Reaction-Diffusion Fronts in Periodically Layered Media

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Abstract

We compute the effective wave front speeds of reaction-diffusion equations in periodically layered media with coefficents that have small amplitude oscillations around a uniform mean state. We compare them with the corresponding wave front speeds in the uniform state. We analyze a one dimensional model where wave propagation is along the layering direction of the medium and a two dimensional shear flow model where wave propagation is orthogonal to the layering direction. We find that the effective wave speed is smaller in the one dimensional model and is larger in the two dimensional model for both bistable cubic and quadratic nonlinearities of the Kolmogorov-Petrovskii-Piskunov form. We derive approximate expressions for the wave speeds in the bistable case.

1 Introduction

We will consider traveling waves for reaction-diffusion equations (R-D)

$$\begin{aligned} u_t &= \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} + f(u) \\ u|_{t=0} &= u_0(x) \end{aligned} \tag{1.1}$$

where $a(x) = (a_{ij}(x))$ is a smooth positive definite matrix, 2π-periodic in each direction in R^n. The nonlinear function $f(u)$ is either the bistable nonlinearity

$$f(u) = u(1-u)(u-\mu) \quad , \qquad \mu \in (0, 1/2) \tag{1.2}$$

or the quadratic nonlinearity

$$f(u) = u(1-u) \tag{1.3}$$

of the Kolmogorov-Petrovskii-Piscunov (KPP) form [11]. Travelling wave solutions of (1.1) have the form $u = U(k \cdot x - ct, x)$, where the direction of propagation $k \in R^n$ is a unit vector, $c = c(k)$ is the speed and the wave profile $U = U(s, y)$, $s = k \cdot x - ct$, $y = x$, satisfies the equation

$$\begin{aligned} (k\partial_s + \nabla_y) \cdot (a(y)(k\partial_s + \nabla_y)U) + cU_s + f(U) = 0 \\ U(-\infty, y) = 0, \;\; U(+\infty, y) = 1, \\ U(s, \cdot) \;\; 2\pi - periodic\ . \end{aligned} \tag{1.4}$$

When $a(x) = I$, the identity, (1.1) is the usual reaction-diffusion equation in a homogeneous medium, the travelling wave solution has the form $U(k \cdot x - ct)$ and (1.4) is an ordinary differential equation for U as a function of s. For a medium with periodic structure it is reasonable to look for wave profiles that have also periodic structure, along with their usual form in the direction of propagation, as is common in homogenization problems [2]. For bistable nonlinearities (1.2), traveling waves and their stability are analyzed in [1],

[7], in the homogeneous case. For KPP nonlinearities they are analyzed in [9], [10], [11]. In [15], Xin showed that if $a(y)$ is close to a constant positive definite matrix, then (1.4) admits a unique solution (U, c) up to a constant shift in s, with $c < 0$. This travelling wave solution is stable in one space dimension. In the multidimensional case it is not known if the travelling waves constructed in [15] are stable.

We see from (1.4) that the wave speed c and wave profile U are coupled and must be determined together. However, in the KPP case the asymptotic speeds can be determined independently of the wave profile and Gartner and Freidlin [10] obtained the following result: If u_0 is nonnegative($\not\equiv 0$) and compactly supported, then there exists a number $c^\star(k)$, the asymptotic speed in the direction k, such that $\lim_{t\to\infty} u(t, ctk) = 1$, if $0 \le c < c^\star(k)$ and $\lim_{t\to\infty} u(t, ctk) = 0$, if $c > c^\star(k)$. To compute $c^\star$, let

$$L_y u = \sum_{i,j=1}^{n} (\partial_{x_j} - y_j)(a_{ij}(y)(\partial_{x_i} - y_i)u) + f_u(0) \tag{1.5}$$

where $y \in R^n$ is any constant vector and u is any smooth 2π-periodic function. Let $\lambda = \lambda(y)$ be its principle eigenvalue with positive eigenfunction. It can be shown that λ is smooth and convex in y. Then

$$c^\star(k) = \inf_{(y,k)>0} \frac{\lambda(y)}{(y, k)} \tag{1.6}$$

In this paper, we calculate approximately the speed of the travelling wave in a periodically layered medium, fluctuating around a constant state with mean one and small variation. We compare the wave speed in the layered medium (the effective wave speed) with that in the constant state and see how the medium affects it. We consider two model problems of propagation in layered media. In section 2 we analyze traveling waves in a two dimensional shear flow where the flow is layered in the y_1 direction and the wave is propagating in the y_2 direction, orthogonal to the direction of layering in the flow. We derive an approximate formula for the speed of the travelling wave, up to second order in the variations of the shear flow. We find that the speed in the two dimensional layered medium is larger than that in the uniform medium. In section 3 we analyze a one dimensional problem where the wave propagates along the layering direction of the medium. As in the 2-D shear flow model the qualitative behavior of the wave speed depends only on the form of the nonlinearity $f(u)$ and not on the detailed properties of the medium. The approximate calculation of the speed requires the determination of a set of constants which are related to the solution of certain second order ordinary differential equations on R^1. Solvability of the ODE's is studied in [15]. Here they are solved numerically using a finite difference method and the constants are then computed. Our numerical results show that the wave speed decreases in the layered medium.

In section 4 we give the corresponding qualitative results for the effective wave speeds in the case of KPP nonlinearity by using formula (1.6). It turns out that the same phenomenon occurs: speedup in the two dimensional shear flow model and slowdown in the one dimensional layered medium. These results are analogous to what happens to the effective diffusivity in the corresponding linear problems. It increases in the two dimensional shear flow case and decreases in the one dimensional case.

2 Wave Speed in a 2-D Shear Flow Model

The reaction-diffusion equation in a two dimensional shear flow is

$$u_t = \triangle u + W \cdot \nabla u + f(u) \tag{2.7}$$
$$W = (0, w(y_1)), \quad \triangle = \partial^2_{y_1 y_1} + \partial^2_{y_2 y_2}$$
$$f(u) = u(1-u)(u-\mu), \quad \mu \in (0, \frac{1}{2})$$

where $w(y_1)$ is a smooth periodic function with period 2π. A travelling wave moving in the y_2 direction has the form

$$u = \varphi(y_2 - ct, y_1) = \varphi(s, y_1), \quad s = y_2 - ct \ .$$

Substituting the above into (2.7), we see that φ satisfies

$$\begin{aligned} \varphi_{ss} + \varphi_{y_1 y_1} + (c + w(y_1))\varphi_s + f(\varphi) = 0 \\ \varphi(-\infty, y_1) = 0, \quad \varphi(+\infty, y_1) = 1, \quad \varphi(s, \cdot)\ 2\pi \ \textit{periodic} \end{aligned} \tag{2.8}$$

and we add a normalization condition

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(0, y_1)dy_1 = 1/2.$$

to make the solution unique. The travelling wave equation (2.8) has also been studied in [5] for a premixed flame propagation model with Neumann boundary conditions on y_1 and combustion nonlinearity. For the derivation of this model and its physical background, we refer to [12] and [14].

We are interested here in the effects of $w(y_1)$ on the speed c. If $w(y_1)$ is a small mean zero periodic function, will c be larger than the speed in the uniform medium or smaller? We expand $w(y_1)$ in terms of a small parameter δ,

$$w(y_1) = \delta w_1(y_1) + \delta^2 w_2(y_1) + \cdots \tag{2.9}$$

where $w_i(y_1)(i = 1, 2, \cdots)$ have mean zero over $[0, 2\pi]$. Now expand the travelling wave and the speed

$$\varphi = \varphi_0 + \delta\varphi_1 + \cdots \tag{2.10}$$

$$c = c_0 + \delta c_1 + \cdots \tag{2.11}$$

and plug these expansions into (2.8). We obtain the following equations up to $O(\delta^2)$

$$O(1) \quad \varphi_{0,ss} + c_0\varphi_{0,s} + f(\varphi_0) = 0 \tag{2.12}$$

$$O(\delta) \quad \varphi_{1,ss} + \varphi_{1,y_1y_1} + c_0\varphi_{1,s} + f'(\varphi_0)\varphi_1 = -(c_1 + w_1(y_1))\varphi_{0,s} \tag{2.13}$$

$$\begin{aligned} O(\delta^2) \quad & \varphi_{2,ss} + \varphi_{2,y_1y_1} + c_0\varphi_{2,s} + f'(\varphi_0)\varphi_2 \\ = -c_1\varphi_{1,s} - c_2\varphi_{0,s} & - w_1\varphi_{1,s} - w_2\varphi_{0,s} - 1/2f''(\varphi_0)\varphi_1^2 \end{aligned} \tag{2.14}$$

From (2.12), we get $\varphi_0 = \varphi_0(s)$, $\varphi_0(0) = 1/2$, and $c_0 = c_0(f)$, solution of the usual travelling wave equation

$$\varphi'' + c\varphi' + f(\varphi) = 0$$
$$\varphi(-\infty) = 0, \ \varphi(0) = 1/2, \ \varphi(+\infty) = 1$$

The solvability condition for (2.13) is

$$\int (c_1 + w_1(y_1))\varphi_{0,s}\psi ds dy_1 = 0$$

where $\psi = e^{c_0 s}\varphi_{0,s}$ spans the null space of the adjoint of the linearized ϕ equation and the integral is over $R^1 \times [0, 2\pi]$. Thus $c_1 = - < w_1(y_1) >= 0$. We use $< \cdot >$ to denote the average of the function inside the bracket over $[0, 2\pi]$. Equation (2.13) then becomes

$$\varphi_{1,ss} + \varphi_{1,y_1y_1} + c_0\varphi_{1,s} + f'(\varphi_0)\varphi_1 = -w_1(y_1)\varphi_{0,s} \tag{2.15}$$

Let $(\cdot, \cdot)$ denote the inner product over $R^1 \times [0, 2\pi]$ divided by 2π. From (2.14) we have

$$c_2(\varphi_{0,s}, \psi) + (w_1\varphi_{1,s}, \psi) + \frac{1}{2}(f''(\varphi_0)\varphi_1^2, \psi) + (w_2\varphi_{0,s}, \psi) = 0 \tag{2.16}$$

The last term on left hand side is equal to zero since w_2 has mean zero and so

$$c_2(\varphi_{0,s}, \psi) = -\frac{1}{2}(f''(\varphi_0)\varphi_1^2, \psi) - (w_1\varphi_{1,s}, \psi) \tag{2.17}$$

Since $f''(\varphi_0) = -6\varphi_0 + 2\mu + 2$,

$$c_2(\varphi_{0,s}, \psi) = ((3\varphi_0 - \mu - 1)\varphi_1^2, \psi) - (w_1\varphi_{1,s}, \psi)$$

with

$$(\varphi_{0,s}, \psi) = \int_{R^1} \varphi_{0,s} e^{c_0 s} \varphi_{0,s} ds$$

and

$$((3\varphi_0 - \mu - 1)\varphi_1^2, \psi) = \int_{R^1} (3\varphi_0 - \mu - 1)\varphi_{0,s} e^{c_0 s} < \varphi_1^2 > ds$$

We solve (2.15) by Fourier series. Let

$$w_1(y_1) = \sum_{m \neq 0} b_m e^{imy_1}$$

and

$$\varphi_1(y_1, s) = \sum_{m \neq 0} a_m(s) e^{imy_1}$$

where we have used the fact that $< \phi_1 >= 0$. We substitute these expansions into (2.15) and collect Fourier coefficients to get

$$a_m'' + c_0 a_m' + (f'(\varphi_0) - m^2) a_m = -b_m \varphi_{0,s} \tag{2.18}$$

Note that $\varphi_{0,s}$ satisfies

$$a_m'' + c_0 a_m' + f'(\varphi_0) a_m = 0$$

and (2.18) has the unique solution

$$a_m = \frac{b_m}{m^2} \varphi_{0,s} \tag{2.19}$$

Thus

$$\begin{aligned} < \varphi_1^2 > &= \sum_{m \neq 0} \frac{|b_m|^2}{m^4} \varphi_{0,s}^2 \\ &= \beta \varphi_{0,s}^2 \end{aligned}$$

where

$$\beta = \sum_{m \neq 0} \frac{|b_m|^2}{m^4}.$$

Let

$$I = \int_{R^1} (\varphi_0 - \frac{\mu+1}{3})\varphi_{0,s}^3 e^{c_0 s} ds,$$

and

$$J = -(w_1 \varphi_{1,s}, \psi) \ .$$

Then

$$c_2 = \frac{3\beta}{\alpha} I + \frac{J}{\alpha}$$

where

$$\alpha = \int_s \varphi_{0,s}^2 e^{c_0 s} ds.$$

By (2.19), we have

$$\varphi_1(s, y_1) = \varphi_{0,s} \sum_{m \neq 0} \frac{b_m}{m^2} e^{imy_1}$$

and so

$$\begin{aligned} J &= -(w_1 \varphi_{1,s}, \psi) = -\int_{R^1} < w_1 \varphi_{1,s} > \psi ds \\ &= -\int_s (\sum_{m \neq 0} \frac{|b_m|^2}{m^2}) \varphi_{0,ss} \varphi_{0,s} e^{c_0 s} ds \end{aligned}$$

Let

$$\gamma = \sum_{m \neq 0} \frac{|b_m|^2}{m^2} \ . \qquad (2.20)$$

Then

$$\begin{aligned} J &= -\gamma \int_{R^1} \varphi_{0,ss} \varphi_{0,s} e^{c_0 s} ds = -\frac{\gamma}{2} \int_{R^1} (\varphi_{0,s}^2)_s e^{c_0 s} ds \\ &= \frac{\gamma}{2} c_0 \int_{R^1} \varphi_{0,s}^2 e^{c_0 s} ds \\ &= \frac{\gamma}{2} \alpha c_0 \end{aligned}$$

Therefore

$$c_2 = \frac{3\beta}{\alpha} I + \frac{\gamma}{2} c_0 \qquad (2.21)$$

Next we compute $I = I(\mu)$. In the case of $f(\varphi) = \varphi(1-\varphi)(\varphi-\mu)$ with $\mu \in (0, \frac{1}{2})$, we have Huxley's explicit expressions for φ_0 and c_0

$$\begin{aligned} \varphi_0 &= (1 + e^{-\frac{s}{\sqrt{2}}})^{-1} \\ c_0 &= \frac{2\mu - 1}{\sqrt{2}} \end{aligned}$$

Let

$$I_1 = \int_{R^1} \varphi_0 \varphi_{0,s}^3 e^{c_0 s} ds$$

and

$$I_2 = \frac{\mu+1}{3} \int_{R^1} \varphi_{0,s}^3 e^{c_0 s} ds$$

so that $I = I_1 - I_2$. We will show that $I_1 = I_2$ and hence that $I = 0$. Since

$$\varphi_{0,s} = \frac{1}{\sqrt{2}} e^{-s/\sqrt{2}} (1 + e^{-s/\sqrt{2}})^{-2}$$

we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} (1 + e^{-s/\sqrt{2}})^{-1} (\frac{1}{\sqrt{2}})^3 exp(-3s/\sqrt{2})(1 + e^{-s/\sqrt{2}})^{-6} e^{c_0 s} ds \\ &= \frac{1}{2\sqrt{2}} \int_{-\infty}^{+\infty} exp(-3s/\sqrt{2})(1 + exp(-s/\sqrt{2}))^{-7} exp(\frac{2\mu - 1}{\sqrt{2}} s) ds \\ &\overset{\eta = s/\sqrt{2}}{=} \frac{1}{2} \int_{-\infty}^{+\infty} exp(-3\eta) exp((2\mu - 1)\eta)(1 + exp(-\eta))^{-7} d\eta \\ &\overset{y = exp(-\eta)}{=} \frac{1}{2} \int_0^{+\infty} y^{-2\mu+3} (1+y)^{-7} dy \\ &\overset{x = y+1}{=} \frac{1}{2} \int_1^{+\infty} (x-1)^{3-2\mu} x^{-7} dx \\ &= \frac{1}{2} \int_1^{+\infty} \frac{1}{x^{4+2\mu}} (1 - 1/x)^{3-2\mu} dx \\ &\overset{z = 1/x}{=} \frac{1}{2} \int_0^1 z^{4+2\mu} (1-z)^{3-2\mu} \frac{dz}{z^2} \\ &= \frac{1}{2} \int_0^1 z^{2+2\mu} (1-z)^{3-2\mu} dz \\ &= \frac{1}{2} B(3 + 2\mu, 4 - 2\mu) \end{aligned} \tag{2.22}$$

where $B = B(x,y)$ is the Beta function. Similarly, we have

$$\begin{aligned} I_2 &= \frac{1}{2}\frac{\mu+1}{3}\int_0^{+\infty} y^{3-2\mu}(1+y)^{-6}dy \\ &= \frac{1}{2}\frac{\mu+1}{3}\int_0^{1} z^{1+2\mu}(1-z)^{3-2\mu}dz \\ &= \frac{1}{2}\frac{\mu+1}{3}B(2+2\mu, 4-2\mu) \end{aligned} \tag{2.23}$$

Using the identities

$$\begin{aligned} B(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\ \Gamma(x+1) &= x\Gamma(x) \end{aligned} \tag{2.24}$$

we obtain

$$\begin{aligned} B(3+2\mu, 4-2\mu) &= \frac{\Gamma(3+2\mu)\Gamma(4-2\mu)}{\Gamma(7)} \\ &= \frac{2+2\mu}{6}\frac{\Gamma(2+2\mu)\Gamma(4-2\mu)}{\Gamma(6)} \\ &= \frac{\mu+1}{3}B(2+2\mu, 4-2\mu) \end{aligned}$$

which implies

$$I_1 = I_2$$

We have shown therefore that $I = 0$ and hence the speed $c(\delta)$ has the expansion

$$c(\delta) = c_0(1 + \frac{\gamma}{2}\delta^2 + \cdots) \tag{2.25}$$

where γ is given by (2.20) and the b_m's are the Fourier coefficients of $w_1(y_1)$.

Our result (2.25) says that waves propagating in a particular (a layered) divergence free, mean zero periodic medium speed up. This phenomenon has been observed in flame propagation through turbulent media, in [6] and [13], and formulas for the turbulent flame speed in terms of the laminar flame speed are similar to (2.25).

3 Wave Speed in a 1-D Layered Medium

In this section, we study the speed of the travelling wave solution of the 1-D bistable reaction-diffusion equation

$$u_t = (a(x)u_x)_x + f(u) \tag{3.26}$$

where $f(u)$ given by (1.2) and $a(x) = 1 + \delta a_1(x)$, with $a_1(x)$ a smooth, 2π-periodic, mean zero function and δ a small parameter. The travelling wave solution has the form $u = u(x - ct, x) = u(s, y)$, where $s = x - ct$, $y = x$. The wave profile u satisfies as a function of s and y the equation

$$\begin{aligned} (\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)u) + cu_s + f(u) = 0 \\ u(-\infty, y) = 0 \quad , \qquad u(+\infty, y) = 1 \\ u(s, \cdot) \quad 2\pi - periodic \qquad < u(0, \cdot) >= 1/2 \end{aligned} \tag{3.27}$$

We write

$$\begin{aligned} u(s, y) &= u(s, y, \delta) = \varphi_0(s) + \delta\varphi_1(s, y) + \cdots \\ c &= c(\delta) = c_0 + \delta c_1 + \frac{\delta^2}{2}c_2 + \cdots \end{aligned}$$

where $\varphi_0(s)$ and c_0 satisfy

$$\begin{aligned} \varphi_{ss} + c_0\varphi_s + f(\varphi) = 0 \\ \varphi(-\infty) = 0,\ \varphi(0) = 1/2,\ \varphi(+\infty) = 1 \end{aligned}$$

Let

$$L_0 v = (\partial_s + \partial_y)^2 v + c_0 v_s + f_u(\varphi_0)v \ .$$

Expanding in powers of δ we find that ϕ_1 satisfies

$$L_0\phi_1 = -c'\varphi_{0,s} - (\partial_s + \partial_y)(a_1(\partial_s + \partial_y)\varphi_0) \tag{3.28}$$

The solvability condition for (3.28) is

$$(-c_1\varphi_{0,s} - (\partial_s + \partial_y)(a_1(\partial_s + \partial_y)\varphi_0, \varphi_{0,s}e^{c_0 s}) = 0$$

or

$$
\begin{aligned}
c_1(\varphi_{0,s}, \varphi_{0,s} e^{c_0 s}) &= -((\partial_s + \partial_y)(a_1(\partial_s + \partial_y)\varphi_0), \varphi_{0,s} e^{c_0 s}) \\
&= -(a_1 \varphi_{0,ss}, \varphi_{0,s} e^{c_0 s}) \\
&= - < a_1 > (\varphi_{0,ss}, \varphi_{0,s} e^{c_0 s}) = 0
\end{aligned}
$$

Thus $c_1 = 0$. Expanding to order δ^2 we find that ϕ_2 satisfies

$$
L_0 \phi_2 = -c_2 \varphi_{0,s} - 2(\partial_s + \partial_y)(a_1(\partial_s + \partial_y)\phi_1) - f_{uu}(\varphi_0)(\phi_1)^2 \tag{3.29}
$$

The solvability condition for (3.29) is

$$
\begin{aligned}
c_2(\varphi_{0,s}, \varphi_{0,s} e^{c_0 s}) &= -(f_{uu}(\varphi_0)(\phi_1)^2 + 2(\partial_s + \partial_y)(a_1(\partial_s + \partial_y)\phi_1), \varphi_{0,s} e^{c_0 s}) \\
&= -(f_{uu}(\varphi_0)(\phi_1)^2 + 2a_1(\partial_s + \partial_y)\phi_{1s}, \varphi_{0,s} e^{c_0 s})
\end{aligned}
$$

where ϕ_1 is given by (3.28) which simplifies to

$$
L_0 \phi_1 = -a_{1,y} \varphi_{0,s} - a_1 \varphi_{0,ss} \tag{3.30}
$$

We solve (3.30) by Fourier series. Let

$$
\begin{aligned}
a_1 &= \sum \beta_m e^{imy}, \quad \beta_0 = 0, \quad \sum m^2 |\beta_m|^2 < +\infty \\
\phi_1 &= \sum \alpha_m e^{imy}, \quad \alpha_0 = 0
\end{aligned}
$$

Then the Fourier coefficients of ϕ_1 satisfy

$$
\begin{aligned}
&\alpha_{m,ss} + (c_0 + 2mi)\alpha_{m,s} + (-m^2 + f_u(\varphi_0))\alpha_m \\
&\qquad = -im\beta_m \varphi_{0,s} - \beta_m \varphi_{0,ss}
\end{aligned} \tag{3.31}
$$

Let q_m satisfy

$$
q_{m,ss} + (c_0 + 2mi)q_{m,s} + (-m^2 + f_u(\varphi_0))q_m = -im\varphi_{0,s} - \varphi_{0,ss} \tag{3.32}
$$

so that $\alpha_m = \beta_m q_m$. Then the second order correction to the speed has the form

$$\begin{aligned} c_2(\varphi_{0,s}, \varphi_{0,s}e^{c_0 s}) = -\sum \int_s |\beta_m|^2 |q_m|^2 f_{uu}(\varphi_0)\varphi_{0,s}e^{c_0 s} ds \\ -2\sum \int_s \varphi_{0,s}e^{c_0 s}|\beta_m|^2(imq_{m,s} + q_{m,ss})ds \\ = -\sum |\beta_m|^2 \int_s (f_{uu}(\varphi_0)\varphi_{0,s}|q_m|^2 e^{c_0 s} + 2\varphi_0 e^{c_0 s}(q_{m,ss} + imq_{m,s}))ds \\ = -\sum |\beta_m|^2 \int_s \varphi_{0,s}e^{c_0 s}(f_{uu}(\varphi_0)|q_m|^2 + 2(q_{m,ss} + imq_{m,s}))ds \end{aligned}$$

Since from (3.32) $\overline{q_m} = q_{-m}$, we get

$$\overline{q_{m,ss} + imq_{m,s}} = q_{-m,ss} - imq_{-m,s} \tag{3.33}$$

and hence

$$c_2(\varphi_{0,s}, \varphi_{0,s}e^{c_0 s}) = -2\sum_{m>0} |\beta_m|^2 B_m \tag{3.34}$$

where

$$B_m = \int_s \varphi_{0,s}e^{c_0 s}(f_{uu}(\varphi_0)|q_m|^2 + 2Re(q_{m,ss} + imq_{m,s}))ds \tag{3.35}$$

Note that the B_m's are independent of $a_1(y)$ and depend only on the nonlinear function f, since (φ_0, c_0) and q_m are determined by solving ODE's involving f. If $q_m = p_m + ir_m$, then (3.32) becomes

$$\begin{aligned} p_{m,ss} + c_0 p_{m,s} - 2mr_{m,s} + (-m^2 + f_u(\varphi_0))p_m = -\varphi_{0,ss} \\ r_{m,ss} + c_0 r_{m,s} + 2mp_{m,s} + (-m^2 + f_u(\varphi_0))r_m = -m\varphi_{0,s} \end{aligned} \tag{3.36}$$

and the B_m's can be written as

$$B_m = \int_{-\infty}^{+\infty} \varphi_{0,s}e^{c_0 s}(f_{uu}(\varphi_0)(p_m^2 + r_m^2) + 2(p_{m,ss} - mr_{m,s}))ds \tag{3.37}$$

which after integration by parts gives

$$B_m = \int_{-\infty}^{+\infty} 2(\varphi_{0,s}e^{c_0 s})_{ss}p_m + 2m(\varphi_{0,s}e^{c_0 s})_s r_m + \varphi_{0,s}e^{c_0 s}f_{uu}(\varphi_0)(p_m^2 + r_m^2)ds \tag{3.38}$$

Since of $c = c_0 + \frac{\delta^2}{2} c_2 + O(\delta^3)$ and c_0 is negative, we see from (3.34) that if the B_m's are positive then the perturbed speed is larger (in absolute value); otherwise, if the B_m's are all negative then the perturbed speed is smaller (in absolute value). If some B_m's are positve and some are negative, then the β_m's, or a_1, must be taken into account.

We use a finite difference method to compute the p_m's and r_m's from (3.36) on a large enough finite interval $[-N, N]$ with zero Dirichlet boundary conditions at the two end points. We use the double precision Linpack routines to solve the linear equations that result from the discretization of the ODE's. Then we calculate the B_m's from (3.38) using a double precision integration routine from Naglab. We also use Huxley's formulas for (φ_0, c_0). We verified that the numerical scheme converges as the grid is refined. We describe here the results of two typical runs. The first is done on the interval $[-10, 10]$ with 500 grid points, and $\mu = 0.15$, where μ is the middle zero point of $f(u) = u(1-u)(u-\mu)$. We compute B_1 to B_{100} and find that they increase slowly from -0.0915 to -0.08053. They are all negative. The second run is done on the interval $[-20, 20]$ with 2000 grid points and $\mu = 0.30$. Now B_1 to B_{500} increase slowly from -0.05981 to -0.03660 and again they are all negative. In all the runs we observed slow convergence of the B_m's to some negative constant.

These numerical results show that the wave speed is smaller in the layered medium.

4 Effective Wave Speeds in the KPP case

In this section, we use formula (1.6) to calculate the effective wave speeds in the model problems of the previous two sections with KPP nonlinearity (1.3). Since $\lambda(y)$ is smooth and convex, the infimum in (1.6) is achieved at a finite point and thus it is enough to analyze λ over a compact set of y. By the Krein-Rutman theorem, λ is a simple eigenvalue of L_y, so it is stable under perturbation. In our calculations we will drop the term $f_u(0)$ in (1.5) because it only shifts λ by a constant.

First we treat the shear flow model. The eigenvalue problem for λ is

$$
\begin{aligned}
L_y u &= (\partial_{x_1} - z_1)^2 u + (\partial_{x_2} - z_2)^2 u + \delta w(y_1)(\partial_{x_2} - y_2)u = \lambda u \\
u &= u(x,\delta) = 1 + \delta u_1(x) + \delta^2 u_2(x) + \cdots \\
\lambda &= \lambda(y,\delta) = \lambda_0(y) + \delta\lambda_1(y) + \delta^2\lambda_2(y) + \cdots
\end{aligned} \tag{4.39}
$$

where $\lambda_0(y) = y_1^2 + y_2^2$. Denoting by ' differentiation with respect to δ, we get from (4.39)

$$
\begin{aligned}
& (\partial_{x_1} - y_1)^2 u' + (\partial_{x_2} - y_2)^2 u' + w(x_1)(\partial_{x_2} - y_2)u + \delta w(x_1)(\partial_{x_2} - y_2)u' \\
= \; & \lambda' u + \lambda u'
\end{aligned} \tag{4.40}
$$

and

$$
\begin{aligned}
& (\partial_{x_1} - y_1)^2 u'' + (\partial_{x_2} - y_2)^2 u'' + 2w(x_1)(\partial_{x_2} - y_2)u' + \delta w(x_1)(\partial_{x_2} - y_2)u'' \\
= \; & \lambda'' u + 2\lambda' u' + \lambda u''
\end{aligned} \tag{4.41}
$$

Letting $\delta = 0$ in (4.40) gives

$$
(\partial_{x_1} - y_1)^2 u' + (\partial_{x_2} - y_2)^2 u' - w(x_1)y_2 = \lambda' + (y_1^2 + y_2^2)u' \tag{4.42}
$$

or

$$
u'_{x_1x_1} + u'_{x_2x_2} - 2y_1 u'_{x_1} - 2y_2 u'_{x_2} - w(x_1)y_2 = \lambda' \tag{4.43}
$$

Averaging over x_1 and using $< w >= 0$, we get

$$
\lambda' = \lambda_1(y) = 0
$$

and hence

$$
\triangle_x u' - 2y \cdot \nabla_x u' = w(x_1)y_2 \tag{4.44}
$$

The only solution of this equation is $u' = u'(x_1)$ with

$$
u'_{x_1x_1} - 2y_1 u'_{x_1} = w(x_1)y_2 \tag{4.45}
$$

Letting $\delta = 0$ in (4.41) gives

$$(\partial_{x_1} - y_1)^2 u'' + (\partial_{x_2} - y_2)^2 u'' + 2w(x_1)(\partial_{x_2} - y_2)u' = \lambda'' + \lambda_0 u'' \tag{4.46}$$

or

$$\triangle_x u'' - 2y \cdot \nabla_x u'' + 2w(x_1)(u'_{x_2} - y_2 u') = \lambda'' \tag{4.47}$$

and averaging over x gives

$$\lambda'' = 2\lambda_2(y) = 2 < w(x_1)(u'_{x_2} - y_2 u') >= (-2y_2) < w(x_1)u' > \tag{4.48}$$

Let us solve (4.45) by Fourier series

$$\begin{aligned} u' &= \sum_{m \neq 0} e^{imx_1} u'_m \\ w(x_1) &= \sum_{m \neq 0} e^{imx_1} \alpha_m \end{aligned}$$

Substituting into (4.45) gives

$$u'_m = \frac{y_2 \alpha_m}{-m^2 - 2my_1 i}$$

and so

$$\begin{aligned} < w(x_1)u' > &= \sum_{m \neq 0} |\alpha_m|^2 \frac{y_2}{-m^2 - 2my_1 i} \\ &= 2y_2 \sum_{m>0} |\alpha_m|^2 Re\{\frac{1}{-m^2 - 2my_1 i}\} \\ &= 2y_2 \sum_{m>0} |\alpha_m|^2 \frac{-m^2}{m^4 + 4m^2 y_1^2} \end{aligned}$$

Therefore by (4.48)

$$\lambda'' = 4y_2^2 \sum_{m>0} |\alpha_m|^2 \frac{1}{m^2 + 4y_1^2} > 0 \tag{4.49}$$

or

$$\lambda(y, \delta) \geq \lambda_0(y) \tag{4.50}$$

which implies that

$$c^\star(\delta) = \inf_{y_2>0} \frac{\lambda(y,\delta)}{y_2} \geq \inf_{y_2>0} \frac{\lambda_0(y)}{y_2} = c^\star(0) > 0 \tag{4.51}$$

This means that the effective wave speed is larger than the corresponding speed in the uniform medium in the shear flow model.

Let us now turn to the 1-D model. The eigenvalue problem for λ is

$$\begin{aligned} L_y u &= (\partial_x - y)((1+\delta a_1)(\partial_x - y)u) = \lambda u \\ u &= 1 + \delta u_1 + \delta^2 u_2 + \cdots \\ \lambda &= y^2 + \delta\lambda_1(y) + \delta^2\lambda_2(y) + \cdots \\ < a_1 > &= 0 \end{aligned} \tag{4.52}$$

Differentiating (4.52) twice with respect to δ, we get

$$(\partial_x - y)(a_1(\partial_x - y)u) + (\partial_x - y)((1+\delta a_1)(\partial_x - y)u') = \lambda' u + \lambda u' \tag{4.53}$$

and

$$2(\partial_x - y)(a_1(\partial_x - y)u') + (\partial_x - y)((1+\delta a_1)(\partial_x - y)u'') = \lambda'' u + 2\lambda' u' + \lambda u'' \tag{4.54}$$

Letting $\delta = 0$ in (4.53), we have

$$-(\partial_x - y)(a_1 y) + (\partial_x - y)^2 u' = \lambda' + y^2 u' \tag{4.55}$$

or

$$-y(a_{1,x} - y a_1) + u'_{xx} - 2y u'_x = \lambda' \tag{4.56}$$

Averaging over x, and using $< a_1 >= 0$, we get

$$\lambda' = \lambda_1(y) = 0$$

We then have

$$u'_{xx} - 2yu'_x = y(a_{1,x} - ya_1) \tag{4.57}$$

Letting $\delta = 0$ in (4.54), we get

$$2(\partial_x - y)(a_1(\partial_x - y)u') + (\partial_x - y)^2 u'' = \lambda'' + y^2 u'' \tag{4.58}$$

Averaging over x gives

$$\lambda'' = (-2y) < a_1(u'_x - yu') > \tag{4.59}$$

Let us solve (4.57) by Fourier series

$$\begin{aligned} a_1 &= \sum_{m \neq 0} e^{imx} \alpha_m \\ u' &= \sum_{m \neq 0} e^{imx} u'_m \end{aligned}$$

Substituting into (4.57) gives

$$u'_m = \frac{y\alpha_m(im - y)}{-m^2 - 2ymi} \tag{4.60}$$

and so

$$\begin{aligned} \lambda'' &= (-2y) < a_1(u'_x - yu') > \\ &= (-2y^2) \sum_{m \neq 0} |\alpha_m|^2 \frac{(im - y)^2}{-m^2 - 2ymi} \\ &= (-4y^2) \sum_{m > 0} |\alpha_m|^2 Re\{\frac{y^2 - m^2 - 2ymi}{-m^2 - 2ymi}\} \\ &= (-4y^2) \sum_{m > 0} |\alpha_m|^2 Re\{\frac{(y^2 - m^2 - 2ymi)(-m^2 + 2ymi)}{m^4 + 4y^2 m^2}\} \\ &= (-4y^2) \sum_{m > 0} |\alpha_m|^2 \frac{m^2 + 3y^2}{m^2 + 4y^2} < 0 \end{aligned} \tag{4.61}$$

which implies

$$\lambda(y, \delta) \leq \lambda(y, 0)$$

or

$$c^\star(\delta) \le c^\star(0)$$

This means that the wave speed in the 1-D layered medium is less than that in the corresponding uniform medium. We conclude that the qualitative behavior of effective wave speeds in the KPP case is the same as that in the bistable case.

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