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006

# CONVECTION OF MICROSTRUCTURE AND RELATED PROBLEMS*

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**Abstract.** We study how two flows that vary spatially in two widely separated scales evolve under the dynamics of Euler's equations. We use a multiple scale approach which we motivate by studying some simpler model problems.

**1. Introduction.** The analysis of flows with rapidly varying structure in space and time, turbulent flows, is a very complex mathematical problem which cannot be approached by direct numerical solution of the Euler or Navier-Stokes equations. It is necessary to reduce somehow the analysis to solving equations that do not have rapidly varying data: averaged equations [1], [2]. The purpose of this paper is to use multiple scale methods to obtain averaged equations for a particular class of flows.

The flows that we analyze convect microstructure. By this we mean flows which at time $t=0$ have the form $u=U(x)+W(x/\varepsilon)$ where $\varepsilon$ is a small dimensionless parameter, the ratio of the small to the large scale. The field $U(x)$ represents the mean flow and $W(y)$ the fluctuations. We assume that $W(y)$ is a stationary (i.e. statistically homogeneous) random field with mean zero. We analyze the evolution of such flows up to times of order $\varepsilon^{-1}$ beyond which the separation of scales cannot be maintained.

In § 2 we discuss transport by a random field and methods for obtaining averaged equations. Some of the problems presented here are well-known. What is not perhaps well-known is that a complete mathematical analysis can be given. This makes precise the conditions under which the averaged equations are valid.

In fluid flow the transport of the velocity field is done by the field itself. Thus, analysis in the manner of § 2 does not carry over in an obvious way, if at all. In § 3 we present an approach to the problem of convection of microstructure by extending suitably the methods of § 2. The basic idea is fairly common in the study of transport phenomena. We imagine that the convecting field is momentarily a known random function, use the usual transport analysis and then update the convecting field suitably.

We attempt here to make these ideas more systematic by introducing the small parameter $\varepsilon$ and constructing an expansion in terms of it. The effective or averaged equations that we obtain in § 3 resemble somewhat the $k-\varepsilon$ model equations [3]. We have solved them numerically in some simple cases and the results are discussed in § 4. Averaged equations not of the $k-\varepsilon$ type have been introduced and used by Saffman [4] and by Saffman and Knight [5], [6], [7].

**2. Transport by a random field.**

**2.1. Turbulent diffusion.** This is a classic problem that has received a good deal of attention [1]. In its simplest form one is given a random velocity field $u(x)$ which is stationary (statistically homogeneous) and one considers the equation

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$$\begin{aligned}&\phi_t(t,x)+\nabla\cdot(u(x)\phi(t,x))=\alpha\,\Delta\phi(t,x),\\&t>0,\qquad x\in\mathbb{R}^3,\\&\phi(0,x)=\phi_0(x).\end{aligned}\tag{2.1}$$

This equation describes convection by $u$ and diffusion of a scalar quantity whose density is $\phi$. We want to derive an equation that describes the evolution of the mean density $\langle\phi(t,x)\rangle$. The bracket $\langle\ \rangle$ denotes ensemble average.

This is impossible in general because the equation for $\langle\phi\rangle$ involves $\langle u\phi\rangle$, that of $\langle u\phi\rangle$ involves $\langle u\phi\phi\rangle$, etc. To obtain a closed equation for $\langle\phi\rangle$, we must consider limiting cases of (2.1) in which simplifications are possible. Several cases can be analyzed depending on the relative sizes of the various effects incorporated in (2.1).

The most frequently studied case is when $u(x)$ is a small perturbation of a uniform field.

$$u(x)=v+\varepsilon\tilde{u}(x),\qquad \langle\tilde{u}(x)\rangle=0,\tag{2.2}$$

and $\alpha=0$ so that there is no molecular diffusion. It was shown in [8; see also references therein] that if $\tilde{u}(x)$ has rapidly decaying correlation functions (i.e. is mixing), then the mean density in relative coordinates after a long time

$$\left\langle\phi\left(\frac{t}{\varepsilon^2},\zeta-\frac{vt}{\varepsilon^2}\right)\right\rangle\tag{2.3}$$

tends to $\bar{\phi}(t,\zeta)$ as $\varepsilon\to0$. The limit density $\bar{\phi}(t,x)$ satisfies the diffusion equation

$$\frac{\partial\bar{\phi}}{\partial t}=\sum_{i,j=1}^{3}a_{ij}(v)\frac{\partial^2\bar{\phi}}{\partial\zeta_i\,\partial\zeta_j},\qquad \bar{\phi}(0,\zeta)=\phi_0(\zeta).\tag{2.4}$$

The diffusion tensor $a_{ij}(v)$ is given by the Kubo formula

$$a_{ij}(v)=\int_0^\infty R_{ij}(vs)\,ds,\tag{2.5}$$

$$R_{ij}(x)=\langle\tilde{u}_i(x+y)\tilde{u}_j(y)\rangle.\tag{2.6}$$

Note that the diffusion tensor is spatially homogeneous. This is a consequence of the statistical homogeneity of $u(x)$ (stationarity).

The conclusion (2.3)-(2.4) can also be stated as follows. The mean density $\langle\phi(t,x)\rangle$ behaves like $\psi(t,x)$ when $\varepsilon$ is small and $\psi(t,x)$ satisfies

$$\begin{aligned}&\psi_t+v\cdot\nabla\psi=\varepsilon^2\nabla\cdot(a\nabla\psi),\\&\psi(0,x)=\phi_0(x).\end{aligned}\tag{2.7}$$

The dependence of the diffusion tensor on the mean drift velocity $v$ is interesting. As $|v|\to0$ it behaves like a constant tensor times $|v|^{-1}$. The limit $|v|\to0$ is therefore singular and perhaps incompatible with diffusive behavior. In [9] Kraichnan presents numerical experiments that show diffusive behavior even when $v=0$ (with $\alpha=0$ as above) provided we are in 3 and not in 2 dimensions. Perturbation calculations in [10] seem to support this observation.

**2.2. Another model of turbulent diffusion.** The result described in the previous section indicates how a closed equation can be obtained for a mean field quantity by perturbation methods when we have a small parameter.

We shall now describe a closure method using a small parameter that is different from the one of § 2.1. It is closely related to the method we use in § 3 but it is not as familiar as the one of § 2.1.

Consider equation (2.1) for $\phi(t, x)$ and assume that the velocity field $u(x)$ satisfies

$$\begin{aligned} &\langle u(x)\rangle = 0, \qquad u(x) \text{ is stationary,} \\ &\nabla \cdot u(x) = 0. \end{aligned} \tag{2.8}$$

We shall moreover assume that we are in three or more dimensions and that the correlation functions

$$R_{ij}(z) = \langle u_i(x+z)u_j(x)\rangle \tag{2.9}$$

have a Fourier transform that is continuous at the origin.

We want to analyze the behavior of $\phi(t, x)$ when $t$ is large and when the initial density $\phi_0$ is slowly varying so that

$$\phi(0, x) = \phi_0(\varepsilon x) \tag{2.10}$$

in (2.1). The slow variation of the initial density is relative to the variations in the velocity field $u$.

If we change space and time scales by letting $t \to t/\varepsilon^2$ and $x \to x/\varepsilon$, problem (2.1) is transformed into

$$\begin{aligned} &\phi_t^\varepsilon + \frac{1}{\varepsilon}\nabla \cdot \left( u\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon \right) = \alpha \Delta \phi^\varepsilon, \\ &\phi^\varepsilon(0, x) = \phi_0(x). \end{aligned} \tag{2.11}$$

We shall show that there is a constant $\alpha_T > 0$, which depends on $\alpha > 0$, such that if $\bar{\phi}(t, x)$ denotes the solution of

$$\begin{aligned} &\bar{\phi}_t = (\alpha + \alpha_T)\Delta\bar{\phi}, \\ &\bar{\phi}(0, x) = \phi_0(x), \end{aligned} \tag{2.12}$$

then $\phi^\varepsilon(t, x) \to \bar{\phi}(t, x)$ in mean square

$$\left\langle \int [\phi^\varepsilon(t, x) - \bar{\phi}(t, x)]^2 \, dx \right\rangle \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.13}$$

Thus, in particular, $\langle \phi^\varepsilon(t, x)\rangle$ behaves like $\bar{\phi}(t, x)$ when $\varepsilon$ is small and a closed equation is obtained asymptotically for the mean density. A basic difference between the present problem and the one of § 2.1 is that the fluctuations in the velocity field are not small here. But they are rapidly varying relative to the initial density $\phi_0$. Moreover, it is not necessary to make assumptions about decaying correlations for $u(x)$ (i.e. mixing) beyond what we state below (2.9). Finally it is important to have divergence free fields, $\nabla \cdot u = 0$. The result is false otherwise.

One more remark should be added before going into the derivation of (2.12). The parameter $\alpha_T$, the turbulent diffusion coefficient, is not given by a simple Kubo formula like (2.5). It will be seen below that it is not possible to compute it explicitly and that it is important that $\alpha$, the molecular diffusion constant, be positive. Is this realistic physically?

With no additional hypotheses on the statistical properties of $u(x)$ it is not possible to show that $\alpha_T$ even has a limit as $\alpha \to 0$. Under some general hypotheses one may be able to show that $\lim_{\alpha\to 0} \alpha_T$ exists. But it will not be known if this limit is positive.

Clearly the most interesting situation arises when the turbulent diffusion coefficient exists and is positive in the absence of any molecular diffusion. A reasonable conjecture is that this will be the case if $u(x)$ is sufficiently mixing (cf. for example [8]). No results of this type seem to be known. We refer again to Kraichnan's [9] numerical experiments that tend to support this conjecture. Velocity fields that are periodic have $\lim \alpha_T = 0$. So randomness is important.

We continue now with the derivation which follows by a multiple scale method ([11], [12] and references therein). We expand the solution of (2.11) in powers of $\varepsilon$ in the form

$$\phi^\varepsilon(t, x) = \bar{\phi}(t, x) + \varepsilon \phi^{(1)}\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon^2 \phi^{(2)}\left(t, x, \frac{x}{\varepsilon}\right) + \cdots. \tag{2.14}$$

We insert this in (2.11) and write $\nabla \phi^\varepsilon$ as $(\nabla_x + \varepsilon^{-1}\nabla_y)\phi$ when acting on functions of $x$ and $x/\varepsilon$. Then we collect coefficients of powers of $\varepsilon$ and obtain the following sequence of problems. The coefficients of $\varepsilon^{-2}$ vanish because the first term on the right of (2.14) does not depend on $x/\varepsilon$. We anticipated this in the ansatz (2.14). The coefficients of $\varepsilon^{-1}$ lead to the equation

$$\alpha \Delta_y \phi^{(1)} - u \cdot \nabla_y \phi^{(1)} - u \cdot \nabla_x \bar{\phi} = 0. \tag{2.15}$$

Let $e_k$, $k = 1, 2, 3$ be the unit vectors in the coordinate directions and let $\chi^k(y)$ satisfy

$$\alpha \Delta_y \chi^k - u(y) \cdot \nabla_y \chi^k - u(y) \cdot e_k = 0. \tag{2.16}$$

If we now put

$$\phi^{(1)}(t, x, y) = \sum_{k=1}^{3} \chi^k(y) \frac{\partial \bar{\phi}(t, x)}{\partial x_k}, \tag{2.17}$$

then (2.15) is satisfied and $\phi^{(1)}$ is determined up to a function of $t$ and $x$ only.

Assuming that (2.16) determines the functions $\chi^k(y)$ and that they are stationary random functions, we continue with the expansion. We return to this assumption later. The coefficients of $\varepsilon^0$ give

$$\alpha \Delta_y \phi^{(2)} - u \cdot \nabla_y \phi^{(2)} - u \cdot \nabla_x \phi^{(1)} + 2\alpha \nabla_x \cdot \nabla_y \phi^{(1)} + \alpha \Delta_x \bar{\phi} - \bar{\phi}_t = 0. \tag{2.18}$$

If (2.18) is to have a solution $\phi^{(2)}(t, x; y)$ which is a stationary random process in $y$ (for each $t$ and $x$ fixed) then the mean value of the last four terms on the right of (2.18) must vanish. This will not imply the existence of a stationary solution but without it one cannot continue. Thus the solvability condition for (2.18) gives, as is usual in multiscale methods, the determining equation for $\bar{\phi}(t, x)$:

$$\bar{\phi}_t = \alpha \Delta_x \bar{\phi} - \langle u \cdot \nabla_x \phi^{(1)} \rangle. \tag{2.19}$$

We may rewrite (2.19) by using the form (2.17) of $\phi^{(1)}$. This gives the equation

$$\frac{\partial \bar{\phi}}{\partial t} = \sum_{i,j=1}^{3} (\alpha \delta_{ij} + \alpha_{Tij}) \frac{\partial^2 \bar{\phi}}{\partial x_i \, \partial x_j} \tag{2.20}$$

where

$$\alpha_{Tij} = -\langle u_i \chi^j \rangle. \tag{2.21}$$

From the defining equation (2.16) for $\chi^j$ and the fact that only the symmetric part of $\langle u_i \chi^j \rangle$ enters in (2.20), we deduce that

$$\alpha_{Tij} = \alpha \langle \nabla_x \chi^i \cdot \nabla_x \chi^j \rangle. \tag{2.22}$$

This shows immediately that $\alpha_T$ is a positive tensor. In (2.12) we assumed tacitly that $\alpha_T$ is a scalar to simplify the statement. In general it will, of course, be a tensor. From (2.22) one also sees, at least formally, the role that $\alpha$ plays. This is even more so in (2.16).

The main part of the analysis is thus the study of (2.16) from which the diffusion tensor emerges. Actually it does not have a stationary solution under the general hypotheses we have here [12]. However, solutions $\chi^k(y)$ exist whose gradients $\nabla\chi^k$ are stationary so $\alpha_{Tij}$ from (2.22) is well defined. The technical details for the complete argument are essentially identical to those in [12].

**2.3. Eddy viscosity for a model Navier–Stokes equation.** In the previous section we saw that closure and the definition of effective diffusivity can be accomplished without assuming that the turbulent velocity field has small fluctuations. The separation of scales between initial data and fluctuating coefficients is enough. This is precisely the approach we will follow in § 3 with the (nonlinear) Euler equations. We will not go as far as getting eddy diffusivities there. This seems to be quite complicated. But we will get the structure of the averaged equations.

It is interesting to note that if we simplify somewhat the real problem of the next section (with viscosity added) we end up with the model equations

$$
\begin{aligned}
&v_t+\left(\frac{1}{\varepsilon}u\left(\frac{x}{\varepsilon}\right)+v\right)\cdot\nabla v+\nabla p=\alpha\,\Delta v,\\
&\nabla\cdot v=0, \qquad (2.23)\\
&v(0,x)=v_0(x).
\end{aligned}
$$

We may think of the vector field $v(t,x)$ as a relative velocity and $p(t,x)$ as the pressure. The random field $u(x/\varepsilon)$ is assumed known.

Applying the method of § 2.2, step by step, we can obtain rigorously a closed equation for $\langle v(t,x)\rangle$ in the limit $\varepsilon\to 0$. If we denote by $\bar{v}$ and $\bar{p}$ the limiting mean velocity and pressure fields, we find that they satisfy the equation

$$
\begin{aligned}
&\bar{v}_t+\bar{v}\cdot\nabla\bar{v}+\nabla\bar{p}=(\alpha+\alpha_T)\,\Delta\bar{v}, \qquad \nabla\cdot\bar{v}=0,\\
&\bar{v}(0,x)=v_0(x).
\end{aligned}
\tag{2.24}
$$

In fact, the eddy diffusivity $\alpha_T$ is obtained exactly as in § 2.2. More details can be found in [11].

From the above examples and remarks we conclude that in many problems where randomness enters in a nonlinear way, as a coefficient in a linear equation or as data in a nonlinear equation, a detailed asymptotic analysis and "closure" of the problem can be given. A good way to do that is multiple scale analysis.

**3. Euler equations with rapidly varying initial data.**

**3.1. Statement of the problem.** We shall consider Euler's equations for three-dimensional ideal flows

$$u_t+(u\cdot\nabla)u+\nabla p=0, \tag{3.1}$$

$$\nabla\cdot u=0 \tag{3.2}$$

where $u(x,t)$ and $p(x,t)$ are the velocity and pressure field respectively. We are interested in the time evolution of flows which initially have the form

$$u(x,0)=U(x)+W\left(\frac{x}{\varepsilon},x\right). \tag{3.3}$$

Here $U(x)$ is a given smooth velocity field in $\mathbb{R}^3$ and $W(y, x)$ is a smooth velocity field that is a periodic or stationary random function of $y$. We think of $U$ as the mean velocity field initially and so we require that $W$ have mean zero

$$\langle W \rangle = 0. \tag{3.4}$$

The angular brackets denote integration with respect to $y$ over a period cell, or ensemble average in the stationary random case. The parameter $\varepsilon > 0$ is the dimensionless ratio of characteristic length scales associated with the fluctuating field $W$ and the mean field $U$. We are interested in the behavior of the solution $u^\varepsilon(x, t)$ of (3.1)-(3.3) for $t > 0$ when $\varepsilon$ is small.

It is not known if (3.1), (3.2) has a solution in some sense for arbitrary data of the form (3.3). For our purposes in this section, one may consider solutions of (3.1), (3.2) when a small viscous term is added. In fact adding to (3.1) a term like $-\varepsilon^\alpha \Delta u$, $\alpha > 2$ does not change the result we shall derive but just complicates the analysis.

Thus, from the physical point of view, this problem is also the study of the effect of a small scale turbulence on the mean flow but, unlike the one studied in § 2, here we have frozen the "turbulence" at one instant of time and wish to study its effect on the mean flow at later times. We will obtain effective equations for the evolution of $u^\varepsilon$ by an asymptotic method.

We look for $u^\varepsilon$, $p^\varepsilon$ in the form

$$\begin{aligned} u^\varepsilon(x, t) &= u(x, t) + w\left(\frac{\theta(x, t)}{\varepsilon}, \frac{t}{\varepsilon}, x, t\right) \\ &\quad + \varepsilon u^{(1)}\left(\frac{\theta(x, t)}{\varepsilon}, \frac{t}{\varepsilon}, x, t\right) + O(\varepsilon^2), \end{aligned} \tag{3.5}$$

$$\begin{aligned} p^\varepsilon(x, t) &= p(x, t) + \pi\left(\frac{\theta(x, t)}{\varepsilon}, \frac{t}{\varepsilon}, x, t\right) \\ &\quad + \varepsilon p^{(1)}\left(\frac{\theta(x, t)}{\varepsilon}, \frac{t}{\varepsilon}, x, t\right) + O(\varepsilon^2) \end{aligned} \tag{3.6}$$

where $\theta(x, t)$ is a Lagrangian coordinate, i.e. the position at time $t$ of the particle transported by the mean flow $u$ from position $x$ at time 0.

In (3.5)-(3.6), $w(y, \tau, x, t)$, $u^{(1)}(y, \tau, x, t)$, $\pi(y, \tau, x, t)$, $p^{(1)}(y, \tau, x, t)$ are periodic or stationary random functions of $y$ and $\tau$ for each $\{x, t\}$ and to distinguish $u$ from $w$ we impose zero $y - \tau$ mean for $w$:

$$\langle w(\cdot, \cdot, x, t) \rangle = 0. \tag{3.7}$$

There are several good reasons for taking such a complicated ansatz. The main one is physical. Indeed the first effect of $U$ on $W$ that one expects is convection: $W$ will be transported by the mean flow. But we shall also see that once (3.5), (3.6) are assumed, then the asymptotics impose

$$\theta_t + u \cdot \nabla \theta = 0, \qquad \theta(x, 0) = x \tag{3.8}$$

which is exactly the equation defining the Lagrangian coordinates.

**3.2. Derivation of the effective equations.** We shall use the following notation for derivatives of a function $v = v(y, x, t)$:

$$\frac{\partial v}{\partial t} = v_{,t}, \quad \frac{\partial v}{\partial x_i} = v_{,i}, \quad \frac{\partial v}{\partial y_i} = v_{,I}, \quad \frac{\partial v}{\partial \tau} = v_{,\tau}. \tag{3.9}$$

We shall use also the "del" operator $\nabla$ for derivatives with respect to $x$ and $\nabla_y$ for derivatives with respect to $y$. With this notation and the summation convention over repeated indices, even when one is a lower case and the other a capital index corresponding to the same letter, we write derivatives of (3.5) as follows.

$$u^{\varepsilon}_{i,t}=\varepsilon^{-1}(w_{i,\tau}+w_{i,K}\theta_{k,t})+u_{i,t}+w_{i,t}+u^{(1)}_{i,\tau}+u^{(1)}_{i,K}\,\theta_{k,t}+O(\varepsilon), \tag{3.10}$$

$$u^{\varepsilon}_{i,j}=\varepsilon^{-1}w_{i,K}\theta_{k,j}+u_{i,j}+w_{i,j}+u^{(1)}_{i,K}\,\theta_{k,j}+O(\varepsilon). \tag{3.11}$$

With similar expressions for derivatives of $p^{\varepsilon}(x,t)$ we insert (3.10) and (3.11) into (3.1) and (3.2) and equate to zero coefficients of powers of $\varepsilon$. This leads to the following sequence of equations.

$$\begin{aligned} &u_\tau+w_{i,K}(\theta_{k,t}+u_j\theta_{k,j})+w_j\theta_{k,j}w_{i,K}+\pi_{,K}\theta_{k,i}=0,\\ &w_{i,K}\theta_{k,i}=0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &u^{(1)}_\tau+u^{(1)}_{i,K}(\theta_{k,t}+u_j\theta_{k,j})+u^{(1)}_j w_{i,K}\theta_{k,j}+w_j u^{(1)}_{i,K}\,\theta_{k,j}\\ &\quad+p^{(1)}_{,K}\theta_{k,i}+(u_j+w_j)(u_{i,j}+w_{i,j})+u_{i,t}+w_{i,t}+p_{,i}+\pi_{,i}=0,\\ &u^{(1)}_{i,K}\,\theta_{k,i}+u_{i,i}+w_{i,i}=0. \end{aligned} \tag{3.13}$$

Let

$$\tilde u^{(1)}_k(y,\tau,x,t)=u^{(1)}_j(y,\tau,x,t)\theta_{k,j}(x,t)(\tilde u^{(1)}_k=\nabla\theta^T u^{(1)}), \tag{3.14}$$

$$\tilde w(y,\tau,x,t)=\theta_t+u\cdot\nabla\theta+\nabla\theta^T w \tag{3.15}$$

and

$$C_{ij}(x,t)=\theta_{i,l}(x,t)\theta_{i,l}(x,t)\qquad (C=\nabla\theta^T\,\nabla\theta); \tag{3.16}$$

then (3.12), (3.13) can be rewritten as

$$\tilde w_\tau+\tilde w\cdot\nabla_y\tilde w+C\nabla_y\pi=0,\qquad \nabla_y\cdot\tilde w=0, \tag{3.17}$$

$$\tilde u^{(1)}_\tau+\tilde w\cdot\nabla_y\tilde u^{(1)}+\tilde u^{(1)}\nabla_y\tilde w+C\nabla p^{(1)}=f,\qquad \nabla_y\cdot\tilde w=g \tag{3.18}$$

where

$$f=-\nabla\theta^T[u_t+w_t+(u+w)\cdot\nabla_x(u+w)+\nabla_x p+\nabla_x\pi], \tag{3.19}$$

$$g=-[\nabla_x\cdot u+\nabla_x\cdot w]. \tag{3.20}$$

Thus (3.17) will be used to define $\tilde w$ and (3.18) to define $\tilde u^{(1)}$; then $w$ and $u^{(1)}$ are computed from (3.14), (3.15). However two difficulties arise.

(i) (3.17) is not sufficient to define a unique periodic or stationary random $\tilde w$; for example $\mathrm{Re}[a+b\exp(ikx)]$ is a solution of (3.17) for all $a$, $b$, $k$ such that $a\cdot k=b\cdot k=0$.

(ii) (3.18) may not be sufficient to define a unique periodic or stationary $\tilde u^{(1)}$ but also $f$ and $g$ must satisfy certain compatibility conditions.

In $R^3$, we have identified three sets of compatibility conditions for $f$ and $g$:

$$\langle f_i+g\tilde w_i\rangle=0,\qquad \langle g\rangle=0, \tag{3.21}$$

$$\langle \tilde w_l C^{-1}_{li}(f_i+g\tilde w_i)\rangle+\langle\pi g\rangle=0, \tag{3.22}$$

$$\varepsilon_{ujk}\theta^{-1}_{i,l}\,\theta^{-1}_{j,m}\langle f_l\tilde w_{m,k}\rangle=0\qquad (\text{i.e. } \langle\nabla\theta^{-T}f\cdot(\nabla\theta\nabla_y)\times(\nabla\theta^{-T}\tilde w)\rangle=0). \tag{3.23}$$

These conditions are obtained by multiplying (3.18) respectively by 1, $C^{-1}w$ and $(\nabla\theta\nabla_y)\times(\nabla\theta^{-T}\tilde w)$ and averaging.

Now we proceed to implement (3.21)-(3.23) when $f$, $g$ are given by (3.19)-(3.20). The first set, (3.21), gives:

$$u_t + u \cdot \nabla u + \nabla \cdot \langle w \otimes w \rangle + \nabla p = 0, \qquad \nabla \cdot u = 0. \tag{3.24}$$

The second set, (3.22), gives:

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\left\langle \frac{|w|^2}{2} \right\rangle + \langle w \otimes w \rangle : \nabla u + \nabla \cdot \left\langle \left(\pi + \frac{1}{2}|w|^2\right) w \right\rangle = 0. \tag{3.25}$$

Finally (3.23) yields

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\langle w \cdot \tilde{\nabla} \times w \rangle + \langle w \otimes \tilde{\nabla} \times w \rangle : \nabla u \\ &+ \nabla_x \cdot \left\langle \left(\pi + \frac{1}{2}|w|^2\right) \nabla_y \times w + \frac{1}{2} w \times w_\tau \right\rangle = 0. \end{aligned} \tag{3.26}$$

We have used the following notation:

$$(a \otimes b)_{ij} = a_i b_j, \qquad A : B = A_{ij} B_{ij}, \quad (\tilde{\nabla})_i = (\nabla \theta \nabla_y)_i = \theta_{j,i} \frac{\partial}{\partial y_j}. \tag{3.27}$$

**3.3. Discussion of the effective equations.** From the previous calculations and in particular (3.25), (3.26), we note that two statistics of $w$ seem to play a special role: the mean kinetic energy:

$$q(x, t) = \tfrac{1}{2}\langle |w|^2 \rangle \tag{3.28}$$

and the mean helicity:

$$r(x, t) = \langle w \cdot \tilde{\nabla} \times w \rangle. \tag{3.29}$$

Thus, the question arises if they can be modulated independently. In other words can we define $\tilde{w}$ to be a periodic or stationary random solution of the Euler-like equation (3.17) with prescribed kinetic energy $q$, helicity $r$ and mean $a$? It seems that the answer is yes if $a = 0$ but we cannot prove it. So for the time being let us just assume that there exists $\tilde{w}(y, \tau, A, q, r)$ ($A$ is $\nabla \theta$) periodic or stationary random in $y - \tau$, solution of

$$\begin{aligned} &\tilde{w}_\tau + \tilde{w} \cdot \nabla_y \tilde{w} + A^T A \nabla_y \pi = 0, \qquad \nabla_y \cdot \tilde{w} = 0, \\ &\langle \tilde{w} \rangle = 0, \quad \tfrac{1}{2}\langle A^{-T}\tilde{w} \cdot A^{-T}\tilde{w} \rangle = q, \quad \langle A^{-T}\tilde{w} \cdot A\nabla_y \times A^{-T}\tilde{w} \rangle = r \end{aligned} \tag{3.30}$$

which depends continuously on the parameters $A$, $q$, $r$. If this is the case, then let us define two second order tensor functions

$$R(A, q, r) = A^{-1}\langle \tilde{w} \otimes \tilde{w} \rangle A^{-T}, \qquad S(A, q, r) = \langle A^{-T}\tilde{w} \otimes A\nabla_y \times A^T \tilde{w} \rangle, \tag{3.31}$$

one vector function

$$d(A, q, r) = \langle (\pi + \tfrac{1}{2}\tilde{w}^T A^{-1} A^{-T} \tilde{w} A^{-T} \tilde{w}) A^{-T} \tilde{w} \rangle \tag{3.32}$$

and one scalar function

$$e(A, q, r) = \langle (\pi + \tfrac{1}{2}(\tilde{w}^T A^{-1} A^{-T} \tilde{w}) A\nabla_y \times A^{-T}\tilde{w} + \tfrac{1}{2}\langle A^{-T} w \times A^{-T} \tilde{w}_\tau \rangle. \tag{3.33}$$

Then from (3.15), (3.24)-(3.26) we find that the mean flow $u$ is obtained by solving a set of equations where the kinetic energy $q$ and helicity $r$ of the oscillating part of the flow appear, along with the Lagrangian coordinate $\theta$:

$$\theta_t + u \cdot \nabla \theta = 0, \tag{3.34}$$

(3.35) $$u_t + u \cdot \nabla u + \nabla \cdot R(\nabla\theta, q, r) + \nabla p = 0, \qquad \nabla \cdot u = 0,$$

(3.36) $$q_t + u \cdot \nabla q + R(\nabla\theta, q, r) : \nabla u + \nabla \cdot d(\nabla\theta, r, q) = 0,$$

(3.37) $$r_t + u \cdot \nabla r + S(\nabla\theta, q, r) : \nabla u + \nabla \cdot e(\nabla\theta, r, q) = 0$$

and the initial conditions are

(3.38) $$\begin{aligned} \theta(x, 0) &= x, \\ u(x, 0) &= U(x), \\ q(x, 0) &= \tfrac{1}{2}\langle |w|^2\rangle, \\ r(x, 0) &= \langle w \cdot \nabla_y \times w\rangle. \end{aligned}$$

The analogy of this system with the $k-\varepsilon$ model for turbulence is striking, except that the rate of dissipated energy $\varepsilon$ has been replaced by the helicity $r$. In the $k-\varepsilon$ model [3], which is used extensively in engineering, $R$ is given by $-\nu_T(\nabla u + \nabla u^T)$ and the eddy viscosity $\nu_T$ is given by

(3.39) $$\nu_T = a\frac{k^2}{\varepsilon}$$

where $k$ and $\varepsilon$ satisfy

(3.40) $$k_{,t} + u \cdot \nabla k - a\frac{k^2}{\varepsilon}\nabla u : (\nabla u + \nabla u^t) + \varepsilon - b\nabla \cdot \left(\frac{k^2}{\varepsilon}\nabla k\right) = 0,$$

(3.41) $$\varepsilon_{,t} + u \cdot \nabla \varepsilon - ck \cdot \nabla u : (\nabla u + \nabla u^t) - \frac{\varepsilon^2}{k} - d\nabla \cdot \left(\frac{k^2}{\varepsilon}\nabla \varepsilon\right) = 0.$$

Here $a$, $b$, $c$, $d$ are constants that are adjusted to fit experimental data, $k$ is the turbulent kinetic energy, and $\varepsilon$ is the rate of turbulent viscous energy.

There is, however, one very important difference between (3.35)-(3.37) the $k-\varepsilon$ models: our model does not (so far) take into account viscous effects.

Indeed (3.35), (3.36) imply conservation of energy:

$$\int_{R^3} \left(\frac{1}{2}u^2 + q\right) dx = \text{constant in } t;$$

so $R$ is not a dissipative tensor, but represents a new kind of interaction which is zero if $w$ is homogeneous ($\nabla \cdot R$ is an exact gradient) and which seems to be more of a hyperbolic character as we shall see in § 4.

Viscous effects of $w$ on $U$ can be obtained from the higher order terms in the ansatz (the $u^{(2)}$ term) and will appear in the mean flow equation as an order $\varepsilon$ term; some preliminary calculations along these lines have been done in [18].

Naturally the analysis of this section is extremely difficult to justify mathematically. The degree of mathematical rigor achieved in the examples of § 2 seems out of reach here. We have merely carried over the formalism and that only to a limited extent because we have not calculated eddy viscosities.

**3.4. Some special cases.** Unless $R$, $S$; $d$, $c$ are known functions of $\nabla\theta$, $q$, $r$, the previous theory is of little practical interest. To compute these functions one has to solve (3.30) for all values of the parameters $A$, $q$, $r$. Since this is a formidable task, we shall look for cases for which (3.30) simplifies.

The first simplification is obtained if $W$ satisfies a stationary Euler equation:

$$w\,\nabla w+\nabla\pi=0,\qquad \nabla\cdot w=0 \tag{3.42}$$

because, in that case, $\tilde{w}$ may be independent of $\tau$. The second simplification is to look for cases where $r$ is identically zero. One way to guarantee this is to assume that $\tilde{w}$ is an odd function of $y$. Then (3.30) reduces to

$$\begin{aligned}&\tilde{w}\,\nabla_y\tilde{w}+C\nabla_y\pi=0,\qquad \nabla_y\cdot\tilde{w}=0,\\&\tilde{w}\ y\text{-periodic and odd},\qquad \tfrac{1}{2}\langle\tilde{w}C^{-1}\tilde{w}\rangle=q,\end{aligned} \tag{3.43}$$

whose general solution is of the form $\sqrt{q}\,(x,t)\tilde{w}(y, C(x,t))$.

However even if $r$ is zero at $t=0$ it will not stay so unless $e$ is small (see (3.37), $S=0$ because $w$ is odd in $y$). Now we notice that $e$ is of degree 3 in $\tilde{w}$ while $\pi$ is only of degree 2; therefore, roughly, $e$ will be small if $\tilde{w}$ (or $\sqrt{q}$) is small. This argument can be formalized by studying the problem

$$\begin{aligned}&u_t^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon+\nabla p^\varepsilon=0,\qquad \nabla\cdot u^\varepsilon=0\\&u^\varepsilon(x,0)=U(x)+\varepsilon^{1/3}\sqrt{q^0}\,(x)\,W\left(\frac{x}{\varepsilon}\right)\end{aligned} \tag{3.44}$$

where $W(y)$ is periodic and odd in $y$, has unit kinetic energy and satisfies the stationary Euler equation.

The appropriate ansatz for this problem is

$$u^\varepsilon(x,t)=u(x,t)+\varepsilon^{1/3}w(y,x,t)+\varepsilon^{2/3}u^{(1)}(y,x,t)+\cdots|_{y=\theta(x,t)/\varepsilon}, \tag{3.45}$$

$$p^\varepsilon(x,t)=p(x,t)+\varepsilon^{2/3}\pi(y,x,t)+\varepsilon p^{(1)}(y,x,t)+\cdots|_{y=\theta(x,t)/\varepsilon}. \tag{3.46}$$

We insert this ansatz into (3.44). A lengthy calculation follows which is similar to the one of § 3.2. The resulting equations, analogous to (3.34)-(3.37) are as follows.

We first construct the field $\tilde{w}'(y,c)$ solution of (3.43) with $q=1$ and define

$$R'(\nabla\theta)=-\nabla\theta^{-1}\langle\tilde{w}(\cdot, C)\rangle\,\nabla\theta^{-T}. \tag{3.47}$$

Then find $\theta$, $u$, $q$ as solutions of the coupled system of equations

$$\theta_t+u\cdot\nabla\theta=0, \tag{3.48}$$

$$u_t+u\cdot\nabla u+\nabla p=\varepsilon^{2/3}\nabla\cdot(q\underline{\underline{R}}'(\nabla\theta)), \tag{3.49}$$

$$q_t+u\cdot\nabla q=R'(\nabla\theta):\nabla u \tag{3.50}$$

with initial conditions

$$\theta(x,0)=x,\quad u(x,0)=U(x),\quad q(x,0)=q^0(x). \tag{3.51}$$

Naturally now the interaction tensor $R$ enters in the mean flow equation as the first order correction term dependent on $\varepsilon$ because the microstructure has energy $\varepsilon^{2/3}q^0$ at $t=0$. More details on this calculation can be found in [18].

**3.5. Summary of results.** We collect here the system of equations that governs the evolution (3.1)-(3.3) via the expansion (3.5)-(3.6). We restrict ourselves to the simplified case that we study further in § 4.

First we solve the *canonical fluctuation problem*

$$\tilde{w}\cdot\nabla_y\tilde{w}+C\nabla_y\tilde{\pi}=0,\qquad \nabla_y\cdot\tilde{w}=0, \tag{3.52}$$

where $C$ is symmetric positive definite matrix near the identity and

$$\frac{1}{2}\langle \tilde{w} C^{-1} \tilde{w} \rangle = 1, \qquad \langle \tilde{w} \rangle = 0, \quad \langle \tilde{\pi} \rangle = 0. \tag{3.53}$$

The initial function $W(y)$ in (3.3) coincides with $\tilde{w}(y; I)$. Once $\tilde{w}(y; C)$ and $\tilde{\pi}$ have been suitably constructed, for example numerically, we define

$$T_{ij}(C) = \langle \tilde{w}_i \tilde{w}_j \rangle. \tag{3.54}$$

Next we have the following coupled system of equations to solve for $u$, $p$, $\theta$ and $q$.

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nabla \cdot \tau, \qquad \nabla \cdot u = 0, \\ &\theta_t + u \cdot \nabla \theta = 0, \\ &q_t + u \cdot \nabla q = \nabla u : \tau \end{aligned} \tag{3.55}$$

together with the initial conditions

$$\begin{aligned} u(x, 0) &= U(x), \\ \theta(x, 0) &= x, \\ q(x, 0) &= q_0(x). \end{aligned} \tag{3.56}$$

Here the stress tensor $\tau$ is a function of $C = \nabla\theta^T \nabla\theta$ and it is computed from the solution $\tilde{w}(y; C)$ of the canonical microstructure problem by the formula

$$\tau = -q \nabla\theta^{-T} T(\nabla\theta^T \nabla\theta) \nabla\theta^{-T}. \tag{3.57}$$

The coupled microscopic-macroscopic system (3.52)-(3.53) and (3.55)-(3.56) is the main result of our analysis in this paper.

## 4. Simple analytical and numerical solutions of the effective equations.

**4.1. Simple macroscopic flows.** Let us assume in this section that a canonical microstructure flow $\tilde{w}(y, C)$, solution of (3.52), (3.53), is known. We will look for a solution of (3.55)-(3.57) corresponding to simple macroscopic flows.

As a first example, consider uniform macroscopic flow in one direction which may be thought of as the mean flow downstream from a turbulence-generating grid. We write

$$u(t, x_1, x_2, x_3) = (u_1, 0, 0)^T \tag{4.1}$$

with $u_1$ constant. Then $\theta$ from (2.46) becomes

$$\theta(t, x_1, x_2, x_3) = (x_1 - u_1 t, x_2, x_3)^T, \tag{4.2}$$

$$\nabla\theta(t, x_1, x_2, x_3) = \text{Identity matrix}, \qquad C = \text{Identity}. \tag{4.3}$$

Thus $w(y, C = I) = W(y)$ and if $W(y)$ is isotropic then (3.54), (3.57) give

$$\tau = -q 2I/3, \tag{4.4}$$

and hence from (3.56)

$$p = 2q/3 \tag{4.5}$$

and

$$q = q_0(x_1 - u_1 t, x_2, x_3). \tag{4.6}$$

Since neither Newtonian viscosity terms or eddy viscosity terms are included in the effective equations (3.55)-(3.56), we see that there is no decay in the mean kinetic turbulent energy $q$.

Similar remarks apply to uniformly rotating flow.

$$u=(-x_2\omega, x_1\omega, 0)^T. \tag{4.7}$$

Consider next flows of the form

$$u(t, x_1, x_2, x_3)=(u_1(t, x_2), 0, 0)^T \tag{4.8}$$

which are plane Poiseuille flows. The solution $\theta$ of (3.55) is

$$\theta(t, x)=\left(x_1-\int_0^t u_1(x_2, \tau)\, d\tau, x_2, x_3\right)^T, \tag{4.9}$$

$$\nabla\theta=\begin{pmatrix} 1 & 0 & 0 \\ -v' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.10}$$

$$C=\nabla\theta^T\nabla\theta=\begin{pmatrix} 1+(v')^2 & -v' & 0 \\ -v' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.11}$$

where $v$ and $v'$ are given by

$$v=\int_0^t u_1\, d\tau, \qquad v'=\frac{\partial}{\partial x_2} v. \tag{4.12}$$

From (3.52) it follows that $\langle\tilde{w}_i\tilde{w}_j\rangle$ is a function of $x_2$ and $t$ only. If $\langle\tilde{w}_2\tilde{w}_3\rangle$ is independent of $x_2$ and if we write

$$\tau_{12}=-qg(v') \tag{4.13}$$

then (3.55) becomes

$$v_{tt}+(qg)_{,2}=0, \qquad q_t+v''gq=0. \tag{4.14}$$

These combine to a single equation

$$v_{tt}+(q_0\, e^{-G(v')}G'(v'))'=0 \tag{4.15}$$

in which $G$ is a primitive of $g$ that is, $G'=g$.

Equation (4.15) is a nonlinear partial differential equation for $v(t, x_2)$. Let

$$F(\xi, \alpha)=q_0(\xi)\frac{d}{d\alpha}e^{-G(\alpha)}. \tag{4.16}$$

Then (4.15) takes the form

$$\begin{aligned} &v_{tt}-(F(\xi, v_\xi))_\xi=0, \\ &v(0, \xi)=0, \qquad v_t(0, \xi)=u_1(0, \xi). \end{aligned} \tag{4.17}$$

Equation (4.17) has the energy identity

$$\frac{\partial}{\partial t}\left[\int\frac{1}{2}v_t^2\, d\xi+\int q_0(\xi)\, e^{-G(v_\xi)}\, d\xi\right]=0. \tag{4.18}$$

The initial value problem for (4.17) will be well posed if for example the energy functional is convex, $e^{-G(\alpha)}$ tends to $+\infty$ as $|\alpha|\to\infty$. This will be the case if $G(\alpha)\to-\infty$

as $|\alpha| \to \infty$ and $G(\alpha)$ is concave. And this is in turn implied by $g'(\alpha) < 0$. Also, the null solution for (4.17) is linearly stable in this case.

When viscous damping is included, it tends to dampen the oscillation in time of solutions of (4.17). When solutions to (4.17) exist, they display oscillating behavior at each spatial point. We return to the discussion of (4.17) following the numerical investigation of microstructure flows.

**4.2. Computation of the canonical microstructure flow.** In order to analyze the macroscopic equations (3.55)-(3.57) we need to know $T_{ij}(C) = \langle \tilde{w}_i \tilde{w}_j \rangle$ which enters into the definition of $\tau$ in (3.57). Thus we must solve numerically in some regularized form the equations for $\tilde{w}$, (3.52), (3.53). We restrict attention to the case where $\tilde{w}$ is an odd function of $y$.

We will look for periodic solutions of (3.51) in the sense of least squares. That is we solve

$$\tilde{w} \cdot \nabla \tilde{w} + C \nabla \tilde{\pi} = 0, \qquad \nabla \cdot \tilde{w} = 0 \quad \text{in } Y = [-\pi, \pi]^3, \tag{4.19}$$

with the normalization

$$\tfrac{1}{2} \langle \tilde{w} C^{-1} \tilde{w} \rangle = 1, \qquad \langle \tilde{w} \rangle = 0, \quad \langle \tilde{\pi} \rangle = 0, \tag{4.20}$$

by minimizing

$$E(\tilde{w}) = \int_Y |\nabla(-\Delta)^{-1}(C^{-1} \nabla \cdot (\tilde{w} \otimes \tilde{w}) + \nabla \tilde{\pi}|^2 \, dy \tag{4.21}$$

with $\tilde{\pi}$ the periodic solution of

$$-\Delta \tilde{\pi} = \nabla \cdot (C^{-1} \tilde{w} \cdot \nabla \tilde{w}) \quad \text{in } Y. \tag{4.22}$$

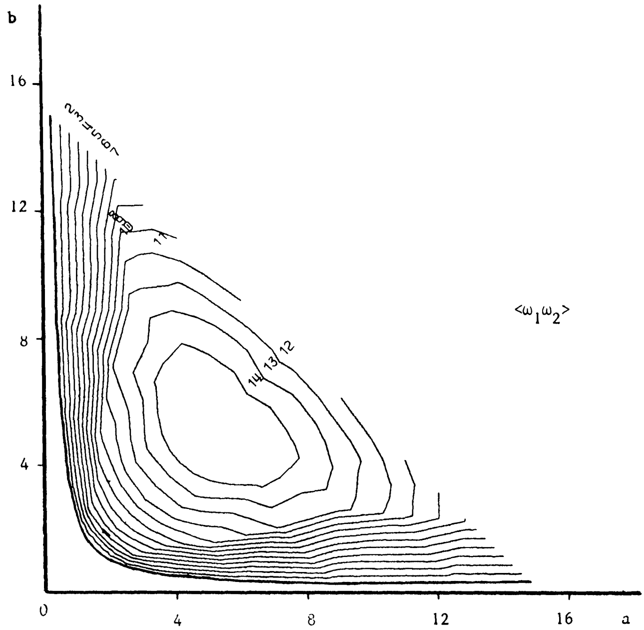
Note that the term in parentheses on the right side of (4.21) with $\tilde{\pi}$ substituted from (4.22) is the residual error for a trial solution of (4.19). We have chosen a weight for the residual error which is in effect the $H^{-1}(Y)$ norm of it. The absolute value denotes vector norm. This kind of regularization and numerical implementation for solutions of (4.19) has been tried successfully in other fluid dynamics problems [14]. This is our motivation for using it here.

The minimization (4.21) is carried out over periodic vector fields $\tilde{w}$ that belong to $L^4(Y)$, and satisfy the constraints (4.20). The numerical computation of the minimum is done by a finite element discretization followed by a conjugate gradient algorithm for finding the minimum. The numerical method is discussed in detail in [15]. We should point out that the minimum of (4.21) is not unique and depends on the initialization of the numerical algorithm.

We have tabulated $T_{ij}(C) = \langle \tilde{w}_i \tilde{w}_j \rangle$ for the numerically determined $\tilde{w}_i$ for matrices $C$ corresponding to two-dimensional flows. For such flows $C = \nabla \theta^T \nabla \theta$ and hence

$$C = \begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } c = \pm\sqrt{1 - ab}. \tag{4.23}$$

The form (4.23) follows from the 2-dimensional character of the flow $u$, the symmetry of $C$ and the fact that $\det C = \det (\nabla \theta)^2 = 1$ from (3.55) and $\nabla \cdot u = 0$. Typical results for $\langle \tilde{w}_i \tilde{w}_j \rangle$ as a function of $a$ and $b$ are shown in Fig. 1. In Fig. 2 a function $G(\alpha)$ in (4.15) is shown and it is obtained for $\langle \tilde{w}_1 \tilde{w}_2 \rangle$ in the special case (cf. (4.11) $a = 1 + \alpha^2$, $b = 1$. Thus from the analysis of § 4.1, Poiseuille flows are stable only if $\alpha \lesssim 0.6$.



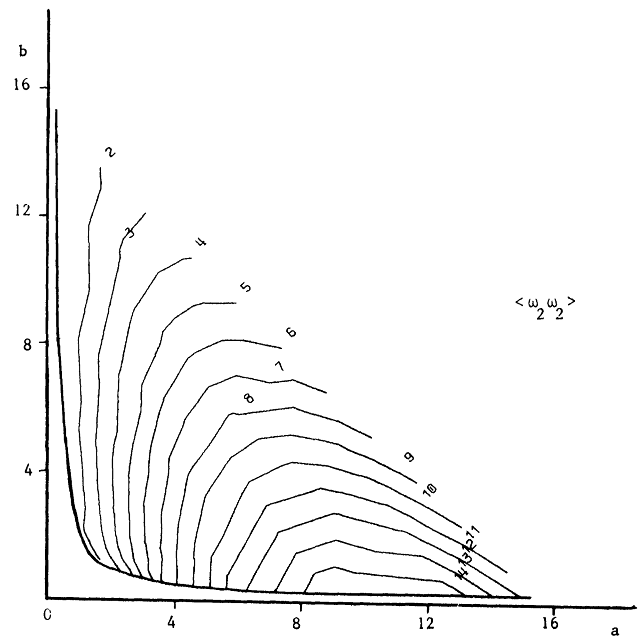




FIG. 1. *Results of the numerical tabulations of the tensor* $\langle \tilde{w} \otimes \tilde{w} \rangle$ *as a function of the parameters a and b of the matrix C corresponding to a two-dimensional mean flow. Other elements* $\langle \tilde{w}_2^2 \rangle$ *and* $\langle \tilde{w}_2 \tilde{w}_3 \rangle$ *can be obtained by symmetry. Computed by C. Begue* [15] *and T. Chacon* [19].

Consequently (cf. (4.12) if $u_1$ is not too far off a constant, the microstructure induces oscillations in the mean flow.

**4.3. Application to the computation of a two-dimensional jet.** We will apply (3.55)-(3.57) to a two-dimensional flow so that all functions depend on $x_1$, $x_2$ and $t$ only and

(4.24) $$u(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t), 0)^T.$$

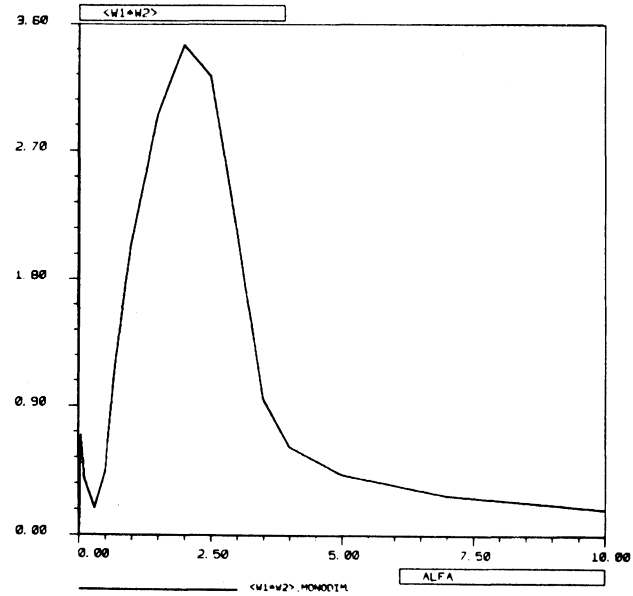



FIG. 2. *Plot of* $-g(x)$ *versus* $\alpha$. *We see that G is approximately concave for* $|\alpha|<2.5$, *except near the origin (this may be due to the numerics). This plot, like the previous one, was computed by minimization of* (6.21) *after discretization with a finite element grid* $17*17*17$ (*see* [19]).

The third equation for $u$ in (3.55) now has the form

$$p_{,3}=\tau_{3,1}+\tau_{3,2}. \tag{4.25}$$

In order that the flow stay two-dimensional, $p_{,3}$ must be zero or, $\tau_{3,1}+\tau_{3,2}$ must be zero. Since $\tau$ by (3.57) depends on $q$ and $\theta$, this will happen generally only if

$$\langle \tilde{w}_1\tilde{w}_3\rangle=\langle \tilde{w}_2\tilde{w}_3\rangle=0. \tag{4.26}$$

Although (4.26) does not hold for all the values of $a$ and $b$ for the $\tilde{w}_j$ that was computed numerically, the two terms in (4.26) are considerably smaller than the other components of $\langle \tilde{w}_i\tilde{w}_j\rangle$. We shall therefore assume that (4.26) holds.

Under these conditions we can write the remaining equations in (3.55) in terms of the stream function $\psi\, e_3$ and vorticity $\omega\, e_3$ as follows.

$$u=\nabla\times\psi\, e_3=(\psi_{,2},-\psi_{,1},0)^T, \tag{4.27}$$

$$-\Delta\psi=\omega \tag{4.28}$$

and

$$\omega_t+u\cdot\nabla\omega=\tau_{21,11}+\tau_{12,21}-\tau_{11,12}-\tau_{12,22}. \tag{4.29}$$

For the numerical computation we have taken initial and boundary conditions as follows:

$$\omega(x,0)=\omega_0(x), \tag{4.30}$$

$$\psi(0,x_2,t)=\psi(L,x_2,t),\qquad \psi(x_1,0,t)=0, \tag{4.31}$$

$$\omega(x_1,1,t)=\psi_1,\qquad \psi_1 \text{ a positive constant}. \tag{4.32}$$

We also assume that $\omega(x_1,x_2,t)$ is periodic in $x_1$ of period $L$. Boundary conditions for $\omega$ on $x_2=0$ and $x_2=1$ are not needed because from (4.32) these boundaries are streamlines and so the normal component of the velocity $u$ vanishes there. System (4.27)-(4.32) is closed by including the equation for $\theta$ and $q$ from (3.55) and evaluating the tensor $\tau$ from (3.57) and the table of values for $\langle \tilde{w}_i \tilde{w}_j \rangle$ (Fig. 2).

The above problem describes a flow in the direction $x_1$, periodically extended with period $L$. The difference in the values of $\psi$ at the boundary $x_2=0$, $x_2=1$ fixes the flux rate. If we take for $\omega_0(x_1, x_2)$ the function

$$\omega_0(x_1, x_2) = 3\psi_1(\delta(x_2-\tfrac{1}{3}) - \delta(x_2-\tfrac{2}{3})), \qquad 0 \leqq x_1 \leqq L, 0 \leqq x_2 \leqq 1 \tag{4.33}$$

this choice corresponds to an initial velocity $u(0, x_1, x_2)$ that is piecewise constant in $x_2$ and uniform in $x_1$.

$$u(0, x_1, x_2) = \begin{cases} 0, & 0 \leqq x_2 < \tfrac{1}{3}, \\ 3\psi_1, & \tfrac{1}{3} \leqq x_2 < \tfrac{2}{3}, \quad 0 \leqq x_1 \leqq L, \\ 0, & \tfrac{2}{3} \leqq x_2 < 1. \end{cases}$$

The initial value for $q$ is zero when $u(0, x) = 0$ and a constant $q_0 > 0$ otherwise.

Problem (4.27)-(4.32) with (3.55) and (3.56) and the initial function (4.33) for $\omega_0$ was solved numerically. Equation (4.28) is solved numerically by the standard 5-point finite difference scheme and the resulting linear system by a relaxation method. Equation (4.29) is decomposed into two equations, since it is linear, one with zero right-hand side and one with zero initial conditions. The one with the initial conditions is solved by transport of discrete vortices. For the other one, as well as for the equations (3.55) for $\theta$ and $q$ we use an explicit finite difference scheme whereby

$$\phi_t + u \cdot \nabla \phi = f$$

is approximated by

$$\phi_{ij}^{n+1} = \pi \phi^n(x_{ij} - u_{ij}^n \Delta t) + f_{ij}^n \Delta t$$

where $x_{ij} = (i \Delta x_1, j \Delta x_2)$, $u_{ij}^n \sim u(x_{ij}, n \Delta t)$ etc. and $\pi$ is a bilinear interpolation operator between the four nearest grid points in space. Note that equation (3.55) for $q$ is linear for $\log q$, since $\tau$ is proportional to $q$. The derivatives on the right side of (4.29) are replaced by symmetric second order differences and $u$ in (4.27) and $\nabla \theta$ in (3.55) are calculated by centered finite differences. Finally the transported $\delta$ functions that appear

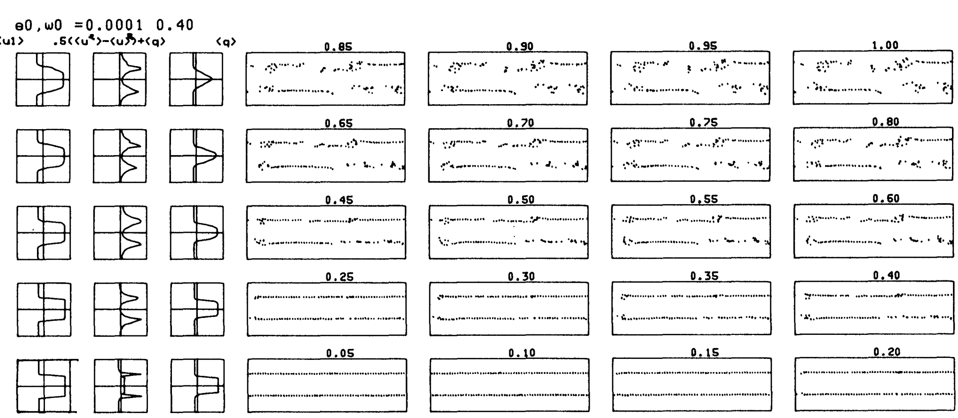



FIG. 3. *Jet flow with very little microstructures in the jet. The first column of square plots shows, as a function of $x_2$ (vertical), the horizontal velocity $u_1$ averaged in the direction $x_1$, in the flow shown on the nearest rectangular plot to the right. The second column of square plots shows the same $x_1$-average of the turbulent kinetic energy. The last column of squares shows the $x_1$-average of the kinetic energy of the microstructure. The rectangular boxes show the evolution of the point vortices at different times.*

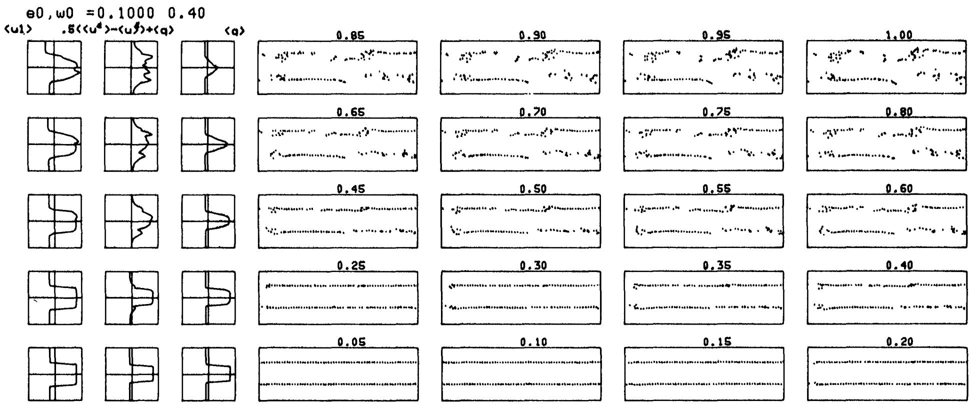



FIG. 4. *Same as in Fig. 3 but with a strong microstructure at time zero uniformly distributed in the middle of the jet. Its influence on the mean flow is hard to see on the distribution of vortices but it is clear on the $x_1$-averaged turbulent kinetic energy which remains flat in the jet even $\langle g\rangle$ decreases.*

on the right side of (4.28) are replaced by masses at the four nearest grid points proportional to the inverse of the distance from the lines connecting the grid points [16].

The results are shown in Figs. 3 and 4. Figure 3 is the solution of our problem with $q_0$ very small $(=10^{-4})$, i.e. with initially a very small amount of microstructure in the flow. Figure 4 shows the results with $q_0=0.1$. The differences are not significant as far as the distribution of the point vortices is concerned. However the velocity profile broadens faster in the second case and the total turbulent energy which is

$$\frac{1}{2}\left[\frac{1}{2}\int_0^L (u^2)\,dx_1-\left(\frac{1}{2}\int_0^L u\,dx_1\right)^2\right]+q$$

is quite different as a function of $x_2$ in the two cases. In the second case the turbulent energy is larger and is spread more evenly in the jet region between the vortices. We also note that $q$ decreases with time. These results seem to be in reasonable qualitative agreement with observations.

More numerical tests are in progress (see [18]) where the hyperbolic character of the coupling between $u$ and $q$ is more visible.

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