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STATISTICS FOR PULSE REFLECTION FROM A RANDOMLY LAYERED MEDIUM*

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Abstract. We consider reflection of a pulse incident on a layered halfspace whose density and bulk modulus vary randomly. We show that when the pulse width is long compared to the average time it takes to travel over one correlation length, the reflected signal is approximately a Gaussian random process. The parameters of this process change slowly on a scale long compared to the pulse width. We give a full characterization of the power spectrum of the Gaussian process in terms of a universal function that does not depend on the medium.

Key words. random media, pulse reflection

AMS(MOS) subject classification. G63

1. Introduction. Consider a layered elastic halfspace whose density and bulk modulus are stationary random functions with given mean values. A plane-parallel pulse disturbance is normally incident on this halfspace from the adjoining homogeneous halfspace so that the pulse reaches the interface at time $t=0$. A reflected signal is generated at the interface at $t>0$. It is a random function of time since it is produced by the random inhomogeneities. What are the statistical properties of this process? This is the question we address in this paper.

Our main result is that when the pulse width of the incident pulse is long compared to the time it takes to travel over one inhomogeneity, the reflected signal is approximately a Gaussian stochastic process. Its parameters vary slowly on time scales of several pulse widths. We show that the power spectrum of this Gaussian process can be expressed in terms of a universal function of frequency that does not depend on the particular random medium. We assume of course that the random perturbations of the density and bulk modulus have some general properties like rapidly decaying correlation functions. The slow modulation of the power spectrum on the longer time scales is given explicitly. The medium properties enter the Gaussian process via the power spectrum through a single scalar parameter, essentially a correlation length. We believe that our result is the first to give precise statistical characterization of pulses reflected by a layered random halfspace starting from first principles, i.e., the wave equation with random coefficients.

Our result is based on an asymptotic analysis for the relevant stochastic equations (formulated in § 2) in the small parameter ε which is the ratio of the time to travel one correlation length to the pulse width. The analysis follows well-known methods which we review briefly in the appendix. The argument we give in § 5 to establish the Gaussian nature of the process is heuristic and more analysis is required to fully substantiate this.

Our attention to the problem of pulse reflection from a layered random halfspace was drawn by the numerical simulation of Richards and Menke [1]. They did not realize that reflected pulses have the structure we give here but one can see that their computations are in qualitative agreement with our results, including the modulation on the long time scales. Furthermore, they computed the Green's function for the

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problem which, in our formulation, is smoothed by the incident pulse shape. We are presently conducting our own numerical simulations and the results and comparison with the theory will be given elsewhere.

For pulse reflection from a statistically homogeneous halfspace with no mean discontinuity at the interface, the reflected pulse contains information only about fluctuations in the medium. What if the density and bulk modulus have some nontrivial mean structure plus random fluctuations? What is the form of the reflected pulse then? Our methods of analysis extend to the special but important case when the mean density and bulk modulus are not constant but piecewise constant. This will also be given in a subsequent paper.

In § 2 we formulate the problem under consideration, introduce the scaling and state the results. We reformulate the problem in the frequency domain since we do our analysis there. In § 3 we carry out the frequency domain asymptotics following [2] and [3]. Some relevant facts about asymptotics for stochastic equations that we need are given in the appendix. In § 4 we analyze further the canonical reflection process, i.e., the process obtained in the limit $\varepsilon \to 0$. It is in this section that the power spectrum is characterized. To get useful numerical information from our theory, one must implement the construction of the power spectrum by numerically solving the problem in § 4.2. This has not been done yet. Finally in § 5 we give a heuristic argument which indicates that the canonical reflection process is Gaussian.

2. Pulse reflection.

2.1. Problem formulation and scaling. We first formulate the problem for the Green's function characterizing the reflected wave at the surface $z=0$ when a plane wave is normally incident from above ($z<0$). It is assumed that the medium is plane stratified, so that the density, ρ, and the bulk modulus, K, are functions of z alone. Let w be the z-component of particle velocity, and let p be the pressure. The linearized momentum and continuity equations (the Euler equations) are

$$\begin{aligned} \rho w_t + p_z &= 0 \\ \frac{1}{K} p_t + w_z &= 0 \end{aligned} \qquad \text{for } t>0,\ z>0. \tag{2.1.1}$$

To obtain the Green's function, we consider an incident δ-function pulse, for constants l_0, ρ_0, c_0

$$\begin{aligned} w &= (l_0/2)\delta(t-z/c_0) \\ p &= (l_0/2)\rho_0 c_0 \delta(t-z/c_0) \end{aligned} \qquad \text{for } t<0. \tag{2.1.2}$$

We assume that ρ, K are stationary random functions of z, which vary on the length scale l_0 that will be specified in the sequel. Furthermore, we take ρ_0 to be the mean value of ρ and K_0 the harmonic mean of K

$$E[\rho]=\rho_0, \qquad \left[E\left[\frac{1}{K}\right]\right]^{-1}=K_0. \tag{2.1.3}$$

These values are chosen because the effective medium theory is valid. For an incident pulse which is long compared to the scale of the inhomogeneities and for propagation distances which are not too large, it can be shown [3] that the solution of (2.1.1), (2.1.2) is well approximated by the solution of the corresponding deterministic problem of pulse propagation in a homogeneous medium with constant parameters ρ_0, K_0. We take for c_0 the corresponding propagation speed

$$c_0=\sqrt{K_0/\rho_0}. \tag{2.1.4}$$

In our formulation of the initial condition (2.1.2), we assume this constant medium for $z<0$.

We express the dependence of ρ and K on ρ_0, K_0 and l_0 by introducing normalized mean zero stochastic processes $\eta(z)$, $\nu(z)$ which vary on a nondimensional length scale. Thus we write

$$\rho(z)=\rho_0[1+\eta(z/l_0)],$$
$$\frac{1}{K(z)}=\frac{1}{K_0}[1+\nu(z/l_0)] \quad \text{where } E[\eta]=E[\nu]=0. \tag{2.1.5}$$

Equations (2.1.1), (2.1.2) may now be put in nondimensional form by the substitutions

$$z=l_0z', \qquad t=\frac{l_0}{c_0}t',$$
$$p(z)=\rho_0c_0^2p'(z'), \qquad w(z)=c_0w'(z'). \tag{2.1.6}$$

For notational convenience we drop primes to obtain the dimensionless equations

$$\begin{aligned} w_t+p_z+\eta(z)w_t&=0,\\ p_t+w_z+\nu(z)p_t&=0, \end{aligned} \qquad z>0, \quad t>0. \tag{2.1.7}$$

$$\begin{aligned} w&=\tfrac{1}{2}\delta(t-z)\\ p&=\tfrac{1}{2}\delta(t-z) \end{aligned} \qquad \text{for } t<0. \tag{2.1.8}$$

We will assume that η and ν are mixing processes (cf. the Appendix). This means that if z_1, z_2 are far apart then $\eta(z_1)$, $\eta(z_2)$ are approximately independent. Furthermore, we will assume that the autocorrelation functions of η and ν decay sufficiently rapidly so that the following integrals converge:

$$\int_0^\infty E[\eta(0)\eta(z)]\,dz<\infty, \qquad \int_0^\infty E[\nu(0)\nu(z)]\,dz<\infty. \tag{2.1.9}$$

The inequalities (2.1.9) can be interpreted as requiring η and ν to have finite (nondimensional) correlation lengths.

We next obtain equations for the downward and upward traveling waves $A(z, t)$, $B(z, t)$ where

$$A(z, t)=w(z, t)+p(z, t), \qquad B(z,t)=w(z, t)-p(z, t). \tag{2.1.10}$$

Let

$$m(z)=(\eta(z)+\nu(z))/2, \qquad n(z)=(\eta(z)-\nu(z))/2. \tag{2.1.11}$$

Then substitution of (2.1.10), (2.1.11) into (2.1.7), (2.1.8) yields

$$\begin{aligned} A_t+A_z+mA_t+nB_t&=0,\\ B_t-B_z+nA_t+mB_t&=0, \end{aligned} \qquad z>0, \quad t>0. \tag{2.1.12}$$

$$B=0, \qquad A=\delta(t-x) \quad \text{for } t<0. \tag{2.1.13}$$

The Green's function for the reflected wave at $z=0$ is then $B(0, t)$. The response at the surface $z=0$ to an incident wavelet of pulse shape f may then be calculated as the time convolution

$$B(0, t)*f=\int_0^t B(0, t-\xi)f(\xi)\,d\xi. \tag{2.1.14}$$

2.2. Statement of the main result. For $\varepsilon > 0$, let the incident wavelet be characterized by the pulse shape $f^\varepsilon(t)$, where

$$f^\varepsilon(t) = f(\varepsilon t). \tag{2.2.1}$$

We will assume that $f(t)$ and its Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{+i\omega t} f(t)\, dt \tag{2.2.2}$$

decay sufficiently rapidly as their argument tends to infinity. For τ, σ of order one as $\varepsilon \downarrow 0$, let

$$t^\varepsilon = \frac{\tau}{\varepsilon^2} + \frac{\sigma}{\varepsilon}. \tag{2.2.3}$$

We define $B^\varepsilon_{\tau,f}(\sigma)$ as the response at the surface $z = 0$ and at the time t^ε to the incident pulse f^ε, when scaled by a factor $\varepsilon^{-1/2}$. Thus

$$\begin{aligned} B^\varepsilon_{\tau,f}(\sigma) &= \frac{1}{\sqrt{\varepsilon}} B(0, t) * f^\varepsilon(t)\big|_{t=t^\varepsilon} \\ &= \frac{1}{\sqrt{\varepsilon}} \int_0^{(\tau/\varepsilon^2)+\sigma/\varepsilon} B\left(0, \frac{\tau}{\varepsilon^2} + \frac{\sigma}{\varepsilon} - \xi\right) f(\varepsilon \xi)\, d\xi. \end{aligned} \tag{2.2.4}$$

Our main result is then as follows:

For τ, f fixed, the stochastic process $B^\varepsilon_{\tau,f}(\cdot)$ converges weakly as $\varepsilon \downarrow 0$ to a stationary, mean zero Gaussian process $B_{\tau,f}(\cdot)$. The power spectral density $S_{\tau,f}(\omega)$ of $B_{\tau,f}$ is given by

$$S_{\tau,f}(\omega) = |\hat{f}(\omega)|^2 \cdot \frac{1}{\tau} \mu(\sqrt{\alpha\tau}\omega) \tag{2.2.5}$$

where

$$\alpha = 2 \int_0^{\infty} E[n(\xi) n(0)]\, d\xi \tag{2.2.6}$$

and μ is a universal function obtained by solving (4.2.3) below.

Remark 1. For $0 < \varepsilon \ll 1$, (2.2.1) expresses the assumption that the duration of the pulse is long compared to the time required to traverse a typical inhomogeneity. That is, ε measures the ratio of the correlation time of the medium to the duration of the pulse. For times of order $O(1/\varepsilon)$ the pulse has traveled, on average $O(1)$ pulse widths, and an effective medium [method of averaging] solution is applicable, as remarked in § 2.1. From (2.2.3), we see that the present result is for much longer times, of order $O(\varepsilon^{-2})$, when, on average, the pulse has traveled many pulse widths.

Remark 2. Note that for the limit process $B_{\tau,f}(\cdot)$, σ plays the role of the time parameter, while τ remains fixed. Thus we are describing the statistics of a *time-windowed process.* More specifically, a time window t^ε is centered at long time τ/ε^2, and a segment of the response of length $O(1/\varepsilon)$ is extracted. The parameter σ measures time differences from the center of the window on the time scale of order $O(1/\varepsilon)$ which is characteristic of the pulse duration; this is also, then, the time scale on which significant correlations of the resulting process $B_{\tau,f}(\sigma)$ occur. Equation (2.2.5) predicts the results of a local Fourier analysis of the time window centered at τ/ε^2.

Remark 3. While the limit process is locally stationary on the $O(\varepsilon^{-1})$ time scale, (2.2.5) gives a long term amplitude decay like $\tau^{-1/2}$ on the $O(\varepsilon^{-2})$ time scale, as well as a shift of power into lower frequencies as τ increases.

2.3. Reformulation in the frequency domain. In order to avoid radiation conditions in the frequency domain, we alter the problem to that of a finite length random slab, occupying $0 \leqq z \leqq L/\varepsilon^2$, and embedded in a homogeneous medium. For m and n uniformly bounded, hyperbolicity of the time domain equations implies finite travel time for signals. Thus the solution of the original problem agrees with that of the finite slab, for times $t < 2L/(\varepsilon^2 C_{\max})$, where $C_{\max}$ is the maximum propagation speed in the random medium. From (2.2.3) the inequality will hold asymptotically as $\varepsilon \downarrow 0$ as long as

$$\tau < \frac{2L}{C_{\max}}. \tag{2.3.1}$$

In particular, the time domain solution will not depend on L at all as long as (2.3.1) is satisfied. We will, however, utilize a frequency domain formulation for which the dependence on L is nontrivial. We therefore denote explicitly the L-dependence of the Fourier transforms A, B by

$$\begin{aligned} A_L(z,\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} A(z,t)\, dt, \\ B_L(z,\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} B(z,t)\, dt. \end{aligned} \tag{2.3.2}$$

Then from (2.2.2), (2.2.4), (2.3.2) we have the frequency domain representation, as long as (2.3.1) is satisfied,

$$B^{\varepsilon}_{\tau,f}(\sigma) = \frac{1}{2\pi\sqrt{\varepsilon}} \int_{-\infty}^{\infty} \exp\left[-i\omega\left(\frac{\tau}{\varepsilon^2}+\frac{\sigma}{\varepsilon}\right)\right]\frac{1}{\varepsilon}\hat{f}\left(\frac{\omega}{\varepsilon}\right) B_L(0,\omega)\, d\omega. \tag{2.3.3}$$

From (2.1.12), (2.3.2) we obtain ordinary differential equations for A_L, B_L

$$\begin{aligned} &\frac{\partial}{\partial z}\begin{bmatrix} A_L \\ B_L \end{bmatrix} = \left\{ i\omega \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i\omega \begin{bmatrix} m & n \\ -n & -m \end{bmatrix}\right\} \begin{bmatrix} A_L \\ B_L \end{bmatrix}, \\ &A_L(0,\omega) = 1, \qquad B_L\left(\frac{L}{\varepsilon^2}, \omega\right) = 0. \end{aligned} \tag{2.3.4}$$

Define the reflection coefficient $R^{\varepsilon}_L(\omega)$ by

$$R^{\varepsilon}_L(\omega) = R_L(0, \varepsilon\omega) \tag{2.3.5}$$

so that (2.3.3) becomes, on setting $\omega = \varepsilon\omega'$ and dropping primes

$$B^{\varepsilon}_{\tau,f}(\sigma) = \frac{1}{2\pi\sqrt{\varepsilon}} \int_{-\infty}^{\infty} e^{-i\omega\tau/\varepsilon} e^{-i\omega\sigma} \hat{f}(\omega) R^{\varepsilon}_L(\omega)\, d\omega. \tag{2.3.6}$$

The equation for R^{ε}_L is obtained by replacing ω by $\varepsilon\omega$ in (2.3.4). It is also convenient to scale z in this equation, and to remove the constant rotation due to the first matrix on the right-hand side. Accordingly, we define

$$\begin{aligned} &\omega = \varepsilon\omega', \qquad z' = \varepsilon^2 z, \\ &A^{\varepsilon} = e^{-i\omega' z'/\varepsilon} A_L, \qquad B^{\varepsilon} = e^{i\omega' z'/\varepsilon} B_L, \end{aligned} \tag{2.3.7}$$

whence (2.3.4) becomes, after dropping primes

$$\begin{aligned}
&\frac{d}{dz}\begin{bmatrix} A^\varepsilon(z,\omega) \\ B^\varepsilon(z,\omega)\end{bmatrix} = \frac{i\omega}{\varepsilon}\begin{bmatrix} m(z/\varepsilon^2) & e^{-2i\omega z/\varepsilon} n(z/\varepsilon^2) \\ -e^{2i\omega z/\varepsilon} n(z/\varepsilon^2) & -m(z/\varepsilon^2)\end{bmatrix}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix}, \\
&A^\varepsilon(0,\omega)=1, \qquad B^\varepsilon(L,\omega)=0.
\end{aligned} \tag{2.3.8}$$

From (2.3.5), (2.3.7) the reflection coefficient is

$$R_L^\varepsilon(\omega) = B^\varepsilon(0,\omega). \tag{2.3.9}$$

In § 3, we will use limit theorems to characterize the statistics of R_L^ε as $\varepsilon \downarrow 0$, based on (2.3.8), (2.3.9). We indicate here how these statistics will be used to analyze the time-domain process $B_{\tau,f}^\varepsilon(\sigma)$.

From (2.3.6), it follows that $B_{\tau,f}^\varepsilon$ has zero mean if R_L^ε has zero mean, which will be shown (cf. § 5.3). From (2.3.6) we also compute the covariance of $B_{\tau,f}^\varepsilon$. Since $B_{\tau,f}^\varepsilon$ is real, and denoting complex conjugate by *, we have

$$\begin{aligned}
E[B_{\tau,f}^\varepsilon(0)B_{\tau,f}^\varepsilon(\sigma)] = &\frac{1}{(2\pi)^2\varepsilon}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2\, e^{-i\omega_1\sigma}\hat{f}(\omega_1)\hat{f}^*(\omega_2) \\
&\cdot e^{-i(\omega_1-\omega_2)\tau/\varepsilon} E[R_L^\varepsilon(\omega_1)R_L^{\varepsilon *}(\omega_2)].
\end{aligned} \tag{2.3.10}$$

In (2.3.10) we make the change of variables $\omega = (\omega_1+\omega_2)/2$, $h=(\omega_2-\omega_1)/\varepsilon$. Let

$$u^\varepsilon(L,h,\omega) = E\left[R_L^\varepsilon\left(\omega-\frac{\varepsilon h}{2}\right)R_L^{\varepsilon *}\left(\omega+\frac{\varepsilon h}{2}\right)\right]. \tag{2.3.11}$$

Then (2.3.10) becomes

$$\begin{aligned}
E[B_{\tau,f}^\varepsilon(0)B_{\tau,f}^\varepsilon(\sigma)] = &\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\sigma}\int_{-\infty}^{\infty} dh\, \hat{f}\left(\omega-\frac{\varepsilon h}{2}\right) \\
&\cdot \hat{f}\left(\omega+\frac{\varepsilon h}{2}\right) e^{i\omega\varepsilon h/2}\, e^{ih\tau} u^\varepsilon(L,h,\omega).
\end{aligned} \tag{2.3.12}$$

It will be shown that $u^\varepsilon(L,h,\omega)$ has a limit $u(L,h,\omega)$ as $\varepsilon\to 0$. Then (2.3.12) becomes, as $\varepsilon \downarrow 0$

$$E[B_{\tau,f}^\varepsilon(0)B_{\tau,f}^\varepsilon(\sigma)] \xrightarrow[\varepsilon\downarrow 0]{} \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\sigma}|\hat{f}(\omega)|^2 \int_{-\infty}^{\infty} dh\, e^{ih\tau}u(L,h,\omega). \tag{2.3.13}$$

Next, from scaling properties of u it will be shown (cf. § 4.3) that for L so large that (2.3.1) is satisfied

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dh\, e^{ih\tau}u(L,h,\omega) = \frac{1}{\tau}\mu(\sqrt{\alpha\tau}\,\omega). \tag{2.3.14}$$

Here $\mu(\omega)$ is a universal function, independent of parameters in the problem, to be determined from the equations for u.

Similar calculations for the higher moments of $B_{\tau,f}^\varepsilon$ show that the limit process is Gaussian as is discussed in § 5.

3. Propagator matrices and their asymptotic properties.

3.1. Propagator matrices. The two-point boundary value problem (2.3.8) can be solved using the fundamental or propagator matrix of the system defined as the solution of

$$
\begin{aligned}
&\frac{d}{dz}Y^\varepsilon(z;\omega)=\frac{i\omega}{\varepsilon}\begin{bmatrix} m(z/\varepsilon^2) & e^{-2i\omega z/\varepsilon}n(z/\varepsilon^2) \\ -e^{2i\omega z/\varepsilon}n(z/\varepsilon^2) & -m(z/\varepsilon^2)\end{bmatrix}Y^\varepsilon(z,\omega),\\
&Y^\varepsilon(0;\omega)=I.
\end{aligned}
\tag{3.1.1}
$$

Here I is the 2×2 identity matrix. The 2×2 matrix Y^ε has determinant 1 since the coefficient matrix on the right of (3.1.1) has trace zero. Moreover if (u, v) is a solution vector of (3.1.1) then (v^*, u^*) is also a solution and they are linearly independent. The propagator matrix thus has the form

$$
Y^\varepsilon(z;\omega)=\begin{bmatrix} a^\varepsilon(z,\omega) & b^\varepsilon(z;\omega) \\ b^{\varepsilon *}(z;\omega) & a^{\varepsilon *}(z;\omega)\end{bmatrix}
\tag{3.1.2}
$$

with $a^\varepsilon(z;\omega)$ and $b^\varepsilon(z;\omega)$ complex-valued random functions of $z\geqq 0$ for each frequency ω. The determinant condition becomes

$$
|a^\varepsilon|^2-|b^\varepsilon|^2=1, \qquad z\geqq 0.
\tag{3.1.3}
$$

This relation means physically that wave energy flux is conserved.

The reflection coefficient $R_L^\varepsilon(\omega)$ in (2.3.9) can be expressed simply in terms of $Y^\varepsilon(L;\omega)$. In fact from (2.3.8) we have

$$
\begin{bmatrix} a^\varepsilon(L;\omega) & b^\varepsilon(L;\omega) \\ b^{\varepsilon *}(L;\omega) & a^{\varepsilon *}(L;\omega)\end{bmatrix}\begin{bmatrix} 1 \\ R_L^\varepsilon(\omega)\end{bmatrix}=\begin{bmatrix} A^\varepsilon(L,\omega) \\ 0\end{bmatrix}.
\tag{3.1.4}
$$

Solving for $R_L^\varepsilon(\omega)$ gives

$$
R_L^\varepsilon(\omega)=-\frac{b^{\varepsilon *}(L;\omega)}{a^{\varepsilon *}(L;\omega)}.
\tag{3.1.5}
$$

3.2. Asymptotics for propagator matrices. The asymptotic behavior of the matrix-valued stochastic process $Y^\varepsilon(z;\omega)$ as $\varepsilon\to 0$ has been studied extensively for a single frequency ω and for several distinct frequencies. Here we will need in addition statistics at nearby frequencies. In the Appendix we discuss general limit theorems for stochastic equations in $\mathbb{R}^d$ of the form

$$
\begin{aligned}
&\frac{dx^\varepsilon(t)}{dt}=\frac{1}{\varepsilon}F\left(x^\varepsilon(t), t, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^2}\right), \qquad t>0,\\
&x^\varepsilon(0)=x.
\end{aligned}
$$

We get all the results we need here by extensions of this theorem. As we remark in the appendix, various forms of this theorem are well known, and our extensions follow from a mild modification of well-known methods. At the level of generality of the theorem in the appendix, the importance of some special features in (3.1.1) is lost. For example, the random coefficients in (3.1.1) vary on the scale z/ε^2 but there are also rapidly varying trigonometric terms on the scale z/ε. This is what produces the window structure of $B_{\tau,f}^\varepsilon(\sigma)$ in (2.2.4).

The main result (cf. [2]) is that the process $Y^\varepsilon(z;\omega)$ converges weakly to a diffusion Markov process $Y(z;\omega)$ as $\varepsilon\to 0$. To understand why the limit is Markovian, consider a typical coefficient such as $\varepsilon^{-1}m(z/\varepsilon^2)$ in (3.1.1). It is a stationary process that has mean zero and covariance

$$E\left\{\frac{1}{\varepsilon}m\left(\frac{z_1}{\varepsilon^2}\right)m\left(\frac{z_2}{\varepsilon^2}\right)\right\}=\frac{1}{\varepsilon^2}\rho_{mm}\left(\frac{z_1-z_2}{\varepsilon^2}\right)$$

where

$$\rho_{mm}(z)=E\{m(z_1+z)m(z_1)\}. \tag{3.2.1}$$

We are assuming throughout that the coefficient processes have rapidly decaying correlations (are mixing in fact; cf. the Appendix). Now it is clear that as $\varepsilon\to 0$

$$\frac{1}{\varepsilon^2}\rho_{mm}\left(\frac{z_1-z_2}{\varepsilon^2}\right)\to\int_{-\infty}^{\infty}\rho_{mm}(s)\,ds\,\delta(z_1-z_2) \tag{3.2.2}$$

in the sense of generalized functions. This means that $\varepsilon^{-1}m(z/\varepsilon^2)$ behaves asymptotically like white noise, and explains why $Y^\varepsilon(t;\omega)$ tends to a diffusion process.

Similar remarks can be made about $\varepsilon^{-1}\,e^{-2i\omega z/\varepsilon}n(z/\varepsilon^2)$, which has the additional trigonometric factor. The presence of the trigonometric factor is not a serious complication in the general theory, as is seen in the Appendix. However it has very important consequences in applications [2] and in particular for the present problem. This is because in computations like (3.2.2) the limits will be zero unless the phases cancel.

We apply now the theorem of the appendix to characterize the limiting diffusion process $Y(z;\omega)$ of the propagator matrices. A convenient way of doing this is as follows [2].

Let η_1, η_2, η_3 be the 2×2 Pauli spin matrices

$$\eta_1=\frac{1}{2}\begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix},\quad \eta_2=\frac{1}{2}\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix},\quad \eta_3=\frac{1}{2}\begin{bmatrix} 0 & i\\ -i & 0\end{bmatrix}. \tag{3.2.3}$$

Equation (3.1.1) can then be written in the form

$$\begin{aligned}&\frac{d}{dz}Y^\varepsilon(z;\omega)=\frac{1}{\varepsilon}\sum_{j=1}^{3}M_j\left(\frac{z}{\varepsilon^2},\frac{z}{\varepsilon};\omega\right)\eta_jY^\varepsilon(z;\omega),\qquad z\geqq 0,\\ &Y^\varepsilon(0,\omega)=I,\end{aligned} \tag{3.2.4}$$

where

$$M_1=2\omega m\left(\frac{z}{\varepsilon^2}\right),\quad M_2=2\omega n\left(\frac{z}{\varepsilon^2}\right)\sin\frac{2\omega z}{\varepsilon},\quad M_3=2\omega n\left(\frac{z}{\varepsilon^2}\right)\cos\frac{2\omega z}{\varepsilon}. \tag{3.2.5}$$

Let Y be a 2×2 complex matrix of the form

$$Y=\begin{bmatrix} a & b\\ b^* & a^*\end{bmatrix}\quad\text{with } |a|^2-|b|^2=1 \tag{3.2.6}$$

and let $g(Y)$ be a smooth function of Y, i.e., a smooth function of a and b such as $g(Y)=-b^*/a^*$ as in (3.1.5). Let

$$u^\varepsilon(z,Y;\omega)=E[g(Y^\varepsilon(z;\omega)Y)],\qquad z\geqq 0 \tag{3.2.7}$$

with Y^ε the solution of (3.1.1). According to the theorem of the appendix, u^ε converges as ε tends to zero to $u(z,Y;\omega)$ which is the solution of the diffusion equation

$$\begin{aligned}&\frac{\partial u}{\partial z}=\underline{L}_\omega u,\qquad z>0,\\ &u(0,Y;\omega)=g(Y).\end{aligned} \tag{3.2.8}$$

The explicit form of the diffusion operator $\underline{L}_\omega$ is best written using the matrices η_1, η_2 and η_3 in (3.2.3). On smooth functions $g(Y)$ we define the differential operations

(3.2.9) $$D_j g(Y) = \lim_{h\downarrow 0} \frac{g(e^{\eta_j h} Y) - g(Y)}{h}, \qquad j = 1, 2, 3.$$

Then

(3.2.10) $$\underline{L}_\omega = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T dt \int_0^\infty ds \sum_{j,k=1}^{3} E[M_j(0, t, \omega) M_k(s, t, \omega)] D_j D_k.$$

Using the explicit expressions (3.2.5) for the coefficients M_j this gives

(3.2.11) $$\underline{L}_\omega = \omega^2 \alpha (D_2 D_2 + D_3 D_3) + \omega^2 \gamma D_1 D_1$$

where

(3.2.12) $$\alpha = 2 \int_0^\infty \rho_{nn}(s)\, ds, \qquad \gamma = 4 \int_0^\infty \rho_{mm}(s)\, ds$$

and

(3.2.13) $$\rho_{nn}(s) = E[n(s+z) n(z)], \qquad \rho_{mm}(s) = E[m(s+z) m(z)].$$

A simple example of how these results can be used is this. To compute

(3.2.14) $$p_2(\omega, L) = \lim_{\varepsilon\downarrow 0} E[|R_L^\varepsilon(\omega)|^2],$$

which is the mean power reflection coefficient for a slab at frequency ω, let

(3.2.15) $$g(Y) = \left|\frac{b}{a}\right|^2 \quad \text{where } Y = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}.$$

Here we use (3.1.5). Then we solve the diffusion equation

(3.2.16) $$\frac{\partial u}{\partial z} = \underline{L}_\omega u, \qquad z > 0,$$
$$u(0, Y; \omega) = g(Y)$$

and set

(3.2.17) $$p_2(\omega, L) = u(L, I; \omega).$$

This calculation is carried out further in § 4.1 where (3.2.16) is reduced to a solvable problem.

3.3. Multifrequency asymptotics. The diffusion operator $\underline{L}_\omega$ defined by (3.2.11) characterizes the statistical properties of the limit propagator matrix $Y(z; \omega)$ for each frequency ω and all $z \geqq 0$. The limit process is a Markov process in z and (3.2.11) is its generator. For the pulse problem we are interested in statistical properties at two nearby frequencies as is seen from (2.3.11) and the discussion in that section. It is useful also to have the asymptotic behavior at distinct nonneighboring frequencies.

Let $\omega_1, \omega_2, \cdots, \omega_N$ be N distinct frequencies whose sums and differences in pairs are also distinct. Consider the matrix-valued process $Y^\varepsilon(z, \omega_1), \cdots, Y^\varepsilon(z, \omega_N)$ jointly. They each satisfy (3.1.1) with different ω_j. The limit theorem of the Appendix applies equally well here as follows.

Let $g(Y_1, Y_2, \cdots, Y_N)$ be a smooth function of N matrix arguments with each matrix of the form (3.2.6) as before. Let

$$u^\varepsilon(z, Y_1, Y_2, \cdots, Y_N, \omega_1, \cdots, \omega_N) = E[g(Y^\varepsilon(z, \omega_1)Y_1, \cdots, Y^\varepsilon(z, \omega_N)Y_N)], \qquad z \geqq 0. \tag{3.3.1}$$

Then u^ε converges as $\varepsilon \to 0$ to $u(z, Y_1, \cdots, Y_N, \omega_1, \cdots, \omega_N)$ which satisfies the diffusion equation

$$\begin{aligned} &\frac{\partial u}{\partial z} = \underline{L}_{\omega_1,\cdots,\omega_N} u, \qquad z > 0, \\ &u(0, Y_1, \cdots, Y_N, \omega_1, \cdots, \omega_N) = g(Y_1, \cdots, Y_N). \end{aligned} \tag{3.3.2}$$

The differential operator $\underline{L}_{\omega_1,\cdots,\omega_N}$ is computed in the same way as in § 3.2. If

$$D_j^{(p)} g(Y_1, \cdots, Y_N) = \lim_{h \downarrow 0} \frac{g(Y_1, \cdots, e^{\eta_j h} Y_p, \cdots, Y_N) - g(Y_2, \cdots, Y_N)}{h}, \qquad p = 1, 2, \cdots, N, \quad j = 1, 2, 3. \tag{3.3.3}$$

Then

$$\underline{L}_{\omega_1,\cdots,\omega_N} = \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T dt \int_0^\infty ds \cdot \sum_{p,q=1}^{N} \sum_{j,k=1}^{3} E[M_j(0, t, \omega_p) M_k(s, t, \omega_q)] \cdot D_j^{(p)} D_k^{(q)}. \tag{3.3.4}$$

Since the frequencies $\omega_1, \omega_2, \cdots, \omega_N$ are assumed distinct (along with their differences in pairs) and positive, the explicit form (3.2.5) of the M_j leads to the following expression:

$$\underline{L}_{\omega_1,\cdots,\omega_N} = \sum_{p=1}^{N} \underline{L}_{\omega_p}^{(p)} + \gamma \sum_{p \neq q = 1}^{N} \omega_p \omega_q D_1^{(p)} D_1^{(q)} \tag{3.3.5}$$

where

$$\underline{L}_{\omega_p}^{(p)} = \omega_p^2 \{\alpha[D_2^{(p)} D_2^{(p)} + D_3^{(p)} D_3^{(p)}] + \gamma D_1^{(p)} D_1^{(p)}\}.$$

The constants α and γ are defined by (3.2.12).

3.4. Asymptotics for propagators at neighboring frequencies. In § 2.3, in particular in expression (2.3.12) for the covariance of the reflected pulse process, we saw that it is necessary to calculate asymptotic properties of the reflection coefficient $R_L^\varepsilon(\omega)$ at two frequencies that differ by $O(\varepsilon)$. As explained in § 3.1, this means that we must calculate asymptotically the joint statistics of $Y^\varepsilon(z, \omega)$ and $Y^\varepsilon(z, \omega + \varepsilon h)$. The equation for $Y^\varepsilon(z, \omega)$ is (3.1.1). The equation for $Y^\varepsilon(z, \omega + \varepsilon h)$ is (3.1.1) with ω replaced by $\omega + \varepsilon h$. We rewrite it

$$\frac{d}{dz} Y^\varepsilon(z, \omega + \varepsilon h) = \frac{i(\omega + \varepsilon h)}{\varepsilon} \cdot \begin{bmatrix} m(z/\varepsilon^2) & e^{-2i\omega z/\varepsilon} e^{-2ihz} n(z/\varepsilon^2) \\ -e^{2i\omega z/\varepsilon} e^{2ihz} n(z/\varepsilon^2) & -m(z/\varepsilon^2) \end{bmatrix} \cdot Y^\varepsilon(z, \omega + \varepsilon h) \tag{3.4.1}$$

because we want to point out that all three scales, z, z/ε and z/ε^2 appear on the right.

Applying the limit theorem of the appendix, in its general form now, leads to the following form of the generator of the limit process $Y(z, \omega)$ and $Y(z, \omega, h)$.

(3.4.2) $$\underline{L}_{z,\omega,h} = \underline{L}_{\omega}^{(1)} + \underline{L}_{\omega}^{(2)} + 2\omega^2 \gamma D_1^{(1)} D_1^{(2)} + 2\alpha\omega^2 \cos 2hz D_2^{(1)} D_2^{(2)}.$$

Here $\underline{L}_{\omega}^{(p)}$ is the operator defined in (3.3.6), $p = 1, 2$, and the $D_j^{(p)}$ by (3.3.3). Note that the operator $\underline{L}_{z,\omega,h}$ depends on z explicitly so that the limit process is inhomogeneous in z.

More specifically, let Y_1 and Y_2 be matrices of the form (3.2.6) and let $g(Y_1, Y_2)$ be a smooth function of them. Let

(3.4.3) $$u^\varepsilon(z, L, Y_1, Y_2; \omega, h) = E[g(Y^\varepsilon(z, L; \omega) Y_1, Y^\varepsilon(z, L; \omega + \varepsilon h) Y_2)].$$

Then as $\varepsilon \to 0$, u^ε converges to $u(z, L, Y_1, Y_2; \omega, h)$ which satisfies the backward Kolmogorov equation

(3.4.4) $$\frac{\partial u}{\partial z} + \underline{L}_{z,\omega,h} u = 0, \qquad 0 \leqq z < L,$$
$$u(L, L, Y_1, Y_2; \omega, h) = g(Y_1, Y_2).$$

Here $Y^\varepsilon(z, L; \omega)$ is the solution of (3.1.1) for $z < L$ with $Y^\varepsilon(z, z; \omega) = I$. Thus,

(3.4.5) $$\lim_{\varepsilon \downarrow 0} E[g(Y^\varepsilon(0, L; \omega), Y^\varepsilon(0, L; \omega + \varepsilon h))] = u(0, L, I, I; \omega, h).$$

We can now identify $u(L, h, \omega)$ in (2.3.13), which is

(3.4.6) $$u(L, h, \omega) = \lim_{\varepsilon \downarrow 0} E\left[R_L^\varepsilon\left(\omega - \frac{\varepsilon h}{2}\right) R_L^{\varepsilon *}\left(\omega + \frac{\varepsilon h}{2}\right)\right].$$

Let Y be a propagator matrix of the form (3.2.6) and let

(3.4.7) $$g(Y) = -\frac{b^*}{a^*}.$$

Then $u(L, h, \omega) = u(0, L, I, I; \omega, h)$ where $u(z, L, Y_1, Y_2; \omega, h)$ satisfies the equation

(3.4.8) $$\frac{\partial u}{\partial z} + \underline{L}_{z,\omega,h} u = 0, \qquad 0 \leqq z < L,$$
$$u(L, L, Y_1, Y_2; \omega, h) = g(Y) g^*(Y).$$

The scaling property (2.3.14) cannot be seen immediately from (3.4.8) and (3.4.2) but will be derived in § 4.3. In particular we show there that the solution of (3.4.8) is independent of the coefficient γ in the operator $\underline{L}_{z,\omega,h}$.

4. The canonical reflection process.

4.1. Polar coordinates. The limit theorem for the propagator matrices in § 3.2 leads to the diffusion equation (3.2.8) which is the backward Kolmogorov equation for the limit of the process $Y^\varepsilon(z; \omega)$. The operator $\underline{L}_\omega$ is given by (3.2.11) with D_1, D_2 and D_3 the differential operators defined by (3.2.9). For actual calculations one must introduce a convenient coordinate system in the space of propagator matrices as follows.

Each Y of the form

(4.1.1) $$Y = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}, \qquad |a|^2 - |b|^2 = 1$$

can be represented by three real variables θ, ϕ and ψ which parametrize the hyperboloid of revolution $|a|^2-|b|^2=1$ in $\mathbb{R}^4$ (2-dimensional complex space)

$$a=e^{i(\phi+\psi+\pi)/2}\cosh\frac{\theta}{2},\qquad b=e^{i(\phi-\psi-\pi)/2}\sinh\frac{\theta}{2} \tag{4.1.2}$$

with

$$0\leqq\theta<\infty,\quad 0\leqq\phi<4\pi,\quad 0\leqq\psi<2\pi. \tag{4.1.3}$$

This range of the variables parametrizes globally in 1-1 fashion the propagator matrices Y of the form (4.1.1).

The differential equation (3.1.1) for the propagator matrix $Y^\varepsilon(z;\omega)$ leads to the following system of differential equations for its polar coordinates $\theta^\varepsilon(z;\omega)$, $\phi^\varepsilon(z;\omega)$, $\psi^\varepsilon(z;\omega)$.

$$\begin{aligned}\frac{d\theta^\varepsilon}{dz}&=\frac{2\omega}{\varepsilon}n\left(\frac{z}{\varepsilon^2}\right)\sin\left(\phi^\varepsilon+\frac{2\omega z}{\varepsilon}\right),\\ \frac{d\phi^\varepsilon}{dz}&=\frac{2\omega}{\varepsilon}\left[m\left(\frac{z}{\varepsilon^2}\right)+n\left(\frac{z}{\varepsilon^2}\right)\coth\theta^\varepsilon\cos\left(\phi^\varepsilon+\frac{2\omega z}{\varepsilon}\right)\right],\\ \frac{d\psi^\varepsilon}{dz}&=-\frac{2\omega}{\varepsilon}n\left(\frac{z}{\varepsilon^2}\right)\operatorname{csch}\theta^\varepsilon\cos\left(\phi^\varepsilon+\frac{2\omega z}{\varepsilon}\right).\end{aligned} \tag{4.1.4}$$

The limit theorems described in the Appendix may be applied to (4.1.4) or to (3.1.1), and they yield the same limit process in the two different representations.

Associated with the diffusion process $Y(z;\omega)$ with generator $\underline{L}_\omega$ given by (3.2.11) we have now the process $(\theta(z;\omega), \phi(z;\omega), \psi(z;\omega))$ in the space (4.1.3). This is also a diffusion process and its backward Kolmogorov equation is obtained from (3.2.16) by using the change of variables (4.1.2). A direct calculation [2] gives the following form for $\underline{L}_\omega$ in terms of (θ,ϕ,ψ).

$$\underline{L}_\omega=\omega^2\alpha\left[\frac{\partial^2}{\partial\theta^2}+\coth\theta\frac{\partial}{\partial\theta}+\left(\coth\theta\frac{\partial}{\partial\phi}-\operatorname{csch}\theta\frac{\partial}{\partial\psi}\right)^2\right]+\omega^2\gamma\frac{\partial^2}{\partial\phi^2}. \tag{4.1.5}$$

The reflection coefficient R_L^ε is given by (3.1.5). Using the parametrization (4.1.2), we get now the expression

$$R_L^\varepsilon(\omega)=e^{i\psi^\varepsilon(z;\omega)}\tanh\frac{\theta^\varepsilon(z;\omega)}{2}. \tag{4.1.6}$$

Here $(\theta^\varepsilon, \phi^\varepsilon, \psi^\varepsilon)$ is the process defined by (4.1.4) associated with Y^ε. In the limit $\varepsilon\to0$ this process converges in distribution to (θ,ϕ,ψ) governed by (4.1.5). Since the coefficients in that operator depend on neither ϕ nor ψ we see that if $g(\theta,\psi)$ is any function of θ and ψ, periodic in $0\leqq\psi<2\pi$, then,

$$P_f(\omega,L)=\lim_{\varepsilon\downarrow0}E[g(\theta^\varepsilon(L;\omega),\psi^\varepsilon(L;\omega))] \tag{4.1.7}$$

is obtained by solving for $u(z,\theta,\psi;\omega)$ in

$$\begin{aligned}\frac{\partial u}{\partial z}&=\omega^2\alpha\left[\frac{\partial^2}{\partial\theta^2}+\coth\theta\frac{\partial}{\partial\theta}+\frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\psi^2}\right]u,\\ u(0,\theta,\psi)&=g(\theta,\psi)\end{aligned} \tag{4.1.8}$$

and then setting

$$P_f(\omega,L)=u(L,0,\cdot). \tag{4.1.9}$$

The dot on the right side of (4.1.9) signifies that at $\theta=0$ the value of the polar angle ψ is undetermined and in fact that u will not depend on it.

Equation (4.1.8) is just the diffusion equation on the hyperbolic disc and can be solved explicitly in terms of hyperbolic Legendre functions. For example, the mean power reflection coefficient $P_2(\omega, L)$ (cf. (3.2.15)-(3.2.17)) corresponds to taking $g=\tanh^2(\theta/2)$ and then [2] we have

$$P_2(\omega, L)=1-2\pi\int_0^\infty e^{-\alpha\omega^2L(t^2+1/4)}\frac{t\sinh\pi t}{\cosh^2\pi t}\,dt. \tag{4.1.10}$$

A simple consequence of the polar coordinate form (4.1.5) of $\underline{L}_\omega$ is the asymptotic independence of $R^\varepsilon(L,\omega_1)$, $R^\varepsilon(L,\omega_2),\cdots,R^\varepsilon(L,\omega_n)$ as $\varepsilon\to 0$ for frequencies ω_1, $\omega_2,\cdots,\omega_n$ that are distinct along with their differences in pairs. From (3.3.5) and the fact that [2]

$$D_s^{(p)}=\frac{\partial}{\partial\phi^{(p)}} \tag{4.1.11}$$

we conclude that the cross frequency terms in (3.3.5) disappear when they act on functions of $(\theta^{(1)},\theta^{(2)},\cdots,\theta^{(n)})$ and $(\psi^{(1)},\psi^{(2)},\cdots,\psi^{(n)})$ only. Hence for any function $g(R)$ on $|R|\leqq 1$ we have

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} E[g(R^\varepsilon(L;\omega_1))\cdots g(R^\varepsilon(L,\omega_n))]\\ &\qquad=E[g(R^\varepsilon(L,\omega_1))]\cdots E[g(R^\varepsilon(L,\omega_n))].\end{aligned} \tag{4.1.12}$$

4.2. Characterization of the local power spectrum. We shall write the backward diffusion (3.4.8) in polar coordinates. This will give us a simpler and more concrete equation for the function $u(L,h,\omega)$ in (2.3.13) and for the canonical power spectrum $\mu(\omega)$ in (2.2.5) and (2.3.14).

It is useful to recall here that the differential operators D_1, D_2 and D_3 defined by (3.2.9) have the following form in terms of the polar coordinates [2].

$$\begin{aligned}D_1&=\frac{\partial}{\partial\phi},\\ D_2&=-\sin\phi\coth\theta\frac{\partial}{\partial\phi}+\sin\phi\,\mathrm{csch}\,\theta\frac{\partial}{\partial\psi}+\cos\phi\frac{\partial}{\partial\theta},\\ D_3&=\cos\phi\coth\theta\frac{\partial}{\partial\phi}-\cos\phi\,\mathrm{csch}\,\theta\frac{\partial}{\partial\psi}+\sin\phi\frac{\partial}{\partial\theta}.\end{aligned} \tag{4.2.1}$$

We can now use formulas (4.2.1) in (3.4.2) and rewrite (3.4.8) in terms of polar coordinates. The form of the operator $\underline{L}_{z,\omega,h}$ can be simplified further by noting that $g=e^{i\psi}\tanh(\theta/2)$ so that the exponential factor $e^{i\psi}$ can be taken out in (3.4.8). Let also

$$\Lambda=\phi_1-\phi_2. \tag{4.2.2}$$

Then (3.4.8) becomes

$$\begin{aligned}0=\frac{\partial u}{\partial z}&+\alpha\omega^2\left\{\frac{\partial^2 u}{\partial\theta_1^2}+\coth\theta_1\frac{\partial u}{\partial\theta_1}+\frac{\partial^2 u}{\partial\theta_2^2}+\coth\theta_2\frac{\partial u}{\partial\theta_2}\right.\\ &\left.\qquad+(\coth^2\theta_1+\coth^2\theta_2)\frac{\partial^2 u}{\partial\Lambda^2}-(\mathrm{csch}^2\,\theta_1+\mathrm{csch}^2\,\theta_2)u\right\}\\ &+2\alpha\omega^2\cos(2hz)\left\{-\cos\Lambda\coth\theta_1\coth\theta_2\frac{\partial^2 u}{\partial\Lambda^2}+\cos\Lambda\frac{\partial^2 u}{\partial\theta_1\,\partial\theta_2}\right.\end{aligned}$$

$$\begin{aligned}(4.2.3)\qquad & -\sin\Lambda\left[\coth\theta_2\frac{\partial^2 u}{\partial\Lambda\,\partial\theta_1}+\coth\theta_1\frac{\partial^2 u}{\partial\Lambda\,\partial\theta_2}\right]\\ &\qquad\qquad -\cos\Lambda\,\mathrm{csch}\,\theta_1\,\mathrm{csch}\,\theta_2 u\bigg\}\\ &+2i\alpha\omega^2\left\{(\coth\theta_2\,\mathrm{csch}\,\theta_2-\coth\theta_1\,\mathrm{csch}\,\theta_1)\frac{\partial u}{\partial\Lambda}\right\}\\ &+2i\alpha\omega^2\cos(2hz)\bigg\{-\cos\phi(\coth\theta_1\,\mathrm{csch}\,\theta_2+\coth\theta_2\,\mathrm{csch}\,\theta_1)\frac{\partial u}{\partial\Lambda}\\ &\qquad\qquad+\sin\phi\left(\mathrm{csch}\,\theta_1\frac{\partial u}{\partial\theta_2}-\mathrm{csch}\,\theta_2\frac{\partial u}{\partial\theta_1}\right)\bigg\}\end{aligned}$$

for $0\leqq z<L$, with terminal condition

$$u(L,L,\theta_1,\theta_2;\sqrt{\alpha}\,\omega,h)=\tanh\frac{\theta_1}{2}\tanh\frac{\theta_2}{2}. \tag{4.2.4}$$

Then $u(L,h,\omega)$ in (2.3.13) is given by

$$u(L,h,\omega)=u(0,L,0,0;\sqrt{\alpha}\,\omega,h) \tag{4.2.5}$$

so that in fact $u(L,h,\omega)$ is a function of $\sqrt{\alpha}\,\omega$: $u=u(L,h,\sqrt{\alpha}\,\omega)$.

4.3. Scaling properties of the local power spectrum. Let

$$\tilde{\mu}(\tau,L,\sqrt{\alpha}\,\omega)=\int_{-\infty}^{\infty}e^{ih\tau}u(L,h,\sqrt{\alpha}\,\omega)\,dh \tag{4.3.1}$$

where $u(L,h,\omega)$ is defined by (4.2.5). For each $\tau>0$, when L is sufficiently large the integral in (4.3.1) is independent of L. This is a consequence of the hyperbolicity discussed in § 2. This is not easy to see directly from (4.2.3)-(4.2.5) and (4.3.1) and we do not at present have a complete argument that gives this result. We can however deduce from (4.2.3)-(4.2.5) and (4.3.1) scaling properties of $\tilde{\mu}$ that simplify its calculation. We will deduce, in fact, (2.3.14) with $\mu(\omega)$ given by

$$\mu(\omega)=\int_{-\infty}^{\infty}e^{ih}u(L,h,\omega)\,dh, \tag{4.3.2}$$

$u(L,h,\omega)$ given by (4.2.5) when $\alpha=1$, and with L large enough so that (4.3.2) is independent of it.

Now it is clear from (4.3.1) that

$$\begin{aligned}\tilde{\mu}(\tau,L,\sqrt{\alpha}\,\omega)&=\frac{1}{\tau}\int_{-\infty}^{\infty}dh\,e^{ih}u\left(L,\frac{h}{\tau},\sqrt{\alpha}\,\omega\right)\\ &=\frac{1}{\tau}\int_{-\infty}^{\infty}dh\,e^{ih}u\left(\frac{L}{\tau},h,\sqrt{\alpha\tau}\,\omega\right)\end{aligned} \tag{4.3.3}$$

because

$$u\left(z,\frac{h}{\tau},\sqrt{\alpha}\,\omega\right)=u\left(\frac{z}{\tau},h,\sqrt{\tau\alpha}\,\omega\right) \tag{4.3.4}$$

as can be seen from (4.2.3). This gives in view of (4.3.2)

$$\tilde{\mu}(\tau,L,\sqrt{\alpha}\,\omega)=\frac{1}{\tau}\mu(\sqrt{\alpha\tau}\,\omega)$$

for $\tau>0$ when L is large enough.

We summarize the necessary facts for the calculation of the canonical, local power spectrum $\mu(\omega)$.

(1) Calculate the solution u of (4.2.3) with $\alpha=1$ as a function of ω, h for L large enough ($L>1$).

(2) Evaluate u at $\theta_1=\theta_2=0$ and $z=0$.

(3) Insert the result in (4.3.2) and compute the integral with respect to h. Alternatively one may compute $\mu(\omega)$ directly by getting an equation for the Fourier integral of u with respect to h.

5. Gaussian law for the canonical reflection process.

5.1. Higher moments of the reflection process. Let $g(\sigma)$ be a test function and define

$$B^{\varepsilon}_{\tau,f}[g]=\int_{-\infty}^{\infty} d\sigma\, g(\sigma)B^{\varepsilon}_{\tau,f}(\sigma) \tag{5.1.1}$$

where $B^{\varepsilon}_{\tau,f}(\sigma)$ is given by (2.2.4). We will show that in the limit $\varepsilon\to 0$, $B^{\varepsilon}_{\tau,f}(\sigma)$ is a Gaussian process by showing that $B^{\varepsilon}_{\tau,f}[g]$ goes, in the limit, to a Gaussian random variable for each test function g. We will do this by computing asymptotically as $\varepsilon\to 0$ all moments of $B^{\varepsilon}_{\tau,f}[g]$.

Let $\hat{g}$ be the Fourier transform of g and let

$$\hat{H}=\hat{g}\hat{f}. \tag{5.1.2}$$

Then from (5.1.1), (5.1.2) and (2.3.6) we have that

$$\begin{aligned} B^{\varepsilon}_{\tau,f}[g]&=\frac{1}{2\pi\sqrt{\varepsilon}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau/\varepsilon}\hat{H}(\omega)R^{\varepsilon}_{L}(\omega)\\ &=\frac{1}{2\pi\sqrt{\varepsilon}}\int_{0}^{\infty} d\omega\, e^{-i\omega\tau/\varepsilon}\hat{H}(\omega)R^{\varepsilon}_{L}(\omega)\\ &\quad+\frac{1}{2\pi\sqrt{\varepsilon}}\int_{0}^{\infty} d\omega\, e^{i\omega\tau/\varepsilon}\hat{H}^{*}(\omega)R^{\varepsilon *}_{L}(\omega). \end{aligned} \tag{5.1.3}$$

The kth moment of $B^{\varepsilon}_{\tau,f}[g]$ is given by

$$\begin{aligned} E\{(B^{\varepsilon}_{\tau,f}[g])^{k}\}&=\frac{1}{\varepsilon^{k/2}(2\pi)^{k}}\sum_{j=0}^{k}\binom{k}{j}\\ &\cdot\int_{0}^{\infty} d\omega_1\cdots\int_{0}^{\infty} d\omega_k \exp-\frac{i\tau}{\varepsilon}\left(\sum_{l=1}^{j}\omega_l-\sum_{m=j+1}^{k}\omega_m\right)\\ &\cdot\prod_{l=1}^{j}\hat{H}(\omega_l)\prod_{m=j+1}^{k}\hat{H}^{*}(\omega_m)E\left\{\prod_{l=1}^{j}R^{\varepsilon}_{L}(\omega_l)\prod_{m=j+1}^{k}R^{\varepsilon *}_{L}(\omega_m)\right\}. \end{aligned} \tag{5.1.4}$$

As (5.1.4) indicates we must compute expected values of products of reflection coefficients at different frequencies, as $\varepsilon\to 0$. We will do this by perturbation theory as was done in earlier sections and is reviewed in the Appendix. It is necessary here, however, to carry out the expansions to higher order in ε to neutralize the singular factor $\varepsilon^{-k/2}$ in (5.1.4).

5.2. Multifrequency asymptotic problem. We will assume here that the coefficients $m(z)$ and $n(z)$ in the propagator equation (3.1.1) or (4.1.4) are components of a stationary Markov process q taking values in some Euclidean space and having Q as its infinitesimal generator. We assume that Q is ergodic, i.e., $Qu=0$ implies that $u=$ constant, and that its adjoint Q^{*} has a one-dimensional null space spanned by the invariant measure $\bar{p}(q)$, i.e., $Q^*\bar{p}=0$. Moreover we will assume that Q^{-1} can be well defined and is bounded when acting on all functions whose integral with respect to $\bar{p}$ vanishes (Fredholm alternative). In the Appendix we refer to the literature where much weaker hypotheses suffice for the analysis of many similar problems.

Let $\omega_1, \omega_2, \cdots, \omega_k$ be k distinct frequencies and let $\theta_j^\varepsilon(z)=\theta^\varepsilon(z;\omega_j)$; $\phi_j^\varepsilon(z)=\phi^\varepsilon(z;\omega_j)$, $\psi_j^\varepsilon(z)=\psi^\varepsilon(z;\omega_j)$, $j=1,2,\cdots,k$ where $(\theta^\varepsilon(z;\omega), \phi^\varepsilon(z;\omega), \psi^\varepsilon(z;\omega))$ is the solution of the stochastic equation (4.1.4). Define the vector of coordinates

$$\mathbf{x}=(\theta_1,\phi_1,\psi_1,\cdots,\theta_k,\phi_k,\psi_k) \tag{5.2.1}$$

and frequencies

$$\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3,\cdots,\omega_k). \tag{5.2.2}$$

The backward equation for the joint Markov process $q^\varepsilon(z)=q(z/\varepsilon^2)$, $\mathbf{x}^\varepsilon(z)$ (solutions of (4.1.4)) is

$$\frac{\partial u^\varepsilon}{\partial z}+\frac{1}{\varepsilon^2}Qu^\varepsilon+\frac{1}{\varepsilon}F(q,\mathbf{x},\boldsymbol{\omega},z/\varepsilon)\cdot\frac{\partial u^\varepsilon}{\partial \mathbf{x}}=0 \tag{5.2.3}$$

where

$$\begin{aligned} F(q,\mathbf{x},\boldsymbol{\omega},\xi)\cdot\frac{\partial}{\partial \mathbf{x}}=\sum_{j=1}^k 2\omega_j\bigg\{&n\sin(\phi_j+2\omega_j\xi)\frac{\partial}{\partial\theta_j}\\ &+[m+n\coth\theta_j\cos(\phi_j+2\omega_j\xi)]\frac{\partial}{\partial\phi_j}\\ &-n\operatorname{csch}\theta_j\cos(\phi_j+2\omega_j\xi)\frac{\partial}{\partial\psi_j}\bigg\}. \end{aligned} \tag{5.2.4}$$

Terminal conditions for u^ε in (5.2.3) are taken as

$$u^\varepsilon|_{z=L}=\prod_{l=1}^j e^{i\psi_l}\tanh\frac{\theta_l}{2}\prod_{m=j+1}^k e^{-i\psi_m}\tanh\frac{\theta_m}{2}. \tag{5.2.5}$$

In view of (4.1.6), we see that

$$u^\varepsilon(0,L,\mathbf{x},q;\boldsymbol{\omega})=E_{\mathbf{x},q}\left\{\prod_{l=1}^j R_L^\varepsilon(\omega_l)\prod_{m=j+1}^k R^{\varepsilon^*}(\omega_m)\right\} \tag{5.2.6}$$

where the expectation on the right is conditional on the coordinates starting from $\mathbf{x}$, q at $z=L$. The quantity of interest is the expectation on the right side of (5.1.4) and it is obtained from (5.2.6) by integrating u^ε with respect to $\bar{p}\,dq$, which we denote by

$$\langle u^\varepsilon(0,L,\mathbf{x},\cdots,\omega)\rangle,$$

and then evaluate this at $\mathbf{x}=0$. Thus

$$\langle u^\varepsilon(0,L;0,\cdot;\omega)\rangle=E\left\{\prod_{l=1}^j R_L^\varepsilon(\omega_l)\prod_{m=j+1}^k R^{\varepsilon^*}(\omega_m)\right\}. \tag{5.2.7}$$

We will analyze (5.2.3) asymptotically as $\varepsilon\to 0$ by perturbation theory using in addition a second time scale

$$\xi=\frac{z}{\varepsilon}. \tag{5.2.8}$$

The asymptotic analysis depends crucially on whether or not the frequencies ω_1, $\omega_2, \cdots, \omega_k$ are in resonance or not (as explained in the next sections). Resonances are important because the integral representation of the kth moment in (5.1.4) requires knowing the behavior of (5.2.7) for all frequencies $\underline{\omega}$ in the positive orthon.

5.3. Leading order term and Gaussian moments. We will first carry out the asymptotic analysis without regard to resonance conditions. We will then see from the expansion what precisely these conditions are and how they affect the expansion. We will then discuss how the expansion is rearranged near a resonance.

We insert into (5.2.3) the two-time expansion

$$u^{\varepsilon}(z, L, \mathbf{x}, q; \boldsymbol{\omega}) = \sum_{l=0}^{\infty} V_l(z, \xi, L, \mathbf{x}, q; \boldsymbol{\omega})\varepsilon^l|_{\xi=z/\varepsilon}, \tag{5.3.1}$$

and equate to zero coefficients of powers of ε. This leads to the following sequence of equations.

$$QV_0=0, \tag{5.3.2}$$

$$QV_1+\left(F\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{\partial}{\partial\xi}\right)V_0=0, \tag{5.3.3}$$

$$QV_2+\left(F\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{\partial}{\partial\xi}\right)V_1+\frac{\partial}{\partial z}V_0=0, \quad \text{etc.} \tag{5.3.4}$$

From (5.3.2) and the ergodicity assumption for the coefficient process q we have that V_0 is independent of q.

Next we multiply (5.3.3) by $\bar{p}$ and integrate dq, i.e., take the $\langle\ \rangle$ of (5.3.3) where the brackets are defined above (5.2.9). Then $\langle QV_1\rangle=0$ and since $\langle F\rangle=0$ also (because m and n have zero mean) we obtain

$$\frac{\partial V_0}{\partial\xi}=0. \tag{5.3.5}$$

Therefore V_0 is independent of ξ as well as q. Let

$$Sg=-Q^{-1}g \tag{5.3.6}$$

acting on functions g such that $\langle g\rangle=0$ and such that $\langle Sg\rangle=0$. More explicit forms of S are discussed in the Appendix and literature cited therein. In terms of S we solve (5.3.3) for V_1 to get

$$V_1=S\left(F\cdot\frac{\partial}{\partial\mathbf{x}}\right)V_0+V_{10} \tag{5.3.7}$$

where V_{10} does not depend on q.

We may now insert (5.3.7) into (5.3.4) and take $\langle\ \rangle$. Since $\langle QV_2\rangle=0$ again, we get

$$\left\langle\left(F\cdot\frac{\partial}{\partial\mathbf{x}}+\frac{\partial}{\partial\xi}\right)S\left(F\cdot\frac{\partial}{\partial\mathbf{x}}\right)V_0\right\rangle+\frac{\partial V_{10}}{\partial\xi}+\frac{\partial V_0}{\partial z}=0. \tag{5.3.8}$$

We have used the fact that $\langle F\rangle=0$.

If $g(\xi)$ is periodic or almost periodic, we let $\bar{g}$ stand for its average

$$\bar{g}=\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T g(\xi)\,d\xi. \tag{5.3.9}$$

Apply the bar operator to (5.3.8). This gives the equation that determines V_0

(5.3.10) $$\frac{\partial V_0}{\partial z}+\overline{\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)S\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)\right\rangle}V_0=0.$$

Final conditions for V_0 at $z=L$ are given by (5.1.7).

The operator $\langle (F\cdot(\partial/\partial \mathbf{x}))S(F\cdot(\partial/\partial \mathbf{x}))\rangle$ is an elliptic second order partial differential operator whose explicit form was given earlier under various conditions. When $\underline{\omega}$ is only a single frequency, this operator is given by (3.2.11) (in terms of the abstract derivatives D_1, D_2, D_3) or by (4.1.5) in terms of the polar coordinates. For N distinct frequencies along with their sums and differences in pairs, the operator is given by (3.3.5). Note that in this case the terms $D_1^{(p)}\,D_1^{(q)}$ (which are the same as $\partial^2/\partial\phi_p\,\partial\phi_q$) in (3.3.5) drop out in (5.3.10) since both the final conditions and coefficients are independent of the ϕ's. Thus, V_0 is in this case a product of the one-frequency solution of (5.3.10). This means that:

> In the case of distinct frequencies the reflection coefficients $R_L^\varepsilon(\omega_1)\cdots R_L^\varepsilon(\omega_k)$ tend to independent random variables as $\varepsilon\to 0$.

This property is not maintained to higher order in ε in (5.3.1).

Another important conclusion that can be drawn from the form of the single frequency leading order operator (cf. (4.1.8)) is that it annihilates the function $e^{i\psi}\tanh(\theta/2)$ which is the final condition (5.2.7) for (5.3.10). On setting $\theta=0$ we see that the mean reflection coefficient is zero to leading order in ε. This fact was used in § 2.3 and in § 5.1 where it was not necessary to subtract the mean of $B_{\tau,f}^\varepsilon[g]$ in (5.1.4).

Let us use the leading term V_0 from (5.3.10) in the integrals on the right-hand side of (5.1.4). Let Ω be a region in the positive orthon in $\mathbb{R}^k$ in which V_0 is a smooth function of $\omega_1,\omega_2,\cdots,\omega_k$. By integration by parts we see that the integral over Ω is of order ε^k and hence makes negligible contributions in (5.1.4). Thus, the only contributions in (5.1.4) due to V_0 must come from regions Ω that contain discontinuities of V_0.

To see where discontinuities of V_0 as a function of $\omega_1,\omega_2,\cdots,\omega_k$ occur, we look for discontinuities in the coefficients of the operator in (5.3.10). We know that discontinuities do not occur in regions where the ω's are distinct. Discontinuities occur because the ξ-averaging operator, denoted by the bar, acts on terms that are a product of sines or cosines of $2\omega_j\xi+\phi_j$. A typical coefficient in (5.3.10) contains for example the ξ average of $\cos(\phi_1+2\omega_1\xi)\cos(\phi_2+2\omega_2\xi)$. When $\omega_1\neq\omega_2$ we obtain

(5.3.11) $$\overline{\cos(\phi_1+2\omega_1\xi)\cos(\phi_2+2\omega_2\xi)}=0.$$

However, when $\omega_1=\omega$, $\omega_2=\omega$ we get

(5.3.12) $$\overline{\cos(\phi_1+2\omega\xi)\cos(\phi_2+2\omega\xi)}=\tfrac{1}{2}\cos(\phi_1-\phi_2).$$

In a neighborhood where one pair of frequencies, say ω_1 and ω_2, are nearly identical we can still integrate by parts on the remaining $k-2$ frequencies which are distinct. So such terms are also negligible in (5.1.4). Implicit here is the assumption that the reflection coefficient at a pair of nearly coalescing frequencies become independent of the remaining $k-2$ distinct frequencies, which are themselves mutually independent, to leading order in ε. We verify this at the end of this section.

The asymptotic behavior of the reflection coefficient at the nearby frequencies was described in § 3.4. Nearby means that $\omega_1=\omega$ and $\omega_2=\omega+\varepsilon h$ with ω and h fixed as $\varepsilon\to 0$. With this information at hand and the asymptotic independence of reflection coefficients for distinct groups of frequencies, we can estimate the contribution of V_0

in (5.1.4). Specifically, an integral over an odd number of frequencies k will be at most of order

$$\frac{1}{\varepsilon^{k/2}}\varepsilon\cdot\varepsilon^{(k-1/2)}=O(\sqrt{\varepsilon})$$

if all but one frequency resonate in pairs. An integral over an even number of frequencies k will contribute only if the frequencies form exactly $k/2$ distinct pairs whereby the exponential in (5.1.4) cancels and the expectation of the reflection coefficients is a product of covariances. Moreover, only the term $j=k/2$ in (5.1.4) meets all these conditions and hence, for k even

$$\lim_{\varepsilon\downarrow 0} E\{(B^{\varepsilon}_{\tau,f}[g])^k\}=\binom{k}{k/2}\left(\frac{k}{2}\right)!\frac{1}{2^{k/2}}(\mu_2)^k \tag{5.3.13}$$

where

$$\mu_2=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}d\omega\,|\hat{H}(\omega)|^2\int_{-\infty}^{\infty}dh\,e^{i\tau h}u(L,h,\omega). \tag{5.3.14}$$

From (5.3.13) we see that $B^{\varepsilon}_{\tau,f}[g]$ is in the limit $\varepsilon\to 0$ Gaussian with second moment μ_2 which agrees with (2.3.13). Of course, so far we have calculated contributions to (5.1.4) due to V_0 in (5.3.1) only. In the next section we argue that all other contributions are negligible as $\varepsilon\to 0$.

Let us verify that to leading order in ε pairs of reflection coefficients at distinct frequencies are statistically independent. A similar argument extends this to pairs at distinct frequencies and at distinct single frequencies. To recompute V_0 when there are k pairs, we set

$$\omega_j=\omega_{j-k}+\varepsilon h_{j-k}\quad\text{for } j=k+1,\cdots,2k. \tag{5.3.15}$$

This leads to the following equation for V_0:

(5.3.16)

$$\frac{\partial V_0}{\partial z}+\alpha\sum_{j=1}^{k}\omega_j^2\{D_2^{(j)}D_2^{(j)}+D_3^{(j)}D_3^{(j)}+D_2^{(j+k)}D_2^{(j+k)}+D_3^{(j+k)}D_3^{(j+k)}+2\cos 2h_jzD_2^{(j)}D_2^{(j+k)}\}$$
$$+\gamma\left|\sum_{j=1}^{k}\omega_j(D_1^{(j)}+D_1^{(j+k)})\right|^2=0.$$

We note that the coefficients in the operator multiplying α in (5.3.16) depend on ϕ's only through the differences $(\phi_{j+k}-\phi_j)$. On the other hand, the operator multiplying γ has the form

$$\left|\sum_{j=1}^{k}\omega_j\left(\frac{\partial}{\partial\phi_j}+\frac{\partial}{\partial\phi_{j+k}}\right)\right|^2, \tag{5.3.17}$$

which annihilates functions of $\phi_{j+k}-\phi_j$. Since the terminal conditions do not depend on the ϕ's, the solution will depend only on the ϕ's through their differences $\phi_{j+k}-\phi_j$. V_0 will thus factor into a product of k independent pair frequency solutions.

5.4. Negligible contributions by higher order terms. We will now compute higher order terms in (5.3.1) starting with V_1. We already know that V_1 has the form (5.3.7) with V_{10} independent of q but still to be determined.

If $g(\xi)$ is a sum of periodic functions of mean $\bar{g}=0$ we define its integral by

$$(I_0 g)(\xi)=\int_0^{\xi} g(s)\,ds$$

and its normalized integral by

$$Ig=I_0 g-\overline{I_0 g}. \tag{5.4.1}$$

Then (5.3.8) can be solved for V_{10} after using (5.3.10) to replace $\partial V_0/\partial z$. Thus

$$V_{10}=V_{100}-I\left\{\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)S\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)V_0\right\rangle-\overline{\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)S\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)V_0\right\rangle}\right\} \tag{5.4.2}$$

where V_{100} is independent of q and ξ. It is clear from the analysis of discontinuities in the coefficients of the operator in (5.3.10) that V_{10} is well defined when the frequencies $\omega_1, \omega_2, \cdots, \omega_k$ are distinct. The integral I generates terms that are sines or cosines divided by $\omega_i-\omega_j$, $i\neq j$. These terms are singular at the resonances considered in the previous section. The reexpansion with a frequency scaling such as (5.3.15) eliminates singularities in V_{10}.

We can now solve (5.3.4) for V_2 using the operator S defined by (5.3.6) to get

$$V_2=S\left[\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)V_1+\frac{\partial V_0}{\partial z}\right]+V_{20} \tag{5.4.3}$$

where V_{20} is independent of q. The next equation after (5.3.4) in the perturbation sequence is

$$QV_3+\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)V_2+\frac{\partial V_1}{\partial z}=0. \tag{5.4.4}$$

Taking the $\langle\ \rangle$ operation in (5.4.4) after using (5.4.3) and (5.3.7) gives

$$\begin{aligned}&\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)S\left[\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)S\left(F\cdot\frac{\partial V_0}{\partial \mathbf{x}}\right)+\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)V_{10}+\frac{\partial}{\partial z}V_0\right]\right\rangle\\&\qquad+\frac{\partial}{\partial \xi}V_{20}+\frac{\partial}{\partial z}V_{10}=0.\end{aligned} \tag{5.4.5}$$

Using (5.4.2) for V_{10} in (5.4.5) and taking the ξ-average gives the equation that determines V_{100}.

$$\frac{\partial V_{100}}{\partial z}+\overline{\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)S\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)\right\rangle}V_{100}+W_1=0. \tag{5.4.6}$$

Here the inhomogeneous term W_1 is given by

$$\begin{aligned}W_1=&\overline{\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)S\left[\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)S\left(F\cdot\frac{\partial V_0}{\partial \mathbf{x}}\right)+\frac{\partial V_0}{\partial z}\right]\right\rangle-\left\langle\left(\mathrm{F}\cdot\frac{\partial}{\partial \mathbf{x}}\right)S\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)\right\rangle}\\&\overline{\cdot I\left[\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{\partial}{\partial \xi}\right)S\left(F\cdot\frac{\partial V_0}{\partial \mathbf{x}}\right)\right\rangle-\left\langle\left(F\cdot\frac{\partial}{\partial \mathbf{x}}\right)S\left(F\cdot\frac{\partial V_0}{\partial \mathbf{x}}\right)\right\rangle\right]}.\end{aligned} \tag{5.4.7}$$

The ξ-average in (5.4.7) extends over the entire right-hand side.

We return now to (5.1.4) and assess the contribution of V_1 to the ω integrals on the right side. Only V_{10} enters since we use $\langle V_1\rangle$ and in view of (5.3.7). We need not consider regions of $\omega_1, \cdots, \omega_k$ where nearby pairs of frequencies occur because their contributions were accounted correctly in the reexpansion process in the previous section. The corresponding first order term in the rearranged expansion makes negligible contribution.

We consider separately the contributions to the ω integrals by V_{100} and the I term in V_{10} given by (5.4.2), outside regions of pair resonances. Recall that $\xi = z/\varepsilon$ so the I term has rapidly varying trigonometric terms whose coefficients are smooth functions of $\boldsymbol{\omega}$. The trigonometric terms are of the form $\exp[i2L(\pm\omega_l \pm \omega_m)/\varepsilon]$, which combine with the exponentials in (5.1.4). The coefficient of each ω_r in the combined exponential cannot vanish when $L > \tau/2$, i.e., when the artificially introduced slab end is located far enough so that it cannot affect the reflected wave up to time τ/ε^2. We can therefore integrate by parts with respect to each ω_r to get a contribution of order $\varepsilon^{k/2+1}$ which is negligible as ε tends to zero. Thus the I term does not contribute in (5.1.4) at the level of V_1. This argument extends to the analogous higher order terms of V_n in (5.3.1).

It remains to check that the contribution of V_{100} in (5.1.4) is also negligible. Our argument extends to the analogous ξ-independent parts in the higher order term in (5.3.1). As with V_0, we must locate the regions in $\boldsymbol{\omega}$ space where V_{100} has discontinuities and integration by parts in (5.1.4) fails. As mentioned above, pair resonances are excluded since they have been assessed already.

The operator acting on V_{100} in (5.4.6) is the same one as in the V_0 equation (5.3.10). Since we are outside pair resonant regions of $\boldsymbol{\omega}$, this operator has constant coefficients in z. The discontinuities in $\boldsymbol{\omega}$ of V_{100} are determined by those of W_1. These can occur only at triple and quadruple resonances, i.e., when sums and differences of three or four ω_j are nearly zero. This follows from (5.4.7) where products of three or four factors F appear in the ξ-averaging. At the level of V_{200} in the next term of the expansion (5.3.1), the corresponding term W_2 has discontinuities at fifth and sixth order resonances, etc.

We must now consider the rearranged expansion near a discontinuity of V_{100}. This will not affect V_0 and the equation (5.4.6) as far as the operator acting on V_{100} is concerned. The computation of W_1 will be affected as will be that of higher order terms. To see how W_1 is affected, let us assume that we have for example six frequencies $\omega_1, \cdots, \omega_6$ that form two resonant triples, i.e., $\omega_1 + \omega_2 - \omega_4$ and $\omega_3 - \omega_5 - \omega_6$ are nearly zero. Set

$$(5.4.8) \qquad \omega_1 + \omega_2 - \omega_4 = \varepsilon h_1, \qquad \omega_3 - \omega_5 - \omega_6 = \varepsilon h_2$$

and consider the contribution of V_{100} to the integral corresponding to $k=6$, $j=3$ in (5.1.4). The rapidly varying exponential becomes $\exp[-i\tau(h_1+h_2)]$. It is therefore no longer rapidly varying in the near resonant case. This is the reason this example was chosen; the absence of a rapidly varying exponential does not make such a term negligible in an obvious way. In fact such a term appears to make a contribution of order one since the singular factor $\varepsilon^{-6/2}$ is cancelled by $\varepsilon^2 \cdot \varepsilon$ which comes from the change of variables (5.4.8) and the fact that V_{100} is an order ε term in the expansion.

To see the form of this seemingly order one term, let us look at the form of W_1 when (5.4.8) is used to evaluate the right side in (5.4.7). W_1 will be a sum of terms that are either independent of h_1, h_2 and z or contain a single trigonometric factor $\cos(2h_l z)$ or $\sin(2h_l z)$, $l = 1, 2$. Thus, the h integration will produce generalized functions of τ which vanish if τ is positive as we assume. They will not make a contribution.

6. Remarks and conclusions. For pulses which are wide compared to a correlation length, the well-known method of averaging can predict the statistics of the reflection process from a randomly stratified half-space, for time scales on which the propagation distances are not too long. The reflection process we have described here characterizes the returned signal at longer times, when the method of averaging is no longer applicable. The resulting limit process, although more mathematically complex in its derivation, has a simple characterization.

The limit process is universal. That is, only a single parameter, an averaged second moment, of the random medium affects the statistics of the process, and this parameter enters only as a scale factor. If a time window is extracted from the returned signal, with the duration of the window on the order of the pulse length, then the process therein is (locally) stationary, Gaussian, with mean zero. Its power spectrum (2.2.5) involves, as well as the pulse shape, the universal function, μ, which we have derived mathematically.

The effect of the position of the time window is explicit in our equations, entering only as a scale factor in the power spectrum. There is a fall-off of the spectrum inversely proportional to time, and a shift to lower frequencies at later times. This shift is in accord with localization theory, which predicts more attenuation of higher frequencies, as they penetrate to lower depths, and is accounted for explicitly in our characterization of the return process.

Appendix. Limit theorems for stochastic equations. We shall review briefly some results in the asymptotic analysis of stochastic equations that are used in this paper. The results can be found in compact but self-contained form in the survey paper [3] where earlier literature is cited. A more recent account is given in Kushner's book [4]. Results specific to system (3.1.1) are given in [2] and [5]. Very general and technically advanced versions of the theorems are found in [6] and [7]. The origin of the asymptotic analysis goes back to the work of Stratonovich [8] and the theorems first proved by Khasminskii [9]. Extensive use of asymptotics for wave propagation problems is made in Klyatskin's book [10].

Systems (3.1.1) or (3.4.1) have in general the form:

$$\frac{dx^\varepsilon(t)}{dt}=\frac{1}{\varepsilon}F(x^\varepsilon(t), t, t/\varepsilon, t/\varepsilon^2, \omega), \qquad x^\varepsilon(0)=x. \tag{A.1}$$

Here $x^\varepsilon(t)$ takes values in the Euclidean space $\mathbb{R}^d$ and $F(x, t, s, \tau, \omega)$ is a random vector function from $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega$ into $\mathbb{R}^d$. The variable ω labels the realizations of the medium and it belongs to a set Ω where $(\Omega, \underline{F}, P)$ is a probability space. Clearly the solution $x^\varepsilon(t)$ depends on ω and the starting point $x \in \mathbb{R}^d$ but this dependence is not shown.

Notice that the scaling by the small parameter $\varepsilon > 0$ has a particular form which is motivated by the specific problem here but arises in many other problems. The important things to note about (A.1) that allow good analysis are the following:

(1) $F(x, t, s, \tau, \omega)$ is a stationary process in τ for each x, t, s fixed.

(2) $F(x, t, s, \tau, \omega)$ is smooth in $x \in \mathbb{R}^d$.

(3) $E[F(x, t, s, \tau, \cdot)] = \int_\Omega F(x, t, s, \tau, \omega)P(d\omega) = 0$.

(4) $F(x, t, s, \tau_1, \omega)$ and $F(x, t, s, \tau_2, \omega)$ at two widely spaced values τ_1 and τ_2 are approximately statistically independent.

Condition (4) is stated in a rough way here because it is somewhat technical and is in any case fully treated in the literature we cited above (the sharpest condition is in [6]).

Let us make some general remarks about the suitability of (1)-(4). Condition (1) says simply that on the fast scale the process is statistically homogeneous. In the examples this says that the random media are locally homogeneous. Spatial and temporal dependence of the statistical properties must necessarily be on a slower scale, as with F above, or else there can be no asymptotic theory. Condition (2) simply assures that (A.1) has a solution. Condition (3) is a *centering* condition. It means that the original physical system is such that it can be brought to the form (A.1) with zero mean on the right. Of course there could be terms like $G(x^\varepsilon(t), t, t/\varepsilon, t/\varepsilon^2, \omega)$ on the right side of (A.1) as well. These are of lower order in ε (no ε^{-1} factor) and can be handled easily. Condition (4) is necessary and it is the only condition that distinguishes a truly random system from, say, a system with periodic coefficients. The results are not valid if (4) is removed or weakened too much.

Now the analysis can go on in the spirit of the perturbation calculations of § 5 or variants of it when we want to economize on assumptions. The main result is as follows. Let $g(x)$ be a smooth function on $\mathbb{R}^d$ and let

$$U_g(x, t, s) = E\left[F(x, t, s, 0, \cdot) \cdot \frac{\partial}{\partial x}\left(F(x, t, s, \tau, \cdot) \cdot \frac{\partial g(x)}{\partial x}\right)\right], \tag{A.2}$$

$$L_t g(x) = \bar{U}_g(x, t) = \lim_{l \uparrow \infty} \frac{1}{l} \int_0^l U_g(x, t, s)\, ds, \tag{A.3}$$

$$u^\varepsilon(t, T, x) = E[g(x^\varepsilon(t, T; x))]. \tag{A.4}$$

Here $x^\varepsilon(t, T; x)$, denotes the solution of (A.1) for $T > t$ with $x^\varepsilon(t, t; x) = x$.

THEOREM. *The process $x^\varepsilon(t)$ defined by the stochastic equation* (A.1) *converges as $\varepsilon \to 0$ weakly to the diffusion process $x(t)$ which has the* (*time-dependent*) *generator L_t given by* (A.3). *In particular, for every smooth function $g(x)$, the function $u^\varepsilon(t, T, x)$ converges as $\varepsilon \to 0$ to $u(t, T, x)$ which solves the final value diffusion equation*

$$\begin{aligned} &\frac{\partial u}{\partial t} + L_t u = 0, \quad t < T, \quad x \in \mathbb{R}^d, \\ &u(T, T, x) = g(x). \end{aligned} \tag{A.5}$$

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