# Transport equations for the Stokes parameters from Maxwell's equations in a random medium*

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Beginning with Maxwell's equations in a random medium and following a perturbation procedure, we obtain transport equations for the Stokes parameters. We compare our equation with Chandrasekhar's transport equation and find that they agree when the random medium is specialized appropriately. We also examine the role of degeneracy in the perturbation analysis.

## 1. INTRODUCTION

Our objective is to analyze the connection between radiative transport theory and the asymptotics of stochastic wave equations. One expects, from physical considerations, in the asymptotic limit of weakly inhomogeneous media with propagation over several correlation lengths and wavelengths small compared with the correlation length, that the appropriate way to describe the transport of field energy is through transport theory. Beginning with Maxwell's equations in a random medium and by following a perturbation procedure for stochastic equations, we obtain transport equations for wave amplitudes (the Stokes parameters or the coherence matrix) in the above mentioned asymptotic limit. By specializing the random medium appropriately we recover Chandrasekhar's transport equations[1] for the Stokes parameters.

In Sec. 2 we discuss the role of degeneracy in the perturbation analysis of linear stochastic equations, since Maxwell's equations are degenerate, and review the basic elements of the perturbation analysis (cf. Ref. 2 and the references cited therein). We give two examples, other than Maxwell's equations, that illustrate the role of degeneracy in determining the form of the ensuing transport equation.

In Sec. 3 we begin with the analysis of Maxwell's equations by transforming them to a form appropriate for the perturbation analysis. In Sec. 4 we apply formally the perturbation procedure of Sec. 2 and arrive at transport equations for the coherence matrix of the wave amplitudes which is simply related to the Stokes parameters. In Sec. 5 we specialize the transport equations appropriately and recover Chandrasekhar's equation. It is interesting to note that for more general random media additional terms appear in the transport equations which are not present in Chandrasekhar's equations.

The derivation of transport equations from stochastic wave equations has received considerable attention (cf. Refs. 3–12 and the references cited therein). In particular, Bourret[8,9] considers problems associated with polarization as we do here in Secs. 3–5. Aside from the formal character of our results, we also do not obtain transport equations in their most general form because we deal with statistically homogeneous random media. The extension of the present analysis to locally statistically homogeneous random media, which lead to general transport equations, requires additional considerations.

## 2. DEGENERATE AND NONDEGENERATE MATRIX PROBLEMS

We shall analyze a simple class of problems in order to illustrate the methods we shall employ and the form of the results that arise. We shall pay particular attention to the role played by degeneracy in the perturbation analysis.

Let $v(t)$ be a complex-valued $n$-dimensional vector function of time satisfying the following system of linear stochastic equations:

$$\frac{dv_p(t)}{dt} = ik_p v_p(t) + \epsilon \sum_{q=1}^{n} \mu_{pq}(t) v_q(t) + \epsilon^2 \sum_{q=1}^{n} \nu_{pq}(t) v_q(t), \quad t > 0,$$

$$v_p(0) = u_{p,0}, \quad p = 1, 2, \ldots, n. \tag{2.1}$$

The coefficients $\mu_{pq}(t)$ and $\nu_{pq}(t)$, $p, q = 1, 2, \ldots, n$, are complex valued stationary processes. We assume that their means are given by

$$E\{\mu_{pq}(t)\} = 0, \quad E\{\nu_{pq}(t)\} = \bar{\nu}_{pq}, \tag{2.2}$$

where $\bar{\nu}_{pq}$, $p, q = 1, 2, \ldots, n$, are constants, and their covariances are defined as follows:

$$\begin{aligned} &E\{\mu_{pq}(t+s)\mu_{p'q'}(s)\} = R^{\cdot,\cdot}_{pq,p'q'}(t), \\ &E\{\mu^*_{pq}(t+s)\mu_{p'q'}(s)\} = R^{*,\cdot}_{pq,p'q'}(t), \\ &E\{\mu_{pq}(t+s)\mu^*_{p'q'}(s)\} = R^{\cdot,*}_{pq,p'q'}(t), \\ &E\{\mu^*_{pq}(t+s)\mu^*_{p'q'}(s)\} = R^{*,*}_{pq,p'q'}(t), \\ &\qquad p, q, p', q' = 1, 2, \ldots, n. \end{aligned} \tag{2.2'}$$

Here $E\{\cdot\}$ denotes mathematical expectation and * denotes complex conjugate. Note that the correlation functions satisfy

$$R^{*,\cdot}_{pq,p'q'}(-t) = R^{\cdot,*}_{p'q',pq}(t), \quad \text{etc.} \tag{2.3}$$

The correlation functions of $\nu_{pq}(t)$ will not be used in the asymptotic analysis and so they are not introduced.

The numbers $k_p$, $p = 1, 2, \ldots, n$, are real. When they are distinct along with their sums and differences we shall call (2.1) a nondegenerate problem. Otherwise we call it degenerate. The dimensionless parameter $\epsilon > 0$ is a measure of the size of the fluctuations. We assume that the noise intensities (cross power spectra at zero frequency)

$$\int_{-\infty}^{\infty} R^{\cdot,\cdot}_{pq,p'q'}(s)\, ds, \quad \text{etc.}, \quad p, q, p'q' = 1, 2, \ldots, n, \tag{2.4}$$

are finite and the integrals converge absolutely. These integrals have the dimension of time and can be thought of, alternatively, as giving a measure of the correlation time (or length) of the fluctuations. The coefficients $\nu_{pq}(t)$ are assumed to have similar properties but, because they enter as $O(\epsilon^2)$ terms in (2.1), they play a less important role in the asymptotic analysis.

Equations (2.1) arise frequently in connection with eigenfunction expansions for random wave propagation problems[10] where $t$ plays the role of the spatial variable in the direction of propagation of the free wave. The initial value problem (2.1) corresponds to the forward scattering or parabolic approximation. The same equations (2.1) arise in quantum mechanics when the Hamiltonian has a random time-dependent perturbation [cf. Eq. (2.20) and Eq. (2.37) also].

We are interested in the behavior of the statistics of the solution $v_p(t)$, $p=1,\ldots,n$ of (2.1) when $\epsilon$ is small. Stochastic effects become significant at times of order $1/\epsilon^2$, i.e., after several correlation times. We shall explain briefly why this is so.

Let

$$\tau=\epsilon^2 t \tag{2.5}$$

and set

$$v^\epsilon(\tau)=v(\tau/\epsilon^2). \tag{2.6}$$

From (2.1) we obtain the following equation for $v^\epsilon(\tau)$.

$$\frac{dv_p^\epsilon(\tau)}{d\tau}=\frac{ik_p}{\epsilon^2}\,v_p^\epsilon(\tau)+\sum_{q=1}^n\frac{1}{\epsilon}\,\mu_{pq}\!\left(\frac{\tau}{\epsilon^2}\right)v_q^\epsilon(\tau)+\sum_{q=1}^n\nu_{pq}\!\left(\frac{\tau}{\epsilon^2}\right)v_q^\epsilon(\tau),$$

$$v_p^\epsilon(0)=u_{p,0},\quad p=1,2,\ldots,n. \tag{2.7}$$

Assume temporarily that the processes $\mu_{pq}(\tau)$ are real and let

$$\mu_{pq}^\epsilon(\tau)=\frac{1}{\epsilon}\,\mu_{pq}\!\left(\frac{\tau}{\epsilon^2}\right). \tag{2.8}$$

We have

$$\begin{aligned}R^\epsilon_{pq,p'q'}(\tau)&=E\{\mu^\epsilon_{pq}(\tau+\sigma)\mu^\epsilon_{p'q'}(\sigma)\}\\&=\frac{1}{\epsilon^2}E\left\{\mu_{pq}\!\left(\frac{\tau+\sigma}{\epsilon^2}\right)\,\mu_{p'q'}\!\left(\frac{\sigma}{\epsilon^2}\right)\right\}\\&=\frac{1}{\epsilon^2}R_{pq,p'q'}\!\left(\frac{\tau}{\epsilon^2}\right).\end{aligned} \tag{2.9}$$

Thus,

$$\lim_{\epsilon\downarrow 0}R^\epsilon_{pq,p'q'}(\tau)=\delta(\tau)\int_{-\infty}^{\infty}R_{pq,p'q'}(s)\,ds, \tag{2.10}$$

which means, roughly, that $\mu^\epsilon_{pq}(\tau)$ tends to white noise with noise intensity the coefficient of the delta function in (2.10). The scaling (2.5) is the only one that will do this or, in other words, we have sought the white noise limit for (2.1) because in this limit stochastic effects become most prominent.

We now return to the systematic analysis of (2.7). When $k_p=0$, $p=1,2,\ldots,n$, we have a fully degenerate problem and the limit as $\epsilon\downarrow 0$ in (2.7) is referred to as the white noise or the diffusion limit. In general, the term $ik_pv_p^\epsilon/\epsilon^2$ in (2.7) must be removed before taking the limit $\epsilon\downarrow 0$. Therefore we pass to the interaction representation and define

$$u_p^\epsilon(\tau)=\exp(-ik_p\tau/\epsilon^2)v_p^\epsilon(\tau),\quad p=1,\ldots,n, \tag{2.11}$$

which satisfies the equation

$$\begin{aligned}\frac{du_p^\epsilon(\tau)}{d\tau}&=\sum_{q=1}^n\exp(-ik_p\tau/\epsilon^2)\frac{1}{\epsilon}\,\mu_{pq}\!\left(\frac{\tau}{\epsilon^2}\right)\exp(ik_q\tau/\epsilon^2)u_q^\epsilon(\tau)\\&\quad+\sum_{q=1}^n\exp(-ik_p\tau/\epsilon^2)\nu_{pq}\!\left(\frac{\tau}{\epsilon^2}\right)\exp(ik_q\tau/\epsilon^2)u_q^\epsilon(\tau),\end{aligned}$$

$$u_p^\epsilon(0)=u_{p,0},\quad p=1,2,\ldots,n. \tag{2.12}$$

We may call the limit $\epsilon\downarrow 0$ in (2.12) the diffusion or white noise limit with averaging, since the rapidly oscillating terms will now be averaged out. Under certain hypotheses we have the following result (cf. Ref. 2). The complex valued process $u_p^\epsilon(\tau)$ converges (weakly) to a diffusion Markov process on $\mathbb{C}^n$, the complex $n$-dimensional space.

To describe the limiting Markov process we proceed as follows.

We define transport coefficients

$$\begin{aligned}a^{\cdot,\cdot}_{pq,p'q'}&=\Delta(k_p-k_q+k_{p'}-k_{q'})\int_0^\infty\exp[-i(k_p-k_q)\sigma]R^{\cdot,\cdot}_{pq,p'q'}(\sigma)\,d\sigma,\\a^{*,\cdot}_{pq,p'q'}&=\Delta(-k_p+k_q+k_{p'}-k_{q'})\int_0^\infty\exp[+i(k_p-k_q)\sigma]\\&\qquad\times R^{*,\cdot}_{pq,p'q'}(\sigma)\,d\sigma,\\a^{\cdot,*}_{pq,p'q'}&=\Delta(-k_p+k_q+k_{p'}-k_{q'})\int_0^\infty\exp[-i(k_p-k_q)\sigma]\\&\qquad\times R^{\cdot,*}_{pq,p'q'}(\sigma)\,d\sigma,\\a^{*,*}_{pq,p'q'}&=\Delta(k_p-k_q+k_{p'}-k_{q'})\int_0^\infty\exp[+i(k_p-k_q)\sigma]R^{*,*}_{pq,p'q'}(\sigma)\,d\sigma,\\b^{\cdot}_{pq}&=\Delta(k_p-k_q)\overline{\nu}_{pq},\\b^{*}_{pq}&=\Delta(k_p-k_q)\overline{\nu}_{pq},\quad pq,p'q'=1,2,\ldots,n.\end{aligned} \tag{2.13}$$

Here $\Delta(k)$ is defined to be zero unless $k=0$ when it is equal to 1.

Let $v=v^r+iv^I$, $v^*=v^r-iv^I$, and $\tilde f(v^r,v^I)$ be a smooth function of $v^r$ and $v^I$. Let

$$f(v,v^*)=\tilde f\left(\frac{v+v^*}{2},\frac{v-v^*}{2i}\right)$$

and define as usual

$$\partial f(v,v^*)=\frac{1}{2}\left(\frac{\partial\tilde f}{\partial v^r}+\frac{\partial\tilde f}{\partial v^I}\right),$$

$$\partial^* f(v,v^*)=\frac{1}{2i}\left(\frac{\partial\tilde f}{\partial v^r}-\frac{\partial\tilde f}{\partial v^I}\right).$$

Let $u=(u_1,u_2,\ldots,u_n)$ be an $n$-dimensional complex vector and let $f(u,u^*)$ be a smooth function in the sense described above. We denote by $\partial_p$ and $\partial_p^*$ partial derivatives with respect to $u_p$ and $u_p^*$ as above and, using the summation convention, we define the operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}f(u,u^*)&=a^{\cdot,\cdot}_{pq,p'q'}u_q\,\partial_pu_{q'}\,\partial_{p'}f(u,u^*)+a^{\cdot,*}_{pq,p'q'}u_q\,\partial_pu^*_{q'}\,\partial^*_{p'}f(u,u^*)\\&\quad+a^{*,\cdot}_{pq,p'q'}u_q^*\,\partial_p^*u_{q'}\,\partial_{p'}f(u,u^*)+a^{*,*}_{pq,p'q'}u_q^*\,\partial_p^*u^*_{q'}\,\partial^*_{p'}f(u,u^*)\\&\quad+b_{pq}u_q\,\partial_pf(u,u^*)+b^*_{pq}u_q^*\,\partial_p^*f(u,u^*)\end{aligned} \tag{2.14}$$

This operator is the infinitesimal generator of the limiting Markov process to which $u^\epsilon(\tau)$ converges that is, $\mathcal{L}_{\mathrm{AD}}$, the formal adjoint of $\mathcal{L}$ in (2.14), is the Fokker–Planck operator for $u^0(\tau)$ [the limit of $u^\epsilon(\tau)$]. The transition probability density of $u^0(\tau)$, $P(\tau,u,u^*)$ satisfies the equation

$$\frac{\partial P}{\partial \tau} = \mathcal{L}_{\mathrm{AD}} P, \quad P(0, u, u^*) = \delta(u - u_0)\delta(u^* - u_0^*). \tag{2.15}$$

We note that $\mathcal{L}$, and hence $\mathcal{L}_{\mathrm{AD}}$, takes real valued functions into real valued functions and that it is a possibly degenerate second order elliptic operator.

From this general convergence result of $u^\epsilon(\tau) \to u^0(\tau)$ one may deduce several interesting consequences.

First, because the stochastic equation (2.12) is linear, we obtain closed equations for moments of each order for $u^0(\tau)$, i.e., as $\epsilon \to 0$. In particular, we obtain closed equations for second and fourth order moments.

Second, and this is important for the applications to Maxwell's equations, the equations for the moments can be obtained independently of the diffusion limit which leads to the Fokker–Planck equation (2.15) and they make sense for partial differential equations (cf. Sec. 4).

Third, the role of the coefficients $\Delta$ in (2.13), i.e., of degeneracy or nondegeneracy, is significant. Let us examine it further.

Suppose that (2.7) is nondegenerate, i.e., the numbers $k_p$ along with their sums and differences are distinct and let us consider the equation for the second order moments of $u^0(\tau) = (u_1^0(\tau), \ldots, u_n^0(\tau))$. Let[13]

$$W_p(\tau) = \lim_{\epsilon \downarrow 0} E\{u_p^\epsilon(\tau) u_p^{\epsilon *}(\tau)\} = E\{u_p^0(\tau) u_p^{0*}(\tau)\},$$
$$p = 1, 2, \ldots, n. \tag{2.16}$$

From (2.14) and (2.15) we obtain a closed system of equations for the $W_p(\tau)$ alone; no cross moments enter.

$$\frac{dW_p(\tau)}{d\tau} = \sum_{q=1}^{n} A_{pq} W_q(\tau) + \left(\sum_{q=1}^{n} B_{pq}\right) W_p(\tau) + C_p W_p(\tau),$$
$$\tau > 0, \quad W_p(0) = u_{p0} u_{p0}^* \quad p = 1, \ldots, n. \tag{2.17}$$

Here we have introduced the following notation:

$$A_{pq} = \int_0^\infty \exp[i(k_p - k_q)\sigma] R^{*,\cdot}_{pq,pq}(\sigma)\, d\sigma + \int_0^\infty \exp[-i(k_p - k_q)\sigma] R^{\cdot,*}_{pq,pq}(\sigma)\, d\sigma,$$
$$B_{pq} = \int_0^\infty \exp[-i(k_p - k_q)\sigma] R^{\cdot,\cdot}_{pq,pq}(\sigma)\, d\sigma + \int_0^\infty \exp[i(k_p - k_q)\sigma] R^{*,*}_{pq,pq}(\sigma)\, d\sigma,$$
$$C_p = \bar{\nu}_{pp} + \bar{\nu}_{pp}^*. \tag{2.18}$$

If we specialize further to the case $\mu_{pq}^*(t) = -\mu_{qp}(t)$, $\nu_{pq}(t) \equiv 0$, we obtain a conservative transport equation[10]

$$\frac{dW_p(\tau)}{d\tau} = \sum_{q=1}^{n} [\tilde{A}_{pq} W_q(\tau) - \tilde{A}_{qp} W_p(\tau)], \quad W_p(0) = u_{p0} u_{p0}^*,$$
$$\tilde{A}_{pq} = \int_{-\infty}^{\infty} \cos(k_p - k_q)\sigma R_{pq,pq}(\sigma)\, d\sigma. \tag{2.19}$$

By conservative we mean that $\sum_{p=1}^n W_p(\tau)$ is independent of $\tau$.

The result (2.17), a closed equation for the expectation of the square moduli of $u_p^\epsilon(\tau)$ in the diffusion limit, is clearly a direct consequence of nondegeneracy. In the completely degenerate case, when all the $k_p$ are equal, cross terms enter and (2.17) [or (2.19)] are not valid.

Degenerate problems arise, usually, when the unperturbed problem (2.1) [or (2.7)] has some internal symmetries. It is natural to assume that the stochastic perturbations posses, in a statistical sense, the same symmetries. Is it then true that the quantities (2.16) satisfy equations analogous to (2.17)? The answer is no in general, i.e., cross moment terms cannot be eliminated, and Maxwell's equation (cf. Secs. 3–5) provide an example where degeneracy of the unperturbed equations precludes the validity of equations such as (2.17). However, we have examples where the answer to the above question is affirmative. We present two such examples, beginning with equations (2.20) and (2.37), respectively.

Let $x \in S^2$, the unit sphere in $R^3$, and consider the stochastic partial differential equation

$$\frac{\partial u(t,x)}{\partial t} = i[\nabla^2 u(t,x) + \epsilon\mu(t,x) u(t,x)], \quad t > 0,$$
$$u(0,x) = u_0(x), \tag{2.20}$$

for the complex-valued function $u(t,x)$. Here, $\nabla^2$ denotes the Laplace–Beltrami operator on $S^2$ and $\mu(t,x)$ is a real, stationary, zero mean, random process, almost surely bounded. We assume that its correlation function is rotationally invariant,

$$E\{\mu(t+s,x)\mu(s,x')\} = R(t, x \cdot x'), \tag{2.21}$$

where $x \cdot x'$ is the dot product of the vector $x, x'$ on $S^2$. Thus, $\mu(t,x)$ is a homogeneous random field on the sphere.

Let $Y_p^l(x)$ denote the normalized spherical harmonics, $p = 0, 1, 2, \cdots, -p \le l \le p$, satisfying

$$\int_{S^2} Y_p^l(x) Y_{p'}^{l'*}(x)\, dS(x) = \delta_{pp'}\delta_{ll'}, \tag{2.22}$$

$$\nabla^2 Y_p^l(x) = -p(p+1) Y_p^l(x), \quad -p \le l \le p. \tag{2.23}$$

We expand the solution $u(t,x)$ of (2.20) in spherical harmonics

$$u(t,x) = \sum_{p=0}^{\infty} \sum_{l=-p}^{p} v_p^l(t) Y_p^l(x). \tag{2.24}$$

This and (2.20) along with the orthogonality property lead to the equation

$$\frac{dv_p^l(t)}{dt} = -ip(p+1)v_p^l(t) + i\epsilon \sum_{q=0}^{\infty} \sum_{m=-q}^{q} \mu_{pq}^{lm}(t) v_q^m(t), \quad t > 0,$$
$$v_p^l(0) = u_{0,p}^l, \quad p = 0, 1, 2, \cdots, \quad -p \le l \le p, \tag{2.25}$$

where

$$\mu_{pq}^{lm}(t) = \int_{S^2} \mu(t,x) Y_p^{l*}(x) Y_q^m(x)\, dS(x). \tag{2.26}$$

This is a problem in the form (2.1) and it is degenerate since $k_p^l = -p(p+1)$ for $-p \le l \le p$. Let

$$\tau = \epsilon^2 t, \quad u_p^l(\tau;\epsilon) = v_p^l(\tau/\epsilon^2) \exp[ip(p+1)\tau/\epsilon^2]. \tag{2.27}$$

We obtain for $u_p^l(\tau;\epsilon)$ a system of the form (2.12):

$$\frac{du_p^l(\tau;\epsilon)}{d\tau} = \frac{1}{\epsilon} \sum_{q=0}^{\infty} \sum_{m=-q}^{q} \exp[ip(p+1)\tau/\epsilon^2] i\mu_{pq}^{lm}(\tau/\epsilon^2) \times \exp[-iq(q+1)\tau/\epsilon^2] u_q^m(\tau;\epsilon),$$
$$\tau > 0, \quad u_p^l(0;\epsilon) = u_{0,p}^l, \quad p = 0, 1, 2, \cdots, \quad -p \le l \le p. \tag{2.28}$$

The system (2.28) is infinite dimensional while the results quoted above have been shown only for finite dimensional systems. Since we are interested here primarily in the algebraic calculations that lead to transport equations of the form (2.19), we shall not pause to provide the necessary justification.

Let us expand the covariance of $\mu$, defined by (2.21), is spherical harmonics:

$$R(t, x\cdot x') = \sum_{r=0}^{\infty} R_r(t)\frac{2r+1}{2\pi} P_r(x\cdot x') = \sum_{r=0}^{\infty}\sum_{s=-r}^{r} R_r(t) Y_r^s(x) Y_r^{s*}(x'). \qquad (2.29)$$

Here we have employed the addition theorem for spherical harmonics. We next formally compute the diffusion operator (2.14) for the present problem. We find that

$$\mathcal{L} = \sum_{\substack{pq,p'q'\\ lm,l'm'}} (a^{\cdot,\cdot\, lm,l'm'}_{pq,p'q'} u_q^m \partial_p^l u_{q'}^{m'} \partial_{p'}^{l'} + a^{*,*\,lm,l'm'}_{pq,p'q'} u_q^{*m}\partial_p^{*l} u_{q'}^{*m'}\partial_{p'}^{*l'} + a^{*,\cdot\, lm,l'm'}_{pq,p'q'} u_q^{*m}\partial_q^{*l} u_{q'}^{m'}\partial_{p'}^{l'} + a^{\cdot,*\,lm,l'm'}_{pq,p'q'} u_q^m \partial_p^l u_{q'}^{*m'}\partial_{p'}^{*l'}), \qquad (2.30)$$

where the coefficients $a$ are given by the following formulas

$$a^{\cdot,\cdot\, lm,l'm'}_{pq,p'q'} = -\Delta(p(p+1)-q(q+1)+p'(p'+1)-q'(q'+1)) \times\sum_{r=0}^{\infty}\int_0^\infty \exp[i(p(p+1)-q(q+1))\sigma]R_r(\sigma)\,d\sigma \times\sum_{s=-r}^{r}\left(\int Y_r^s Y_p^{l*} Y_q^m\right)\left(\int Y_r^{s*} Y_{p'}^{l'*} Y_{q'}^{m'}\right),$$

$$a^{*,\cdot\, lm,l'm'}_{pq,p'q'} = \Delta(-p(p+1)+q(q+1)+p'(p+1)-q'(q'+1)) \times\sum_{r=0}^{\infty}\int_0^\infty \exp[-i(p(p+1)-q(q+1))\sigma R_r(\sigma)\,d\sigma \times\sum_{s=-r}^{r}\left(\int Y_r^s Y_p^{l} Y_q^{m*}\right)\left(\int Y_r^{s*} Y_{p'}^{l'*} Y_{q'}^{m'}\right),$$

$$a^{\cdot,*\, lm,l'm'}_{pq,p'q'} = \Delta(-p(p+1)+q(q+1)+p'(p'+1)-q'(q'+1)) \times\sum_{r=0}^{\infty}\int_0^\infty \exp[i(p(p+1)-q(q+1))\sigma]R_r(\sigma)\,d\sigma \times\sum_{s=-r}^{r}\left(\int Y_r^s Y_p^{l*} Y_q^m\right)\left(\int Y_r^{s*} Y_{p'}^{l'} Y_{q'}^{m'*}\right),$$

$$a^{*,*\, lm,l'm'}_{pq,p'q'} = -\Delta(p(p+1)-q(q+1)+p'(p'+1)-q'(q'+1)) \times\sum_{r=0}^{\infty}\int_0^\infty \exp[-i(p(p+1)-q(q+1))\sigma]R_r(\sigma)\,d\sigma \times\sum_{s=-r}^{r}\left(\int Y_r^s Y_p^{l} Y_q^{m*}\right)\left(\int Y_r^{s*} Y_{p'}^{l'} Y_{q'}^{m'*}\right). \qquad (2.31)$$

Here we have used again the function $\Delta$ introduced in (2.13), which has value 1 or zero according as the argument vanishes or not, and integrals of the form

$$\int_{S^2} Y_r^s(x) Y_p^l(x) Y_q^{m*}(x)\,dS(x), \qquad (2.32)$$

which can be expressed in terms of Clebsh—Gordan coefficients. It is sufficient for our purposes here to note that the integral (2.32) is zero unless

$$s+l-m=0 \quad\text{and}\quad |p-q|\le r\le p+q, \qquad (2.33)$$

which are the selection rules.

As with (2.16) and (2.17) we can use (2.30) to obtain closed equations for

$$W_\gamma^\nu(\tau) = \lim_{\epsilon\downarrow 0} E\{u_\gamma^\nu(\tau;\epsilon)u_\gamma^{*\nu}(\tau;\epsilon)\},$$
$$\tau\ge 0,\quad \gamma=0,1,2,\cdots,\quad -\gamma\le\nu\le\gamma. \qquad (2.34)$$

Because of the form of the coefficients (2.31) and the selection rules (2.33), a direct consequence of the rotational invariance of the covariance of $\mu(t,x)$, we find that the $W_\gamma^\nu$ satisfy the following transport equations:

$$\frac{dW_\gamma^\nu(\tau)}{d\tau} = \sum_{q=0}^{\infty}\sum_{m=-q}^{q}[A_{\gamma q}^{\nu m}W_q^m(\tau) - A_{q\gamma}^{m\nu}W_\gamma^\nu(\tau)],$$
$$\tau>0,\quad W_\gamma^\nu(0) = u_{0\gamma}^\nu u_{0\gamma}^{*\nu}, \qquad (2.35)$$

where

$$A_{\gamma q}^{\nu m} = \sum_{r=|\gamma-q|}^{\gamma+q}\left|\int Y_r^{m-\nu}Y_\gamma^\nu Y_q^{*m}\right|^2 \times 2\int_0^\infty \cos(\gamma(\gamma+1)-q(q+1))\sigma R_r(\sigma)\,d\sigma, \qquad (2.36)$$

and

$$A_{\gamma q}^{\nu m}\ge 0,\quad A_{\gamma q}^{\nu m} = A_{q\gamma}^{m\nu},$$
$$\gamma, q = 0,1,2,\cdots,\quad -\gamma\le\nu\le\gamma,\quad -q\le m\le q.$$

We have shown therefore, at least for the truncated system (2.28), that despite the degeneracy of the unperturbed problem (rotational invariance of $\nabla^2$), statistical homogeneity of the random perturbations restores the validity of the transport equations (2.35) in the diffusion limit.

Another example, quite similar to the one just described, is the following. Let $g$ denote an element of $SU(2)$, the group of $2\times 2$ unitary unimodular matrices, and let $D^2$ denote the Laplace—Beltrami (Casimir) operator on $SU(2)$. Consider the stochastic differential equation

$$\frac{\partial u(t,g)}{\partial t} = i[D^2 u(t,g) + \epsilon\mu(t,g)u(t,g)],\quad t>0,$$
$$u(0,g) = u_0(g), \qquad (2.37)$$

which is analogous to (2.20).

We proceed now in the same way as above. Let $t_{lm}^p(g)$ denote the matrix elements of the $(2p+1)$-dimensional representation of $SU(2)$, $p=0,\frac12,1,\frac32,\cdots$, $-p\le l,m\le p$, $l,m$ integers or half-odd integers together with $p$. We note here the following properties[14,15]:

unitarity $\quad t_{lm}^p(g^{-1}) = t_{lm}^{p*}(g), \qquad (2.38)$

orthogonality $\quad \int_{SU(2)} t_{l'm'}^{p'}(g)t_{lm}^{p*}(g)\,dg = (2p+1)^{-1}\delta_{pp'}\delta_{ll'}\delta_{mm'}. \qquad (2.39)$

eigenfunction of $D^2$ $\quad D^2 t_{lm}^p(g) = -p(p+1)t_{lm}^p(g), \qquad (2.40)$

addition theorem $\quad t_{lm}^p(g_1g_2) = \sum_{\nu=-p}^{p} t_{l\nu}^p(g_1)t_{\nu m}^p(g_2). \qquad (2.41)$

We expand the solution of (2.37) in terms of the representation $t_{lm}^p(g)$

$$u(t,g) = \sum_q\sum_{\lambda,\eta}\sqrt{2q+1}\,u_q^{\lambda\eta}(t)t_{\lambda\eta}^q(g), \qquad (2.42)$$

so, on using (2.38), we obtain

$$\frac{du_p^{lm}(t)}{dt} = -ip(p+1)u_p^{lm}(t) + i\epsilon\sum_q\sum_{\lambda\eta}\mu_{pq}^{lm,\lambda\eta}(t)u_q^{\lambda\eta}(t),$$
$$u_q^{\lambda\eta}(0) = u_{0,q}^{\lambda\eta}. \qquad (2.43)$$

Here and in (2.42) the sum is over $q=0,\frac12,1,\frac32,\cdots$ and

$-q \leqslant \lambda, \eta \leqslant q$ with integer increments. The coefficients $\mu_{pq}^{lm,\lambda\eta}(t)$ are defined by

$$\mu_{pq}^{lm,\lambda\eta}(t)=\sqrt{(2p+1)(2q+1)}\int_{SU(2)} t_p^{lm*}(g)t_q^{\lambda\eta}(g)\mu(t,g)\,dg. \tag{2.44}$$

Passing to the interaction representation, as in (2.28), we let

$$\tau=\epsilon^2 t,\quad u_p^{lm}(\tau;\epsilon)=v_p^{lm}(\tau/\epsilon^2)\exp[ip(p+1)\tau/\epsilon^2] \tag{2.45}$$

and obtain

$$\frac{du_p^{lm}(\tau;\epsilon)}{d\tau}=\frac{1}{\epsilon}\sum_q\sum_{\lambda\eta}\exp[ip(p+1)\tau/\epsilon^2]i\mu_{pq}^{lm,\lambda\eta}(\tau/\epsilon^2)$$
$$\times\exp[-iq(q+1)\tau/\epsilon^2]u_q^{\lambda\eta}(\tau;\epsilon),$$
$$\tau>0,\quad u_p^{lm}(0;\epsilon)=u_{0,p}^{lm}. \tag{2.46}$$

Before evaluating the diffusion limit we must specify the properties of the random function $\mu(t,g)$. It has mean zero, as above, and covariance

$$E\{\mu(t+s,g_1)\mu(s,g_2)\}=R(t,g_1,g_2). \tag{2.47}$$

We assume that $R$ has the following invariance properties:

$$R(t,g_1,g_2)=R(t,e,g_1^{-1}g_2)=R(t,e,g_2g_1^{-1}), \tag{2.48}$$

where $e$ denotes the identity element of $SU(2)$. Thus $R(t,e,g)$ is a function which is constant on conjugacy classes

$$R(t,e,g_2)=R(t,e,g_1g_2g_1^{-1}). \tag{2.49}$$

It has therefore the expansion[15]

$$R(t,g_1,g_2)=R(t,e,g_1^{-1}g_2)=\sum_{r=0}^{\infty}R_r(t)t_{00}^r(g_1^{-1}g_2)$$
$$=\sum_{r=0}^{\infty}\sum_{\nu=-r}^{r}R_r(t)t_{0\nu}^r(g_1^{-1})t_{\nu0}^r(g_2)$$
$$=\sum_{r=0}^{\infty}\sum_{\nu=-r}^{r}R_r(t)t_{\nu0}^{r*}(g_1)t_{\nu0}^r(g_2). \tag{2.50}$$

Here we have used (2.38) and (2.41).

The limiting diffusion operator, analogous to (2.30), can be obtained just as before but the new coefficients $a$ have 12 instead of 8 indices. We shall not write them explicitly. We proceed directly to the transport equations corresponding to (2.35) by employing the selection rules for the Clebsh–Gordan coefficients.[15]

Let

$$W_\gamma^{\nu\lambda}(\tau)=\lim_{\epsilon\downarrow0}E\{u_\gamma^{\nu\lambda}(\tau;\epsilon)u_\gamma^{\nu\lambda*}(\tau;\epsilon)\},\quad \tau\geqslant0,$$
$$\gamma=0,\tfrac12,1,\tfrac32,\cdots,\quad -\gamma\leqslant\nu,\lambda\leqslant\gamma. \tag{2.51}$$

Then, $W_\gamma^{\nu\lambda}(\tau)$ satisfy the following transport equations:

$$\frac{dW_\gamma^{\nu\lambda}(\tau)}{d\tau}=\sum_q\sum_{ml}[A_{\gamma q}^{\nu\lambda,ml}W_q^{ml}(\tau)-A_{q\gamma}^{ml,\nu\lambda}W_\gamma^{\nu\lambda}(\tau)],$$
$$\tau>0,\quad W_\gamma^{\nu\lambda}(0)=u_{0,\gamma}^{\nu\lambda}u_{0,\gamma}^{\nu\lambda*}, \tag{2.52}$$

where

$$A_{\gamma q}^{\nu\lambda,ml}=\sum_{r=|\gamma-q|}^{\gamma+q}(2p+1)(2q+1)\left|\int t_{m-\nu,n-\lambda}^r(g)t_{mn}^r(g)t_{\nu\lambda}^{q*}(g)\,dg\right|^2$$
$$\times2\int_0^\infty\cos(\gamma(\gamma+1)-q(q+1))\sigma R_r(\sigma)\,d\sigma, \tag{2.53}$$
$$A_{\gamma q}^{\nu\lambda,ml}\geqslant0,\quad A_{\gamma q}^{\nu\lambda ml}=A_{q\gamma}^{ml\nu\lambda},$$
$$\gamma,q=0,\tfrac12,1,\tfrac32,\cdots,\quad -\gamma\leqslant\nu,\lambda\leqslant\gamma,\quad -q\leqslant m,l\leqslant q.$$

In (2.53) the summation is over integers or half-odd integers depending on whether $|\gamma-q|$ is an integer or half an odd integer.

## 3. PRELIMINARY TRANSFORMATIONS OF MAXWELL'S EQUATIONS IN A STATISTICALLY HOMOGENEOUS MEDIUM

In this section we shall treat Maxwell's equations in a nonconducting medium whose dielectric constant[16] $\epsilon$ and magnetic permeability $\mu$ are random positive definite symmetric tensors. The mean values (ensemble averages) of $\epsilon$ and $\mu$ will be constant multiples of the unit tensor $I$ so that

$$E\{\epsilon(\mathbf{x},t)\}=\bar\epsilon I,$$
$$E\{\mu(\mathbf{x},t)=\bar\mu I, \tag{3.1}$$

$\bar\epsilon,\bar\mu$ are constants. The deviations of $\epsilon$ and $\mu$ from their means are small stationary tensor-valued random processes in space and time which are of order $\gamma$,[16] where $\gamma$ is a small parameter. The exact nature of these random processes will be specified later.

Maxwell's equations in a nonconducting medium are

$$\frac1c\partial_t\mathbf{D}=\nabla\wedge\mathbf{H},$$
$$\frac1c\partial_t\mathbf{B}=-\nabla\wedge\mathbf{E}, \tag{3.2}$$

$$\mathbf{D}=\epsilon\mathbf{E},$$
$$\mathbf{B}=\mu\mathbf{H}. \tag{3.3}$$

From (3.2) we may deduce that the further equations

$$\nabla\cdot\mathbf{D}=0,$$
$$\nabla\cdot\mathbf{B}=0, \tag{3.4}$$

hold for all times if they hold initially. Here E, H are the electric displacement and magnetic induction, and $c$ is the speed of light *in vacuo*.

When $\epsilon,\mu$ are independent of $t$ (3.2), (3.3) imply

$$\frac1c\epsilon^{1/2}\partial_t(\epsilon^{1/2}\mathbf{E})=\nabla\wedge\mathbf{H},$$
$$\frac1c\mu^{1/2}\partial_t(\mu^{1/2}\mathbf{H})=-\nabla\wedge\mathbf{E}, \tag{3.5}$$

where $\epsilon^{1/2}$, $\mu^{1/2}$ are the positive square roots of $\epsilon$, $\mu$. On making the substitution

$$\tilde{\mathbf{E}}=\epsilon^{1/2}\mathbf{E},\quad\tilde{\mathbf{H}}=\epsilon^{1/2}\mathbf{H}, \tag{3.6}$$

Eqs. (3.5) become

$$\partial_t\tilde{\mathbf{E}}=c\epsilon^{-1/2}\nabla\wedge(\mu^{-1/2}\tilde{\mathbf{H}}),$$
$$\partial_t\tilde{\mathbf{H}}=-c\mu^{-1/2}\nabla\wedge(\epsilon^{-1/2}\tilde{\mathbf{E}}). \tag{3.7}$$

In what follows we shall take (3.7) to be true even when $\epsilon$, $\mu$ do depend upon $t$ since we are primarily interested in the case where the time derivatives of $\epsilon$, $\mu$ are small, and (3.7) has the convenient property of energy conservation. Thus we may define the energy density $\mathcal{E}$ by

$$\mathcal{E}=\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}, \tag{3.8}$$

which may also be written

$$\mathcal{E} = \tilde{\mathbf{E}}^2 + \tilde{\mathbf{H}}^2. \tag{3.9}$$

Energy conservation follows from (3.5) since $\partial_t \mathcal{E}$ is a divergence

$$\partial_t \mathcal{E} = 2c\nabla\cdot(\mathbf{H}\wedge\mathbf{E}). \tag{3.10}$$

Let us define the matrices $A^j$ by

$$(A^j)_{ik} = \epsilon_{ijk}, \tag{3.11}$$

where $\epsilon_{ijk}$ is the alternating symbol, and the matrix $A(\mathbf{k})$, $\mathbf{k} = (k_1, k_2, k_3)$, by

$$A(\mathbf{k}) = A^j k_j, \tag{3.12}$$

with summation implied. Then, if $x_1, x_2, x_3$ are Cartesian coordinates, $\partial_j = \partial/\partial x_j$, $\partial = (\partial_1, \partial_2, \partial_3)$, we may write (3.7) in the form

$$\begin{aligned} \partial_t \tilde{\mathbf{E}} &= c\epsilon^{-1/2} A(\partial)(\mu^{-1/2}\tilde{\mathbf{H}}), \\ \partial_t \tilde{\mathbf{H}} &= -c\mu^{-1/2} A(\partial)(\epsilon^{-1/2}\tilde{\mathbf{E}}), \end{aligned} \tag{3.13}$$

which, on differentiating the products, becomes

$$\begin{aligned} \partial_t \tilde{\mathbf{E}} &= c\epsilon^{-1/2} A^j \mu^{-1/2} \partial_j \tilde{\mathbf{H}} + c\epsilon^{-1/2} A^j (\partial_j \mu^{-1/2})\tilde{\mathbf{H}}, \\ \partial_t \tilde{\mathbf{H}} &= -c\mu^{-1/2} A^j \epsilon^{-1/2} \partial_j \tilde{\mathbf{E}} - c\mu^{-1/2} A^j (\partial_j \epsilon^{-1/2})\tilde{\mathbf{E}}. \end{aligned} \tag{3.14}$$

Let us now be more explicit about the dependence of the random processes $\epsilon$, $\mu$ upon $\mathbf{x}, t$ and the small parameter $\gamma$. With no loss in generality and to avoid very lengthy calculations we shall assume that

$$\begin{aligned} \epsilon^{-1/2}(\mathbf{x}, t) &= \epsilon^{(0)} I + \gamma\epsilon^{(1)}(\mathbf{x}, t), \\ \mu^{-1/2}(\mathbf{x}, t) &\equiv I, \end{aligned} \tag{3.15}$$

where $\epsilon^{(0)}$ is a constant scalar, $\epsilon^{(1)}$ is a random symmetric tensor valued process with zero mean and stationary in $x, t$. Thus, for instance, the correlations

$$E\{\epsilon^{(1)}_{uv}(\mathbf{x}, t)\epsilon^{(1)}_{u'v'}(\mathbf{x}', t')\} \tag{3.16}$$

depend upon $\mathbf{x}, t, \mathbf{x}', t'$ only through $\mathbf{x}-\mathbf{x}'$, $t-t'$. On substituting (3.15) in (3.14), we obtain

$$\begin{aligned} \frac{1}{c}\partial_t \tilde{\mathbf{E}} &= \epsilon^{(0)} A^j \partial_j \tilde{\mathbf{H}} + \gamma\epsilon^{(1)} A^j \partial_j \tilde{\mathbf{H}} \\ \frac{1}{c}\partial_t \tilde{\mathbf{H}} &= -A^j \partial_j \tilde{\mathbf{E}} - \gamma A^j \epsilon^{(1)} \partial_j \tilde{\mathbf{E}} - A^j(\partial_j \epsilon^{(1)})\tilde{\mathbf{E}}. \end{aligned} \tag{3.17}$$

We shall next Fourier transform (3.17) with respect to $\mathbf{x}$. Denote by $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}$, $\hat{\epsilon}^{(1)}$, the Fourier transforms of $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ and $\epsilon^{(1)}$ with $\mathbf{k}$ the transform variable. Then

$$\begin{aligned} \frac{1}{c}\partial_t \hat{\mathbf{E}}(\mathbf{k}, t) &= -ik_j \epsilon^{(0)} A^j \hat{\mathbf{H}}(\mathbf{k}, t) - i\gamma \int J(\mathbf{k}, \mathbf{s}, t)\hat{\mathbf{H}}(\mathbf{s}, t)\, d\mathbf{s}, \\ \frac{1}{c}\partial_t \hat{\mathbf{H}}(\mathbf{k}, t) &= ik_j A^j \epsilon^{(0)} \hat{\mathbf{E}}(\mathbf{k}, t) + i\gamma \int K(\mathbf{k}, \mathbf{s}, t)\hat{\mathbf{E}}(\mathbf{s}, t)\, d\mathbf{s}, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} J(\mathbf{k}, \mathbf{s}, t) &= \hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s}, t) A^j s_j, \\ K(\mathbf{k}, \mathbf{s}, t) &= A^j k_j \hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s}, t). \end{aligned} \tag{3.19}$$

The integrals in (3.18) are over all of $R^3$.

In order to use the perturbation formalism of the next section, which is formally identical to the one employed in Sec. 2, we must eliminate the terms $O(1)$ as $\gamma \to 0$, on the right side of (3.18). In preparation we first diagonalize the $O(1)$ term which may be written

$$i\epsilon^{(0)}\begin{pmatrix} 0 & -A(\mathbf{k}) \\ A(\mathbf{k}) & 0 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{pmatrix}. \tag{3.20}$$

The $6\times 6$ block matrix in (3.20) is Hermitian [since $A(\mathbf{k})$ is skew] with eigenvalues $|\mathbf{k}|, |\mathbf{k}|, -|\mathbf{k}|, -|\mathbf{k}|, 0, 0$. The matrix of normalized eigenvectors may be taken to be

$$\tilde{\mathbf{T}}(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{i}(\mathbf{k}) & -\mathbf{j}(\mathbf{k}) & \mathbf{i}(\mathbf{k}) & \mathbf{j}(\mathbf{k}) & \sqrt{2}\hat{\mathbf{k}} & 0 \\ \mathbf{j}(\mathbf{k}) & \mathbf{i}(\mathbf{k}) & -\mathbf{j}(\mathbf{k}) & \mathbf{i}(\mathbf{k}) & 0 & \sqrt{2}\hat{\mathbf{k}} \end{pmatrix} \tag{3.21}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction $\mathbf{k}$ and $\mathbf{i}(\mathbf{k})$, $\mathbf{j}(\mathbf{k})$, $\hat{\mathbf{k}}$ form a right-handed, orthonormal triple of column vectors in $R^3$. The dependence of $\mathbf{i}$ and $\mathbf{j}$ on $\mathbf{k}$ will sometimes not be shown explicitly in the sequel.

On writing

$$\Lambda = \mathrm{diag}(1, 1, -1, -1, 0, 0) \tag{3.22}$$

and setting

$$\begin{pmatrix} \hat{\mathbf{E}}(\mathbf{k}, t) \\ \hat{\mathbf{H}}(\mathbf{k}, t) \end{pmatrix} = \tilde{\mathbf{T}}(\mathbf{k}) u(\mathbf{k}, t), \tag{3.23}$$

(3.18) becomes

$$\begin{aligned} \frac{1}{c}\partial_t u(\mathbf{k}, t) &= i\epsilon^{(0)}|\mathbf{k}|\Lambda u(\mathbf{k}, t) \\ &\quad + \gamma \int \tilde{T}^T(\mathbf{k})\mathcal{K}(\mathbf{k}, \mathbf{s}, t)\tilde{T}(\mathbf{s}) u(\mathbf{s}, t)\, d\mathbf{s} \end{aligned} \tag{3.24}$$

where

$$\mathcal{K}(\mathbf{k}, \mathbf{s}, t) = \begin{pmatrix} 0 & -iJ(\mathbf{k}, \mathbf{s}, t) \\ iK(\mathbf{k}, \mathbf{s}, t) & 0 \end{pmatrix} \tag{3.25}$$

The $O(1)$ term in (3.24) may be eliminated by the transformation

$$u = \exp(iv|k|\Lambda t)\tilde{w} \tag{3.26}$$

where we have defined

$$v \equiv c\epsilon^{(0)} \tag{3.27}$$

as the speed of wave propagation when $\gamma = 0$. Denoting by $\hat{\epsilon}^{(1)}$ again the quantity $c\hat{\epsilon}^{(1)}$, we have

$$\begin{aligned} \partial_t \tilde{w}(\mathbf{k}, t) &= \int \exp(-iv|k|\Lambda t)\tilde{T}(\mathbf{k})\mathcal{K}(\mathbf{k}, \mathbf{s}, t)\tilde{T}(\mathbf{s}) \\ &\quad \times \exp(iv|s|\Lambda t)\tilde{w}(\mathbf{s}, t)\, d\mathbf{s}. \end{aligned} \tag{3.28}$$

Equation (3.28) is a $6\times 6$ system. We shall reduce it first to a $4\times 4$ system and then further to a $2\times 2$ system. Here we have used the tilde on $w$ so that at a later stage we may use plain $w$ for a different but related quantity.

Consider the divergence equations (3.4). These imply that the components of $\hat{E}$ and $\hat{H}$ along $\mathbf{k}$ do not contribute to the zero-order nor to the first-order terms; they contribute only $O(\gamma^2)$ terms. This contribution has coefficients with mean zero and, according to the perturbation analysis of the next section, it makes no contribution in the asymptotic limit we are seeking.

Thus, in (3.28) the last two components of

$$u(\mathbf{k}, t) = \tilde{T}^T(\mathbf{k})\begin{pmatrix} \hat{\mathbf{E}}(\mathbf{k}, t) \\ \hat{\mathbf{H}}(\mathbf{k}, t) \end{pmatrix}$$

may be dropped and we may treat (3.20) as a $4\times 4$ system

$$\partial_t \bar{w}(\mathbf{k},t) = \gamma \int \exp(-iv|\mathbf{k}|\bar{\Lambda}t)\bar{T}^T(\mathbf{k})K(\mathbf{k},\mathbf{s},t)\bar{T}(\mathbf{s}) \times \exp(iv|\mathbf{s}|\bar{\Lambda}t)\bar{w}(\mathbf{s},t)\,d\mathbf{s}, \tag{3.29}$$

where

$$\bar{w} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tilde{w}, \tag{3.30}$$

$$\bar{T}(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{i}(\mathbf{k}) & -\mathbf{j}(\mathbf{k}) & \mathbf{i}(\mathbf{k}) & \mathbf{j}(\mathbf{k}) \\ \mathbf{j}(\mathbf{k}) & \mathbf{i}(\mathbf{k}) & -\mathbf{j}(\mathbf{k}) & \mathbf{i}(\mathbf{k}) \end{pmatrix}, \tag{3.31}$$

and

$$\bar{\Lambda} = \mathrm{diag}(1, 1, -1, -1). \tag{3.32}$$

Our next step is to make use of the fact that it is not necessary to distinguish between

$$\begin{pmatrix} \bar{w}_1(\mathbf{k}) \\ \bar{w}_2(\mathbf{k}) \end{pmatrix} \text{ and } \begin{pmatrix} \bar{w}_3(-\mathbf{k}) \\ \bar{w}_4(-\mathbf{k}) \end{pmatrix}$$

which both represent waves travelling in the $+\mathbf{k}$ direction when $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ are multiplied by $\exp(-i\mathbf{k}\cdot x)$ for the synthesis of the inverse Fourier transform. In fact, Eqs. (3.18) have the property that if

$$\begin{pmatrix} \hat{\mathbf{E}}^*(\mathbf{k}) \\ \hat{\mathbf{H}}^*(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{E}}(-\mathbf{k}) \\ \hat{\mathbf{H}}(-\mathbf{k}) \end{pmatrix} \tag{3.33}$$

is true at $t=0$, it remains true for all time. But (3.33) is true initially since we assume $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ are real fields. This symmetry is reflected in (3.29) by the fact that

$$\begin{pmatrix} \bar{w}_3(\mathbf{k}) \\ \bar{w}_4(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} \bar{w}_1^*(-\mathbf{k}) \\ \bar{w}_2^*(-\mathbf{k}) \end{pmatrix}, \tag{3.34}$$

so that we may eliminate $\bar{w}_3$ and $\bar{w}_4$ in favor of $\bar{w}_1$ and $\bar{w}_2$.

Let $w(\mathbf{k},t)$ denote the first two components of $\bar{w}(\mathbf{k},t)$ and let us omit the bar so that

$$w(\mathbf{k},t) = \begin{pmatrix} w_1(\mathbf{k},t) \\ w_2(\mathbf{k},t) \end{pmatrix}. \tag{3.35}$$

Let $T(\mathbf{k})$ and $T^\perp(\mathbf{k})$ be given by

$$T(\mathbf{k}) = (\mathbf{i} \;\; -\mathbf{j}), \quad T^\perp(\mathbf{k}) = (\mathbf{j} \;\; \mathbf{i}) \tag{3.36}$$

where as before $\mathbf{i}, \mathbf{j}, \hat{\mathbf{k}}$ form a right-handed orthonormal triple. Implicit in the notation $T(\mathbf{k})$ is that $\mathbf{i}$ and $\mathbf{j}$ are functions of $\mathbf{k}$ which we now make definite. Let $(\theta, \phi)$ be spherical polar angles relative to some fixed reference frame. Thus, if

$$\hat{\mathbf{k}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},$$

then

$$\mathbf{i}(\mathbf{k}) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \mathbf{j}(\mathbf{k}) = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{3.37}$$

It is clear that

$$\mathbf{i}(-\mathbf{k}) = \mathbf{i}(\mathbf{k}), \quad \mathbf{j}(-\mathbf{k}) = -\mathbf{j}(\mathbf{k}), \tag{3.38}$$

so that (3.31) and (3.36) imply

$$\bar{T}(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} T(\mathbf{k}) & T(-\mathbf{k}) \\ T^\perp(\mathbf{k}) & T^\perp(-\mathbf{k}) \end{pmatrix}. \tag{3.39}$$

The matrix $\bar{T}^T(\mathbf{k})K(\mathbf{k},\mathbf{s},t)\bar{T}(\mathbf{s})$ in (3.29) may be written in partitioned form. Then, using (3.34) the first two rows of (3.39) take the form

$$\begin{aligned} \partial_t w(\mathbf{k},t) = &\int \exp[-iv(|\mathbf{k}| - |\mathbf{s}|)t][-iT(\mathbf{k})J(\mathbf{k},\mathbf{s},t)T^\perp(\mathbf{s}) \\ &+ iT^I(\mathbf{k})K(\mathbf{k},\mathbf{s},t)T(\mathbf{s})]w(\mathbf{s},t)\,d\mathbf{s} \\ &+ \int \exp[-iv(|\mathbf{k}| + |\mathbf{s}|)t] \\ &\times[-iT(\mathbf{k})J(\mathbf{k},\mathbf{s},t)T^\perp(-\mathbf{s})]w^*(-\mathbf{s},t)\,d\mathbf{s} \end{aligned} \tag{3.40}$$

where

$$T^I = (T^\perp)^T. \tag{3.41}$$

From (3.19) it follows that

$$T^T(\mathbf{k})J(\mathbf{k},\mathbf{s},t)T^\perp(\mathbf{s}) = T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s},t)A^j s_j T^\perp(\mathbf{s}).$$

But

$$T^T(\mathbf{k})A^j k_j = |\mathbf{k}|\,T^I(\mathbf{k}), \quad A^j s_j T^\perp(\mathbf{s}) = |\mathbf{s}|\,T(\mathbf{s}).$$

Thus

$$T^T(\mathbf{k})J(\mathbf{k},\mathbf{s},t)T^\perp(\mathbf{s}) = |s|\,T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s},t)T(\mathbf{s}), \tag{3.42}$$

and similarly

$$\begin{aligned} T^I(\mathbf{k})K(\mathbf{k},\mathbf{s},t)T(\mathbf{s}) &= -|\mathbf{k}|\,T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s},t)T(\mathbf{s}), \\ T^T(\mathbf{k})J(\mathbf{k},\mathbf{s},t)T^\perp(-\mathbf{s}) &= |\mathbf{s}|\,T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s},t)T(-\mathbf{s}), \\ T^I(\mathbf{k})K(\mathbf{k},\mathbf{s},t)T(-\mathbf{s}) &= -|\mathbf{k}|\,T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s},t)T(-\mathbf{s}). \end{aligned} \tag{3.43}$$

On using (3.42) and (3.43) in (3.40) and replacing $\mathbf{s}$ by $-\mathbf{s}$ in the second integral, we obtain the equation for $w(\mathbf{k},t)$ in the desired form:

$$\begin{aligned} \partial_t w(\mathbf{k},t) = &-i\gamma\int(|\mathbf{k}| + |\mathbf{s}|)\exp[-iv(|\mathbf{k}| - |\mathbf{s}|)t] \\ &\times T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}-\mathbf{s},t)T(\mathbf{s})w(\mathbf{s},t)\,d\mathbf{s} \\ &-i\gamma\int(|\mathbf{k}| - |\mathbf{s}|)\exp[-iv(|\mathbf{k}| + |\mathbf{s}|)t] \\ &\times T^T(\mathbf{k})\hat{\epsilon}^{(1)}(\mathbf{k}+\mathbf{s},t)T(\mathbf{s})w^*(\mathbf{s},t)\,d\mathbf{s}. \end{aligned} \tag{3.44}$$

Recall that $w(\mathbf{k},t)$ is given by (3.35). Summarizing the above transformations, $w_1(\mathbf{k},t)$ and $w_2(\mathbf{k},t)$ are related to $\hat{\mathbf{E}}(\mathbf{k},t)$, $\hat{\mathbf{H}}(\mathbf{k},t)$ as follows:

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{k},t) = &\frac{\exp(iv|\mathbf{k}|t)}{\sqrt{2}}(\mathbf{i}(\mathbf{k})w_1(\mathbf{k},t) - \mathbf{j}(\mathbf{k})w_2(\mathbf{k},t)) \\ &+ \frac{\exp(-iv|\mathbf{k}|t)}{\sqrt{2}}(\mathbf{i}(-\mathbf{k})w_1^*(-\mathbf{k},t) - \mathbf{j}(-\mathbf{k})w_2^*(-\mathbf{k},t)) \\ \hat{\mathbf{H}}(\mathbf{k},t) = &\frac{\exp(iv|\mathbf{k}|t)}{\sqrt{2}}(\mathbf{j}(\mathbf{k})w_1(\mathbf{k},t) + \mathbf{i}(\mathbf{k})w_2(\mathbf{k},t)) \\ &+ \frac{\exp(-iv|\mathbf{k}|t)}{\sqrt{2}}(\mathbf{j}(-\mathbf{k})w_1^*(-\mathbf{k},t) + \mathbf{i}(-\mathbf{k})w_2^*(-\mathbf{k},t)). \end{aligned} \tag{3.45}$$

Let

$$\phi(\mathbf{k},\mathbf{x},t) = v|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x}, \tag{3.46}$$

$$\mathcal{E}(\mathbf{k},\mathbf{x},t) = \frac{\exp[i\phi(\mathbf{k},\mathbf{x},t)]}{\sqrt{2}}(\mathbf{i}(\mathbf{k})w_1(\mathbf{k},t) - \mathbf{j}(\mathbf{k})w_2(\mathbf{k},t)). \tag{3.47}$$

Then we may rewrite (3.45) in the form

$$\hat{\mathbf{E}}(\mathbf{k},t)\exp(-i\mathbf{k}\cdot\mathbf{x}) = \hat{\mathcal{E}}(\mathbf{k},\mathbf{x},t) + \hat{\mathcal{E}}^*(-\mathbf{k},\mathbf{x},t), \tag{3.48}$$

$$\hat{\mathbf{H}}(\mathbf{k},t)\exp(-i\mathbf{k}\cdot\mathbf{x}) = \hat{\mathbf{k}}\wedge\hat{\mathcal{E}}(\mathbf{k},\mathbf{x},t) - \hat{\mathbf{k}}\wedge\hat{\mathcal{E}}^*(-\mathbf{k},\mathbf{x},t). \tag{3.49}$$

From (3.49) it follows that the magnetic vector $\hat{\mathbf{H}}$ need not be considered independently, so we shall restrict attention to the electric vector. The representation (3.48) of the electric field $\hat{\mathbf{E}}(\mathbf{k},t)\exp(-i\mathbf{k}\cdot\mathbf{x})$ corresponds to a forward propagating wave $\hat{\mathcal{E}}(\mathbf{k},\mathbf{x},t)$ and backward propagating one along $\hat{\mathbf{k}}$. Since in the inverse Fourier transform we integrate $\hat{E}(\mathbf{k},t)\exp(-i\mathbf{k}\cdot\mathbf{x})$ over $\mathbf{k}$, the total wave with wavevector $\mathbf{k}$ is a sum of the forward wave for $\mathbf{k}$ and the backward wave for $-\mathbf{k}$:

$$2\,\mathrm{Re}\,\hat{\mathcal{E}}(\mathbf{k},\mathbf{x},t) = \hat{\mathcal{E}}(\mathbf{k},\mathbf{x},t) + \hat{\mathcal{E}}^*(\mathbf{k},\mathbf{x},t). \tag{3.50}$$

We shall end this section by displaying the relation between the electric wave with vector $\mathbf{k}$ given by (3.50) and the Stokes parameters (Ref. 1, pp. 28–34). Let

$$w_1(\mathbf{k},t) = |w_1(\mathbf{k}.t)|\exp[i\theta_1(\mathbf{k},t)],$$

$$w_2(\mathbf{k},t) = |w_2(\mathbf{k},t)|\exp[i\theta_2(\mathbf{k},t)]. \tag{3.51}$$

With this definition, (3.47) and (3.50) yield

$$2\,\mathrm{Re}\,\hat{\mathcal{E}} = \mathbf{i}(\sqrt{2}\,|w_1|\cos(\phi+\theta_1)) + \mathbf{j}(\sqrt{2}\,|w_2|\cos(\phi+\theta_2+\pi)). \tag{3.52}$$

Define $I$, $Q$, $U$, $V$ as follows:

$$I = 2(|w_1|^2 + |w_2|^2), \quad Q = 2(|w_1|^2 - |w_2|^2),$$

$$U = 4|w_1||w_2|\cos(\theta_1 - \theta_2 - \pi),$$

$$V = 4|w_1||w_2|\sin(\theta_1 - \theta - \pi). \tag{3.53}$$

With this definition we have the identity

$$ww^\dagger = \begin{pmatrix} w_1w_1^* & w_1w_2^* \\ w_2w_1^* & w_2w_2^* \end{pmatrix} = \frac{1}{4}\begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}. \tag{3.54}$$

In (3.53) and (3.54) $I$, $Q$, $U$ and $V$ are random functions of $\mathbf{k}$ and $t$. Equation (3.54) is the desired relationship between the Stokes parameters and the coherence matrix $ww^\dagger$. Relations (3.53) correspond to Chandrasekhar's equations 159 and 160 (Ref. 1, p. 29). Reference should be made to this work for further discussion of their significance.

## 4. EVOLUTION EQUATION FOR THE COHERENCE MATRIX

In this section we shall apply the perturbation formalism of Sec. 2, to be described shortly for the present problem, to obtain an evolution equation for the expectation of the coherence matrix

$$W(\mathbf{k},t) = w(\mathbf{k},t)w^\dagger(\mathbf{k},t), \tag{4.1}$$

as $\epsilon \to 0$, where $w(\mathbf{k},t)$ satisfies (3.44). Before describing the relevant asymptotic limit we shall derive stochastic equations for the coherence matrix.

We write (3.44) in the form

$$\partial_t w = -i\gamma(\mathcal{K}_1 w + \mathcal{K}_2 w^*), \tag{4.2}$$

where $\mathcal{K}_i$, $i=1,2$, is the operator which takes one vector function $w(\mathbf{k},t)$ with another $\mathcal{K}_i w$ given by

$$(\mathcal{K}_i w)(\mathbf{k},t) = \int K_i(\mathbf{k},\mathbf{s},t)w(\mathbf{s},t)\,d\mathbf{s}, \quad i=1,2, \tag{4.3}$$

with

$$K_1(\mathbf{k},\mathbf{s},t) = (|\mathbf{k}|+|\mathbf{s}|)\exp[-iv(|\mathbf{k}|-|\mathbf{s}|)t] \times T^T(\mathbf{k})\hat{\epsilon}(\mathbf{k}-\mathbf{s},t)T(\mathbf{s}),$$

$$K_2(\mathbf{k},\mathbf{s},t) = (|\mathbf{k}|-|\mathbf{s}|)\exp[-iv(|\mathbf{k}|+|\mathbf{s}|)t] \times T^T(\mathbf{k})\hat{\epsilon}(\mathbf{k}+\mathbf{s},t)T(\mathbf{s}). \tag{4.4}$$

Here and in the sequel we shall omit the superscript on $\hat{\epsilon}$ since it is not necessary. We note that in (4.2) both $w$ and $w^*$ occur so it is convenient to adjoin to (4.2) its complex conjugate to get

$$\partial_t\begin{pmatrix} w \\ w^* \end{pmatrix} = \gamma\begin{pmatrix} -i\mathcal{K}_1 & -i\mathcal{K}_2 \\ i\mathcal{K}_2^* & i\mathcal{K}_1^* \end{pmatrix}\begin{pmatrix} w \\ w^* \end{pmatrix}. \tag{4.5}$$

Now consider the (outer) product

$$\begin{pmatrix} w(\mathbf{k},t) \\ w^*(\mathbf{k},t) \end{pmatrix}(w^\dagger(\mathbf{k}',t)w^T(\mathbf{k}',t)) = \begin{pmatrix} ww^\dagger & ww^T \\ w^*w^T & w^*w^T \end{pmatrix} = \begin{pmatrix} W & V \\ V^* & W^* \end{pmatrix}, \tag{4.6}$$

say, where

$$W^\dagger(\mathbf{k},\mathbf{k}',t) = W(\mathbf{k}',\mathbf{k},t), \quad V^*(\mathbf{k},\mathbf{k}',t) = V(\mathbf{k}',\mathbf{k},t). \tag{4.7}$$

Then, on using (4.5), we see that

$$\partial_t\begin{pmatrix} W & V \\ V^* & W^* \end{pmatrix} = \gamma\begin{pmatrix} -i\mathcal{K}_1 & -i\mathcal{K}_2 \\ i\mathcal{K}_2^* & i\mathcal{K}_1^* \end{pmatrix}\begin{pmatrix} W & V \\ V^* & W^* \end{pmatrix} + \gamma\begin{pmatrix} W & V \\ V^* & W^* \end{pmatrix}\begin{pmatrix} i\mathcal{K}_1^\dagger & -i\mathcal{K}_2^T \\ i\mathcal{K}_2^\dagger & -i\mathcal{K}_1^T \end{pmatrix}. \tag{4.8}$$

Here, an operator $\mathcal{K}$ acting on the left operates on $W$, $V$, etc., as functions of $k$ but a $\mathcal{K}$ acting on the right operators on the $k'$ dependence. Let

$$\mathcal{W}(\mathbf{k},\mathbf{k}',t) = \begin{pmatrix} W(\mathbf{k},\mathbf{k}',t) & V(\mathbf{k},\mathbf{k}',t) \\ V^*(\mathbf{k},\mathbf{k}',t) & W^*(\mathbf{k},\mathbf{k}',t) \end{pmatrix},$$

$$\tilde{\mathcal{K}} = \begin{pmatrix} -i\mathcal{K}_1 & -i\mathcal{K}_2 \\ i\mathcal{K}_2^* & i\mathcal{K}_1^* \end{pmatrix}. \tag{4.9}$$

Then (4.8) may be written

$$\partial_t\mathcal{W} = \gamma(\tilde{\mathcal{K}}(t)\mathcal{W} + \mathcal{W}\tilde{\mathcal{K}}^\dagger(t)) \equiv \gamma\mathcal{K}(t)\mathcal{W}, \tag{4.10}$$

say.

We recall that in (4.10) $\gamma$ is a small parameter and $\mathcal{K}(t)$ is a random linear operator such that

$$E\{\tilde{\mathcal{K}}(t)\} = 0, \tag{4.11}$$

which follows from the assumption that $\epsilon^{(1)}$ in (3.15) has mean zero. We are therefore in a situation formally similar to (2.1) with $\nu_{pq}\equiv 0$ and the $O(1)$ terms removed by an exponential transformation. Now of course we are dealing with integral operators rather than matrices. The formalism of Sec. 2 can be applied to (4.10) as described in Ref. 17, for example. This perturbation analysis is the Markovian limit of the smoothing approximation.[18,19] A theorem characterizing the asymptotic behavior of solutions to operator equations such as (4.10) is given in Ref. 20 but we shall proceed formally here.

Let $\tau = \gamma^2 t$ and set

$$\mathcal{W}^\gamma(\tau) = \mathcal{W}(\tau/\gamma^2), \tag{4.12}$$

where the arguments $\mathbf{k}, \mathbf{k}'$ are suppressed. Then, according to the procedures described in Refs. 17—20, as $\gamma \to 0$,

$$E\{\mathcal{W}^{\gamma}(\tau)\} \to \overline{\mathcal{W}}(\tau)$$

and

$$\partial_\tau \overline{\mathcal{W}} = \overline{\mathcal{K}\mathcal{K}}\,\overline{\mathcal{W}} \tag{4.13}$$

where

$$\overline{\mathcal{K}\mathcal{K}} = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T \int_0^t E\{\mathcal{K}(t)\mathcal{K}(t')\}\,dt'\,dt. \tag{4.14}$$

In (4.14) the integrand is the operator composed of $\mathcal{K}(t')$ acting first and followed by $\mathcal{K}(t)$. Departing somewhat from the framework of Sec. 2, we shall assume that initially $\tilde{\mathbf{E}}(\mathbf{x}, 0)$ and $\tilde{\mathbf{H}}(\mathbf{x}, 0)$ are stationary random fields with mean zero and statistically independent of the random inhomogeneities. This leads to initial data for $\overline{\mathcal{W}}$ which contain delta functions. However, since the inhomogeneities are also stationary, the kernel of the operator $\overline{\mathcal{K}\mathcal{K}}$ will also contain delta functions in a consistent manner so that (4.13) makes sense as we show below.

The explicit calculation of the operator $\overline{\mathcal{K}\mathcal{K}}$ for (4.10) with $\tilde{\mathcal{K}}$ given by (4.9) and $\mathcal{K}_i$, $i = 1, 2$, by (4.3), (4.4) is very lengthy but straightforward. We give a sample calculation in the Appendix. We shall now state the result.

First we note the consequences of stationarity assumed about the random symmetric tensor $\epsilon(\mathbf{x}, t)$ in (3.15) (with the superscript omitted). Let

$$E\{\epsilon_{uv}(\mathbf{x}', t')\epsilon_{u'v'}(\mathbf{x}' - \mathbf{y}, t' - t)\} = R_{uv,\,u'v'}(\mathbf{y}, t),$$
$$u, v, u', v' = 1, 2, 3. \tag{4.15}$$

Then, we have the following:

$$\begin{aligned}
&E\{\hat{\epsilon}_{uv}(\mathbf{k}, t')\hat{\epsilon}_{u'v'}(\mathbf{s}, t' - t)\}\\
&\quad = \int\int \exp(i\mathbf{k}\cdot\mathbf{x} + i\mathbf{s}\cdot\mathbf{x}')E\{\epsilon_{uv}(\mathbf{x}, t')\epsilon_{u'v'}(\mathbf{x}', t' - t)\}\,d\mathbf{x}\,d\mathbf{x}'\\
&\quad = \int\int \exp[i(\mathbf{k} + \mathbf{s})\cdot\mathbf{x}]R_{uv,\,u'v'}(\mathbf{y}, t)\exp(-i\mathbf{s}\cdot\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}\\
&\quad = (2\pi)^3\delta(\mathbf{k} + \mathbf{s})\int R_{uv,\,u'v'}(\mathbf{y}, t)\exp(-i\mathbf{s}\cdot\mathbf{y})\,d\mathbf{y}\\
&\quad = (2\pi)^3\delta(\mathbf{k} + \mathbf{s})\hat{R}_{uv,\,u'v'}(-\mathbf{s}, t)\\
&\quad = (2\pi)^3\delta(\mathbf{k} + \mathbf{s})\hat{R}_{uv,\,u'v'}(\mathbf{k}, t).
\end{aligned} \tag{4.16}$$

Let $S^+$ and $S^-$ be defined as follows:

$$\begin{aligned}
S^+_{uv,\,u'v'}(\mathbf{k}, \omega) &= (2\pi)^3 \int_0^\infty \hat{R}_{uv,\,u'v'}(\mathbf{k}, t)\exp(-iv\omega t)\,dt\\
&= (2\pi)^3 \int\int_0^\infty R_{uv,\,u'v'}(\mathbf{y}, t)\exp[i(\mathbf{k}\cdot\mathbf{y} - v\omega t)]\,d\mathbf{y}\,dt,\\
S^-_{uv,\,u'v'}(\mathbf{k}, \omega) &= (2\pi)^3 \int_{-\infty}^0 \hat{R}_{uv,\,u'v'}(\mathbf{k}, t)\exp(-iv\omega t)\,dt\\
&= (2\pi)^3 \int\int_{-\infty}^0 R_{uv,\,u'v'}(\mathbf{y}, t)\exp[i(\mathbf{k}\cdot\mathbf{y} - v\omega t)]\,d\mathbf{y}\,dt,\\
&u, v, u', v' = 1, 2, 3.
\end{aligned} \tag{4.17}$$

Let also

$$S_{uv,\,u'v'} = S^+_{uv,\,u'v'} + S^-_{uv,\,u'v'}, \tag{4.18}$$

which is the space—time power spectral tensor.

As a result of the time averaging in the definition (4.14) of $\overline{\mathcal{K}\mathcal{K}}$, $\overline{W}$, the top left $2\times 2$ block in $\overline{\mathcal{W}}$, decouples from the rest of $\overline{\mathcal{W}}$. In addition, if we assume that $\overline{W}(0, \mathbf{k}, \mathbf{k}') = \overline{W}_0(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}')$, which corresponds to stationary initial fields, then this form of $\overline{W}$ is preserved for $\tau > 0$, i.e., $\overline{W}(\tau, \mathbf{k}, \mathbf{k}') = W(\tau, \mathbf{k})\delta(\mathbf{k} - \mathbf{k}')$. We shall continue to denote the "diagonal" terms by $\overline{W}(\tau, \mathbf{k})$ as indicated already. Furthermore, we find that (4.13) leads to the following evolution equation for $\overline{W}(\tau, \mathbf{k}) = \overline{W}_{pp'}(\tau, \mathbf{k})$, $p, p' = 1, 2$ (summation implied):

$$\begin{aligned}
\partial_\tau \overline{W}_{pp'}(\tau, \mathbf{k}) = {} & \int (|\mathbf{k}| + |\mathbf{s}|)^2 T_{up}(\mathbf{k})T_{vq}(\mathbf{s})T_{u'p'}(\mathbf{k})T_{v'q'}(\mathbf{s})\\
& \times S_{uv,\,u'v'}(\mathbf{k} - \mathbf{s}, |\mathbf{k}| - |\mathbf{s}|)\overline{W}_{qq'}(\tau, \mathbf{s})\,d\mathbf{s}\\
& + \int (|\mathbf{k}| - |\mathbf{s}|)^2 T_{up}(\mathbf{k})T_{vq}(\mathbf{s})T_{u'p'}(\mathbf{k})T_{v'q'}(\mathbf{s})\\
& \times S_{uv,\,u'v'}(\mathbf{k} + \mathbf{s}, |\mathbf{k}| + |\mathbf{s}|)\overline{W}^*_{qq'}(\tau, \mathbf{s})\,d\mathbf{s}\\
& - \int T_{up}(\mathbf{k})T_{vr}(\sigma)T_{u'r}(\sigma)T_{v'q}(\mathbf{k})[(|\mathbf{k}| + |\sigma|)^2\\
& \times S^+_{uv,\,u'v'}(\mathbf{k} - \sigma, |\mathbf{k}| - |\sigma|) + (|\mathbf{k}| - |\sigma|)^2\\
& \times S^+_{uv,\,u'v'}(\mathbf{k} + \sigma, |\mathbf{k}| + |\sigma|)]\,d\sigma\cdot\overline{W}_{qp'}(\tau, \mathbf{k})\\
& - \int T_{up'}(\mathbf{k})T_{vr}(\sigma)T_{u'r}(\sigma)T_{v'q'}(\mathbf{k})[(|\mathbf{k}| + |\sigma|)^2\\
& \times S^{+*}_{uv,\,u'v'}(\mathbf{k} - \sigma, |\mathbf{k}| - |\sigma|) + (|\mathbf{k}| - |\sigma|)^2\\
& \times S^{+*}_{uv,\,u'v'}(\mathbf{k} + \sigma, |\mathbf{k}| + |\sigma|)]\,d\sigma\cdot\overline{W}_{pq'}(\tau, \mathbf{k}).
\end{aligned} \tag{4.19}$$

This evolution equation is the main result of this section. In the next section we shall specialize (4.19) and compare it with Chandrasekhar's[1] equations. In the remainder of this section we shall verify that Eq. (4.19) has certain properties which are necessary for a reasonable transport equation. These are:

(a) Under the "reversibility" hypothesis

$$S_{uv,\,u'v'} = S^{+*}_{u'v',\,uv} + S^{+*}_{uv,\,u'v'} = S^*_{uv,\,u'v'} \tag{4.20}$$

which follows if

$$R_{uv,\,u'v'}(\mathbf{y}, t) = R_{uv,\,u'v'}(-\mathbf{y}, -t), \tag{4.21}$$

i.e., $\epsilon(\mathbf{x}, t)$ is reversible, Eq. (4.19) conserves "total energy":

$$\partial_\tau \int (\overline{W}_{11}(\tau, \mathbf{k}) + \overline{W}_{22}(\tau, \mathbf{k}))\,d\mathbf{k} = 0. \tag{4.22}$$

(b) For each $\mathbf{k}$ the positive definiteness of $\overline{W}(\tau, \mathbf{k})$ is preserved.

Let us show that (4.22) holds. Let us take $\partial_\tau$ under the integral sign. Then it is just the left member of (4.19) contracted over $p$, $p'$ and integrated on $\mathbf{k}$. The right member similarly contracted and integrated is

$$\begin{aligned}
& \int\int (|\mathbf{k}| + |\mathbf{s}|)^2 T_{up}(\mathbf{k})T_{vq}(\mathbf{s})T_{u'p}(\mathbf{k})T_{v'q'}(\mathbf{s})\\
& \quad \times S_{uv,\,u'v'}(\mathbf{k} - \mathbf{s}, |\mathbf{k}| - |\mathbf{s}|)\overline{W}_{qq}(\tau, \mathbf{s})\,d\mathbf{s}\,d\mathbf{k}\\
& + \int\int (|\mathbf{k}| - |\mathbf{s}|)^2 T_{up}(\mathbf{k})T_{vq}(\mathbf{s})T_{u'p}(\mathbf{k})T_{v'q'}(\mathbf{s})\\
& \quad \times S_{uv,\,u'v'}(\mathbf{k} + \mathbf{s}, |\mathbf{k}| + |\mathbf{s}|)\overline{W}^*_{qq'}(\tau, \mathbf{s})\,d\mathbf{s}\,d\mathbf{k}\\
& - \int\int (|\mathbf{k}| + |\mathbf{s}|)^2 T_{up}(\mathbf{k})T_{vr}(\sigma)T_{u'r}(\sigma)T_{v'q}(\mathbf{k})\\
& \quad \times S^+_{uv,\,u'v'}(\mathbf{k} - \sigma, |\mathbf{k}| - |\sigma|)\overline{W}_{qp}(\tau, \mathbf{k})\,d\sigma\,d\mathbf{k}\\
& - \int\int (|\mathbf{k}| - |\mathbf{s}|)^2 T_{up}(\mathbf{k})T_{vr}(\sigma)T_{u'r}(\sigma)T_{v'q}(\mathbf{k})\\
& \quad \times S^+_{uv,\,u'v'}(\mathbf{k} + \sigma, |\mathbf{k}| + |\sigma|)\overline{W}_{qp}(\tau, \mathbf{k})\,d\sigma\,d\mathbf{k}\\
& - \int\int (|\mathbf{k}| + |\sigma|)^2 T_{up}(\mathbf{k})T_{vr}(\sigma)T_{u'r}(\sigma)T_{v'q}(\mathbf{k})\\
& \quad \times S^{+*}_{uv,\,u'v'}(\mathbf{k} - \sigma, |\mathbf{k}| - |\sigma|)\overline{W}_{pq}(\tau, \mathbf{k})\,d\sigma\,d\mathbf{k}\\
& - \int\int (|\mathbf{k}| - |\sigma|)^2 T_{up}(\mathbf{k})T_{vr}(\sigma)T_{u'r}(\sigma)T_{v'q}(\mathbf{k})\\
& \quad \times S^{+*}_{uv,\,u'v'}(\mathbf{k} + \sigma, |\mathbf{k}| + |\sigma|)\overline{W}_{pq}(\tau, \mathbf{k})\,d\sigma\,d\mathbf{k}
\end{aligned} \tag{4.23}$$

We shall show that the third and fifth integrals combine to cancel the first and similarly the fourth and sixth cancel the second.

First $\overline{W}^*_{pq} = \overline{W}_{qp}$ and so we may combine the third and fifth terms to get

$$-\int\int (|\mathbf{k}| + |\sigma|)^2 T_{up}(\mathbf{k}) T_{vr}(\sigma) T_{u'r}(\sigma) T_{v'q}(\mathbf{k}) \times (S^+_{uv,u'v'} + S^{+*}_{v'u',vu})(\mathbf{k}-\sigma, |\mathbf{k}| - |\sigma|) \overline{W}_{pq}(\tau, \mathbf{k})\, d\sigma\, d\mathbf{k}. \quad (4.24)$$

On renaming $\mathbf{k}$ as $\mathbf{s}$, $\sigma$ as $\mathbf{k}$, $r$ as $p$, $p$ as $q'$, $u$ as $v'$, $v'$ as $v$ and $u'$ as $u$, we get

$$-\int\int (|\mathbf{k}| + |\mathbf{s}|)^2 T_{v'q'}(\mathbf{s}) T_{u'p}(\mathbf{k}) T_{up}(\mathbf{k}) T_{vq}(\mathbf{k}) \times (S^+_{v'u'uv} + S^{+*}_{vuu'v'})(\mathbf{s}-\mathbf{k}, |\mathbf{s}| - |\mathbf{k}|) \overline{W}_{qq'}(\tau, \mathbf{s})\, d\mathbf{k}\, d\mathbf{s}. \quad (4.25)$$

But $S^+$ is symmetric in its first two subscripts since it inherits this symmetry from the symmetry of $\epsilon(\mathbf{x}, t)$. Thus, in view of (4.20) the cancellation with the first term in (4.23) follows.

Let us next show that (4.19) preserves $\overline{W}(\tau, \mathbf{k})$ as a positive definite hermitian matrix. It is clear using (4.20) that (4.19) preserves hermiticity. Let us rewrite (4.19) in the more convenient form

$$\partial_\tau \overline{W}(\tau, \mathbf{k}) = \int S_1[\mathbf{k}, \mathbf{s}, \overline{W}(\tau, \mathbf{s})]\, d\mathbf{s} + \int S_2[\mathbf{k}, \mathbf{s}, \overline{W}^*(\tau, \mathbf{s})]\, d\mathbf{s} - A(\mathbf{k})\overline{W}(\tau, \mathbf{k}) - \overline{W}(\tau, \mathbf{k}) A^+(\mathbf{k}), \quad (4.26)$$

where the operators $S_1$ and $S_2$ are linear in the third slot and $A(\mathbf{k})$ is a matrix. Transferring the last two terms to the left in (4.26), we get

$$\partial_\tau Z(\tau, \mathbf{k}) = \exp[-A(\mathbf{k})\tau] \int S_1[\mathbf{k}, \mathbf{s}, \overline{W}(\tau, \mathbf{s})]\, d\mathbf{s}\, \exp[-A^\dagger(\mathbf{k})\tau] + \exp[-A(\mathbf{k})\tau] \int S_2[\mathbf{k}, \mathbf{s}, \overline{W}^*(\tau, \mathbf{s})]\, d\mathbf{s} \times \exp[-A^\dagger(\mathbf{k})\tau], \quad (4.27)$$

where

$$Z(\tau, \mathbf{k}) = \exp[A(\mathbf{k})\tau] \overline{W}(\tau, \mathbf{k}) \exp[A^\dagger(\mathbf{k})\tau]. \quad (4.28)$$

From (4.28) it follows that $\overline{W}$ is nonnegative if and only if $Z$ is. Moreover, $\partial_\tau Z$ is nonnegative if and only if

$$\int S_1[\mathbf{k}, \mathbf{s}, \overline{W}(\tau, \mathbf{s})]\, d\mathbf{s} + \int S_2[\mathbf{k}, \mathbf{s}, \overline{W}^*(\tau, \mathbf{s})]\, d\mathbf{s} \quad (4.29)$$

is nonnegative. Thus it is enough that (4.29) is nonnegative if $\overline{W}$ is. We proceed to show this next for the second term in (4.29) since the other one follows in the same way.

Let $\xi$ be an arbitrary vector. We shall show that for each $\mathbf{k}$ and $\mathbf{s}$

$$\xi S_2[\mathbf{k}, \mathbf{s}, \overline{W}^*(\tau, \mathbf{s})] \xi^\dagger \geq 0, \quad (4.30)$$

provided that $\overline{W}$ is nonnegative. If $\overline{W}$ is nonnegative $\overline{W}^*$ can be expressed as a sum of two terms each of which is nonnegative

$$\overline{W}^* = \eta\eta^\dagger + \zeta\zeta^\dagger$$

where $\eta$ and $\zeta$ are eigenvectors normalized so that $|\eta|^2$ and $|\zeta|^2$ are the (positive) eigenvalues of $\overline{W}^*$. From the linearity of $S_2$ in the third slot, it is enough to consider

$$\xi S_2[\mathbf{k}, \mathbf{s}, \eta\eta^\dagger] \xi^\dagger. \quad (4.31)$$

Writing this explicitly from (4.19), we get

$$(|\mathbf{k}| - |\mathbf{s}|)^2 T_{up}(\mathbf{k}) \xi_p T_{vq}(\mathbf{s}) \eta^*_q T_{u'p'}(\mathbf{k}) \xi^*_{p'} T_{v'q'}(\mathbf{s}) \eta_{q'} \times S_{uvu'v'}(\mathbf{k}+\mathbf{s}, |\mathbf{k}| + |\mathbf{s}|). \quad (4.32)$$

Let $T(\mathbf{k})\xi = \boldsymbol{\alpha}$ and $T(\mathbf{s})\eta^* = \boldsymbol{\beta}$. Then to show (4.32) is positive it is enough to show

$$\alpha_u \beta_v \alpha^*_{u'} \beta^*_{v'} S_{uvu'v'}(\mathbf{k}+\mathbf{s}, |\mathbf{k}| + |\mathbf{s}|) \geq 0. \quad (4.33)$$

But $S$ is a power-spectral tensor and therefore by Bochner's theorem (4.33) is true. The proof of preservation of nonnegativity for $\overline{W}$ in (4.19) is complete.

## 5. THE TIME-HOMOGENEOUS ISOTROPIC CASE

Let us assume that the random inhomogeneous $\epsilon(\mathbf{x}, t)$ in (3.15) do not depend on $t$ and that in (4.15) we have

$$R_{uv,u'v'}(\mathbf{y}, t) = \delta_{uv}\delta_{u'v'} R(|\mathbf{y}|). \quad (5.1)$$

From (4.17) and (4.18) it follows that

$$\begin{aligned} S_{uv,u'v'}(\mathbf{k}, w) &= (2\pi)^3 \int_{-\infty}^{\infty} R(|\mathbf{y}|) \exp[i(\mathbf{k}\cdot\mathbf{y} - \omega v t)]\, d\mathbf{y}\, dt\, \delta_{uv}\delta_{u'v'} \\ &= (2\pi)^5 \frac{\delta(\omega)}{v} \int_0^\infty \int_0^\pi R(r) \exp(i|\mathbf{k}| r\cos\theta) \times r^2 \sin\theta\, d\theta\, dr\, \delta_{uv}\delta_{u'v'} \\ &= 2\pi \delta_{uv}\delta_{u'v'} \frac{\delta(\omega)}{v} 2\cdot(2\pi)^4 \int_0^\infty \frac{r \sin|\mathbf{k}| r}{|\mathbf{k}|} R(r)\, dr \\ &= 2\pi\delta_{uv}\delta_{u'v'} \frac{\delta(\omega)}{v} \tilde{S}(|k|), \end{aligned} \quad (5.2)$$

say, and similarly

$$S^+_{uv,u'v'}(\mathbf{k}, w) = \delta_{uv}\delta_{u'v'} \left(\pi \frac{\delta(\omega)}{v} + \frac{i}{\omega v}\right) \tilde{S}(|k|). \quad (5.3)$$

In (5.3) $1/\omega$ denotes the generalized function corresponding to the principal value and this will be the meaning of the singular integrals in (5.4) below.

Using polar coordinates in $\mathbf{s}$ so that $\mathbf{s} = |\mathbf{s}|\xi$ and $d\mathbf{s} = d\Omega(\hat{\mathbf{s}})|\mathbf{s}|^2 d|\mathbf{s}|$ we rewrite (4.19) for $S$ and $S^+$ given by (5.2) and (5.3). After a few rearrangements we obtain

$$\begin{aligned} \partial_\tau \overline{W}_{pp'}(\tau, \mathbf{k}) = 2\pi &\int_{|\mathbf{k}|=|\mathbf{s}|} \frac{4|\mathbf{k}|^4}{v} T_{up}(\hat{\mathbf{k}}) T_{uq}(\hat{\mathbf{s}}) T_{u'p'}(\hat{\mathbf{k}}) T_{u'q'}(\hat{\mathbf{s}}) \times \tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}} - \hat{\mathbf{s}}|) \overline{W}_{qq'}(\tau, |\mathbf{k}|\hat{\mathbf{s}})\, d\Omega(\hat{\mathbf{s}})\, d|\mathbf{s}| \\ &- \pi \frac{4|\mathbf{k}|^4}{v} \int_{|\mathbf{k}|=|\sigma|} T_{up}(\hat{\mathbf{k}}) T_{ur}(\hat{\sigma}) T_{u'r}(\hat{\sigma}) T_{u'q}(\hat{\mathbf{k}}) \tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}} - \hat{\sigma}|)\, d\Omega(\hat{\sigma}) \overline{W}_{qp'}(\tau, \mathbf{k}) \\ &+ \frac{i}{v} \int_0^\infty \frac{(|\mathbf{k}| + |\sigma|)^2 |\sigma|^2}{|\mathbf{k}| - |\sigma|} \int T_{up}(\hat{\mathbf{k}}) T_{ur}(\hat{\sigma}) T_{u'r}(\hat{\sigma}) T_{u'q}(\hat{\mathbf{k}}) \tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}} - \hat{\sigma}|)\, d\Omega(\hat{\sigma})\, d|\sigma|\, \overline{W}_{qp'}(\tau, \mathbf{k}) \end{aligned}$$

$$+\frac{i}{v}\int_0^\infty \frac{(|\mathbf{k}|-|\sigma|)^2|\sigma|^2}{|\mathbf{k}|+|\sigma|}\int T_{up}(\hat{\mathbf{k}})T_{ur}(\hat{\sigma})T_{u'r}(\hat{\sigma})T_{u'q}(\hat{\mathbf{k}})\tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}}-\hat{\sigma}|)\,d\Omega(\sigma)\,d|\sigma|\,\overline{W}_{qp'}(\tau,\mathbf{k})$$

$$-\pi\frac{4|\mathbf{k}|^4}{v}\int T_{up'}(\hat{\mathbf{k}})T_{ur}(\hat{\sigma})T_{u'r}(\hat{\sigma})T_{u'q'}(\hat{\mathbf{k}})\tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}}-\hat{\sigma}|)\,d\Omega(\hat{\sigma})\,d|\sigma|\,\overline{W}_{pq'}(\tau,\mathbf{k})$$

$$-\frac{i}{v}\int_0^\infty \frac{(|\mathbf{k}|+|\sigma|)^2|\sigma|^2}{|\mathbf{k}|-|\sigma|}\int T_{up'}(\hat{\mathbf{k}})T_{ur}(\hat{\sigma})T_{u'r}(\hat{\sigma})T_{u'q'}(\hat{\mathbf{k}})\tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}}-\hat{\sigma}|)\,d\Omega(\hat{\sigma})\,d|\sigma|\,\overline{W}_{pq'}(\tau,\mathbf{k})$$

$$-\frac{i}{v}\int_0^\infty \frac{(|\mathbf{k}|-|\sigma|)^2|\sigma|^2}{|\mathbf{k}|+|\sigma|}\int T_{up'}(\hat{\mathbf{k}})T_{ur}(\hat{\sigma})T_{u'r}(\hat{\sigma})T_{u'q'}(\hat{\mathbf{k}})\tilde{S}(|\mathbf{k}|\,|\hat{\mathbf{k}}-\hat{\sigma}|)\,d\Omega(\sigma)\,d|\sigma|\,\overline{W}_{pq'}(\tau,\mathbf{k}). \qquad (5.4)$$

Let us define the $2\times 2$ matrix

$$L=(L_{pq}(\mathbf{k},\mathbf{s}))=(T_{up}(\mathbf{k})T_{uq}(\mathbf{s})). \qquad (5.5)$$

We note that

$$L^T(\mathbf{k},\sigma)=L(\sigma,\mathbf{k}). \qquad (5.6)$$

Furthermore, recalling that $T(\hat{\mathbf{k}})$ is given by (3.36) and i, j by (3.37), it is easily verified that

$$\int L(\hat{\mathbf{k}},\hat{\sigma})L(\hat{\sigma},\hat{\mathbf{k}})\,d\Omega(\sigma)=\frac{8\pi}{3}I. \qquad (5.7)$$

For $|\mathbf{k}|$ fixed assume that $\tilde{S}(\alpha)\equiv\tilde{S}$ (constant) say, for $0\le\alpha\le 2|\mathbf{k}|$ and $S(\alpha)=0$ for $\alpha$ large. In this case (5.4) simplifies considerably because the imaginary terms cancel with the help of (5.7). Thus, for this choice of power spectrum $\tilde{S}$ we have

$$\partial_\tau\overline{W}(\tau,\mathbf{k})=\frac{64\pi^2|\mathbf{k}|^4\tilde{S}}{3v}\left(\frac{1}{4\pi}\int_{|\mathbf{k}|=|\mathbf{s}|}L(\hat{\mathbf{k}},\hat{\mathbf{s}})\overline{W}(\tau,|\mathbf{k}|\hat{\mathbf{s}})\times L^T(\hat{\mathbf{k}},\hat{\mathbf{s}})\,d\Omega(\mathbf{s})-\overline{W}(\tau,\mathbf{k})\right). \qquad (5.8)$$

The evolution equation (5.8) is a transport equation for the average coherence matrix $\overline{W}(\tau,\mathbf{k})$ in the asymptotic limit of long times or distances of propagation and weak inhomogeneities. We have assumed throughout statistical spatial homogeneity and so the size of the wavenumber $|\mathbf{k}|$ did not play any role in the asymptotics. If we interpret $\tau$ in (5.8) as optical distance, rather than time, then (5.8) coincides (up to normalization) with Chandrasekhar's equation (2.12) (Ref. 1, p. 40) including the $|\mathbf{k}|^4$ dependence of the constant, which appears as the $\lambda^{-4}$ in Chandrasekhar, originating from Rayleigh's law of scattering. This follows immediately from the relation

$$\overline{W}(\tau,\mathbf{k})=\frac{1}{4}\begin{pmatrix}\bar{I}+\bar{Q} & \bar{U}+i\bar{V}\\ \bar{U}-I\bar{V} & \bar{I}-\bar{Q}\end{pmatrix}, \qquad (5.9)$$

corresponding to (3.54) between the average coherence matrix and the mean values of the Stokes parameters $\bar{I}, \bar{Q}, \bar{U}, \bar{V}$, and the definition (5.5) of $L$.

It is interesting to note that Rayleigh's law of scattering that gave rise to Eq. (2.12) in Ref. 1 corresponds here to assumption (5.1) and the further assumption on $\tilde{S}$ stated below (5.7). If this latter assumption is not satisfied, however, the form of (5.8) changes rather drastically since we have to go back to (5.4). Thus, if the power spectrum $\tilde{S}(\alpha)$ of the inhomogeneities varies significantly in the region $0\le\alpha\le 2|\mathbf{k}|$, Eq. (5.8) is no longer valid.

When the assumption of statistical spatial homogeneity is replaced by (i) local spatial statistical homogeneity, (ii) $|\mathbf{k}|$ large compared to the reciprocal characteristic length of these variations then, we expect that (5.8) should be replaced by a space–time transport equation of the local average coherence matrix $\overline{W}(\tau,\mathbf{k},\mathbf{x})$. Note that for an asymptotic theory of this sort the size of the wavenumber of the primary fields plays an important role, in addition to $\gamma$, the size of the inhomogeneities. As we mentioned in the introduction, such a result requires additional considerations and will not be given here.

We note finally that the imaginary terms in (5.4) may be interpreted as representing residual phase retardation effects. This interpretation is motivated by the similar appearance these terms have to the real part of the effective propagation constant in the dispersion theory for the mean fields.[18]

## APPENDIX. THE OPERATOR $\overline{KK}$ OF (4.14)

In (4.14) we defined the operator

$$\overline{KK}=\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_0^t E\{K(t)K(t')\}\,dt'\,dt. \qquad (A1)$$

Here

$$K(t)W=\tilde{K}(t)W+W\tilde{K}^T(t), \qquad (A2)$$

where

$$W(\mathbf{k},\mathbf{k}',t)=\begin{pmatrix}W(\mathbf{k},\mathbf{k}',t) & V(\mathbf{k},\mathbf{k}',t)\\ V^*(\mathbf{k},\mathbf{k}',t) & W^*(\mathbf{k},\mathbf{k}',t)\end{pmatrix}, \qquad (A3)$$

$$\tilde{K}=\begin{pmatrix}-iK_1 & -iK_2\\ iK_2^* & iK_1^*\end{pmatrix}, \qquad (A4)$$

$K_1$, $K_2$ being operators defined on $2\times 2$ matrix functions $W(k)$ by

$$(K_1W)(\mathbf{k})=\int(|\mathbf{k}|+|\mathbf{s}|)\exp[-iv(|\mathbf{k}|-|\mathbf{s}|)t]\times T^T(\mathbf{k})\hat{\epsilon}(\mathbf{k}-\mathbf{s},t)T(\mathbf{s})W(\mathbf{s})\,d\mathbf{s},$$

$$(K_2W)(\mathbf{k})=\int(|\mathbf{k}|-|\mathbf{s}|)\exp[-iv(|\mathbf{k}|+|\mathbf{s}|)t]\times T^t(\mathbf{k})\hat{\epsilon}(\mathbf{k}+\mathbf{s},t)T(\mathbf{s})W(\mathbf{s})\,d\mathbf{s}. \qquad (A5)$$

Using (A2) we have

$$K(t)K(t')W=\tilde{K}(t)\tilde{K}(t')W+W\tilde{K}^T(t')\tilde{K}^+(t)+K(t)WK^+(t')+K(t')WK^T(t), \qquad (A6)$$

where $W$ is a dummy operand and is regarded as deterministic when taking expectations for (A1).

To illustrate the calculation we shall consider only the contribution of the first term on the right of (A6). From (A4)

$$\tilde{K}(t)\tilde{K}(t')=\begin{pmatrix}-K_1(t)K_1(t')+K_2(t)K_2(t') & -K_1(t)K_2(t')+K_2(t)K_1(t')\\ K_2^*(t)K_1(t')-K_1^*(t)K_2(t') & K_2(t)K_2(t')-K_1(t)K_1(t')\end{pmatrix} \tag{A7}$$

For the purposes of (A1) we need

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_0^t E\{\tilde{K}(t)\tilde{K}(t')\}dt'\,dt. \tag{A8}$$

The top left block of this is

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_0^t E\{-K_1(t)K_1(t')+K_2(t)K_2(t')\}dt'\,dt. \tag{A9}$$

From (A5) the first term in (A9) is a linear integral operator acting upon $2\times 2$ matrices whose kernel is the matrix function

$$\begin{aligned}-\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_0^t\int_{R^3}&(|\mathbf{k}|+|\boldsymbol{\sigma}|)(|\boldsymbol{\sigma}|+|\mathbf{s}|)\\ &\times\exp(-iv[(|\mathbf{k}|-|\boldsymbol{\sigma}|)t+(|\boldsymbol{\sigma}|-|\mathbf{s}|)t'])\\ &\times E\{T^T(\mathbf{k})\hat{\epsilon}(\mathbf{k}-\boldsymbol{\sigma},t)T(\boldsymbol{\sigma})T^T(\boldsymbol{\sigma})\hat{\epsilon}(\boldsymbol{\sigma}-\mathbf{s},t')T(\mathbf{s})\}d\boldsymbol{\sigma}\,dt'\,dt.\end{aligned} \tag{A10}$$

The $pq$ entry of this may be written

$$\begin{aligned}-\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_0^T\int_{R^3}&(|\mathbf{k}|+|\boldsymbol{\sigma}|)(|\boldsymbol{\sigma}|+|\mathbf{s}|)\\ &\times\exp(-iv[(|\mathbf{k}|-|\boldsymbol{\sigma}|)t+(|\boldsymbol{\sigma}|-|\mathbf{s}|)(t-\tau)])\\ &\times T_{up}(\mathbf{k})T_{ur}(\boldsymbol{\sigma})T_{u'r}(\boldsymbol{\sigma})T_{v'q}(\mathbf{s})\\ &\times E\{\hat{\epsilon}_{uv}(\mathbf{k}-\boldsymbol{\sigma},t)\hat{\epsilon}_{u'v'}(\boldsymbol{\sigma}-\mathbf{s},t-\tau)\}d\boldsymbol{\sigma}\,dt\,d\tau,\end{aligned} \tag{A11}$$

where we have changed variable of integration from $t'$ to $\tau=t-t'$ and inverted the order of integration. From (4.16) we may write this as

$$\begin{aligned}&-(2\pi)^3\int(|\mathbf{k}|+|\boldsymbol{\sigma}|)^2\delta(\mathbf{k}+\mathbf{s})T_{up}(\mathbf{k})T_{vr}(\boldsymbol{\sigma})T_{u'r}(\boldsymbol{\sigma})T_{v'q}(\mathbf{s})\\ &\times\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int_\tau^T\hat{R}_{uv,u'v'}(\mathbf{k}-\boldsymbol{\sigma},\tau)\exp[iv(|\mathbf{k}|-|\boldsymbol{\sigma}|)\tau]\,dt\,d\tau\,d\boldsymbol{\sigma}\\ &\quad=-(2\pi)^3\delta(\mathbf{k}-\mathbf{s})\int(|\mathbf{k}|+|\boldsymbol{\sigma}|)^2T_{up}(\mathbf{k})T_{vr}(\boldsymbol{\sigma})T_{u'r}(\boldsymbol{\sigma})T_{v'q}(\mathbf{s})\\ &\qquad\times S^+_{uv,u'v'}(\mathbf{k}-\boldsymbol{\sigma},|\mathbf{k}|-|\boldsymbol{\sigma}|)\,d\boldsymbol{\sigma}.\end{aligned} \tag{A12}$$

This result contributes the fourth term on the right in (4.19).

This is the contribution of $-K_1(t)K(t')$ in (A7). The contributions of the other terms may be calculated in a similar way. In like manner every part of the various terms in (A6) may be calculated.

It is found that the operator $\overline{KK}$ of (A1) is reducible in that the blocks containing $W$, $W^*$ in (A3) are invariant under the action $\overline{KK}$, and so are the blocks containing $V$, $V^*$. [It is found, for instance, that the off-diagonal blocks in (A7) are zero.] This implies that the equation for the evolution of $W$, $W^*$ decouples from that for $V$, $V^*$. We have written only Eq. (4.19) for $W$, $W^*$ since $V$, $V^*$ are not related to the Stokes parameters.

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