Robustness of time-reversal focusing in changing random environments

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Abstract

We study the stability of spatial focusing and temporal compression properties in time reversal when the medium is different for the forward and backward propagation stages. We consider two different models for multiple scattering: a medium containing randomly distributed discrete scatterers and a waveguide whose interior has a refractive index that fluctuates continuously. Numerical simulations of both models show strong spatial focusing and time compression despite the change in the medium between the forward and backward propagation stages. However, the peak amplitude of the time reversed signal diminishes as the change in the medium between the forward and backward propagation stages increases. Nevertheless, the spatial focusing and temporal compression is destroyed only when changes in the medium are large.

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I. INTRODUCTION

Recently, several studies have addressed time reversal of acoustic or electromagnetic waves in a multiple scattering medium [1–9]. In a basic time reversal experiment, there are two stages: forward propagation and backward propagation. For the forward propagation stage, a localized source emits a short pulse that propagates through a medium. A time reversal mirror records this signal over a sufficiently long time window. For the backward propagation stage, the time reversal mirror re-emits the time reversed version of the recorded signal back into the same medium. This signal retraces the paths it took in the forward propagation stage thereby focusing in space and time at the original source location. The spatial focusing of the back-propagated wave is sharper for a random medium than for a homogeneous medium. Moreover, the spatial focusing and temporal compression are statistically stable [6, 10]. This fundamental phenomenon is understood well physically and mathematically. Moreover, time reversal has many important applications in several areas such as underwater sound [1, 5], ultra-sound [11, 12], imaging [13–15], and communications [16–19], among others.

The three remarkable properties of time reversal are spatial focusing, temporal compression and statistical stability. Because of spatial focusing and temporal compression, the signal power is peaked at the source location and decays rapidly away from it in space and time. Because of statistical stability, spatial focusing and temporal compression do not depend on the individual realization of the random medium. Statistical stability is very important for applications with significant multiple scattering of waves. A lack of statistical stability does not mean that the properties of time reversal deteriorate, but that they also change unpredictably with realizations of the medium, and therefore, they are useless.

For several applications, it is important to consider the case in which the medium changes between the forward and backward propagation stages. In other words, the time-reversed signals propagate through a medium in the backward propagation stage that is different from the medium in the forward stage. In principle, this change will break the time reversal symmetry of the problem, i.e. the time reversed signals are unable to trace their way back to the source. This change will diminish the focusing in space and time at the source.

Several researchers have considered this problem experimentally. Tourin et al [20] performed two sets of ultrasound experiments. In one set of experiments, they perturbed the scattering medium, comprised of a distribution of steel rods immersed in water, by removing
some of the rods during the backward propagation stage. For the other set of experiments, they varied the ambient temperature of the medium during the backward propagation stage. For both sets of experiments, they observed robust focusing of the time reversed signal. However, they found that the peak amplitude at the focal spot diminishes as the perturbation increases. It was only for a very large perturbation that they observed a complete “breaking” of the time reversal peak. Kim et al [21] have shown the degradation of spatial focusing in the changing ocean. In their experiment, they conducted the backward propagation stages at several different times after the forward propagation stage (e.g. 16 minutes, 1 day and 1 week). They studied the degradation in spatial focusing in the context of underwater acoustic communications. In particular, they found that a stable focus was limited to either minutes or days depending on the frequency band they used.

In addition, several researchers have considered this problem theoretically. Snieder et al [22] show that time reversal under wave motion is much more stable than under particle motion when either the initial conditions or the scattered positions change between the forward and backward propagation stages. They give critical length perturbations for both, wave motion and particle motion. Alfaro Vigo et al [23] have addressed analytically and numerically the statistical stability properties for the refocused signal in one-dimensional changing media. For their analysis, they use the acoustic wave equation. Bal and Verastegui [24] have studied statistical stability of refocusing in three dimensional, weakly fluctuating random media that change between the forward and backward propagation stages. For their analysis, they study high frequency wave propagation in the radiative transport and diffusion regimes. Bal and Ryzhik [25] do a similar analysis but for the parabolic wave equation. In that work, they quantify theoretically how the focusing quality of the back-propagated signal degrades as the correlation between the media in the forward and backward stages increases. Recently, Mehta and Snieder [26] use the two dimensional wave equation to study theoretically and computationally time reversal in uniform media whose velocity changes between forward and backward propagation stages. They determine explicitly a shift in the location of the refocused pulse caused by the change in the medium. They propose a novel method to estimate the change in velocity given this shift.

In this paper, we study time reversal for changing media using two different models for wave propagation and multiple scattering: (i) an unbounded medium containing randomly distributed point scatterers, and (ii) guided wave propagation in a medium with a refractive
index that fluctuates continuously. For the discrete random medium, we solve the self-consistent Foldy-Lax system of equations. This model has also been used by Snieder et al. [22] to study time reversal stability of wave propagation. For the random waveguide, we use the phase screen method for solving the wave propagation problem in the interior of the waveguide.

We show that these two simple models provide results that are consistent with the experimental and theoretical results described above. The Foldy-Lax system models the experiments by Tourin et al. [20] and outdoor wireless communications [19], while the random waveguide models the experiments done by Kim et al. [21] and indoor wireless communications [19]. In particular, we show that spatial focusing and temporal compression are robust to perturbations in the medium. However, the peak of the focusing diminishes as the strength of the perturbations increase. We derive a radiative transfer equation for the wave correlation that explicitly shows the deterioration of time reversal properties as function of the perturbation strength. It shows that the change of the medium leads to an effective albedo which is only a function of the perturbation strength. Even more, a discussion about statistical stability is carried out. We show that time reversal is still self-averaging when the perturbations are not large in the broadband regime. However, in a narrowband regime the change of the medium leads to a loss of statistical stability. Besides investigating the role of the signal bandwidth, we study the influence of the time-reversal mirror aperture and the impact of the amount of multiple scattering on the robustness of the focusing properties of the reversed signal.

The remainder of this paper is organized as follows. In Section II, we introduce the Foldy-Lax equations for the discrete scattering medium and the phase screen method for the waveguide model. Section III contains results from our numerical experiments. In Sections IV and V we show analytically the effect of changing the medium. Section VI contains the conclusions.

II. MODELS FOR MULTIPLE SCATTERING

We consider two different models for multiple scattering. One model involves a random distribution of point scatterers in a homogeneous medium. We study this problem using the self-consistent theory due to Foldy [27] and Lax [28]. The other model involves guided
wave propagation in a medium whose refractive index fluctuates randomly. We study this problem using the phase screen method [19, 29–31].

A. Discrete multiple scattering medium

We consider a discrete multiple scattering medium that is composed of $M$ scatterers distributed randomly in an otherwise uniform medium (see Fig. 1). Wave propagation and multiple scattering in such a medium can be formulated in terms of the so-called Foldy-Lax equations [27, 28, 32]. We consider here scalar waves and isotropic point scatterers. We assume that the particles are sufficiently far apart from one another so that we can use the far-field approximation for the free-space propagator. Moreover, we ignore self-interacting fields.

Consider $M$ point scatterers located at positions $\xi_1, \xi_2, \ldots, \xi_M$. Furthermore, consider a time-harmonic incident field evaluated at position $x$ due to a source at position $y$,

$$\hat{\psi}^i(x; y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

with wavenumber $k = \omega/c$, where $c$ is the constant wave speed. We suppress the dependence on frequency $\omega$ in the notation unless it is necessary. The total field $\hat{\psi}$ can be written as the sum of the incident and scattered waves:

$$\hat{\psi}(x; y, \xi_1, \ldots, \xi_M) = \hat{\psi}^i(x; y) + \sum_{j=1}^{M} \hat{\psi}_j^s(x; \xi_1, \ldots, \xi_M).$$

Here, $\hat{\psi}_j^s(x; \xi_1, \ldots, \xi_M)$ denotes the scattered field at position $x$ due to the scatterer at position $\xi_j$. It is given by

$$\hat{\psi}_j^s(x; \xi_1, \ldots, \xi_M) = G_0(x - \xi_j)\tau_j \hat{\psi}_j^e(\xi_1, \ldots, \xi_M),$$

with

$$G_0(x - \xi_j) = \frac{e^{ik|x-\xi_j|}}{4\pi|x-\xi_j|}$$

denoting the free-space propagator, $\tau_j$ denoting the scattering amplitude, and $\hat{\psi}_j^e$ denoting the exciting field at the scatterer located at $\xi_j$.

Because we ignore self-interacting fields, the exciting field $\hat{\psi}_j^e$ is equal to the sum of the incident field $\hat{\psi}^i$ at $\xi_j$ and the scattered fields at $\xi_j$ due to all scatterers except for the one
at $\xi_j$. Moreover, because we assume that the scatterers are sufficiently far apart from one another, $\hat{\psi}_j^e$ is given by

$$
\hat{\psi}_j^e(\xi_1, \ldots, \xi_M) = \hat{\psi}_j^i(\xi_j; y) + \sum_{m=1, m \neq j}^{M} G_0(\xi_j - \xi_m) \tau_m \hat{\psi}_m^e(\xi_1, \ldots, \xi_M). 
$$

(5)

Eq. (5) is a self-consistent system of $M$ equations for the $M$ unknown exciting fields, $\hat{\psi}_1^e, \ldots, \hat{\psi}_M^e$. This system can be written in matrix form and solved numerically by standard techniques if the number of scatterers $M$ is small. For $M$ large, this system can be solved efficiently using the Fast Multipole Method [33–35]. This method has computational complexity that scales as either $O(M \log M)$ or $O(M)$ depending on the implementation. In contrast, traditional methods have a computational complexity that is $O(M^3)$ (see Ref. [34] for more details). Regardless, upon solution of (5), we use (3) and (2) to compute the scattered field at any position (we neglect the incident field). First, we compute forward propagation stage to obtain the signals received at the time reversal mirror (TRM). After time reversing those signals, we compute the backward propagation and study the resulting spatial focusing and temporal compression.

**B. Continuous multiple scattering medium**

We consider a continuous multiple scattering medium that is a two dimensional waveguide whose interior has a weakly fluctuating refractive index. In this random waveguide, there are two distinct mechanisms for multipathing: waves reflect off of the waveguide boundaries as well as scatter multiply due to the random fluctuations. Fig. 2 shows a sketch of this waveguide and two signal paths propagating from the source at $y$ to an element of the time-reversal array located at $x_p$. The signals reaching $x_p$ experience both of these two multipathing effects.

We assume that the average refractive index is $\langle n \rangle = 1$. We model the random inhomogeneities in the refractive index as

$$
n(x) = \sqrt{1 + \sigma \mu \ell},
$$

(6)

with $\sigma$ denoting the standard deviation or strength, $\mu$ denoting a mean-zero, isotropic, Gaussian correlated random function, and $\ell$ denoting the correlation length. We compute realizations of $\mu$ through the summation of random series.
We use the phase screen method to compute the random space-time field \([29–31]\). This method ignores the weak backscattering and propagates only the forward propagating normal modes. Simulating propagation along the waveguide involves combining a sequence of steps from one screen to the next. Let \( \mathbf{x} = (x, z) \), with \( x \in (-L_x/2, L_x/2) \) denoting the coordinate that runs across the width of the waveguide, and \( z \in (0, L_z) \) denoting the coordinate that runs along the length of the waveguide. Equi-spaced phase screens are placed inside the waveguide separated by distance \( \Delta z = L_z/N_z \) with \( N_z \) denoting the number of phase screens. We define the field over the phase screen \( \{(x, z) : z = 0\} \) that contains the source position \( \mathbf{y} \) as \( \hat{\psi}(x, 0) \). Using that field, we seek \( \hat{\psi}(x, \Delta z) \) at the next phase screen. From that field, we seek field at the subsequent phase screen, \( \hat{\psi}(x, 2\Delta z) \), and so on. We propagate the field to advance one phase screen by computing the field propagated through an uniform waveguide followed by a random phase correction. In what follows, we describe the propagation of a known field to the next phase screen.

Consider the field \( \hat{\psi}(x, z_1) \) defined over a phase screen \( \{(x, z) : z = z_1\} \). We use that field to compute the field \( \hat{\psi}(x, z_2) \) over the next phase screen \( \{(x, z) : z = z_2\} \) with \( z_2 = z_1 + \Delta z \). Because the waveguide is periodic, we represent \( \hat{\psi}(x, z_1) \) in terms of its Fourier series:

\[
\hat{\psi}(x, z_1) = \sum_{m=-\infty}^{\infty} \hat{\Psi}_m(z_1) e^{i\beta_m x},
\]

with

\[
\hat{\Psi}_m(z_1) = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} \hat{\psi}(x, z_1) e^{-i\beta_m x} dx,
\]

and \( \beta_m = 2\pi m/L_x \). To propagate this field to \( z_2 = z_1 + \Delta z \), we compute

\[
\hat{\psi}(x, z_2) = e^{ik\tilde{\mu}(x)\Delta z/2} \sum_{m=-\infty}^{\infty} \hat{\Psi}_m(z_1) e^{ik\Delta z \sqrt{1 - \beta_m^2/k^2} + i\beta_m x}.
\]

Eq. (9) involves two distinct steps: (i) propagation of the field in a uniform waveguide encapsulated in the factor \( e^{ik\Delta z \sqrt{1 - \beta_m^2/k^2}} \) and (ii) adding a random phase correction encapsulated in the factor \( e^{ik\tilde{\mu}(x)\Delta z/2} \). Here, the random phase correction \( \tilde{\mu}(x) \) denotes a path-integral of the fluctuation \( \mu(x, z) \).

For the forward propagation stage, this procedure is repeated over all phase screens to compute the field at the TRM plane. Upon time reversing the signals over the TRM, we repeat this procedure in reverse for the backward propagation stage.
III. NUMERICAL RESULTS

A. Discrete scattering medium.

A set of numerical experiments has been carried out for different two dimensional changing environments (the free-space propagator given in (4) is adjusted accordingly). A single numerical experiment can be described as follows. A pulse

$$f(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \hat{f}(\omega) d\omega,$$

emanating from a point source $\mathbf{y}$, illuminates an array of $N = 45$ transducers located at positions $\mathbf{x}_p$, $p = 1, \cdots, N$ (see Fig 1). Because we are interested in studying the properties of the refocused signal that has scattered at least on time within the medium, we consider a non-line-of-sight situation. The refocused properties due to the incident fields are substantially the same as that in a homogeneous medium. Therefore, we study only scattered fields and neglect contributions due to incident fields. This correspond to a physical situation in which the scatterers have a finite size and block the line-of-sight.

The transducers are half wavelength apart. Between the source and the array there is a medium of dimensions $10 \times 10$ containing $M = 400$ point scatterers that are randomly distributed. The scattering amplitude of each scatterer is $|\tau_j| = 0.5$ for all $j$. Both the source and the array are separated from the medium edges by a distance of $1 \text{m}$. Measurements span the bandwidth 4-6 GHz with sampling rate of 2.5 MHz. The recorded signals

$$\psi(\mathbf{x}_p, \mathbf{y}, t) = \int_{-\infty}^{\infty} e^{-i\omega t} \hat{\psi}(\mathbf{x}_p, \mathbf{y}, \omega) d\omega$$

last long enough so there is negligible additional scattered energy. At this stage of the numerical experiment, the medium may undergo a perturbation. To evaluate the impact of the perturbations we compute the back-propagated field

$$\Gamma(\mathbf{y}^s, \mathbf{y}, t) = \int_{-\infty}^{\infty} e^{-i\omega t} \hat{\Gamma}(\mathbf{y}^s, \mathbf{y}, \omega) d\omega$$

at a point $\mathbf{y}^s$ within a square grid around the source location $\mathbf{y}$. The square grid spans a region of $14\lambda_0 \times 14\lambda_0$, where $\lambda_0$ is the wavelength of the mid frequency of measurements (5 GHz). The grid resolution is $\lambda_0/2$. In Eq. (12),

$$\hat{\Gamma}(\mathbf{y}^s, \mathbf{y}, \omega) = \hat{f}(\omega) \sum_{p=1}^{N} \hat{\psi}(\mathbf{y}^s; \mathbf{x}_p, \omega) \hat{\psi}(\mathbf{x}_p; \mathbf{y}, \omega).$$
We have estimated the amount of multiple scattering, defined by
\[
\frac{\int_{-\infty}^{\infty} (|\Gamma(y^s, y, t)| - |\Gamma(y^s, y, t)|_{\text{single}}) \, dt}{\int_{-\infty}^{\infty} |\Gamma(y^s, y, t)| \, dt},
\]
(14)
to be approximately equal to 0.25 in our simulations. \( |\Gamma(y^s, y, t)|_{\text{single}} \) is computed in Eq. (14) by neglecting the sum in Eq. (5).

Due to the reciprocity symmetry of the wave propagation \( \hat{\psi}(x^p; y, \omega) = \hat{\psi}(y; x^p; \omega) \), and therefore, Eq. (13) can be written as
\[
\hat{\Gamma}(y^s, y, \omega) = \overline{f(\omega)} \sum_{p=1}^{N} \hat{\psi}(y^s; x^p, \omega) \overline{\hat{\psi}(y; x^p; \omega)}.
\]
(15)

Since \( \hat{\psi}(y; x^p; \omega) \) is a random function which depends on the scatterers positions, for a particular realization of the medium we can write
\[
\hat{\psi}(y^s; x^p, \omega) \overline{\hat{\psi}(y; x^p; \omega)} = \hat{\Gamma}_p(y^s, y, \omega) + n_p(y^s, y, \omega),
\]
(16)
where the ensembled-averaged
\[
\hat{\Gamma}_p(y^s, y, \omega) = < \hat{\psi}(y^s; x^p, \omega) \overline{\hat{\psi}(y; x^p; \omega)} >
\]
(17)
represents the correlation function of the backpropagated field at \( y^s \) and the backpropagated field at \( y \). In Eq. (16), \( n_p(y^s, y, \omega) \) is a zero-mean random function representing the fluctuations for a particular realization of the medium.

Note that (12) can be viewed as an estimator of the correlation function of scattered waves, and the medium can be viewed as a natural correlator [20, 36]. The time reversal mirror sends the reversed signal back through the medium autocorrelating it. Talk here about whether or not time or space averages are equivalent with ensemble averages?. The exploration of most of all posible phase-states is effectively achieved through the summation of the frequency spectrum in (12)?.

We will study the properties of the correlation function (17) as the medium changes between the forward and backward stages in the next sections.

1. **Spatial focusing**

   We begin this numerical investigation by examining the spatial focusing properties of the time reversed wave in an unperturbed random medium, in an homogeneous medium and in
a perturbed medium. Fig. 3 contains one example from this study. As expected, we see in the upper left plot the enhanced spatial focusing in usual time reversal (when the forward and back-propagating waves travel through the same medium). Enhanced focusing in time reversal in random media means that the cross range resolution is much smaller than that in time reversal in a homogeneous medium (shown in the upper right plot). The focal spot size in a random medium is distinctly smaller than in a homogeneous one. In addition, we show in the bottom left plot the case when the back-propagating waves are re-emitted by the array into a different random medium that is fully uncorrelated with respect to the one used in the forward propagation. We observe that the refocused energy when the waves propagate backward through a different medium lies within the region defined by time reversal in a homogeneous medium. We see, however, that the magnitude of the recompressed field has decreased by a factor close to 5. In the bottom right plot we show the refocused fields for imaging where the back propagation is done in a homogeneous medium (this is equivalent to Kirchhoff migration) [13–15]. In this case, the focal spot size is similar to time reversal in a homogeneous medium but the magnitude of the peak has decreased.

In Fig. 4, we show the amplitude of these refocused fields as a function of cross-range at the refocus plane (at range = 0 in Fig. 3). The solid curve is for time reversal in an unperturbed discrete random medium. The dashed curve is for time reversal in a homogeneous medium. The dot-dashed curve is for time reversal in which the media for the forward and backward propagation stages are completely uncorrelated. The dotted curve is for Kirchhoff migration. As expected, only time reversal for the unperturbed discrete random medium shows sharp spatial focusing.

2. Temporal compression

In Fig. 5 we show the time dependence of the signals. In the left column, we show the signals transmitted through a homogeneous medium (Fig. 5 (a)), and a discrete random medium (Figs. 5 (c) and (e)) received at the center of the arrays. In the right column, we show the compressed pulses at the original source position after time reversal in a homogeneous medium (Fig. 5 (b)), in an unperturbed random medium (Fig. 5 (d)), and in a completely uncorrelated random medium (Fig. 5 (f)). These plots show that the signal corresponding to true time reversal (Fig. 5 (d)) is compressed in time more than the others. This is so because
true physical time reversal uses full knowledge of the realization of the random medium.

3. Statistical stability

The third important issue is statistical stability. In other words, the time reversal process (when the medium is known) is self-averaging in broadband regimes. Because of statistical stability, the super-resolution observed in Figs. 3 and 5 do not depend on the individual realization of the random medium [6, 10].

To investigate the statistical instability phenomenon, we have decreased the number of transducers in the array from 45 to 23 in our numerical simulations. For that case, we expect the resolution of the spatial focusing to be diminished. In Fig. 6, we show the refocused fields for an unperturbed random medium (left) and for a homogeneous medium (right) when we use 23 transducers in the array. In comparing these results for the homogeneous medium with that shown in Fig. 3, we observe that the resolution of the spatial focusing decreases by a factor of 2.

We show in the top row of Fig. 7 results for three different realizations of the medium into which the back-propagating signal is re-emitted. Fig. 7 clearly shows that a reliable estimate of the source location cannot be achieved if the true medium is replaced by another one.

The bottom row of this figure corresponds to Kirchhoff migration for three different realizations of the random medium. Even though there is a focal spot, we observe that its position changes for each of the three different realizations. This change is due to the fact that Kirchhoff migration is not statistically stable [15].

4. Small perturbations to the medium

Despite of the results shown in the previous figures, it is natural to expect that the remarkable refocusing properties of the time-reversed waves will remain, to some extent, if the changes in the environment are small enough. To measure the impact of the strength of the perturbations in the medium on the refocusing properties, we have conducted the following numerical experiments. We have randomly perturbed the original location of the particles uniformly in direction and normally in radius with a standard deviation equal to
Here, $\sigma$ is measured in units of the central wavelength $\lambda$. From top to bottom and from left to right Fig. 8 shows the results for $\sigma = 0.0$ (unperturbed medium), $\sigma = 0.5$, $\sigma = 1.0$, $\sigma = 1.5$, $\sigma = 2.0$, and $\sigma = 2.5$. In this numerical experiment there are 45 transducers in the time reversal mirror, and the signal bandwidth is 2 GHz centered at 5 GHz. We observe that for these values of $\sigma$, the peak at the source location losses a significant amount of power. However, the peak remains centered at the source location and does not destabilizes. For this signal bandwidth and this number of transducers, larger perturbations are needed to destabilize the refocused pulse. To investigate the lost of statistical stability we plot in the next two figures the results corresponding to a shorter bandwidth and a smaller time reversal mirror.

Figure 9 depicts the backpropagated signal for different values of $\sigma$ when the signal bandwidth is reduced to 0.5 GHz. In addition to the decrease of peak’s power as the strength of the perturbations increases, this figure shows the appearence of speakles for some realizations. This means that there is a loss of statistical stability as the bandwidth deacreases and, therefore, these images cannot be use as usefull estimates for the source location. Figure 10 shows the results when the number of transducers are reduced to 23. This figure clearly show a faster destabilization than in Figs. 8 and 9 as the strength of the perturbations increases.

Fig. 11 quantifies the impact of the perturbations on the amplitude of the peak intensity of the recompressed wave when we use 45 (left), 23 (center), and 11 (right) transducers. We observe that the value of the peak intensity rapidly decreases until it saturates for a perturbation strength $\sigma$ larger than 1.0. This saturation value is approximately the same for which we see lost of the statistical stability of time reversal when the media in the forward and backward propagation stages of the time reversal process were uncorrelated (bottom-left plot of Fig. 3). Our simulations show the same decay rate as Fig. 11 when we change the number of transducers in the array.

It now remains to investigate the effect of the amount of multiple scattering on the decay rate of the peak intensity value after time reversal. In Fig. 12 we have plotted these values as a function of $\sigma$ for different scattering amplitudes $\tau$. We observe in this figure that the decay rate is faster for $|\tau| = 0.75$ (dot-dashed line), than for $|\tau| = 0.65$ (dashed line) and $|\tau| = 0.5$ (solid line). This is so, as we will see in section V, because the contributions to the decay of the correlation function of the backpropagated field (17) accumulates for each scattering event.
B. Waveguide model.

We investigate the stability of time reversal with a changing medium using the waveguide model described in Section II.B. For our simulations, the waveguide width is 2.56 m and the length of the waveguide is 50 m. The fluctuation is an isotropic, Gaussian correlated random function of space. The random fluctuation has RMS height $h = 0.05$ and correlation length $\ell = 1.5$ m. The signal bandwidth is $B = 200$ MHz centered at 2.5 GHz. The sampling rate is $\Delta f = 250$Hz. The time reversal mirror (TRM) is centered in the waveguide and is 40 cm (approximately 3 central wavelengths) wide.

The waveguide model is different from the discrete scattering model in that there are two mechanisms for multipathing: reflection by the waveguide walls and multiple scattering by the weakly random fluctuations. Multipathing due to reflections off of the waveguide walls is enough to ensure a tight refocusing of time reversed signals [11, 12]. However, this refocused signal has spatial side-lobes due to the periodicity of the waveguide domain. The weakly random medium inside the waveguide suppresses these side-lobes, and does not disrupt the refocusing.

In Fig. 13 we show the refocused time reversed signals as a function of cross-range and time at the source location. All signals are normalized to have a peak amplitude of unity. In the top left plot, we show the refocused signal for time reversal. The refocused signal is sharp in cross-range and time about the source location. In the top right plot, we show the time reversed signal in a homogeneous waveguide. Again, the time reversed signal sharply focuses in space and time about the source location. However, we observe two, distinct side-lobes. For the weakly random waveguide shown in the top left plot of Fig. 13, we observe that these side-lobes are largely suppressed. Next, we study the case in which the back-propagating waves are emitted by the array into a different random medium that is fully uncorrelated with the one used for the forward propagation. One example from this study appears in the lower left plot of Fig. 13. Similar to the results for the discrete scattering medium above, we observe no refocusing. Rather than perturbing the entire medium for the backward propagation, we have also studied the case in which only a portion of the waveguide has changed. In particular, we study the case in which the central 10 of the 50 phase screens are removed for the backward propagation stage. The lower right plot shows an example result from this experiment. Here, we observe a sharp refocus of the signal in
space and time. However, there is more signal clutter away from the source location in comparison with the upper left and right plots. Nonetheless, we find that the refocusing of time reversed signals remains stable even under this perturbation just as we observed for the discrete scattering medium.

IV. ANALYSIS OF THE DEGRADATION OF THE SPATIAL FOCUSING AND TEMPORAL COMPRESSION USING THE BORN APPROXIMATION

We analyze time reversal in a discrete scattering medium using the Born approximation. From that analysis, we determine the degradation of spatial focusing due to changes in the medium for the backward propagation stage. We do this analysis in the frequency domain. From that analysis, we determine what occurs in the time domain.

A. Time reversal in an unperturbed discrete random medium

Let $U^B(x_p; y)$ denote the single scattered field at the TRM element at position $x_p$ due to the source at $y$ for the forward propagation stage. We have suppressed the dependence on $\omega$ to simplify this presentation. According to the Born approximation, $U^B(x_p; y)$ is given by

$$U^B(x_p; y) = \sum_{j=1}^{M} \tau_j G_0(x_p - \xi_j)G_0(\xi_j - y),$$

(18)

with $G_0$ defined in (4).

Let the back-propagated scattered field $\Gamma^B(y^s; y)$ be given as the sum:

$$\Gamma^B(y^s; y) = \sum_{p=1}^{N} V^B_p(y^s)$$

(19)

with $V^B_p(y^s)$ denoting the back-propagated scattered field due to element $p$ of the TRM at position $x_p$. According to the Born approximation, $V^B_p(y^s)$ is given by

$$V^B_p(y^s) = \sum_{j=1}^{M} \tau_j G_0(y^s - \xi_j)G_0(\xi_j - x_p)U^B(x_p; y).$$

(20)
Substituting (18) into (20) yields

\[ V_B^p(y^*) = \sum_{j=1}^{M} |\tau_j|^2 G_0(y^* - \xi_j) \overline{G_0(\xi_j - y)} G_0(\xi_j - x_p) G_0(\xi_j - x) \]

\[ + \sum_{j=1}^{M} \sum_{k \neq j} \tau_j \tau_k G_0(y^* - \xi_j) \overline{G_0(\xi_k - y)} G_0(\xi_j - x_p) G_0(\xi_j - x). \]  

(21)

For time reversal, the first term in (21) matters most for spatial refocusing and temporal compression because it involves coherent phase cancellations. The second term involves the accumulation of disparate phases due to scattering.

**B. Time reversal in a perturbed discrete random medium**

Now suppose that the locations of the scatterers has changed for the backward propagation stage from \( \xi_j \) to \( \xi_j + \Delta \xi_j \). The random perturbations \( \Delta \xi_j \) are independent and identically distributed. We assume that they are normal with \( \langle \Delta \xi_j \rangle = 0 \) and \( \langle \Delta \xi_j^2 \rangle = \sigma^2 \).

For this case, \( V_p^s(y^*) \) is given by

\[ V_p^B(y^*) = \sum_{j=1}^{M} |\tau_j|^2 G_0(y^* - \xi_j - \Delta \xi_j) \overline{G_0(\xi_j - y)} G_0(\xi_j + \Delta \xi_j - x_p) G_0(\xi_j - x). \]

\[ + \sum_{j=1}^{M} \sum_{k \neq j} \tau_j \tau_k G_0(y^* - \xi_j - \Delta \xi_j) \overline{G_0(\xi_k - y)} G_0(\xi_j + \Delta \xi_j - x_p) G_0(\xi_j - x_k). \]  

(22)

For the same reasons stated above, the first term in (22) matters most for spatial refocusing and temporal compression. To focus our attention on the degradation of spatial focus affected by the perturbations, we study the expression

\[ v(y^*; x_p, \xi_j, \Delta \xi_j) = G_0(y^* - \xi_j - \Delta \xi_j) \overline{G_0(\xi_j - y)} G_0(\xi_j + \Delta \xi_j - x_p) G_0(\xi_j - x). \]  

(23)

and seek its ensemble average.

Assuming that \( |\Delta \xi_j| \ll |\xi_j - x_p| \), we find that

\[ |x_p - \xi_j - \Delta \xi_j| - |x_p - \xi_j| \sim -\Delta \xi_j \cdot \Omega, \]

(24)

where \( \Omega = \frac{(x_p - \xi_j)}{|x_p - \xi_j|} \). Moreover, by assuming that \( |y - y^*| \ll |y^* - \xi_j| \), we find that

\[ |y^* - \xi_j - \Delta \xi_j| - |\xi_j - y| \sim -\Delta \xi_j \cdot \Omega' - (y - y^*) \cdot \frac{(y - y^*)}{|y - y^*|}, \]

(25)

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where \( \Omega' = \frac{y^s - \xi_j}{|y^s - \xi_j|} \). Upon substitution of (4) into (23) using (24) and (25), we obtain

\[
v(y^s; x_p, \xi_j, \Delta \xi_j) \sim \frac{\exp \left[ -i k (y - y^s) \cdot \frac{y^s - \xi_j}{|y^s - \xi_j|} \right]}{(4\pi)^4 |x_p - \xi_j|^2 |\xi_j - y^s|^2} \exp \left[ -i k \Delta \xi_j \cdot (\Omega - \Omega') \right]. \quad (26)
\]

Using this asymptotic expression, we average over all random perturbations \( \Delta \xi_j \) and find readily that

\[
\langle v(y^s; x_p, \xi_j) \rangle \sim \frac{\exp \left[ -i k (y - y^s) \cdot \frac{y^s - \xi_j}{|y^s - \xi_j|} \right]}{(4\pi)^4 |x_p - \xi_j|^2 |\xi_j - y^s|^2} e^{-k^2 |\Omega - \Omega'|^2 \sigma^2}. \quad (27)
\]

The expression given above is the same that would be found for the non-perturbed problem except for the factor of \( e^{-k^2 |\Omega - \Omega'|^2 \sigma^2} = e^{-2k^2(1-\Omega \cdot \Omega') \sigma^2} \). This factor shows explicitly how perturbing the locations of the scatterers affects the spatial focusing and temporal compression. We find that the spatial refocusing and temporal compression degrades as a Gaussian as the strength of the perturbation (given by \( \sigma \)) increases. This result is consistent with what we have observed in our numerical simulations in that we have found that severe degradation of the spatial focusing and temporal compression occur when \( \sigma = O(\lambda) \). We will show in the next section, where we study the effect of multiple scattering, that the factor \( e^{-k^2 |\Omega - \Omega'|^2 \sigma^2} \) is the responsible of the deterioration of the refocusing properties of the reversed signal, and that it plays the role of an effective albedo in the multiple scattering process.

V. ANALYSIS OF THE DEGRADATION OF THE SPATIAL FOCUSING AND TEMPORAL COMPRESSION USING MULTIPLE SCATTERING THEORY

In this section we derive a radiative transport equation for the wave correlation function. The derivation is based on the work by Foldy [27], Lax [28], and Twersky [37] who described the two first moments of the field in a multiple scattered medium, and provided a bridge between transport and multiple scattering theories. Ishimaru [38] considered later the case of multiple scattering by particles in motion.

Let us consider \( M \) point scatterers at positions \( \xi_j, j = 1, \ldots, M \), distributed accordingly to a probability density function \( P(\xi_1, \xi_2, \ldots, \xi_M) \). \( P(\xi_1, \xi_2, \ldots, \xi_M) \, d\xi_1 \, d\xi_2 \ldots \, d\xi_M \) gives the probability of finding the first scatterer within a volume \( d\xi_1 \) about \( \xi_1 \), the second scatterer within a volume \( d\xi_2 \) about \( \xi_2 \), and so on. In the case in which the scatterer density is low, and the position of each scatterer is independent of the positions of the others,

\[
P(\xi_1, \xi_2, \ldots, \xi_M) = p(\xi_1)p(\xi_2) \ldots p(\xi_M), \quad (28)
\]
where \( p(\xi_j) d\xi_j \) represents the probability of finding the scatterer \( j \) within a volume \( d\xi_j \) about \( \xi_j \). The number of scatterers per unit volume is then defined by the number density \( \rho(\xi) = M p(\xi) \). In this case, the ensemble average of a quantity \( f \) is given by

\[
< f > = \frac{1}{M} \int f(\xi) \rho(\xi) \, d\xi . \tag{29}
\]

Using (29) and (3), let us now take the ensemble average of Eq. (2):

\[
< \hat{\psi}(x; y) > = \hat{\psi}^i(x; y) + \sum_{j=1}^{M} \tau_j \int G_0(x - \xi_j) \, \hat{\psi}_j^e(\xi_j) \, \rho(\xi_j) \, d\xi_j = \hat{\psi}^i(x; y) + \tau_j \int G_0(x - \xi_j) \, \hat{\psi}_j^e(\xi_j) \, \rho(\xi_j) \, d\xi_j , \tag{30}
\]

where \(<>_j\), which means an average with the scatterer \( j \) fixed, is only a function of \( \xi_j \). \(< \hat{\psi}_j^e(\xi_j) >_j \) represents the field acting on scatterer \( j \) averaged over all the possible positions of the other scatterers. Under the approximation \(< \hat{\psi}_j^e(\xi_j) >_j \approx < \hat{\psi}(\xi_j) >\) we can write

\[
< \hat{\psi}(x; y) > = \hat{\psi}^i(x; y) + \tau_j \int G_0(x - x') \, \hat{\psi}(x') \, \rho(x') \, dx' . \tag{31}
\]

This approximation means that \(< \hat{\psi}_j^e(x') >_j \) is replaced by the average field that would exist at \( x' \) if the scatterer is not present. The physical significance of this approximation is that all paths which go through the same scatterer more than one time are neglected [37]. It can be shown that these paths are negligible compared to those going through different scatterers when \( M \to \infty \). We note that (31) is consistent with the first order smoothing approximation to the Dyson equation [39].

By applying the operator \([\Delta + k^2]\) to the integral equation (31), it is easy to show that \(< \hat{\psi}(x; y) > \) satisfy the wave equation in a continuous medium in which the effective wavenumber \( K \) differs from its value \( k \) when there are no scatterers according to [27]

\[
K^2 = k^2 + 4\pi \rho \tau . \tag{32}
\]

The analysis to obtain the ensemble average of the correlation function (17), \( \hat{\Gamma}_p(y^s, y) \), is more cumbersome. We give here the main result and we refer to [27, 28, 32, 37] for the details. Appropriate ensemble averaging of the wave equation leads to the following integral equation for the covariance:

\[
< \hat{\psi}(y^s; x_p) \bar{\hat{\psi}}(\bar{y}; x_p) > = < \hat{\psi}(y^s; x_p) > < \bar{\hat{\psi}}(\bar{y}; x_p) > + |\tau|^2 \int L(y^s, y'; y') \, < \hat{\psi}(y'; x_p) \bar{\hat{\psi}}(y'; x_p) > \, \rho(y') \, dy' \tag{33}\]
where the kernel $L(y^s, y; y')$ satisfy the integral equation
\[
L(y^s, y; y') = G_{\text{eff}}(y^s - y') G_{\text{eff}}(y - y') + \frac{1}{(4\pi)^2} \int \int (\tau_1 + k''^2) + \tau_1 y'' + k''^2) L(y'', y'''; y') \times \nonumber \\
G_{\text{eff}}(y^s - y'') G_{\text{eff}}(y - y''') dy'' dy''' .
\]

In Eq. (34),
\[
G_{\text{eff}}(y - y') = \frac{e^{iK|y-y'|}}{4\pi|y-y'|}
\]
is the effective propagator in the multiple scattering medium, and $K$ is given by (32).

**A. Time reversal in an unperturbed discrete random medium**

Let us assume that the medium has not change between the forward and the backward propagation stages of the time reversal process. In the far field approximation, the kernel in Eq. (33) can be represented by the first term $G_{\text{eff}} G_{\text{eff}}$ in the right hand side of (34). Therefore,
\[
\hat{\Gamma}_p(y^s, y) = \langle \hat{\psi}(y^s; x_p) \rangle < \hat{\psi}(y; x_p) > + |\tau|^2 \int G_{\text{eff}}(y^s - y') G_{\text{eff}}(y - y') < |\hat{\psi}(y'; x_p)|^2 > \rho(y') dy'
\]

The integral term represents the wave at $y^s$ generated from the coherent wave at $y'$, and the wave at $y$ generated from the backpropagated wave at $y'$.

We now introduce the center coordinate system
\[
y_c = \frac{1}{2} (y^s + y); \quad y_d = (y^s - y),
\]
and the quantity $I(y_c, \Omega)$ given by the Fourier transform
\[
\hat{\Gamma}_p(y_c, y_d) = \int I(y_c, \Omega) e^{iK_R \Omega \cdot y_d} d\Omega.
\]

In (38), the correlation function is written as an expansion of plane waves of wave number $K_R = \text{Re}(K)$ propagating in a distribution of directions specified by the unit vector $\Omega$. $I(y_c, \Omega)$ is therefore the angular spectrum of plane waves. $I(y_c, \Omega)$ is the specific intensity in the radiative transfer theory and has the meaning of the energy density at position $y_c$ travelling in direction $\Omega$. The integration in Eq. (38) is done over the unit sphere. Eq. (38) provides the link between wave propagation in a multiple scattering medium and
the radiative transfer theory. We note here that, strictly speaking, this relation is only valid
when the correlation function varies slowly respect to \( y_c \).

Similarly, an expansion for the product of the average fields can be written as [32]:

\[
\langle \hat{\psi}(y^*; x_p) \rangle = \langle \hat{\psi}(y; x_p) \rangle = \int I_{coh}(y_c, \Omega) e^{iK_R \cdot y_c} d\Omega.
\]  

(39)

Upon substitution of (38) and (39) in Eq. (36), and making use of the following approx-
imation in the phases and magnitude that appear in \( G_{eff}(y^* - y') \) and \( \overline{G_{eff}(y - y')} \)

\[
|y^* - y'| \approx |y_c - y'| + \Omega \cdot y_d/2
\]

\[
|y - y'| \approx |y_c - y'| - \Omega \cdot y_d/2
\]

\[
\frac{1}{|y^* - y'|} \approx \frac{1}{|y - y'|} \approx \frac{1}{|y_c - y'|}
\]  

(40)

we find that the specific intensity satisfy the integral equation

\[
I(y_c, \Omega) = I_{coh}(y_c, \Omega) + \frac{\rho|\tau|^2}{(4\pi)^2} \int \int \frac{e^{-\rho \sigma t|y_c - y'|}}{|y_c - y'|} I(y', \Omega') dy' d\Omega',
\]  

(41)

where \( \rho \sigma t = 2 \text{Im}(K) \). \( \sigma t \) is the extinction coefficient. From (41), we find readily the known
radiative transfer equation in the form

\[
\Omega \cdot \nabla I(y_c, \Omega) = -\rho \sigma t I(y_c, \Omega) + \frac{\rho|\tau|^2}{(4\pi)^2} \int I(y_c, \Omega') d\Omega'.
\]  

(42)

Appropriate modifications of (42) for a perturbed medium are derived in the next subsection.

B. Time reversal in a perturbed discrete random medium

If the locations of the scatterers have changed in the backward propagation stage, the
correlation function, and therefore the specific intensity as well, should also depend on a
quantity \( \sigma \) that measures the correlation between the media in the forward and backward
propagation stages. To describe mathematically this correlation we define the joint number
density

\[
\rho(y', y'') = \rho(y'_c) w(y'|y''),
\]  

(43)

where \( y'_c = (y' + y'')/2 \). In Eq. (43) \( w(y'y') \) is the conditional probability, i.e., given one
scatterer at position \( y \) in the forward propagation stage, \( w \) is the probability of finding it at \( y' \) in the backward propagation stage. In the backward propagation stage we assumed that
the location of the particles have changed uniformly in direction and normally in radius, so the conditional probability is described by the normal distribution

\[ w(y'|y'') \equiv w(|y'_d|) = \left( \frac{k^2}{8 \pi^3 \sigma^2} \right)^{3/2} e^{-|y'_d|^2 k^2/(8 \pi^2 \sigma^2)}, \]  

where the standard deviation \( \sigma \) is measured in units of the wavelength \( \lambda = 2\pi/k \), and \( y'_d = (y' - y'') \). Note that when \( \sigma \rightarrow 0 \) the media are completely correlated, and when \( \sigma \rightarrow \infty \) the media are completely decorrelated.

When the medium have changed in the backward propagation stage Eq. (36) has to be generalized leading to

\[ \hat{\Gamma}_p(y_c, y_d; \sigma) = < \hat{\psi}(y^s; x_p) > < \overline{\hat{\psi}(y; x_p)} > + |\tau|^2 \int \int G_{eff}(y^s - y') G_{eff}(y - y'') \rho(y', y'') \hat{\Gamma}_p(y'_c, y'_d; \sigma) \, dy' \, dy'', \]  

where \( \hat{\Gamma}_p(y'_c, y'_d; \sigma) = < \hat{\psi}(y'; x_p) \overline{\hat{\psi}(y''; x_p)} > \). Accordingly, relation (38) has to be generalized to

\[ \hat{\Gamma}_p(y_c, y_d; \sigma) = \int I(y_c, \Omega; \sigma) e^{iK_R \cdot y_d} \, d\Omega. \]  

To obtain and equation for \( I(y_c, \Omega; \sigma) \) we use similar approximations than in the previous subsection:

\[ |y^s - y'| \approx |y_c - y'_c| + \Omega \cdot (y_d - y'_d)/2, \]
\[ |y - y''| \approx |y_c - y'_c| - \Omega \cdot (y_d - y'_d)/2, \]
\[ \frac{1}{|y^s - y'|} \approx \frac{1}{|y - y''|} \approx \frac{1}{|y_c - y'_c|} \]  

Substituting Eqs. (47) into Eq. (45) we obtain

\[ \hat{\Gamma}_p(y_c, y_d; \sigma) = < \hat{\psi}(y^s; x_p) > < \overline{\hat{\psi}(y; x_p)} > + \frac{|\tau|^2}{(4\pi)^2} \int \int \frac{e^{iK_R \cdot y_d - iK_R \cdot y'_c \cdot \rho(y_c - y'_c)}}{|y_c - y'_c|^2} \rho(y'_c, y'_d) \hat{\Gamma}_p(y'_c, y'_d; \sigma) \, dy'_c \, dy'_d, \]  

Making use of Eqs. (43) and (46) we can write

\[ \hat{\Gamma}_p(y_c, y_d; \sigma) = < \hat{\psi}(y^s; x_p) > < \overline{\hat{\psi}(y; x_p)} > + \frac{|\tau|^2}{(4\pi)^2} \int \int \frac{e^{iK_R \cdot y_d - iK_R \cdot y'_c \cdot \rho(y_c - y'_c)}}{|y_c - y'_c|^2} \rho(y'_c) R(K_R (\Omega - \Omega'); \sigma) I(y'_c, \Omega'; \sigma) \, dy'_c \, d\Omega', \]  

where we have defined the characteristic function

\[ R(q; \sigma) = \int w(|y'_d|) e^{-i\mathbf{q} \cdot y'_d} \, dy'_d. \]  

\[ \text{(50)} \]
Finally, using Eqs. (39) and (46) and following the same steps as in the previous subsection, (49) become the radiative transfer equation for the correlation function in a perturbed medium:

\[
\Omega \cdot \nabla I(y_c, \Omega; \sigma) = -\rho \sigma I(y_c, \Omega; \sigma) + \frac{\rho |\tau|^2}{(4\pi)^2} \int R(K_R (\Omega - \Omega'); \sigma) I(y'_c, \Omega'; \sigma) d\Omega'.
\]

(51)

Note that if \( w(|y'_d|) \) in Eq. (51) is given by (44), then

\[
R(q; \sigma) = \left( \frac{k^2}{8\pi^3 \sigma^2} \right)^{3/2} \int e^{-|y_d|^2 k^2/(8\pi^2 \sigma^2)} e^{-i q \cdot y'_d} dy'_d = e^{-|q|^2 \sigma^2}.
\]

(52)

Since the the argument of the characteristic function in Eq. (51) is \( q = K_R (\Omega - \Omega') \), \(|q|^2 = 2K_R^2 (1 - \Omega \cdot \Omega') \), and therefore

\[
R(K_R (\Omega - \Omega'); \sigma) = e^{-2K_R^2 \sigma^2} e^{2K_R \sigma^2 \Omega \cdot \Omega'}.
\]

(53)

Due to the change of the medium in the backward propagation stage, the radiative transfer equation (42) with isotropic scattering is transform into the radiative transfer equation (51) with albedo \( e^{-2K_R^2 \sigma^2} \) and anisotropic scattering. We note that the effect of the albedo in the radiative transfer equation is accumulative. As a consequence, the higher the order of multiple scattering involved in the transport process, the higher the absorption is. In other words, the deterioration of the focusing properties of the time reversal signal depends on the number of interactions the waves have had with the perturbed scatterers. This result is in agreement with Fig 12.

Note that when \( \sigma \to 0 \) in Eq. (51) we recover (42), and that these results are consistent with those obtain in the previous section using the Born approximation.

It is also important to note that the absorption depends on the wavenumber \( K_R \), so that higher frequencies are filtered out. This is why the recompressed peak at the source location dissapear faster than backpropagated energy at its sides as the perturbation \( \sigma \) increases.

We mention here that the derivation of the correlation transfer equation (51) follows the same steps as the one given by Ishimaru for the transport equation when the particles are in motion [32]. These type of equations have been use in diffusive wave spectroscopy to study motions by means of correlation functions [40, 41].
VI. SUMMARY

In summary, we have studied the stability of spatial focusing and temporal compression in time reversal when the medium changes in between the forward and backward propagation stages. With numerical simulations of a discrete random medium and a random waveguide, we have shown that the spatial focusing and temporal compression are robust to perturbations in the medium. However, the strength of the focusing diminishes as the strength of the perturbations increase. In addition, we have shown that the focusing is destroyed for changes in the medium that are larger than the wavelength on average. Using multiple scattering theory for wave propagation in discrete random media, we have computed explicitly this degradation as a function of the strength of the perturbation. The theoretical results are in good agreement with our numerical simulations. These results may be important for imaging and communications in cluttered media.

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LIST OF FIGURES

FIG. 1: The computational setup for the discrete random medium simulations. The source, located at position $y$, illuminates a medium comprised of point scatterers distributed randomly in a uniform medium. The signal is recorded at the array of aperture $a$. At this stage, the medium may change. The signal is time-reversed and re-emitted into the new medium.

FIG. 2: The computational setup for the random waveguide model. Two rays from the source at $\mathbf{y}$ propagating to an element of the time reversal array located at $\mathbf{x}_p$. Signals propagating in this waveguide undergo multipathing due to reflection from walls and scattering by inhomogeneities.

FIG. 3: Amplitude of the refocused fields, $|\Gamma(y^s, y, t = 0)|$, shown as a function of range and cross-range in meters. Top left: true time reversal, the back-propagating waves are emitted into the same random media. Top right: time reversal in an homogeneous medium. Bottom left: time reversal in fully uncorrelated changing media. Bottom right: Kirchhoff migration (back-propagation is done in a homogeneous medium). The vertical array has 45 transducers that are one half wavelength apart. The center frequency of the measurements is 5 GHz, so that the central wavelength is $\lambda_0 = 0.06$ m. The grid spans a region of $14 \lambda_0 \times 14 \lambda_0$.

FIG. 4: Cross sections along the transverse direction of the re-focused fields shown in Fig. 3. Unperturbed random medium (solid line), homogeneous medium (dashed line), fully uncorrelated random medium (dot-dashed line), and for Kirchhoff migration (dotted line). The vertical array has 45 transducers. Center frequency equal to 5 GHz.

FIG. 5: (a) Signal transmitted through a homogeneous medium and received at the center of the array. (b) Compressed impulse response recorded at the source position after time reversal in the homogeneous medium. (c) Signal transmitted through the multiple scattering random medium and recorded at the center of the array. (d) Compressed impulse response after time reversal in the same random medium. (e) Same as (c). (f) Compressed impulse response after time reversal in a changing medium. The vertical array has 45 transducers. Center frequency equal to 5 GHz.

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FIG. 6: Refocused fields $|\Gamma(y^*, y, t = 0)|$ for time reversal in an unperturbed random medium (left) and in an homogeneous medium (right). The vertical array has 23 transducers. The grid spans a region of $14 \lambda_0 \times 28 \lambda_0$.

FIG. 7: Top: Refocused fields $|\Gamma(y^*, y, t = 0)|$ for three different perturbations of the random medium in which the wave is re-emitted from the array. Bottom: Kirchhoff migration for three different realizations of the random medium in which the wave is emitted from the source. The vertical array in these numerical experiments has 23 transducers. Center frequency equal to 5 GHz. The grid spans a region of $14 \lambda_0 \times 28 \lambda_0$.

FIG. 8: Refocused fields $|\Gamma(y^*, y, t = 0)|$ for time reversal in different correlated media. Top row from left to right: $\sigma = 0.0$ (unperturbed medium), $\sigma = 0.5$, and $\sigma = 1.0$. Bottom row from left to right: $\sigma = 1.5$, $\sigma = 2.0$, and $\sigma = 2.5$. The vertical array has 45 transducers. The signal bandwidth is equal to 2 GHz centered at 5 GHz. The grid spans a region of $14 \lambda_0 \times 14 \lambda_0$ with resolution $\lambda/4$.

FIG. 9: Refocused fields $|\Gamma(y^*, y, t = 0)|$ for time reversal in different correlated media. Top row from left to right: $\sigma = 0.0$ (unperturbed medium), $\sigma = 0.5$, and $\sigma = 1.0$. Bottom row from left to right: $\sigma = 1.5$, $\sigma = 2.0$, and $\sigma = 2.5$. The vertical array has 45 transducers. The signal bandwidth is equal to 0.5 GHz centered at 5 GHz. The grid spans a region of $14 \lambda_0 \times 14 \lambda_0$ with resolution $\lambda/4$.

FIG. 10: Refocused fields $|\Gamma(y^*, y, t = 0)|$ for time reversal in different correlated media. Top row from left to right: $\sigma = 0.0$ (unperturbed medium), $\sigma = 0.5$, and $\sigma = 1.0$. Bottom row from left to right: $\sigma = 1.5$, $\sigma = 2.0$, and $\sigma = 2.5$. The vertical array has 23 transducers. The signal bandwidth is equal to 0.5 GHz centered at 5 GHz. The grid spans a region of $14 \lambda_0 \times 14 \lambda_0$ with resolution $\lambda/4$.

FIG. 11: Peak intensity at the focal plane as function of $\sigma$ for 45 transducers (left), 23 transducers (center), and 11 transducers (right) in the vertical array. Center frequency equal to 5 GHz.
FIG. 12: Peak intensities (normalized to 1) at the focal plane as function of $\sigma$, for $|\tau| = 0.5$ (solid line), $\tau = 0.65$ (dashed line), and $\tau = 0.75$ (dot-dashed line). We use 45 transducers (left), and a signal bandwidth equal to 0.5 GHz centered at 5 GHz.

FIG. 13: Refocused fields for the waveguide simulations. Top left: time-reversal in the same weakly random medium for the forward and backward propagation stages. Top right: time-reversal in a waveguide composed of a homogeneous medium. Bottom left: time-reversal in a waveguide with two uncorrelated weakly random media for the forward and backward propagation stages. Bottom right: time-reversal in a waveguide in the same weakly random medium, except that the randomness of middle 15 m of the waveguide has been removed for the backward propagation stage.

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