Combining these relations gives the exact expression
\[
\psi_N = \ln 2 - \frac{1}{2N} \times \sum_{i<j} [(1 + \langle s_i s_j \rangle) \ln (1 + \tanh \beta J_{ij}) \\
+ (1 - \langle s_i s_j \rangle) \ln (1 - \tanh \beta J_{ij})] \\
+ (1/N) \sum_{i<j} J_{ij} \int_0^\beta d\beta' \langle s_i s_j \rangle - \tanh \beta' J_{ij}) .
\] (3.8)

Since \( \langle s_i s_j \rangle \geq \tanh \beta J_{ij} \), the integrand is non-negative\( ^6 \) for all pairs \( i, j \) such that \( 0 \leq i < j \leq N \); consequently,
\[
\psi_N \geq \ln 2 - \frac{1}{2N} \times \sum_{i<j} [(1 + \langle s_i s_j \rangle) \ln (1 + \tanh \beta J_{ij}) \\
+ (1 - \langle s_i s_j \rangle) \ln (1 - \tanh \beta J_{ij})] .
\] (3.9)

The double sum over \( i < j \) has been replaced by the sum over the set \( \{0, N\} \) since for \( i, j \notin \{0, N\}, J_{ij} = 0 = \ln (1 + \tanh \beta J_{ij}). \) The proof is now completed by using the fact that, for \( 0 \leq u \leq v < 1, 
\[
\int (u, v) \equiv -[(1 + u) \ln (1 + v) + (1 - u) \ln (1 - v)] \\
\geq \int (u, u). \] (3.10)

The latter inequality is obtained by adding the convexity relations\( ^6 \)
\[
p_i \ln p_i - p_i \ln q_i - p_i + q_i \geq 0,
\] for \( i = 1, 2, \) with \( p_i = 1 + u, p_2 = 1 - u, q_i = 1 + v, \) and \( q_2 = 1 - v. \)

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\* A summary of these results was given by the author at the 24th Yeshiva University Statistical Mechanics Meeting on 3 December 1970.
\* For an elementary discussion of relevant convexity relations see, e.g., H. Falk, Am. J. Phys. 38, 858 (1970).
\* For the nearest-neighbor Ising ring in zero external field, the integrand \( \rightarrow 0 \) in the limit \( N \rightarrow \infty \) for zero field and positive temperature.

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Motion of a Particle in a Random Field*  
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The motion of a particle in a random force field is investigated using the diffusion approximation.

1. INTRODUCTION

In this note we extend the analysis of Frisch\(^1\) and others,\(^2-7\) concerning the motion of a particle in a random field. We use the diffusion approximation, which is based on the assumption that the fluctuations of the field are weak and that they are weakly correlated at widely separated points. The equations of the diffusion approximation have been derived using a two-time method in\(^8\) which, in addition, some other applications are given. The results in this note can be viewed as a generalization of the Ornstein–Uhlenbeck theory of Brownian motion.\(^9\) Here the starting point is a stochastic equation much more complicated than Langevin's equation which is the basis of that theory. Consequently, the analysis must be confined to suitable approximations.

2. FORMULATION AND ANALYSIS OF THE PROBLEM

Let \( r(t) \) be the position vector at time \( t \) of a particle of unit mass in space. We shall assume that its motion for \( t > 0 \) is governed by the equations
\[
\frac{dr}{dt} = v(t), \quad r(0) = r_0,
\]
\[
\frac{dv}{dt} = \epsilon F(r) - \epsilon^2 \beta v, \quad v(0) = v_0.
\] (2.1)
Here \( \mathbf{v}(t) \) is the velocity vector, \( \mathbf{F}(\mathbf{r}) \) is a random force field, \( \epsilon \) is a small parameter characterizing the size of the fluctuations, and \( \beta \) is a parameter characterizing the frictional forces. The random force field is assumed to have mean zero and a homogeneous and isotropic covariance tensor:

\[
E[\mathbf{F}(\mathbf{r})] = 0,
\]

\[
E[\mathbf{F}_i(\mathbf{r})\mathbf{F}_j(\mathbf{r}')] = \delta_{ij}N(|\mathbf{r} - \mathbf{r}'|), \quad i, j = 1, 2, 3.
\]  

The symbol \( E[\cdot] \) denotes ensemble averaging.

Let us introduce the function \( \rho \) defined by

\[
\rho(t, \mathbf{r}, \mathbf{v}, \mathbf{r}_0, v_0) = \delta(\mathbf{r} - \mathbf{r}(t))\delta(\mathbf{v} - \mathbf{v}(t)).
\]

Then \( \rho \) satisfies the stochastic Liouville equation

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \left( \epsilon \mathbf{F}(\mathbf{r}) - \epsilon^2 \beta \mathbf{v} \right) \cdot \nabla \rho = 0,
\]

\[
\rho(0, \mathbf{r}, \mathbf{v}, \mathbf{r}_0, v_0) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(\mathbf{v} - v_0).
\]

Here \( \nabla_r \) and \( \nabla_v \) denote the gradient operator on the space and velocity variables, respectively.

We shall consider \( E[\rho(t, \mathbf{r}, \mathbf{v}, \mathbf{r}_0, v_0)] \), which is the transition probability density in phase space at time \( t \). To find it, we shall follow2,6 and use the diffusion approximation. This approximation is valid provided that \( \epsilon \) is small, \( t \) is large, and \( N(\mathbf{r}) \) decays rapidly with increasing \( r \). Let us denote the approximate transition probability density by \( f \). Then, as is shown in the above mentioned works and with some minor modifications due to the friction term in (2.1), \( f \) satisfies the equation

\[
\frac{\partial f}{\partial \tau} = \frac{1}{2} \frac{\partial^2 f}{\partial v^2} \bigg( \frac{1}{v} - \frac{1}{v^2} \bigg) - \frac{\partial}{\partial v} \left[ \left( \frac{1}{2v^2} - \alpha \right) f \right],
\]

\[
v \geq 0, \quad \tau > 0,
\]

\[
f(0, v, v_0) = \delta(v - v_0), \quad v = |\mathbf{v}|, \quad v_0 = |\mathbf{v}_0|,
\]

\[
D = 2 \int_0^\infty N(\mathbf{r}) \, dr, \quad \alpha = \frac{\beta}{D}, \quad \epsilon^2 D t.
\]

The fact that \( f \) is only a function of the modulus of the velocity is a consequence of the homogeneity and isotropy condition (2.3). In the remainder of this paper we shall solve (2.6) and discuss briefly the result.

Let us observe that (2.6) is a forward or Fokker-Plank equation in \( v \geq 0 \) with diffusion constant \( 1/v \) and drift constant \( (1/2v^2) - \alpha v \). According to Feller's theory of boundary conditions, two \( 0 \) and \( \infty \) are not regular, and therefore there exists exactly one fundamental solution common to both the forward and backward equation. Moreover, no boundary conditions are necessary. We shall therefore analyze the backward equation corresponding to (2.6), which is

\[
\frac{\partial f}{\partial \tau} = \frac{1}{2} \frac{\partial^2 f}{\partial v^2} + \frac{1}{2} \frac{\partial f}{\partial v} - xv \frac{\partial f}{\partial v}, \quad v_0 \geq 0,
\]

\[
f(0, v, v_0) = \delta(v - v_0).
\]

Let us introduce new variables as follows:

\[
\zeta_0 = e^{-3\alpha r}^{\frac{3}{8}} \zeta, \quad \sigma = (1 - e^{-3\alpha r})/3a.
\]

Then our equation transforms to

\[
\frac{\partial f}{\partial \sigma} = \frac{1}{2} \frac{\partial^2 f}{\partial \zeta^2} + \frac{1}{2} \frac{\partial f}{\partial \zeta}.
\]

The solution of (2.11) can be obtained readily by using the Hankel transform or observing that it corresponds to radial Browning motion in two dimensions. After reverting to the original variables, we obtain the following result:

\[
f(\tau, v, v_0) = \frac{2xv_0^2}{1 - e^{-3\alpha r}} \exp \left( \frac{-3\alpha r (v^2 + v_0^2 e^{-3\alpha r})}{1 - e^{-3\alpha r}} \right)
\]

\[
	imes I_0 \left( \frac{3\alpha r v^2 e^{-3\alpha r}}{1 - e^{-3\alpha r}} \right), \quad v, v_0 \geq 0.
\]

Here \( I_0 \) is the modified Bessel function of the first kind. The expression (2.12) is the diffusion approximation to the transition probability density of the velocity modulus, given the initial velocity modulus at time zero. As we have pointed out, this approximation is valid for \( t \) large and \( \epsilon \) small, with \( \tau \) arbitrary but finite. It is the main result of this paper.

If \( \alpha r \) is also large, then (2.12) can be approximated further by the following density function:

\[
f_0(v) = 2xv_0^2 \exp \left( -\frac{3}{8} \alpha r v^2 \right), \quad v \geq 0.
\]

The transition density (2.12) can be thought of as an equilibrium density for the original problem and so can (2.13) if, in addition, \( \alpha r \gg 1 \). The difference between the two is important because the second is independent of the initial velocity modulus \( v_0 \), while the first does depend on \( v_0 \) and also on the parameter \( \tau \).

Let us compute the mean square of the velocity modulus \( E[v^2] \). For this purpose let \( u \) and \( \psi \) be defined by

\[
u = (v_0^2 e^{-3\alpha r})^{\frac{1}{2}}, \quad \psi = (1 - e^{-3\alpha r})/3a.
\]

Then we have

\[
E[v^2] = \int_0^\infty v^2 f(\tau, v, v_0) \, dv
\]

\[
= (\psi)^2 (2\psi)^{\frac{3}{2}} \Gamma(\frac{3}{2}) e^{-u^2/2\psi} \psi^{\frac{1}{2}} F_1(\frac{3}{2}, 1, u^2/2\psi).
\]
p. 72. Finally, in the approximation $\tau \alpha \gg 1$, (2.15) reduces to
\[ E_{\infty}(\alpha^2) = (3/2\gamma)^{3/2} \Gamma(3/2). \] (2.16)

This last result can also be obtained from (2.13) directly.

In Figs. 1–8 we have plotted $f(\alpha)$ from (2.12) and $f_{\infty}(\alpha)$ from (2.13) for various values of $\alpha$, $t_0$, and $\tau$.

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