A stochastic Gaussian beam*

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We consider the propagation of a Gaussian beam in a strongly focusing medium with random deviations from uniformity. We compute the intensity and intensity fluctuations on the beam axis and the mean power of the fundamental mode when the random inhomogeneities are weak and the distance between source and observation point is large. We also compute the mean power transferred to each higher mode.

1. INTRODUCTION

Several investigators[1,2] have considered Gaussian beams in homogeneous or inhomogeneous deterministic media. We consider here a stochastic Gaussian beam modeled so that the following conditions hold:

(i) The beam propagates in a strongly focusing axisymmetric medium with random deviations from axial uniformity.
(ii) The random inhomogeneities are weak and the source is far from the observation point.

Our analysis is based on an explicit representation of the field of the beam in terms of a stochastic process which satisfies a stochastic differential equation. We analyze this stochastic process in the limit of weak inhomogeneities and long distances between source and observation point by using the method one of us[3] employed previously to study wave propagation in a slab of random medium.

In Sec. 2 we formulate the problem. In Sec. 3 we analyze the above mentioned stochastic differential equation. Sections 4, 5, 6, and 7 contain the main results which are as follows. The expected value of the beam intensity on its axis remains constant even at large distances from the source, but the fluctuations grow exponentially with distance from the source. The expected value of the power in the fundamental mode, normalized to 1 at the source, decays with distance from the source. Finally we give a general formula for the average power transfered to each higher mode.

The above results can be generalized to nonaxisymmetric (nonorthogonal[1]) beams by using certain group theoretical methods[4] developed by Burridge and Papanicolaou[5] for a random slab problem. We shall present these results elsewhere.

The related problem of a Gaussian beam propagating through a system of lenses with random imperfections has been treated by Steier[6], using the methods of geometrical optics.

2. FORMULATION OF THE PROBLEM

Let $e^{i\omega t}\,\Psi(x,\tilde{y},\tilde{z})$ be a time harmonic complex-valued scalar field satisfying the reduced wave equation

$$\partial_x^2\Psi + \partial_{\tilde{y}}^2\Psi + \partial_{\tilde{z}}^2\Psi + k^2 n^2(x,\tilde{y},\tilde{z})\Psi = 0, \qquad i = \sqrt{-1}, \tag{2.1}$$

where $x, \tilde{y}, \tilde{z}$ are Cartesian coordinates, $\partial_x, \partial_{\tilde{y}}, \partial_{\tilde{z}}$ denote partial derivatives, k is the free space wavenumber, and n is the index of refraction.

We shall suppose that, for each x, n^2 attains a maximum value of approximately unity on the x axis and we shall restrict attention to wave propagation with large k in the x direction and confined to the neighborhood of the x axis. Under these conditions it is useful to make the so-called parabolic approximation in solving (2.1).

This approximation may be obtained as follows: Write

$$y = k^{1/2}\tilde{y}, \quad z = k^{1/2}\tilde{z}, \quad \Psi(x,\tilde{y},\tilde{z}) = e^{-ikx}\psi(x,y,z). \tag{2.2}$$

Inserting (2.2) into (2.1) yields

$$-2i\,\partial_x\psi + k^{-1}\partial_x^2\psi + (\partial_y^2 + \partial_z^2)\psi + k[n^2(x, k^{-1/2}y, k^{-1/2}z) - 1]\psi = 0. \tag{2.3}$$

When k is large, we may neglect the term $k^{-1}\partial_x^2\psi$ and consider the initial value problem (with suitable initial condition):

$$-2i\,\partial_x\psi + \partial_y^2\psi + \partial_z^2\psi + k(n^2 - 1)\psi = 0, \quad x > 0, \qquad \psi(0,y,z) = \psi_0(y,z) \quad \text{(given)}. \tag{2.4}$$

This is the parabolic approximation. It is also called the forward scattering approximation. Note that (2.4) also governs wave propagation in the negative x direction provided that we multiply Ψ by $e^{-i\omega t}$ instead of $e^{i\omega t}$. We shall use this fact in Sec. 4.

Let us now make some further assumptions about n^2, but first let us expand n^2 as a Taylor series in $\tilde{y}, \tilde{z}$ up to quadratic terms:

$$n(x,\tilde{y},\tilde{z}) = a(x) - [b_{11}(x)\tilde{y}^2 + 2b_{12}(x)\tilde{y}\tilde{z} + b_{22}(x)\tilde{z}^2] = a(x) - k^{-1}[b_{11}(x)y^2 + 2b_{12}(x)y\,z + b_{22}z^2], \tag{2.5}$$

where $a(x)$ is approximately 1 and the quadratic form is positive definite. The linear terms in $\tilde{y}, \tilde{z}$ have been neglected in (2.5) since we wish n^2 to have a maximum on the x axis.

We shall now restrict (2.5) to be axisymmetric about the x axis and allow $a(x)$ and $b(x)$ to be random functions of x as follows:

$$n^2(x,\tilde{y},\tilde{z}) = a(x) - k^{-1}b(x)(y^2 + z^2), \tag{2.6}$$

with

$$a(x) = 1 + \epsilon\alpha(x), \; b(x) = b_0 + \epsilon\beta(x). \tag{2.7}$$

Here ϵ is a small parameter and $\alpha(x), \beta(x)$ are stationary stochastic processes with expected value zero.

If the random medium is not strongly focusing, i.e., if the term $-k^{-1}(y^2 + z^2)$ is not present in (2.6), then it is natural to assume $\alpha = \alpha(x,y,z)$, a random field. When the field ψ is confined by the focusing to a narrow cylinder about the x-axis, then our assumption (2.6) is a reasonable one. The problem without focusing has been treated by Klyatskin and Tatarskii.[7] The problem with focusing and $\alpha = \alpha(x,y,z), \beta = \beta(x,y,z)$ has been analyzed by one of the authors,[8] but here we seek more detailed information which is difficult to obtain by the method presented there. We now proceed with the formulation of the present problem.

Using (2.6) and (2.7), we rewrite (2.4):

$$i\partial_x\psi = \tfrac{1}{2}[(\partial_y^2 + \partial_z^2) - b_0(y^2 + z^2)]\psi + \epsilon[\tfrac{1}{2}k\alpha(x) - \tfrac{1}{2}(y^2 + z^2)\beta(x)]\psi, \quad \psi(0,y,z) = \psi_0(y,z). \qquad (2.8)$$

By redefining dependent and independent variables we may take $b_0 = 1$ as we do in the sequel.

We shall choose ψ_0 to be the fundamental mode of the unperturbed problem and so

$$\psi_0(0,y,z) = (1/\sqrt{\pi})\, e^{-(y^2+z^2)/2}. \qquad (2.9)$$

The orthonormal modes $h_{pq}(y,z)$, $p,q = 0,1,2,\cdots$, satisfy the eigenvalue problem

$$\tfrac{1}{2}[\partial_y^2 + \partial_z^2 - (y^2 + z^2)]h_{pq}(y,z),$$
$$h_{pq}(y,z) = [\pi 2^p p!\, 2^q q!]^{-1/2} H_p(y) H_q(z) e^{-(y^2+z^2)/2},$$
$$\lambda_{pq} = -(p + q + 1), \quad p,q = 0,1,2,\cdots. \qquad (2.10)$$

Here $H_p(y)$ denotes the p^{th} Hermite polynomial,

$$H_p(y) = (-1)^p e^{y^2}\left(\frac{d^p}{dy^p} e^{-y^2}\right), \qquad (2.11)$$

and we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{pq}(y,z)h_{p'q'}(y,z)\,dy\,dz = \delta_{pp'}\delta_{qq'}. \qquad (2.12)$$

Thus the initial field (2.9) equals $h_{00}(y,z)$, the fundamental mode.

From (2.8) and (2.9) it follows that the solution of (2.8) is

$$\psi(x,y,z) = \frac{1}{\sqrt{\pi}} \exp\left(-\int_0^x \mu(\xi)d\xi - \frac{i\epsilon k}{2}\int_0^x \alpha(\xi)d\xi\right) \times \exp[-\tfrac{1}{2}i\mu(x)(y^2 + z^2)], \qquad (2.13)$$

where $\mu(x)$ is a complex-valued random function satisfying the equation

$$\frac{d\mu(x)}{dx} + \mu^2(x) + [1 + \epsilon\beta(x)] = 0, \quad x \ge 0, \quad \mu(0) = -i. \qquad (2.14)$$

Define $u(x)$ as the complex-valued solution of

$$\frac{d^2u(x)}{dx^2} + [1 + \epsilon\beta(x)]u(x) = 0, \quad x \ge 0,$$
$$u(0) = 1, \quad \frac{du(0)}{dx} = -i. \qquad (2.15)$$

Then we have that

$$\mu(x) = \frac{1}{u(x)}\frac{du(x)}{dx}, \qquad (2.16)$$

$$\psi(x,y,z) = \frac{1}{\sqrt{\pi}u(x)} \exp\left(\frac{-i\epsilon k}{2}\int_0^x \alpha(\xi)d\xi - \frac{i}{2}\mu(x)(y^2 + z^2)\right). \qquad (2.17)$$

Thus the random field ψ is completely determined by the random function $u(x)$.

Our objective is to determine the statistical characteristics of $u(x)$, hence of ψ, given the statistical characteristics of $\alpha(x)$ and $\beta(x)$. We shall do this in the limit of large x and small ϵ with $\epsilon^2 x$ of order one. In particular we shall obtain the first two moments of the beam intensity $J(x)$ on the beam axis

$$J(x) = |\psi(x,0,0)|^2 = 1/(\pi|u(x)|^2), \qquad (2.18)$$

and the expected value of $Q_{00}(x)$, the squared modulus of the amplitude of the fundamental mode

$$Q_{00}(x) = \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x,y,z)\frac{e^{-(y^2+z^2)/2}}{\sqrt{\pi}}\,dy\,dz\right|^2,$$
$$Q_{00}(0) = 1. \qquad (2.19)$$

We also study the transfer of power to the higher modes.

Note that $J(x)$ and $Q_{00}(x)$ do not depend on $\alpha(x)$ since it only affects the phase of ψ.

3. THE STATISTICAL PROBLEM FOR $u(x)$

In this section we analyze the stochastic differential equation (2.15). Let $A(x)$ and $B(x)$ be complex-valued random functions and define

$$u(x) = A(x)\cos x + B(x)\sin x,$$
$$\dot{u}(x) = -A(x)\sin x + B(x)\cos x. \qquad (3.1)$$

Here the dot stands for d/dx. On using (3.1) in (2.15) and rearranging the result, we obtain the following system of equations for $A(x)$ and $B(x)$

$$\begin{pmatrix}\dot{A}(x)\\ \dot{B}(x)\end{pmatrix} = \epsilon\begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{pmatrix}\begin{pmatrix}0 & 0\\ -\beta(x) & 0\end{pmatrix}\begin{pmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{pmatrix}\begin{pmatrix}A(x)\\ B(x)\end{pmatrix},$$
$$\begin{pmatrix}A(0)\\ B(0)\end{pmatrix} = \begin{pmatrix}1\\ -i\end{pmatrix}. \qquad (3.2)$$

Let $M(x,x')$, $x \ge x'$, denote the fundamental solution matrix of (3.2), i.e., the matrix-valued solution of (3.2) with initial condition $M(x',x') = I$, the identity matrix. Since the matrix multiplying $(A(x), B(x))$ on the right side of (3.2) has trace zero we have

$$\det M(x,x') = 1, \quad x \ge x'. \qquad (3.3)$$

Thus $M(x,x')$ is a real 2×2 unimodular matrix-valued process; it belongs in $Sl(2,R)$, the group of all such matrices.

Define the matrices b_1, b_2, b_3 by

$$b_1 = \tfrac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad b_2 = \tfrac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad b_3 = \tfrac{1}{2}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}. \qquad (3.4)$$

Since $M \in Sl(2,R)$, we can write[5]

$$M(x,0) = e^{b_1\chi} e^{b_2\theta} e^{b_1\phi}. \qquad (3.5)$$

Here $\chi(x)$, $\theta(x)$, $\phi(x)$ are random functions. The parametrization (3.5) is analogous to the Euler-angle parametrization of $SU(2)$, the group of 2×2 unitary unimodular matrices, except that here χ, θ, ϕ vary over the ranges

$$0 \le \chi < 4\pi, \quad 0 \le \phi < 2\pi, \quad 0 \le \theta < \infty. \tag{3.6}$$

Now we insert (3.5) into (3.2) and, after some computation, obtain the following system of equations:

$$\begin{aligned} \dot{\chi} &= \epsilon\beta(x)[1 + \cos(2x + \chi)\coth\theta], \\ \dot{\theta} &= \epsilon\beta(x)\sin(2x + \chi), \\ \dot{\phi} &= -\epsilon\beta(x)\csc\theta\cos(2x + \chi), \\ \theta(0) &= 0, \quad \chi(0) + \phi(0) = 0, \quad \chi(0) - \phi(0) \\ &\text{arbitrary.} \end{aligned} \tag{3.7}$$

Let us denote expected values by $E\{\ \}$. As in Sec. 2 we assume that

$$E\{\beta(x)\} = 0, \tag{3.8}$$

$$E\{\beta(x)\beta(x')\} = R(x - x'). \tag{3.9}$$

$R(x)$ is the covariance of the process $\beta(x)$. Let us also assume that

$$|\beta(x)| \le 1 \tag{3.10}$$

almost surely. Under these circumstances and a few other additional assumptions on $\beta(x)$ we may use formally a result of R. Z. Hashminskii[9] to obtain the asymptotic behavior of the processes $\chi(x), \theta(x), \phi(x)$ defined by (3.7). We say formally because one condition in that theorem cannot be satisfied namely, the right sides of (3.7) are not bounded as a function of χ, θ, ϕ in the range (3.6). By making additional restrictions on $\beta(x)$ however, the following analysis can be made rigorous.[10]

The above mentioned result is as follows: Let

$$\tau = \epsilon^2 x, \quad \chi^{(\epsilon)}(\tau) = \chi(\tau/\epsilon^2), \quad \theta^{(\epsilon)}(\tau) = \theta(\tau/\epsilon^2), \quad \phi^{(\epsilon)}(\tau) = \phi(\tau/\epsilon^2). \tag{3.11}$$

Then if $f(\chi, \theta, \phi)$ is any bounded smooth function of its arguments, the conditional expectation

$$P^{(\epsilon)}(\tau; \chi, \theta, \phi) = E\{f(\chi^{(\epsilon)}(\tau), \theta^{(\epsilon)}(\tau), \phi^{(\epsilon)}(\tau))\}, \tag{3.12}$$

given $\chi^{(\epsilon)}(0) = \chi, \theta^{(\epsilon)}(0) = \theta, \phi^{(\epsilon)}(0) = \phi$ converges as $\epsilon \to 0$ to $P^{(0)}(\tau; \chi, \theta, \phi)$, which satisfies the diffusion equation

$$\begin{aligned} \frac{\partial P^{(0)}}{\partial \tau} &= \gamma\left(\frac{\partial^2}{\partial\theta^2} + \coth\theta\frac{\partial}{\partial\theta} + \frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}\right)P^{(0)} \\ &+ \left(\int_0^\infty R(s)ds + \gamma\coth^2\theta\right)\frac{\partial^2 P^{(0)}}{\partial\chi^2} \\ &- \tfrac{1}{2}\int_0^\infty R(s)\sin 2s\, ds\ (1 - \csc^2\theta)\frac{\partial P^{(0)}}{\partial\chi}. \\ P^{(0)}(0; \chi, \theta, \phi) &= f(\chi, \theta, \phi), \quad \gamma = \tfrac{1}{2}\int_0^\infty R(s)\cos 2s\, ds. \end{aligned} \tag{3.13}$$

In the next section we shall see that the quantities of principal interest to us do not depend on χ. This leads to a very significant simplification in (3.13) since the terms involving the χ derivatives can be ignored on the right side of (3.13). The differential operator in the θ, ϕ variables in (3.13) is the Laplace–Beltrami operator in the Lobachevski plane with θ, ϕ as polar coordinates.[11]

This could have been predicted heuristically from the second and third equations in (3.7) since the third may be written in the form $\sinh\theta\dot{\phi} = -\epsilon\beta(x)\cos(2x + \chi)$. Thus, the "radial velocity" $\dot{\theta}$ and the "transverse velocity" $\sinh\theta\dot{\phi}$ are proportional to $\sin(2x + \chi)$ and $-\cos(2x + \chi)$, respectively. Since x is a fast varying variable in the above limit while χ is slowly varying, we would expect these equations to approximate a Brownian motion which is characterized by the fact that all directions of infinitesimal motion are equally likely. The first and second equations of (3.13), on the other hand, would lead us not to expect Brownian motion in the (θ, χ) plane.

The Laplace–Beltrami operator is self-adjoint relative to the volume element

$$\sinh\theta\, d\theta d\phi \tag{3.14}$$

and hence, because $\theta(0) = 0$, it suffices to have the fundamental solution of (3.13) (with the χ derivatives absent) which is initially concentrated at $\theta = 0$. This is given by[3]

$$P(\tau, \theta, \phi) = \frac{e^{-\gamma\tau/4}}{4\sqrt{2}(\pi\gamma\tau)^{3/2}}\int_\theta^\infty \frac{\rho e^{-\rho^2/4\gamma\tau}d\rho}{\sqrt{\cosh\rho - \cosh\theta}}, \quad \theta \ge 0, \quad 0 \le \phi < 2\pi. \tag{3.15}$$

This function is the transition probability density, relative to the volume element (3.14), of $\theta^{(\epsilon)}(\tau), \phi^{(\epsilon)}(\tau)$ given $\theta(0) = 0$, $\phi(0)$ arbitrary, in the limit $\epsilon \to 0, \tau$ fixed. Thus if $f(\theta, \phi)$ is a bounded smooth function of θ and ϕ, we have

$$\lim_{\epsilon\to 0} E\{f(\theta^{(\epsilon)}(\tau), \phi^{(\epsilon)}(\tau))\} = \int_0^\infty \left(\int_0^{2\pi} f(\theta, \phi)d\phi\right)P(\tau, \theta)\sinh\theta\, d\theta. \tag{3.16}$$

Here and hereafter $E\{\ \}$ denotes expectation conditional on $\theta(0) = 0$. In (3.16) we have also used the fact that P of (3.16) is independent of ϕ, i.e., $\phi^{(\epsilon)}(\tau)$ is uniformly distributed in $(0, 2\pi)$ in the limit $\epsilon \to 0$, given $\theta(0) = 0$.

We now proceed with the application of (3.15), (3.16) to the beam problem.

4. MEAN INTENSITY OF THE BEAM ON ITS AXIS

First we alter the original formulation (2.8) of the problem for the field $\psi(x, y, z)$ in the following way. According to (2.4) the source of the beam is located on the plane $x = 0$, and $J(x)$ in (2.18) is the intensity of the beam at $(x, 0, 0)$. Thus $J(x)$ is considered a function of the observation point. From (3.1) and the definition of the matrix $M(x, 0)$ we have

$$\begin{pmatrix} u^+(x) \\ \dot{u}^+(x) \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} M_{11}(x,0) & M_{12}(x,0) \\ M_{21}(x,0) & M_{22}(x,0) \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{4.1}$$

where now we have written u^+ for u to indicate that it corresponds to waves propagating in the positive x direction. We shall later use superscript $^-$ to indicate propagation in the negative x direction.

If we locate the source at the point x and replace $e^{i\omega t}$ by $e^{-i\omega t}$, then the beam points in the negative x direction. We may seek the intensity at $x = 0$ as a function of x, the location of the source. With this point of view we have

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} M_{11}(x,0) & M_{12}(x,0) \\ M_{21}(x,0) & M_{22}(x,0) \end{pmatrix} \begin{pmatrix} u^- & (x) \\ \dot{u}^- & (x) \end{pmatrix}, \tag{4.2}$$

where $u^-(x)$ and $\dot{u}^-(x)$ are defined by (4.2). Thus, $J^-(x) = (1/\pi)|u^-(x)|^{-2}$ is the beam intensity at (0, 0, 0) considered as a function of the location of the source. Similar remarks hold for $Q_{00}^-(x)$. Hereafter we shall adopt this point of view. While $u^-(x), J^-(x)$, and $Q_{00}^-(x)$ are physical quantities distinct from those denoted by $u(x), J(x)$, and $Q_{00}(x)$ in Sec. 2, $J(x)$ and $Q_{00}(x)$ have the same moments in both cases.

Let us now express $J^-(x)$ in terms of the functions χ, θ, ϕ of (3.5). From (4.2), (3.4), and (3.5) it follows that

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos(x+\chi/2) & \sin(x+\chi/2) \\ -\sin(x+\chi/2) & \cos(x+\chi/2) \end{pmatrix} \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} \times \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix} \begin{pmatrix} u^-(x) \\ \dot{u}^-(x) \end{pmatrix}, \tag{4.3}$$

and hence

$$u^-(x) = e^{i[(\chi(x)/2)+x]}\{\cos[\phi(x)/2]e^{-\theta(x)/2} + i\sin[\phi(x)/2]e^{\theta(x)/2}\}, \tag{4.4}$$

$$\dot{u}^-(x) = e^{i[(\chi(x)/2)+x]}\{\sin[\phi(x)/2]e^{-\theta(x)/2} - i\cos[\phi(x)/2]e^{\theta(x)/2}\}, \tag{4.5}$$

$$J^-(x) = (1/\pi)\{\cos^2[\phi(x)/2]e^{-\theta(x)} + \sin^2[\phi(x)/2]e^{\theta(x)}\}^{-1}. \tag{4.6}$$

For comparison we note the corresponding form for $J^+(x)$:

$$J^+(x) = (1/\pi)[\cos^2(x+\chi/2)e^{\theta(x)} + \sin^2(x+\chi/2)e^{-\theta(x)}]^{-1}.$$

To find the expected value of $J^-(x)$ in the limit $x \to \infty$, $\epsilon \to 0$, $\epsilon^2 x = \tau$ fixed, we use (4.6) in (3.16) and obtain

$$\lim_{\epsilon\to 0} E\left\{J^-\left(\frac{\tau}{\epsilon^2}\right)\right\} = \int_0^\infty \left[\int_0^{2\pi} \frac{1}{\pi}\left(\cos^2\frac{\phi}{2}e^{-\theta} + \sin^2\frac{\phi}{2}e^{\theta}\right)^{-1} d\phi\right] P(\tau,\theta)\sinh\theta\, d\theta. \tag{4.7}$$

Here $P(\tau,\theta)$ is given by (3.15). The angular integral in (4.7) is elementary. We have

$$\int_0^{2\pi}\left(\cos^2\frac{\phi}{2}e^{-\theta} + \sin^2\frac{\phi}{2}e^{\theta}\right)^{-1} d\phi = 2\pi, \qquad \theta \geq 0. \tag{4.8}$$

Thus, since $P(\tau,\theta)$ is a probability density, (4.7) and (4.8) yield

$$\lim_{\epsilon\to 0} E\{J^-(\tau/\epsilon^2)\} = 1/\pi = J(0). \tag{4.9}$$

This is the main result of this section.

The result (4.9) is somewhat surprising. Because the operator on the right side of (2.8) is self-adjoint we have, for any $x \geq 0$,

$$\int_{-\infty}^\infty\int_{-\infty}^\infty |\psi(x,y,z)|^2\,dy\,dz = \int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{-(y^2+z^2)}}{\pi}\,dy\,dz = 1. \tag{4.10}$$

Thus, in addition to (4.10) which is true for all $x \geq 0$ without taking expected values, the expected value of the intensity at a point on the beam axis tends to the constant $1/\pi$ when the fluctuations are weak and the generator is far away. This is a consequence of the special form (2.6), (2.7) of the strongly focusing index of refraction.

We could, in principle, obtain (4.9) by using the methods of Secs. 3 and 4 in a previous work of one of the authors.[8] This, however, is not a simple matter. Our analysis bypasses these difficulties because it exploits the explicit representation (2.13) of the field ψ. This representation is in turn a consequence of the form (2.6), (2.7) for $n^2(x,y,z)$.

5. FLUCTUATIONS OF THE INTENSITY

In this section we compute the expected value of the square of the difference of the beam intensity $J(x)$ from its expected value

$$\lim_{\epsilon\to 0} E(\{[J^-(\tau/\epsilon^2)] - E[J^-(\tau/\epsilon^2)]\}) = \lim_{\epsilon\to 0} E\{[J^-(\tau/\epsilon^2)]^2\} - \pi^{-2}. \tag{5.1}$$

From (4.6), (3.15) and (3.16) we have

$$\lim_{\epsilon\to 0} E\left\{\left[J^-\left(\frac{\tau}{\epsilon^2}\right)\right]^2\right\} = \int_0^\infty\left\{\pi^2\int_0^{2\pi}\left(\cos^2\frac{\phi}{2}e^{-\theta} + \sin^2\frac{\phi}{2}e^{\theta}\right)^{-2} d\phi\right\} P(\tau,\theta)\sinh\theta\, d\theta. \tag{5.2}$$

The angular integral is again elementary:

$$\int_0^{2\pi}\left(\cos^2\frac{\phi}{2}e^{-\theta} + \sin^2\frac{\phi}{2}e^{\theta}\right)^{-2} d\phi = 2\pi\cosh\theta, \qquad \theta \geq 0. \tag{5.3}$$

Thus, using (5.3) in (5.2) we obtain

$$\begin{aligned}\lim_{\epsilon\to 0} E\left\{\left[J^-\left(\frac{\tau}{\epsilon^2}\right)\right]^2\right\} &= \frac{e^{-\gamma\tau/4}\pi^{-2}}{2\sqrt{2\pi}(\gamma\tau)^{3/2}} \times \int_0^\infty \cosh\theta\sinh\theta\int_\theta^\infty \frac{\rho e^{-\rho^2/4\gamma\tau}\,d\rho}{\sqrt{\cosh\rho - \cosh\theta}}\,d\theta \\ &= \frac{e^{-\gamma\tau/4}\pi^{-2}}{2\sqrt{2\pi}(\gamma\tau)^{3/2}}\int_0^\infty \rho e^{-\rho^2/4\gamma\tau}\int_0^\rho \frac{\cosh\theta\sinh\theta\,d\theta}{\sqrt{\cosh\rho - \cosh\theta}}\,d\rho \\ &= \frac{e^{-\gamma\tau/4}\pi^{-2}}{2\sqrt{2\pi}(\gamma\tau)^{3/2}} \times\int_0^\infty \rho e^{-\rho^2/4\gamma\tau}\,\frac{2\sqrt{2}}{3}(1+2\cosh\rho)\sinh\frac{\rho}{2}\,d\rho \\ &= \pi^{-2}e^{2\gamma\tau}.\end{aligned} \tag{5.4}$$

We have therefore

$$\lim_{\epsilon\to 0} E\{(J^-(\tau/\epsilon^2) - E[J^-(\tau/\epsilon^2)])^2\} = \pi^{-2}(e^{2\gamma\tau} - 1). \quad (5.5)$$

This is the main result of this section. Its physical meaning should be clear. Even though the expected value of the intensity $E\{J(x)\}$ is approximately constant when ϵ is small, x is large, and $\tau = \epsilon^2 x$ is of order one, the expected value of its fluctuation grows exponentially in the parameter τ at a rate equal to 2γ where γ is given by (3.13). The fluctuations in the intensity are, of course, zero on the plane of the source and (5.5) is indeed zero when $\tau = 0$.

6. MEAN POWER OF THE FUNDAMENTAL MODE

In this section we compute the expected value of the power $Q_{00}(x)$ in the fundamental mode, given by (2.19), in the familiar limit.

By using (2.17), (2.19) and (4.4), (4.5), we find that

$$Q_{00}^-(x) = 2[\cosh\theta(x) + 1]^{-1}. \quad (6.1)$$

On using (6.1) and (3.15) in (3.16) it follows that

$$\lim_{\epsilon\to 0} E\left\{Q_{00}^-\left(\frac{\tau}{\epsilon^2}\right)\right\} = \frac{e^{-\gamma\tau/4}}{\sqrt{2\pi}\,(\gamma\tau)^{3/2}} \int_0^\infty \sinh\theta[\cosh\theta + 1]^{-1} \times \int_\theta^\infty \frac{\rho e^{-\rho^2/4\gamma\tau}\,d\rho}{\sqrt{\cosh\rho - \cosh\theta}}\,d\theta$$

$$= \frac{e^{-\gamma\tau/4}}{\sqrt{2\pi}\,(\gamma\tau)^{3/2}} \times \int_0^\infty \rho e^{-\rho^2/4\gamma\tau} \int_0^\rho \frac{\sinh\theta\, d\theta}{(\cosh\theta + 1)\sqrt{\cosh\rho - \cosh\theta}}\,d\rho$$

$$= \frac{4e^{-\gamma\tau/4}}{\sqrt{\pi}} \int_0^\infty \frac{\rho^2 e^{-\rho^2}d\rho}{\cosh(\rho\sqrt{\gamma\tau})}. \quad (6.2)$$

Formula (6.2) is the desired result. In the limit under consideration the expected value of the power in the fundamental mode decays exponentially with τ. For $\tau = 0$ the expected value equals 1 of course. Numerical integration of (6.2) yields the graph shown in Fig. 1. The function (6.2) arises also in a different context[3,12] and its graph was obtained there.

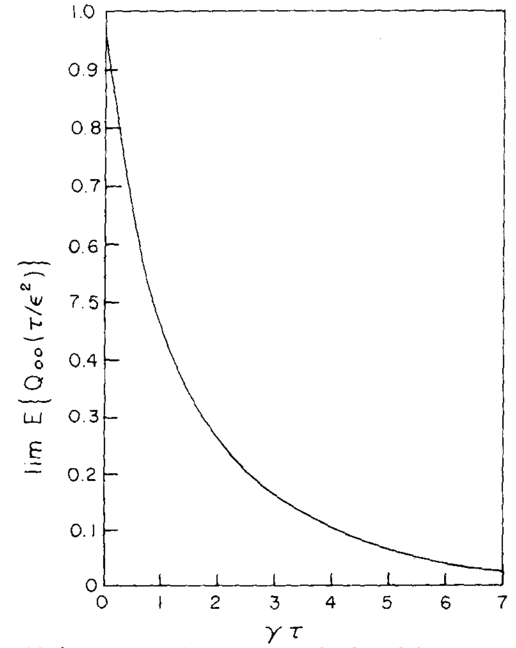

FIG. 1. Here we plot the expected value of the power remaining in the fundamental mode at the point $x = \tau/\epsilon^2$, in the limit $\epsilon \to 0$ versus $\gamma\tau$. See formula (6.2).

Finally, we note the result (6.2) and the graph shows in a very explicit manner how power leaks out of the excited (fundamental) mode into the other (higher) modes.

7. EXPECTED VALUE OF THE POWER IN HIGHER MODES

In previous sections we have been concerned only with the total field on the beam axis and with the power in the fundamental mode. However, one may also consider the higher modes of propagation. We calculate here the expected value of the power in each higher mode in the familiar asymptotic limit. We also derive an expression for the generating function for the modal energy distribution.

We define the $(2p, 2q)$ amplitude of the field by

$$I_{2p,2q}^-(x) = \int_{R^2} h_{2p,2q}(y,z)\psi^-(x,y,z)\,dy\,dz, \qquad p,q = 0,1,2,\ldots, \quad (7.1)$$

where the basis functions $\{h_{2p,2q}\}$ are given by (2.10)–(2.11). The power in the $(2p, 2q)$th mode is then defined as

$$Q_{2p,2q}^-(x) = |I_{2p,2q}^-(x)|^2, \qquad p,q = 0,1,2,\ldots. \quad (7.2)$$

We need only consider these modes since all others vanish. Utilizing the integral identity for Hermite polynomials,[13]

$$\int_{-\infty}^\infty e^{-t^2} H_{2p}(xt)\,dt = \sqrt{\pi}\,\frac{(2p)!}{p!}\,(x^2 - 1), \quad (7.3)$$

we calculate $Q_{2p,2q}^-(x)$ in terms of u^- and $\dot{u}^-$:

$$Q_{2p,2q}^-(x) = 4^{-p-q+1}\binom{2p}{p}\binom{2q}{q} \times\Big[\Big(|u^-|^2 + |\dot{u}^-|^2 + i\,(\dot{u}^-(u^-)^* - (\dot{u}^-)^* u^-)\Big)^{-1} \times \left(\frac{|u^-|^2 + |\dot{u}^-|^2 - i(\dot{u}^-(u^-)^* - (\dot{u}^-)^* u^-)}{|u^-|^2 + |\dot{u}^-|^2 + i(\dot{u}^-(u^-)^* - (\dot{u}^-)^* u^-)}\right)^{p+q}\Big]. \quad (7.4)$$

Here we have used (2.16), (2.17) for ψ^- and we have expressed the factorials in terms of binomial coefficients. * stands for complex conjugate. When we use (4.4) and (4.5) in (7.4), we obtain the following expression:

$$Q_{2p,2q}^-(x) = 2^{2(p+q)+1}\binom{2p}{p}\binom{2q}{q}(\cosh\theta + 1)^{-1}\tanh^{2(p+q)}\frac{\theta}{2}. \quad (7.5)$$

While we could calculate the expected value of $Q_{2p,2q}$, we prefer first to sum over the degenerate modes corresponding to the same $r \equiv p + q$. Thus, we fix r and define $\tilde{Q}_r(x)$ by

$$\tilde{Q}_r(x) = \sum_{p+q=r} Q_{2p,2q}(x), \qquad r = 0,1,2,\cdots. \quad (7.6)$$

When we employ the identities

$$\sum_{p+q=r} \frac{1}{4^r}\binom{2p}{p}\binom{2q}{q} = \sum_{p+q=r} (-1)^r \binom{-1/2}{p}\binom{-1/2}{p}$$
$$= (-1)^r \binom{-1}{r}, \quad (7.7)$$

we find that

$$\tilde{Q}_r(x) = 2(-1)^r \binom{-1}{r} (\cosh\theta + 1)^{-1} \tanh^{2r}\frac{\theta}{2}\,. \quad (7.8)$$

Next we use (7.8) and (3.15) in (3.16) and obtain

$$\lim_{\epsilon\to 0} E\left\{\tilde{Q}_r\left(\frac{\tau}{\epsilon^2}\right)\right\} = \frac{1}{2\sqrt{2\pi}} \frac{e^{-\gamma\tau/4}}{(\gamma\tau)^{3/2}} \times \int_\theta^\infty \left[\frac{2(-1)^r \; {-1 \atop r} \; \tanh^{2r}(\theta/2)}{\cosh\theta + 1}\right] \times \int_0^\infty \frac{\rho e^{-\rho^2/4\gamma\tau} d\rho}{\sqrt{\cosh\rho - \cosh\theta}} \sinh\theta \, d\theta$$
$$= \frac{e^{-\gamma\tau/4}}{2\sqrt{2\pi}(\gamma\tau)^{3/2}} \int_0^\infty \rho e^{-\rho^2/4\gamma\tau} F_r(\rho)\, d\rho, \quad (7.9)$$

where $F_r(\rho)$ is given by

$$F_r(\rho) = 2(-1)^r \binom{-1}{r} \int_0^\rho \frac{\tanh^{2r}(\theta/2)\sinh\theta\, d\theta}{(\cosh\theta + 1)\sqrt{\cosh\rho - \cosh\theta}}. \quad (7.10)$$

The qualitative behavior of the expected value of the power in the rth mode may be obtained from (7.9). For fixed $r > 0$, the expected value is zero initially ($\tau = 0$) and increases with increasing τ until a maximum is reached, after which it decays. For fixed τ, the expected value of the power in the rth mode decreases with increasing r. (7.9) and (7.10) are the main results of this section. The function $F_r(\rho)$ of (7.10) may be simplified through the change of variable $T = \tanh^2(\theta/2)$,

$$F_r(\rho) = \sqrt{2}\,(-1)^r \binom{-1}{r} \int_0^{\tanh^2(\rho/2)} \times \frac{T^r\, dT}{\sqrt{1-T}\sqrt{(1-T)\cosh^2(\rho/2) - 1}}. \quad (7.11)$$

In this form the integral can be evaluated by repeated integration by parts using formula (231, 7a).[14]

It is also possible to define a modal generating function for which the integrations are elementary. Specifically we define G by

$$G(x;z) = \sum_{r=0}^\infty \tilde{Q}_r(x) z^r. \quad (7.12)$$

On using (7.9) we find that

$$\lim_{\epsilon\to 0} E\left\{G\left(\frac{\tau}{\epsilon^2}; z\right)\right\} = \frac{e^{-\gamma\tau/4}}{2\sqrt{2\pi}(\gamma\tau)^{3/2}} \int_0^\infty \rho e^{-\rho^2/4\gamma\tau} F(\rho;z)\, d\rho, \quad (7.13)$$

where $F(\rho;z)$ is given by

$$F(\rho;z) = \sum_{r=0}^\infty F_r(\rho) z^r. \quad (7.14)$$

We now insert (7.11) in (7.14) and obtain

$$F(\rho;z) = \sqrt{2}\int_0^{\tanh^2(\rho/2)} \frac{1}{\sqrt{1-T}\sqrt{(1-T)\cosh^2(\rho/2) - 1}} \times \left\{\sum_{r=0}^\infty \binom{-1}{r}(-1)^r (zT)^r \, dT\right\}$$
$$= \sqrt{2}\int_0^{\tanh^2(\rho/2)} \frac{1}{\sqrt{1-T}\sqrt{(1-T)\cosh^2(\rho/2) - 1}} \times \frac{dT}{1 - zT}. \quad (7.15)$$

This last integral is known; formula (221, 7b) yields[14]

$$F(\rho;z) = \frac{\sqrt{2}}{\sqrt{(1-z)[(1-z)S+1]}} \log\Big[1 + 2(1-z)S - 2 \times \sqrt{(1-z)S[(1-z)S+1]}\Big], \quad S = \sinh^2(\rho/2). \quad (7.16)$$

Thus, we have obtained a representation of the expected value of the modal generating function as a single integral over ρ, (7.13). Clearly the expected value of the power in the rth mode may be obtained by differentiation as follows:

$$\lim_{\epsilon\to 0} E\left\{\tilde{Q}_r\left(\frac{\tau}{\epsilon^2}\right)\right\} = \frac{e^{-\gamma\tau/4}}{2\sqrt{2\pi}(\gamma\tau)^{3/2}} \int_0^\infty \rho e^{-\rho^2/4\gamma\tau} \frac{1}{r!} \times \left[\frac{\partial^r F(\rho;z)}{\partial z^r}\right]_{z=0} d\rho. \quad (7.17)$$

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[1] J. A. Arnaud, Bell Syst. Tech. J. **49**, 2311 (1970).
[2] G. A. Deschamps, "Beam Optics and Complex Rays," URSI Symposium on Electromagnetic Waves, Stresa (1968).
[3] G. C. Papanicolaou, SIAM J. Appl. Math. (Soc. Ind. Appl. Math.) **21**, 13 (1971).
[4] R. Burridge, D. McLaughlin, and G. C. Papanicolaou, (to appear).
[5] R. Burridge and G. C. Papanicolaou, Commun. Pure Appl. Math., (to appear).
[6] W. H. Steier, Bell Syst. Tech. J. **45**, 451 (1966).
[7] B. N. Klyatskin and B. N. Tatarskii, Radiofizika **13**, 1061 (1970).
[8] G. C. Papanicolaou, J. Math. Phys. (submitted).
[9] R. Z. Hashminskii, Theor. Probability Appl. **11**, 390 (1966).
[10] G. C. Papanicolaou and R. Hersh, Indiana Univ. Math. J. **21**, 815 (1972).
[11] C. Caratheodory, *Conformal representation* (Cambridge U.P., Cambridge, 1958).
[12] J. A. Morrison, G. C. Papanicolaou, and J. B. Keller, Commun. Pure Appl. Math. **24**, 473 (1971).
[13] W. Magnus, F. Oberhetinger, and R. P. Soni, *Formulas and theorems for the special function of mathematical physics* (Springer, New York, 1966).
[14] W. Gröbner and N. Hofreiter, *Integraltafel* (Springer, Vienna, 1949).