yields (9.1) (after the preliminary smoothing is removed).

Compactness of Q_x^ϵ, with $|x| \leqslant M$, means that for any $\delta > 0$ there is a compact set K_δ in $X = C([0,\infty); R^d)$ such that

$$(9.3) \qquad Q_x^\epsilon(K_\delta, \omega) \geqslant 1 - \delta$$

for all $\epsilon > 0$, $\omega \in \Omega$ and $|x| \leqslant M$. Thus, to prove (8.6) it suffices to prove

$$(9.4) \qquad \lim_{\epsilon \downarrow 0} \int_\Omega dP(\omega) \Big| \int_{|x| \leqslant M} dx \varphi(x) \int_{K_\delta} F(\zeta) Q_x^\epsilon(d\zeta, \omega) - \int_{|x| \leqslant M} dx \varphi(x) \int_{K_\delta} F(\zeta) Q_x(d\zeta) \Big|^2 = 0$$

or, since $\varphi \in L^2(R^d)$,

$$(9.5) \qquad \lim_{\epsilon \downarrow 0} \int_\Omega dP(\omega) \int_{|x| \leqslant M} dx \Big| \int_{K_\delta} F(\zeta) Q_x^\epsilon(d\zeta, \omega) - \int_{K_\delta} F(\zeta) Q_x(d\zeta) \Big|^2 = 0.$$

Now $F(\zeta)$ is a bounded continuous function on X so in particular $F \in C(K_\delta)$ (the space of bounded continuous functions on the compact set K_δ). Finite linear combinations of functions in $C(K_\delta)$ of the form

$$(9.6) \qquad f_1(x(t_1)) \ldots f_N(x(t_N)) \qquad (0 \leqslant t_1 < t_2 < \ldots < t_N < \infty,$$

with $f_j(x)$ in $C_0^\infty(R^d)$ form an algebra in $C(K_\delta)$ that contains the constant functions and separates points. This algebra is thus dense in $C(K_\delta)$ by the Stone – Weierstrass theorem. It suffices then to prove (9.5) with F of the form (9.6). But then (9.5) will follow if we can show that for each $T < \infty$ and for each $f(x) \in C^\infty(R^d)$ that decays to zero as $|x| \to \infty$, say, exponentially fast we have

$$(9.7) \qquad \lim_{\epsilon \downarrow 0} \sup_{0 \leqslant t \leqslant T} \int_\Omega dP(\omega) \int_{R^d} dx \, |E^{Q_x^\epsilon}(f(x(t))) - E^{Q_x}(f(x(t)))|^2 = 0.$$

Here we use the fact that $u(t, x) = E^{Q_x}\{f(x(t))\}$ satisfies the heat equation

$$(9.8) \qquad \frac{\partial u}{\partial t} = \mathscr{L} u \qquad (t > 0, \; u(0, x) = f(x))$$

and has smooth and decaying solution as $|x| \to \infty$ when $f(x)$ is also smooth and decaying as $|x| \to \infty$.

Let

$$(9.9) \qquad u^\epsilon(t,x,\omega) = E^{Q^\epsilon_x}\{f(x(t))\}.$$

This function is a suitable weak solution of the diffusion equation

$$(9.10) \qquad \frac{\partial u^\epsilon}{\partial t} = \mathscr{L}^\epsilon_\omega u^\epsilon \qquad (t>0,\ u^\epsilon(0,x,\omega) = f(x)),$$

with $\mathscr{L}^\epsilon_\omega$ given by (8.3). Since $\mathscr{L}^\epsilon_\omega$ is in divergence form we consider (9.10) in the following manner.

Let $\mathscr{H} = L^2(\Omega, \mathscr{F}, P)$, $H = L^2(R^d; \mathscr{H})$ and $H^1 = H^1(R^d; \mathscr{H})$ with the usual inner products (cf. Section 2). On H^1 which is dense in H the operator $A^\epsilon = -\mathscr{L}^\epsilon_\omega$ defines a nonnegative, symmetric bilinear form. We denote the Friedrichs extension of A also by A^ϵ and note that it is a densely defined, selfadjoint nonnegative operator on H. Let A be the operator defined by $-\mathscr{L}$ on H (which is essentially the Laplacian). From Theorem 3 we know that for each $\alpha > 0$

$$(9.11) \qquad (A^\epsilon + \alpha)^{-1} \to (A+\alpha)^{-1}$$

strongly in H, as $\epsilon \downarrow 0$. Let T^ϵ_t and T_t be, respectively, the semigroup defined by $\exp(-A^\epsilon t)$ and $\exp(At)$ via the spectral theorem for the selfadjoint operators A^ϵ and A. From the strong resolvent convergence (9.11) it follows that the semigroups converge

$$(9.12) \qquad T^\epsilon_t \to T_t,$$

strongly in H, as $\epsilon \downarrow 0$, uniformly in finite t intervals. But (9.12) is precisely the statement (9.7) which was to be shown. The proof Theorem 5 is complete.

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