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MEAN FIELD AND GAUSSIAN APPROXIMATION FOR PARTIAL DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS*

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Abstract. After discussing in several contexts how mean field and fluctuation approximations arise and can be used, we give a simple method by which the analysis of the approximations can be carried out.

1. Introduction. For many linear or nonlinear boundary value problems with rapidly varying random coefficients a first approximation to the solution is provided by simply averaging these coefficients. Then the fluctuation of the solution about that of the averaged problem has asymptotically a Gaussian distribution. Our purpose here is to show in some specific cases how this mean field and fluctuation approximation can be analyzed easily by a method which has proved useful in other contexts [1], [2]. Frequently [2] the first approximation does not come by averaging the coefficients and it is likely but unknown at present that fluctuations for the problem in [2] are not Gaussian.

In the next section we formulate the questions to be considered for two-point boundary value problems. A detailed analysis of this is given in [3]. In § 3 we formulate and analyze heuristically a nonlinear boundary value problem. The method of [1] can be used for such problems, but to simplify the presentation, we give here the details for the Laplace operator with random potential, in §§ 4 and 5.

2. Two-point boundary value problems. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $F(u, x, y, \omega)$ a function from $\mathbb{R}^n \times [a, b] \times \mathbb{R}^1 \times \Omega$ into $\mathbb{R}^n$, which is a stationary process in (y, ω) for each u and x. Let $\varepsilon > 0$ be a small parameter characteristic of the scale of fluctuations and consider the two-point boundary value problem

$$\frac{du^\varepsilon(x, \omega)}{dx} = F\left(u^\varepsilon(x, \omega), x, \frac{x}{\varepsilon}, \omega\right), \qquad a < x < b, \tag{2.1}$$

$$H(u^\varepsilon(a, \omega), u^\varepsilon(b, \omega)) = 0. \tag{2.2}$$

Here H is defined on $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$ and represents nonlinear boundary conditions. The question is what is the behavior of the $\mathbb{R}^n$-value process $u^\varepsilon(x, \omega)$ for ε small. It should be noted that since (2.1), (2.2) is a nonlinear problem we do not know that it has a solution. The existence question is part of the problem.

Motivated by the well-known method of averaging for initial value problems, we associate with (2.1), (2.2) the averaged problem

$$\frac{d\bar{u}(x)}{dx} = \bar{F}(\bar{u}(x), x), \qquad a < x < b, \tag{2.3}$$

$$H(\bar{u}(a), \bar{u}(b)) = 0, \tag{2.4}$$

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where

$$\bar{F}(u,x)=E\{F(u,x,y,\cdot)\}. \tag{2.5}$$

Note that $\bar{F}$ does not depend on y by stationarity. The basic hypothesis regarding the boundary value problem is that (2.3), (2.4) has a solution. In addition, it is assumed that the variational problem

$$\frac{dV(x)}{dx}=\frac{\partial\bar{F}}{\partial u}(\bar{u}(x),x)V(x), \qquad a<x<b, \tag{2.6}$$

$$MV(a)=NV(b)=0 \tag{2.7}$$

has only the trivial solution $V\equiv 0$. Here M and N are $n\times n$ matrices given by

$$\begin{aligned} M&=\frac{\partial H(u,v)}{\partial u}\bigg|_{u=\bar{u}(a),v=\bar{u}(b)},\\ N&=-\frac{\partial H(u,v)}{\partial v}\bigg|_{u=\bar{u}(a),v=\bar{u}(b)}. \end{aligned} \tag{2.8}$$

It is also assumed that F is twice differentiable in u and that it is an ergodic stationary process.

It now follows, either by the methods of [3] or those of [1] (which will be presented later for a partial differential equation), that there is an $\varepsilon_0>0$ such that for each $0<\varepsilon\leqq\varepsilon_0$ there is a set $\Omega_\varepsilon\subset\Omega$ such that $P(\Omega_\varepsilon)\to 1$ as $\varepsilon\to 0$ and (2.1), (2.2) has a unique solution $u^\varepsilon(x,\omega)$ for $\omega\in\Omega_\varepsilon$. Moreover

$$\sup_{\omega\in\Omega_\varepsilon}\sup_{a\leqq x\leqq b}|u^\varepsilon(x,\omega)-\bar{u}(x)|\to 0 \quad \text{as } \varepsilon\to 0. \tag{2.9}$$

One next looks at the fluctuations $\zeta^\varepsilon(x,\omega)=(u^\varepsilon(x,\omega)-\bar{u}(x))\varepsilon^{-1/2}$, $\omega\in\Omega_\varepsilon$, just as in the case of initial value problems. It is found that, under suitable additional properties of F, such as mixing, ζ^ε converges weakly to the Gaussian process

$$\zeta(x)=\int_a^b G(x,y)\,dw(y). \tag{2.10}$$

Here $G(x,y)$ is the Green's function of the linear problem (2.6), (2.7),

$$\begin{aligned} \frac{d}{dx}G(x,y)&=\frac{\partial\bar{F}}{\partial u}(\bar{u}(x),x)G(x,y)+I\delta(x-y),\\ MG(a,y)&=NG(b,y)=0, \end{aligned} \tag{2.11}$$

and $w(x)$ is a zero mean Gaussian process with independent increments and covariance

$$E\{w(x)w^T(x)\}=\int_a^x A(\bar{u}(s),s)\,ds, \tag{2.12}$$

with

$$A(u,x)=\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_0^T E\{[F(u,x,s)-\bar{F}(u,s)][F(u,x,\sigma)-\bar{F}(u,\sigma)]^T\}\,d\sigma ds. \tag{2.13}$$

In order to understand why ζ^ε should formally behave like ζ of (2.10), one simply obtains a differential equation for ζ^ε and expands to $O(\sqrt{\varepsilon})$. This is done in the next section.

3. A nonlinear boundary value problem. Let $\mathcal{O}$ be a bounded subset of $\mathbb{R}^n$ and let F be a function defined on $\mathbb{R}\times\mathcal{O}\times\mathbb{R}^n\times\Omega$ into $\mathbb{R}$. Consider the boundary value problem

$$\begin{aligned} &\Delta u^\varepsilon(x,\omega)+F\left(u^\varepsilon(x,\omega),x,\frac{x}{\varepsilon},\omega\right)=0, \qquad x\in\mathcal{O},\\ &u^\varepsilon(x,\omega)=0, \qquad x\in\partial\mathcal{O}. \end{aligned} \tag{3.1}$$

We assume that $F(u,x,y,\omega)$ is a stationary random field in (y,ω) for each u and x and set

$$\bar{F}(u,x)=E\{F(u,x,y,\cdot)\}. \tag{3.2}$$

We also assume that the averaged problem

$$\Delta\bar{u}(x)+\bar{F}(\bar{u}(x),x)=0,\quad x\in\mathcal{O},\qquad \bar{u}(x)=0,\quad x\in\partial\mathcal{O} \tag{3.3}$$

has a solution for which the associated variational problem

$$\begin{aligned} &\Delta z(x)+V(x)z(x)=0, \qquad x\in\mathcal{O},\\ &z(x)=0, \qquad x\in\partial\mathcal{O} \end{aligned} \tag{3.4}$$

has only the trivial solution $z(x)\equiv 0$. Here

$$V(x)=\frac{\partial\bar{F}}{\partial u}(\bar{u}(x),x). \tag{3.5}$$

The case in which (3.4) has nontrivial solutions corresponds to bifurcation and is analyzed in [1].

Now as in § 2 one shows that (3.1) has a suitable solution for $\omega\in\Omega_\varepsilon$ and $P(\Omega_\varepsilon)\to 1$ as $\varepsilon\to 0$ while $u^\varepsilon(x,\omega)\to\bar{u}(x)$ in $L^2(\mathcal{O})$ for each ω in Ω_ε.

It should be noted that not every problem with rapidly oscillating coefficients has solutions close to the ones of the problem with averaged coefficients. For example, when $\bar{F}\equiv 0$ the solution of

$$\Delta u^\varepsilon+\frac{1}{\varepsilon}F\left(u,x,\frac{x}{\varepsilon},\omega\right)=0, \qquad u^\varepsilon|_{\partial\mathcal{O}}=0$$

tends to the solution of

$$\Delta\bar{u}+G(\bar{u},x)=0, \qquad \bar{u}|_{\partial\mathcal{O}}=0,$$

where now $G(u,x)$ is given by

$$\frac{1}{\Gamma_n}\int_{\mathbb{R}^n}\frac{1}{|z|^{n-2}}E\left\{\frac{\partial F}{\partial u}(u,x,z)F(u,x,0)\right\}dz,$$

where Γ_n is the surface area of the unit sphere in $\mathbb{R}^n$. This is readily identified as a once-iterated form of averaging familiar from the ordinary differential equations case [4 and references there].

In more complicated problems such as systems, or in cases where the parameter ε appears in a singular manner in several places, the correct form of the associated averaged problem may not be easily recognizable. In such cases, formal multiple scale expansions help in sorting out terms properly. An example may be found in [2] or in [5] for periodically oscillating coefficients.

To analyze the fluctuations of (3.1) let

$$\zeta^\varepsilon(x,\omega)=\varepsilon^{-\gamma}(u^\varepsilon(x,\omega)-\bar u(x)), \tag{3.6}$$

where $\gamma>0$ will be specified later. Using (3.1) and (3.3) as well as (3.5), one obtains an equation for $\zeta^\varepsilon(x,\omega)$ which when expanded in ε^γ gives

$$\begin{aligned}-(\Delta+V(x))\zeta^\varepsilon(x,\omega)&=\varepsilon^{-\gamma}\Big(F\Big(\bar u(x),x,\frac{x}{\varepsilon},\omega\Big)-\bar F(\bar u(x),x)\Big)\\&\quad+\left(\frac{\partial F\Big(\bar u(x),x,\frac{x}{\varepsilon},\omega\Big)}{\partial u}-\frac{\partial\bar F(\bar u(x),x)}{\partial u}\right)(u^\varepsilon(x,\omega)-\bar u(x))+\cdots.\end{aligned} \tag{3.7}$$

The first and hardest step is to show that all terms but the first on the right side of (3.7) do not contribute in the limit (we take this up in the next section). It then must be shown that for a suitable γ the process

$$\eta^\varepsilon(x,\omega)=\varepsilon^{-\gamma}\int_{\mathcal O}G(x,y)\Big(F\Big(\bar u(y),y,\frac{y}{\varepsilon},\omega\Big)-\bar F(\bar u(y),y)\Big)\,dy \tag{3.8}$$

tends to a Gaussian process when F is stationary and mixing. In (3.8) $G(x,y)$ is the kernel of $-(\Delta+V)^{-1}$, i.e., the Green's function of the linearized problem (3.4). From the nature of the singularity of G, which is like that of $(-\Delta)^{-1}$, one finds readily by calculating the covariance of η^ε that the correct scaling γ is

$$\gamma=\frac{n}{2}\quad\text{for } n<4,\qquad \gamma=2\quad\text{for } n>4. \tag{3.9}$$

For $n=4$ instead of $\varepsilon^{-\gamma}$, we must use the normalization $\varepsilon^{-2}(\log\varepsilon^{-1})^{-1/2}$.

The result is then that $\zeta^\varepsilon(x,\omega)$ and $\eta^\varepsilon(x,\omega)$ have the same limiting distribution and that the one for $\eta^\varepsilon(x,\omega)$ is Gaussian since it is just the integral of a given mixing process. There is a very desirable separation of difficulties to be noted. The asymptotic equivalence of ζ^ε and η^ε is primarily a PDE question with a few probabilistic considerations. On the other hand, the asymptotic normality of η^ε is a more or less standard problem in probability.

4. Laplace equation with random potential. To present in detail the method of analysis we have in mind, we shall study the following equation in $\mathbb{R}^3$:

$$L^\varepsilon u^\varepsilon(x)=(-\Delta+\lambda+v^\varepsilon(x))u^\varepsilon(x)=f(x). \tag{4.1}$$

Here $\lambda>0$ is fixed, $f(x)$ is any $L^2(\mathbb{R}^3)$ function with compact support and $v^\varepsilon(x)$ is a stationary random process which is a Poisson superposition of positive bumps. The Poisson density is $O(\varepsilon^{-3})$ and the support of the bumps $O(\varepsilon)$.

More precisely,

$$v^\varepsilon(x)=\sum_i v\Big(\frac{x-y_i^{(\varepsilon)}}{\varepsilon}\Big)=\int v\Big(\frac{x-y}{\varepsilon}\Big)\,\gamma_{\rho/\varepsilon^3}(dy), \tag{4.2}$$

where $0\leqq v(y)\in C^\infty$ is the fixed shape of each bump and $\gamma_\rho(A)$, $A\subset\mathbb{R}^3$, is the stationary Poisson process with average density ρ.

For every $\varepsilon>0$ (4.1) defines a process $u^\varepsilon(x)$ since the potential is bounded and nonnegative for all realizations. It is convenient in the following to regard $v^\varepsilon(x)$ as a random field. With each $f \in L^2(\mathbb{R}^3)$ put

$$v^\varepsilon(f) = \int f(x) v^\varepsilon(x)\, dx.$$

Then for $\alpha > 0$

$$E\{e^{-\alpha v^\varepsilon(f)}\} = \exp\left\{\frac{\rho}{\varepsilon^3}\int_{\mathbb{R}^3}\left[\exp\left(-\alpha\int v\left(\frac{x-y}{\varepsilon}\right)f(y)\,dy - 1\right)\right]dx\right\},$$

and by differentiation we have

$$E\{v^\varepsilon(x)\} = \rho\bar{v}, \qquad \bar{v} = \int v(y)\,dy, \tag{4.3}$$

$$E\{(v^\varepsilon(x)-\rho\bar{v})(v^\varepsilon(y)-\rho\bar{v})\} = \rho\int v(s)v\left(s - \frac{x-y}{\varepsilon}\right)ds \equiv R_\varepsilon(x-y), \tag{4.4}$$

$$\begin{aligned} E\left\{\prod_{i=1}^4 (v^\varepsilon(x_i)-\rho\bar{v})\right\} &= \int \prod_{i=1}^4 v\left(\frac{x_i - y}{\varepsilon}\right)\frac{\rho}{\varepsilon^3}\,dy \\ &= R_\varepsilon(x_1-x_2)R_\varepsilon(x_3-x_4) + R_\varepsilon(x_1-x_3)R_\varepsilon(x_2-x_4) \\ &\quad + R_\varepsilon(x_1-x_4)R_\varepsilon(x_2-x_3). \end{aligned} \tag{4.5}$$

Finally we note that for the random field $v^\varepsilon(x)$ the central limit theorem is valid. Put

$$\tau^\varepsilon(x) = \frac{v^\varepsilon(x)-\rho\bar{v}}{\varepsilon^{3/2}}. \tag{4.6}$$

Then $\tau^\varepsilon(x)$ converges in distribution to a Gaussian random field with mean zero and covariance ρ times the identity operator. That is, for each $f \in L^2$ the random variable $\tau^\varepsilon(f)$ converges to a Gaussian random variable with covariance equal to $\rho\|f\|_{L^2}^2$.

For the asymptotic analysis of (4.1), we do not need to assume such a specific form of v^ε as (4.2). Mixing and 4 moments suffice.

THEOREM 1. *Let $\bar{u}(x)$, $x \in \mathbb{R}^3$ be the solution of*

$$(L\bar{u})(x) = (-\Delta + \lambda + \rho\bar{v})\bar{u}(x) = f(x). \tag{4.7}$$

Then

$$E\{\|u^\varepsilon - \bar{u}\|_p\} \to 0, \qquad 1 \leqq p \leqq \infty, \tag{4.8}$$

as $\varepsilon \downarrow 0$.

THEOREM 2. *Let $w(x)$ be the Gaussian random field with mean zero and covariance*

$$E\{w(f)w(g)\} = (f, Kg) = \int f(x)K(x,y)g(y)\,dydx,$$

where the kernel $K(x,y)$ is given by

$$K(x,y) = \rho\|v\|_{L^2}^2 \int |\bar{u}(z)|^2 G(x-z)G(y-z)\,dz,$$

where $G(x-y)$ is the kernel of $(-\Delta+\lambda+\rho\bar{v})^{-1} = L^{-1}$. Then the random field

$$\xi^\varepsilon(x) = \varepsilon^{-3/2}(u^\varepsilon(x) - \bar{u}(x)) \tag{4.9}$$

converges to $w(x)$ *in distribution as* $\varepsilon \to 0$. *In fact, if*

$$\eta^\varepsilon(x) = -L^{-1}(\bar{u}\tau^\varepsilon)(x), \tag{4.10}$$

then ξ^ε *and* η^ε *are asymptotically equivalent,*

$$E\{\|\xi^\varepsilon - \eta^\varepsilon\|_{L^2}^2\} \to 0 \quad as\ \varepsilon \to 0. \tag{4.11}$$

5. Proof of Theorems 1 and 2. The basic starting point, as in [1], is to introduce the process

$$\chi^\varepsilon(x, \omega) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\frac{1}{|x-z|} - \frac{1}{|z|} \right) (v^\varepsilon(z, \omega) - \rho \bar{v})\, dz. \tag{5.1}$$

Note that χ^ε is well defined but is not stationary.

LEMMA 1. *For every* g *with finite norms on the right below, there are constants* c_1, c_2, c_3 *independent of* ε *such that*

$$E\{\|g\chi^\varepsilon\|_{L^2}^2\} \leqq c_1 \varepsilon^{5/2} \|g\sqrt{1+|x|^2}\|_2^2, \tag{5.2}$$

$$E\{\|g\chi^\varepsilon\|_{L^4}^4\} \leqq c_2 \varepsilon^5 \|g(1+|x|^4)^{1/4}\|_{L^4}^4, \tag{5.3}$$

$$E\{\|g\nabla\chi^\varepsilon\|_{L^2}^2\} \leqq c_3 \varepsilon^2 \|g\|_{L^2}^2. \tag{5.4}$$

Note further that $\nabla\chi^\varepsilon$ *is a stationary process and that* χ^ε *satisfies almost everywhere the Poisson equation*

$$\Delta\chi^\varepsilon + (v^\varepsilon - \rho\bar{v}) = 0. \tag{5.5}$$

Proof. These are generalizations of the estimates given in [1]. For completeness we show that

$$E\{(\chi^\varepsilon(x))^2\} \leqq B_1 \varepsilon^{5/2}(1+|x|^2),$$

from which (5.2) follows.

By Parseval's theorem and the Fourier transform of (4.4),

$$\begin{aligned}
E\{(\chi^\varepsilon(x))^2\} &= \rho \int \frac{|e^{ik\cdot x} - 1|^2}{|k|^4} \varepsilon^3 |\hat{v}(\varepsilon k)|^2\, dk \\
&\leqq \rho \int_{|k|<\varepsilon^{1/2}} \frac{|k|^2|x|^2}{|k|^4} \varepsilon^3 |\hat{v}(\varepsilon k)|^2\, dk + \rho \int_{|k|>\varepsilon^{1/2}} \frac{4}{|k|^4} \varepsilon^3 |\hat{v}(\varepsilon k)|^2\, dk \\
&\leqq |x|^2 \varepsilon^2 \rho \int_{|k|<\varepsilon^{3/2}} \frac{|\hat{v}(k)|^2}{|k|^2}\, dk + 4\varepsilon^4 \rho \int_{|k|>\varepsilon^{3/2}} \frac{|\hat{v}(k)|^2}{|k|^4}\, dk \\
&\leqq B_1 \varepsilon^{5/2}(1+|x|^2),
\end{aligned}$$

which completes the proof.

Let us next define the process $z^\varepsilon(x)$ by

$$u^\varepsilon(x) - \bar{u}(x) = \chi^\varepsilon(x)\bar{u}(x) + z^\varepsilon(x). \tag{5.6}$$

We note that by the maximum principle

$$|u^\varepsilon(x)| \leqq |\tilde{u}(x)|,$$

where $\tilde{u}(x)$ is the solution of $(-\Delta+\lambda)\tilde{u} = f$. Thus since f has compact support, $u^\varepsilon(x)$ decreases at infinity faster than $e^{-\sqrt{\lambda}|x|}$. Let $H^2(\mathbb{R}^3)$ denote the Sobolev space of functions with square integrable derivatives up to second order.

LEMMA 2. *We have the estimate*

$$E\{\|z^\varepsilon\|_{H^2}^2\} \leqq c\varepsilon^2. \tag{5.7}$$

Remark. From this lemma and the Sobolev inequality in $\mathbb{R}^3$ it follows that

$$E\{\|z^\varepsilon\|_\infty^2\} \leqq c\varepsilon^2. \tag{5.8}$$

Now since $\bar{u}(x)$ is bounded and has bounded derivatives, it follows from (5.4) and the Sobolev inequality that we also have

$$E\{\|\chi^\varepsilon \bar{u}\|_{H^2}^2\} \leqq c\varepsilon^2, \qquad E\{\|\chi^\varepsilon \bar{u}\|^2\} \leqq c\varepsilon^2. \tag{5.9}$$

Proof of Lemma 2. First we note that z^ε satisfies the equation

$$-\Delta z^\varepsilon + \lambda z^\varepsilon + v^\varepsilon z^\varepsilon = 2\nabla\chi^\varepsilon \cdot \nabla \bar{u} - \chi^\varepsilon f - \chi^\varepsilon \bar{u}(v^\varepsilon - \rho \bar{v}). \tag{5.10}$$

Since λ and v^ε are positive, $(-\Delta + \lambda + v^\varepsilon)^{-1}$ is a bounded map from L^2 to H^2 independently of ε. So we must show that the right side of (5.10) has average L^2 norm less than a constant times ε^2. But this follows readily from Lemma 1. The proof is complete.

In the following, we need an estimate of $z^\varepsilon(x)$ weighted at infinity.

LEMMA 3. *For each $\lambda > 0$ there is an α, $0 < \alpha < \lambda$, such that if $h(x) \in C^\infty$ is defined by*

$$h(x) = \begin{cases} 1 & \text{if } |x| < R, \\ |x| & \text{if } |x| > 2R \end{cases}$$

for some R, then

$$\tilde{z}^\varepsilon(x) = e^{\alpha h(x)} z^\varepsilon(x)$$

satisfies

$$E\{\|\tilde{z}^\varepsilon\|_{H^2}^2\} \leqq c\varepsilon^2. \tag{5.11}$$

Proof. As in the previous lemma, we obtain by differentiation an equation of the form (5.10) for $\tilde{z}^\varepsilon$ with a few additional terms involving derivatives of h. These can be controlled with α and then Lemma 1 does the rest.

Proof of Theorem 1. This is now straightforward, because we have in fact shown that

$$E\{\|u^\varepsilon - \bar{u}\|_\infty^2\} \leqq 2E\{\|\chi^\varepsilon \bar{u}\|_\infty^2\} + 2E\{\|z^\varepsilon\|_\infty^2\} \leqq c\varepsilon^2,$$

and since u^ε and $\bar{u}$ decrease at infinity, (4.8) holds for all $p \geqq 1$. The proof is complete.

The decay properties of Lemma 3 give in addition the estimates

$$E\{\|e^{\alpha h}(u^\varepsilon - \bar{u})\|_\infty^2\} \leqq c\varepsilon^2, \tag{5.12}$$

$$E\{\|e^{\alpha h}(\nabla(u^\varepsilon - \bar{u}))\|_2^2\} \leqq c\varepsilon^2. \tag{5.13}$$

To complete the proof of Theorem 2, we find next the equation satisfied by $\xi^\varepsilon(x)$ of (4.9):

$$(-\Delta + \lambda + \rho\bar{v})\xi^\varepsilon = -\frac{(v^\varepsilon - \rho\bar{v})}{\varepsilon^{3/2}}(u^\varepsilon - \bar{u}) - \frac{(v^\varepsilon - \rho\bar{v})\bar{u}}{\varepsilon^{3/2}}. \tag{5.14}$$

With the notation (4.6), (4.7) and (4.10) it follows that

$$\xi^\varepsilon - \eta^\varepsilon = -L^{-1}[\tau^\varepsilon(u^\varepsilon - \bar{u})]. \tag{5.15}$$

It is now clear that Theorem 2 will be proved if we show that (4.11) holds (asymptotic equivalence), which by (5.15) means that we need to show

$$E\{\|L^{-1}[\tau^\varepsilon(u^\varepsilon-\bar{u})]\|_{L^2}\}\to 0 \qquad \text{as } \varepsilon\to 0. \tag{5.16}$$

To prove (5.16), let $G(x)=(4\pi|x|)^{-1}\exp(-\sqrt{+\rho\bar{v}}|x|)$, which is the kernel of L^{-1}. Integrating by parts and using (5.5), we obtain

$$\begin{aligned} &E\{\|L^{-1}(\tau^\varepsilon(u^\varepsilon-\bar{u}))\|_{L^2}\} \\ &=E\left\{\left\|\int G(x-y)\varepsilon^{-3/2}(\nabla\cdot\nabla\chi^\varepsilon(y))(u^\varepsilon(y)-\bar{u}(y))\,dy\right\|_{L^2}\right\} \\ &\leqq E\left\{\left\|\int \nabla_y G(x-y)\cdot\varepsilon^{-3/2}\nabla\chi^\varepsilon(y)(u^\varepsilon(y)-\bar{u}(y))\,dy\right\|_{L^2}\right\} \\ &\quad +E\left\{\left\|\int G(x-y)\varepsilon^{-3/2}\nabla\chi^\varepsilon(y)\cdot\nabla(u^\varepsilon(y)-\bar{u}(y))\,dy\right\|_{L^2}\right\}. \end{aligned} \tag{5.17}$$

Let A and B denote the two terms on the right of the inequality in (5.17). Repeated application of Young's inequality ($\|f*g\|_{L^2}\leqq\|f\|_{L^1}\|g\|_{L^2}$) and Schwarz's inequality gives

$$\begin{aligned} A&\leqq\||\nabla G|\|_{L^1}E\{\|\varepsilon^{-3/2}|\nabla\chi^\varepsilon|\,e^{-\alpha h}e^{\alpha h}(u^\varepsilon-\bar{u})\|_{L^2}\} \\ &\leqq\||\nabla G|\|_{L^1}E\{\|\varepsilon^{-3/2}|\nabla\chi^\varepsilon|\,e^{-\alpha h}\|_{L^2}\|e^{\alpha h}(u^\varepsilon-\bar{u})\|_\infty\} \\ &\leqq\||\nabla G|\|_{L^1}E^{1/2}\{\|\varepsilon^{-3/2}|\nabla\chi^\varepsilon|\,e^{-\alpha h}\|_{L^2}^2\}\cdot E^{1/2}\{\|e^{\alpha h}(u^\varepsilon-\bar{u})\|_{L^\infty}^2\} \\ &\leqq c\varepsilon^{1/2}, \end{aligned}$$

and similarly,

$$B\leqq c\varepsilon^{1/2}.$$

Theorem 2 is thus proved.

In conclusion, we should point out that the analysis of (4.1) and the theorems of § 4 can be done in many other ways, for example, using probabilistic methods via the Feynman–Kac formula. The method we have used has the advantage that it works equally well for very general linear or nonlinear problems. It has, of course, limitations, namely, that for problem (4.1) one cannot treat the much more difficult hard core case [6]–[9].

An interesting question we have not considered is whether the approach we followed here can be sharpened to yield information about the behavior of the spectrum of L^ε for ε small.

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