Market Influence of Portfolio Optimizers

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Abstract

We study the feedback effects induced by portfolio optimizers on the underlying asset prices. Through their interaction with reference traders, who trade based on some aggregate incomes process, they are assumed to move asset prices away from the standard log-normal model. With market clearing as our main constraint, we solve analytically for the approximate dynamics of the asset price assuming that the wealth of the portfolio optimizers is small relative to the total market capitalization of the stock. We also calculate numerically the influence of portfolio optimizers when their wealth is not so small. There is good agreement between the numerical and analytical results when the wealth of the optimizers is small. We find that portfolio optimizers influence the price of the risky asset so as to decrease its volatility. The optimal allocation to the risky asset also changes as a result of the portfolio optimizers' actions. In general, it is advantageous to hold more of the risky asset, relative to the log normal Merton model.

Keywords: Hamilton-Jacobi-Bellman equation, feedback, portfolio optimization.

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1 Introduction

The feedback effects caused by market participants on underlying prices of risky assets have received considerable attention in recent years. Modelling of feedback effects has typically involved economies in which there are two types of investors. One of these participates in such a manner that, were the other group not present, the underlying risky asset follows a standard log-normal price process. The second group acts in a way so as to disrupt the simple log-normal model of asset prices. This second group generally trades to hedge portfolios of derivatives. Most of the literature has focussed on feedback effects that result from such hedging. We instead look at feedback effects that arise from the actions of portfolio optimizers.

There have been many works that have looked at feedback effects in recent years. Föllmer and Schweizer (1993) consider "information traders" who believe in a value of the asset and noise traders whose demands come from hedging. From this they derive models of the asset price from the equilibrium that is achieved through the groups' interactions. Frey and Stremme (1997) present a continuous time model that splits up reference traders who would uphold the basic Black-Scholes model if they could, and program traders who act so as to hedge their portfolios of options. This setup was used to explain increases in market volatility of the underlying when there are program traders. This setup was also used by Schönbucher and Wilmott (2000). While Frey and Stremme use their model to look at option pricing, Schönbucher and Wilmott look at the price dynamics of the underlying share. They show that in incomplete markets with low liquidity, discontinuities in the price of the underlying may arise. Platen and Schweizer (1998) use a simple random process for the demand of the reference traders and, using a market-clearing condition, they determine the dynamics of the stock price. Sircar and Papanicolaou (1998) also use a market-clearing condition but introduce an aggregate incomes process for the reference traders. This aggregate incomes process is the source of uncertainty in the market and reference traders act based on the income available to them and the realization of the share price. Their paper showed that hedging strategies increase the volatility of the underlying asset.

In this paper we use the methods of Sircar and Papanicolaou (1998) but it is worth noting some other approaches to feedback models. Whereas Sircar and Papanicolaou (1998) allow the market participants to interact and thereby force certain price processes on underlying assets because of clearing constraints, other works have tried to model price impact. Cvitanic and Ma (1996) define a large agent's impact function on the underlying price process. From this, they obtain option prices in terms of solutions to forward-backward stochastic differential equations. The price impact model was used by Jonsson and Keppo (2002) to derive a nonlinear partial differential equation for option prices in the presence of large traders. Like Sircar and Papanicolaou (1998), they write the option price as a solution to a nonlinear partial differential equation. Interestingly, they find that the more the large agent sells call options, the higher is the unit call option price because of the hedging in the underlying that pushes underlying prices up.

In this paper we focus on the feedback effects that are induced by portfolio optimizers. In continuous time, portfolio optimization was first studied by Merton (1971). Using log-normal models of asset prices, he constructed an optimal portfolio of holdings in stocks and bonds given the utility functions of the optimizers. In particular, he found that for constant relative risk aversion (CRRA) utilities, the proportion of wealth invested in the risky asset is a constant. Relevant deviations from this basic setup include one by Cuoco and Cvitanic (1998) who look at optimal consumption and investment problems for a large investor whose portfolio choices affect the instantaneous returns of the traded asset. Using martingale and duality techniques, they solve for the optimal policies. However, their work uses an exogenously defined price-impact function of the large investor and its effect is only seen on expected returns, not on volatility. Brennan and Schwartz (1989) used a CRRA utility for a expected utility maximizing investor in a one-period model. In that model, the other investor had a simple portfolio insurance strategy that was known to all in advance. Their model shows that volatility increases by up to a few percent in such a market.

We look at the correction to Merton's result when the portfolio maximizers interact with other traders so as to influence the price of the underlying. Our optimal portfolio investor, however, acts with reference traders to influence a price in continuous time. Neither strategies nor price impacts

are known beforehand. Instead, at each instant in time, demand and supply of assets are equated. Moreover, we get corrections to not just the volatility but also the optimal holdings of the portfolio optimizing investor.

We find that the volatility of the underlying decreases in normal market conditions (where the Merton optimal portfolio has a fraction between 0 and 1 of wealth invested in the risky asset), which is an effect that has not been seen before in papers involving feedback effects. Interestingly, we also find that under these conditions optimal holdings should increase as the wealth of the portfolio investors increases. We suggest that these results are due to a conservation of maximal expected utility. In other words, portfolio optimizers who are price takers have the same maximal expected utility to first order as portfolio optimizers whose demands actually influence the prices of the underlying asset.

2 A Model for the Market Influence of Portfolio Optimizers

We suppose, as in Frey and Stremme (1997), that there is an aggregate stochastic incomes process, Y_t , which satisfies:

$$dY_t = \mu(Y_t, t)dt + \eta(Y_t, t)dB_t,$$

where B_t is the usual Brownian motion, and the dynamics above are given in the physical (i.e. not risk-neutral) measure. Reference traders are assumed to trade based on the income available to them at a particular time. We will make this more explicit when we look at the agents in the economy.

We further suppose that

$$\begin{aligned}\mu &= \mu_1 y \quad \text{and} \\ \eta &= \eta_1 y.\end{aligned}\tag{1}$$

In other words, we specialize to the case where the aggregate incomes process is a geometric

Brownian Motion. We do this to make our subsequent analysis tractable.

There is one risky asset in the economy. Its price process is assumed to have the following dynamics:

$$dX_t = \alpha X_t dt + \sigma X_t dB_t$$

where B_t is the same Brownian motion as in the incomes process. We do not know α or σ *a priori*. In other words, we assume that there is only one source of risk in the economy, one that arises from the stochastic incomes process. All other risky prices are derived from this stochastic incomes process and from the interactions of the various market participants.

There is one other asset in the market: the risk-free money-market account. The price process of one dollar invested in such an account is deterministic and is given by:

$$d\beta_t = r\beta_t dt,$$

where r is a constant, risk-free interest rate.

There are two agents in the economy. Reference traders trade according to their income and their observation of the price. Their demand is given by $D(X_t, Y_t, t)$.

The portfolio optimizers represent another part of the economy. We suppose that their initial wealth is given by w . We will later assume that this wealth is small in a specific sense. The portfolio optimizers, who put $\pi = \pi(X_t, W_t, t)$, a fraction of their wealth, into the risky asset, would then have a wealth process given by:

$$dW_t = W_t(\pi(\alpha - r) + r)dt + W_t\pi\sigma dB_t.$$

This is a result of the self-financing condition. Our portfolio optimizers, on aggregate, act so as to optimize a utility at time T given by:

$$u(w) = \frac{w^\lambda}{\lambda}.$$

This utility is a constant relative risk-aversion (CRRA, with relative risk aversion given by $-\lambda$) utility. Portfolio optimizers do not make any trading decisions based on the incomes process. Instead, they trade solely on the realizations of the risky asset price and their wealth.

Finally, we establish the market-clearing constraints. Let us suppose that there is a constant supply of risky asset in the economy: S_0 . If we let $\Phi(x, w, t)$ be the demand of the portfolio optimizers for the risky asset, we may write the total demand as:

$$G(x, y, w, t) = D(x, y, t) + \Phi(x, w, t). \quad (2)$$

Furthermore, equating demand and supply tells us that in order for the market to be at equilibrium:

$$G(X_t, Y_t, W_t, t) \equiv S_0. \quad (3)$$

The objective of the portfolio optimizers is simple. They wish to maximize their expected utility at some terminal time, T . Under their control is the fraction of their wealth that they wish to put in the risky asset. Formally, their objective may be encoded in a value function:

$$V(x, w, t) = \sup_{\pi} E_{x,w,t} [u(W_T)]. \quad (4)$$

It is well-known that, given the dynamics of X_t and the dynamics of W_t , the function V satisfies the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} V_t + \sup_{\pi} \left[\frac{1}{2} V_{ww} \pi^2 \sigma^2 w^2 + V_w (\pi(\alpha - r) + r)w + V_{xw} \sigma^2 \pi x w \right] \\ + V_x \alpha x + \frac{1}{2} V_{xx} \sigma^2 x^2 = 0. \end{aligned} \quad (5)$$

$$V(x, w, T) = u(w)$$

It is important to note here that α and σ are functions of x and w , as is π . This coupling of the wealth and risky asset processes introduces an additional layer of complexity over the standard Merton portfolio optimization results.

Differentiating Equation (5) with respect to π in order to find its optimal value in terms of the value function gives us:

$$\pi = - \frac{V_w (\alpha - r)w + V_{xw} \sigma^2 x w}{V_{ww} \sigma^2 w^2} \quad (6)$$

In general, this optimal policy depends on both the value of the underlying risky asset and the wealth of the portfolio optimizer. Moreover, π may be related to the Φ introduced in Equation

(2). Since π represents the fraction of wealth invested in the risky asset, and Φ is the optimizer's demand for the risky asset, we must have:

$$\Phi = \frac{\pi W_t}{X_t} \quad (7)$$

In summary, the problem is as follows. We wish to solve for the optimal policy of the portfolio optimizers using Equation (5), while also making sure the market-clearing constraint in Equation (3) is satisfied. To do this, we need to solve for the value function, for which we first need to know the dynamics of X_t and W_t . Let us suppose we start with some guess at the dynamics. We may, for example, suppose X_t and W_t follow the dynamics that they would in the absence of the portfolio optimizers. The solved value function then gives us the optimal holdings, π , through Equation (6), which we use in Equation (3) in order to update X_t . We then need to calculate the dynamics of X_t (i.e. find out its drift and volatility) before repeating the process. The mixture of calculating forward (in time) paths of a stochastic process while solving the Hamilton-Jacobi-Bellman equation backwards (in time) is not simple to track analytically.

Here we approach the problem in two ways. In Sections 3 and 4, we reduce the complexity of the problem by considering the fraction of portfolio optimizing assets to be small. We will look at the situation where the portfolio optimizers' starting wealth is considered to be small, proportional to a small parameter, ρ . Our dynamic problem may thus be linearized and the results of this analysis are presented in detail. In Section 5, we formulate and solve numerically a Hamilton-Jacobi-Bellman Equation for the value function that is obtained by incorporating the market clearing constraint into it. Our numerical method is based on a finite-difference scheme for the PDE combined with policy iteration to get the optimal policy. We then discuss and compare the results we obtain numerically to the small parameter expansion solution in Section 6.

3 The Wealth of the Portfolio Optimizers as a Small Parameter

3.1 The Small Parameter Model of Feedback Effects

As we have stated, solving for the dynamics of X_t and W_t would be difficult given the coupling that results from Equation (3) in combination with Equations (6-7). In order to proceed, we therefore introduce a small parameter, ρ . This parameter serves to indicate the relative value of the total wealth of the portfolio optimizers and hence the portfolio optimizers' relative demand to the demand of the reference traders. We assume this proportion is small on a global scale. Specifically, we assume that the initial value of the wealth of the portfolio optimizers is proportional to ρ , and so we write their wealth as ρw .

Although we have introduced this small parameter at the global market-clearing level, the portfolio optimizer's microeconomic decision of how much to invest in the risky asset is independent of this parameter. In other words, Equation (5) does not change its form because to do so would mean that the individual decision is based on aggregate wealth, which seems unreasonable. Mathematically speaking, ρ dependence is eliminated at the level of the PDE because of the PDE's homogeneity in w . Hence, π is unaffected by the small parameter (except perhaps in the parameters α and σ , which will be made clear), whereas Φ is now given by:

$$\Phi(X_t, \rho W_t, t) = \rho \frac{\pi W_t}{X_t},$$

and our clearing condition is given by:

$$D(X_t, Y_t, t) + \Phi(X_t, \rho W_t, t) \equiv S_0. \quad (8)$$

This formalizes our desire to force the portfolio optimizers' demand to be small relative to the demand of the reference traders.

We now further suppose that our asset price process actually has parameters α and σ given by:

$$\alpha = \alpha_0 + \rho\alpha_1 \quad \text{and}$$

$$\sigma = \sigma_0 + \rho\sigma_1.$$

Here again we would like to stress that α_0 , α_1 , σ_0 and σ_1 all depend on both the risky asset price and the wealth of the portfolio optimizers. However, in writing the equations above, we assume that the effects of the portfolio optimizers are linear in ρ on both the drift and the diffusion of X_t . In addition, the effect of the optimization may now be seen in Equations (5) and (6) through the parameters α and σ .

Finally, we suppose that we may expand the portfolio holdings as:

$$\pi = \pi_0 + \rho\pi_1,$$

where both π_0 and π_1 are functions of x and w also.

In the rest of this section, we solve for the effects of our portfolio optimizers on the asset price in the limit as $\rho \rightarrow 0$. We find corrections to the drift and diffusion of the risky asset.

3.2 Restrictions on the Asset Price Process

As in Sircar and Papanicolaou (1998), we look at the effects that our model has on the risky asset price. If we assume that we may invert Equation (8), then we may write X_t as $X_t = \psi(Y_t, \rho W_t, t)$. We may Taylor expand in ρW_t (since this is proportional to the small parameter ρ) and discard higher-order terms to obtain:

$$X_t = H(Y_t, t) + \rho K(Y_t, t)W_t, \tag{9}$$

where $H(Y_t, t) = \psi(Y_t, 0, t)$ and $K(Y_t, t) = \psi_2(Y_t, 0, t)$. The subscript here denotes a partial derivative with respect to the second variable. Using Itô's Formula, we may then determine the dynamics

of X_t as:

$$\begin{aligned}
dX_t = & (H_t + \frac{1}{2}H_{yy}\eta^2 + H_y\mu + \rho K(\pi(\alpha - r) + r)W_t \\
& + \frac{1}{2}\rho^2 K\pi^2\sigma^2 W_t^2 + \rho W_t K_y\mu + \rho W_t K_t \\
& + \frac{1}{2}\rho W_t K_{yy}\eta^2 + \pi\sigma\rho W_t K_y\eta)dt \\
& + (H_y\eta + \rho W_t K_y\eta + \rho K W_t\pi\sigma)dB_t
\end{aligned}$$

As $\rho \rightarrow 0$ (i.e. $\rho W_t \rightarrow 0$), we have:

$$dX_t = (H_t + \frac{1}{2}H_{yy}\eta^2 + H_y\mu)dt + H_y\eta dB_t.$$

Assuming the trading strategy of the reference traders is constant through time and that they therefore only really trade based on X_t and Y_t , we may then suppose $H_t = 0$.

In order for X_t to be a geometric Brownian Motion also, which we hope to have as our base case, we know that H must satisfy certain equations in the specific case where we suppose Y_t is a geometric Brownian Motion also, as in Equation (1). In particular, H must satisfy:

$$H_y\eta_1 y = \sigma_0 x \tag{10}$$

$$H_y\mu_1 y + \frac{1}{2}H_{yy}\eta_1^2 y^2 = \alpha_0 x \tag{11}$$

Furthermore $H_y = -\frac{D_y}{D_x}$ from Equation (3) and this implies, using Equation (11) that:

$$D(x, y) = f(y^\gamma/x)$$

where $\gamma = \frac{\sigma_0}{\eta_1}$. This is exactly the same expression for D that was obtained in the work by Sircar and Papanicolaou (1998). We can think of γ as a volatility multiplication factor. When we take f to be linear, as we will eventually do, this factor helps us relate the volatility of the incomes process, η_1 , and the volatility of the underlying asset. We must be able to do this because Y_t is the only source of uncertainty in the economy, and in the absence of other traders, X_t is completely determined by it.

3.3 The First-Order Correction for the Risky Asset Process

Given the demand function we have determined, we may revisit Equation (9) to determine a process for X_t . Firstly, we note that Equation (2) tells us (with the appropriate restriction) that:

$$f\left(\frac{Y_t^\gamma}{X_t}\right) + \rho \frac{\pi W_t}{X_t} = S_0.$$

Setting $X_t = H + \rho K W_t$, and discarding terms of higher order in ρ as we go, we get:

$$\frac{1}{S_0} f\left(\frac{Y_t^\gamma}{H + \rho K W_t}\right) + \frac{\pi_0 \rho W_t}{(H + \rho K W_t) S_0} = 1.$$

Multiplying both sides by $H + \rho K W_t$, this gives us:

$$\frac{1}{S_0} (H + \rho K W_t) f\left(\frac{Y_t^\gamma}{H + \rho K W_t}\right) + \rho \frac{\pi_0 W_t}{S_0} = H + \rho K W_t.$$

So, rearranging and expanding (and assuming that f is a differentiable function), we may write:

$$\frac{1}{S_0} \left(H f\left(\frac{Y_t^\gamma}{H}\right) + \rho K W_t f\left(\frac{Y_t^\gamma}{H}\right) - \rho \frac{Y_t^\gamma}{H} K W_t f'\left(\frac{Y_t^\gamma}{H}\right) \right) + \rho \frac{\pi_0 W_t}{S_0} = H + \rho K W_t.$$

Equating expressions of different scales, we get:

$$H S_0 = H f\left(\frac{Y_t^\gamma}{H}\right),$$

which implies that H solves:

$$S_0 = f\left(\frac{Y_t^\gamma}{H}\right). \quad (12)$$

At the ρ scale, Equation (12) tells us that:

$$K = \frac{\pi_0}{\frac{Y_t^\gamma}{H} f'(Y_t^\gamma/H)}. \quad (13)$$

The preceding analysis tells us that, to first order in ρ , X_t does actually depend on the aggregate income Y_t . In other words, α_1 is a function of Y_t . In order to simplify the analysis so that we do not have to find an explicit formula for Y_t in terms of X_t , we choose the same functional form for f that was used in Sircar and Papanicolaou (1998). Namely, we choose f to be linear in its argument. We suppose henceforth that f is given by $f(z) = \kappa z$, where we let $\kappa = \tilde{\kappa} S_0$.

From Equation (2), we may solve for X_t :

$$X_t = \kappa Y_t^\gamma + \rho \frac{\pi_0 W_t}{S_0} \quad (14)$$

and therefore K is just:

$$K = \frac{\pi_0}{S_0}.$$

We note here that this may also be derived from Equation (13) because if $f(Y_t^\gamma/H) = S_0$ and f is linear, then $\frac{Y_t^\gamma}{H} f'(Y_t^\gamma/H) = S_0$ also.

Using this, we may obtain expressions for both α_1 and σ_1 . In particular, looking at the expression for the drift of X_t to first order, we may write:

$$dX_t = d(\kappa Y_t^\gamma) + \rho \frac{\pi_0}{S_0} dW_t,$$

which, on expansion, yields:

$$dX_t = \kappa Y_t^\gamma (\gamma \mu_1 + \frac{1}{2} \eta_1^2 \gamma (\gamma - 1)) dt + \rho (\frac{\pi_0}{S_0} W_t (\pi_0 (\alpha_0 - r) + r)) dt + \kappa Y_t^\gamma (\gamma \eta_1) dB_t + \rho \frac{\pi_0}{S_0} W_t \pi_0 \sigma_0 dB_t. \quad (15)$$

Equation (14) may then be reapplied and this tells us that:

$$\alpha_0 = \gamma \mu_1 + \frac{1}{2} \eta_1^2 \gamma (\gamma - 1) \quad (16)$$

and

$$\sigma_0 = \gamma \eta_1. \quad (17)$$

Equating coefficients of order ρ gives:

$$\alpha_1 = \frac{\pi_0 W_t}{X_t S_0} ((\pi_0 - 1)(\alpha_0 - r))$$

and looking at the volatility to first order in ρ gives us:

$$\sigma_1 = \sigma_0 (\pi_0 - 1) \frac{\pi_0 W_t}{X_t S_0}.$$

Notice that both these expressions for the first-order corrections are linear in W_t . In normal market conditions (and over a wide range of possible α_0), these equations mean that portfolio optimizers

tend to decrease the volatility of the underlying asset (since we assume that $0 < \pi_0 < 1$ under such normal conditions). They also tend to decrease the average growth rate of the stock, although by not as much as they do the volatility because $\alpha_0 - r < \sigma_0$ in most situations.

It is worth noting, however, what happens to the volatility in situations where the Merton optimal portfolio holdings are greater than 1 or less than 0. These portfolio holdings are possible in situations where the excess rate of return of the stock is large compared to the variance of the stock's returns or when the excess rate of return is negative, respectively. When $\pi_0 > 1$, $\sigma_1 > 0$, while when $\pi_0 < 0$, $\sigma_1 > 0$ also. This implies that portfolio optimizers tend to increase the volatility of the risky asset when this asset has a large positive excess return or when it has a negative excess return. This is the main result so far. We have, in effect, managed to decouple the solution of the optimal portfolio holdings from the evaluation of the parameters that determine the dynamics of the underlying risky asset. We are therefore able to find these parameters without ever knowing the correction to the optimal portfolios.

4 Small Feedback Effects on the Optimal Holdings

4.1 Small corrections to the Hamilton-Jacobi-Bellman Equation

We proceed to analyze the effects on the optimal portfolio allocations that the optimizers should place in the risky asset. In order to solve for the corrected optimal holdings of the portfolio optimizers, we need to look at the value function associated with their investment problem. We have already seen that this value function solves Equation (5), and that the feedback effects come in through the parameters α and σ in that equation. The HJB equation for the value function

becomes:

$$\begin{aligned} \sup_{\pi} V_t + \frac{1}{2} V_{ww} \pi^2 (\sigma_0 + \rho \sigma_1)^2 w^2 + V_w (\pi (\alpha_0 + \rho \alpha_1 - r) + r) w \\ + V_x \alpha x + \frac{1}{2} V_{xx} (\sigma_0 + \rho \sigma_1)^2 x^2 + V_{xw} (\sigma_0 + \rho \sigma_1)^2 \pi x w = 0. \end{aligned} \quad (18)$$

$$V(x, w, T) = u(w)$$

To solve the coupled equation above, we look at the limit of the equation as $\rho \rightarrow 0$. We expand the value function V as:

$$V = V_0(w, t) + \rho V_1(x, w, t). \quad (19)$$

The first term of our expansion does not have any dependence on x because we assume that feedback effects come in only as a correction term. Therefore, the value function, to principal order, is just a function of w and t and we will need to check this *a posteriori*.

If we take Equation (19) to be the correct expansion for V , then to first order in ρ , we may write, as before,:

$$\pi = \pi_0 + \rho \pi_1$$

where now we have explicit formulas for π_0 and π_1 :

$$\begin{aligned} \pi_0 &= -\frac{V_{0,w}(\alpha_0 - r)w}{V_{0,ww}\sigma_0^2 w^2} \\ \pi_1 &= -\frac{\alpha_1 w V_{0,w} + V_{1,w}(\alpha_0 - r)w + V_{1,xw}\sigma_0^2 x w}{V_{0,ww}\sigma_0^2 w^2} + \frac{(V_{0,w}(\alpha_0 - r)w)(2V_{0,ww}\sigma_0\sigma_1 w^2 + V_{1,ww}\sigma_0^2 w^2)}{(V_{0,ww}\sigma_0^2 w^2)^2} \end{aligned} \quad (20)$$

Equation (20) is a result of some simple expansions to first order. Namely, from our optimal holding expression in Equation (6) and the expansion we used in Equation (19), we know that:

$$\pi = -\frac{(V_{0,w} + \rho V_{1,w})(\alpha_0 + \rho \alpha_1 - r)w + \rho V_{1,xw}(\sigma_0 + \rho \sigma_1)^2 x w}{(V_{0,ww} + \rho V_{1,ww})(\sigma_0 + \rho \sigma_1)^2 w^2} \quad (21)$$

$$= -\frac{V_{0,w}(\alpha_0 - r)w + \rho(V_{1,w}(\alpha_0 - r)w + V_{1,xw}\sigma_0^2 x w + V_{0,w}\alpha_1 w)}{V_{0,ww}\sigma_0^2 w^2 + V_{1,ww}\sigma_0^2 w^2 + 2\sigma_0\sigma_1 w^2 V_{0,ww}} \quad (22)$$

Equation (22) follows from Equation (21) after throwing away higher order terms in both the numerator and the denominator. After this, Equation (20) is a simple application of the expansion below to the expression in Equation (22):

$$\frac{a + \rho b}{c + \rho d} = \frac{a}{c} + \rho \left(\frac{b}{c} - \frac{ad}{c^2} \right).$$

Substituting our expression for π in terms of A and C back into Equation (5) gives us a new equation that the value function must solve:

$$\begin{aligned} (V_0 + \rho V_1)_t + \frac{1}{2}(V_{0,ww} + \rho V_{1,ww})(\pi_0^2 + 2\rho\pi_0\pi_1)(\sigma_0^2 + 2\sigma_0\sigma_1\rho)w^2 \\ + (V_{0,w} + \rho V_{1,w})((\pi_0 + \rho\pi_1)(\alpha_0 + \rho\alpha_1 - r) + r)w + \rho V_{1,x}(\alpha_0 x) \\ + \frac{1}{2}\rho V_{1,xx}\sigma_0^2 x^2 + \rho V_{1,xw}\sigma_0^2 \pi_0 x w = 0. \end{aligned} \quad (23)$$

Looking at the terms that are independent of ρ in Equation (23), we obtain:

$$V_{0,t} + \frac{1}{2}V_{0,ww}\pi_0^2\sigma_0^2 w^2 + V_{0,w}(\pi_0(\alpha_0 - r) + r)w = 0. \quad (24)$$

Given the particular form of the demand function, D , that we assumed in Section 3.2, we know that α_0 and σ_0 are constants. Equation (24) then reduces exactly to Merton's problem. This means that our guess of the form for V_0 was correct because the value function under Merton's theory is independent of the price of the asset, as long as the asset price follows log-normal dynamics with constant coefficients. This will be shown in the next section.

We obtain a more complicated equation for V_1 by equating terms of order ρ :

$$\begin{aligned} V_{1,t} + V_{0,ww}\pi_0^2\sigma_0\sigma_1 w^2 + V_{0,ww}\pi_0\pi_1\sigma_0^2 w^2 + \frac{1}{2}V_{1,ww}\pi_0^2\sigma_0^2 w^2 \\ + V_{0,w}(\pi_0\alpha_1 w + \pi_1(\alpha_0 - r)w) + V_{1,w}(\pi_0(\alpha_0 - r) + r)w + V_{1,x}(\alpha_0 x) \\ + \frac{1}{2}V_{1,xx}\sigma_0^2 x^2 + V_{1,xw}\sigma_0^2 \pi_0 x w = 0. \end{aligned} \quad (25)$$

4.2 Solving for the Value Functions in the Small Parameter Case

We first solve the zeroth order equation, which is just Merton's problem. We assume that the terminal condition for our portfolio optimizer is given by the CRRA utility we assumed earlier:

$$V(x, w, T) = u(w) = \frac{w^\lambda}{\lambda}$$

where λ is just a measure of risk-aversion of the portfolio optimizer. Typically, we suppose that $0 \leq \lambda < 1$. The case $\lambda = 0$ is actually the log utility, $u(w) = \log(w)$. A utility function of CRRA type allows us to compute an explicit form for $V_0(w, t)$ since $V_0(w, t)$ solves:

$$V_{0,t} - \frac{1}{2} \frac{V_{0,w}^2 (\alpha_0 - r)^2}{V_{0,ww} \sigma_0^2} + r w V_{0,w} = 0. \quad (26)$$

The solution to Equation (26) is just the usual solution to Merton's optimization problem:

$$V_0(w, t) = \frac{w^\lambda}{\lambda} \exp \left(-\lambda \left(\frac{1}{2} \left(\frac{\alpha_0 - r}{\sigma_0} \right)^2 \frac{1}{\lambda - 1} - r \right) (T - t) \right) \quad (27)$$

and this tells us that:

$$\pi_0 = \frac{\alpha_0 - r}{\sigma_0^2 (1 - \lambda)}. \quad (28)$$

Using our calculated values for V and A from Equations (27) and (28), we may write:

$$\pi_1 = -\frac{\pi_0^2 w}{x S_0} (\pi_0 - 1) + V_{1,w} w \frac{\pi_0}{k(t) w^\lambda} - V_{1,xw} x w \frac{1}{k(t) w^\lambda (\lambda - 1)} - V_{1,ww} w^2 \frac{\pi_0}{k(t) w^\lambda (\lambda - 1)}, \quad (29)$$

where

$$k(t) = \exp \left(-\lambda \left(\frac{1}{2} \left(\frac{\alpha_0 - r}{\sigma_0} \right)^2 \frac{1}{\lambda - 1} - r \right) (T - t) \right).$$

We may now simplify Equation (25).

$$\begin{aligned} V_{1,t} + V_{1,w} w (\pi_0 (\alpha_0 - r) + r) + V_{1,xw} x w (\pi_0 \sigma_0^2) + \frac{1}{2} V_{1,ww} w^2 \pi_0^2 \sigma_0^2 \\ + V_{1,xx} x \alpha_0 + \frac{1}{2} V_{1,xx} \sigma_0^2 x^2 = 0 \end{aligned} \quad (30)$$

Since the equation has zero terminal condition and has no inhomogeneous term, its solution is just $V_1(x, w, t) = 0$ everywhere. This is a very surprising consequence of the feedback effect. We

have shown that the first-order correction to the value function is zero, which means that portfolio optimizers do not increase their expected utility (to first order) in a market with feedback effects (compared to a market where they are simply price-takers). The risky asset price dynamics change so as to conserve the maximal expected utility of the portfolio optimizers.

From Equation (29), we may compute a correction to the optimal asset allocation.

$$\pi_1 = -\frac{\pi_0 w}{x S_0}(\pi_0 - 1). \quad (31)$$

Notice that this correction to optimal asset allocation depends both on the risky asset price and also on the wealth of the portfolio optimizers. Moreover, the asset allocation is time-dependent.

5 Numerical Computation of the Feedback Effects

So far, our analysis has been limited to small relative wealth so that we could linearize the problem. However, there is no reason, *a priori*, to believe that this small parameter approximation is close to the exact solution. We revisit that question now. We first formulate a global Hamilton-Jacobi-Bellman equation assuming the same form for the demand function that we used in Section 3. We then structure a numerical scheme to solve this equation. Finally, we present some numerical results to show that this small parameter approximation is indeed valid.

5.1 Formulation of a Global Hamilton-Jacobi-Bellman Equation

Equation (5) is the HJB equation that determines the value function. We now incorporate the market-clearing equation into this PDE. Although it was possible to do this earlier, we did not as our clearing condition in Equation (3) was not in the simplified form that is now available with our particular form of the reference traders' demand. Although the global HJB equation we derive below is new, its linearization with small ρ is consistent with our previous formulation.

Let us again suppose that $f(z) = \kappa z$ is the demand of the reference traders. We may therefore write the clearing constraint as:

$$\kappa \frac{Y_t^\gamma}{X_t} + \rho \frac{\pi_t W_t}{X_t} = S_0. \quad (32)$$

Rearranging Equation (32) gives us:

$$X_t = \kappa Y_t^\gamma + \rho \frac{\pi_t W_t}{S_0}. \quad (33)$$

Assuming that W_t is self-financing, we may write the dynamics of X_t as:

$$dX_t = d(\kappa Y_t^\gamma) + \rho \frac{\pi_t}{S_0} dW_t,$$

which, following upon expansion and using the identities given in Equations (16-17), yields:

$$dX_t = \kappa Y_t^\gamma \alpha_0 dt + \rho \left(\frac{\pi_t}{S_0} W_t (\pi_t (\alpha - r) + r) \right) dt + \kappa Y_t^\gamma \sigma_0 dB_t + \rho \frac{\pi_t}{S_0} W_t \pi_t \sigma dB_t.$$

Substituting $\kappa Y_t^\gamma = X_t - \rho \frac{\pi_t W_t}{S_0}$, which we obtain from Equation (32), lets us write the dynamics for X_t as:

$$\begin{aligned} dX_t &= (X_t - \rho \frac{\pi_t}{S_0} W_t) \alpha_0 dt + \sigma_0 (X_t - \rho \frac{\pi_t}{S_0} W_t) dB_t + \rho \frac{\pi_t}{S_0} (\pi_t (\alpha - r) + r) W_t dt + \sigma \rho \frac{\pi_t}{S_0} \pi_t W_t dB_t \\ &= \left(\alpha_0 X_t - \rho \alpha_0 \frac{\pi_t}{S_0} W_t + \rho \frac{\pi_t}{S_0} W_t (\alpha - r) + \rho r \frac{\pi_t}{S_0} W_t \right) dt \\ &\quad + \left(\sigma_0 X_t - \sigma_0 \rho \frac{\pi_t}{S_0} W_t + \rho \sigma \frac{\pi_t}{S_0} \pi_t W_t \right) dB_t \end{aligned}$$

We have also assumed, however, that $dX_t = \alpha X_t dt + \sigma X_t dB_t$, which means that we may rearrange the equations above to solve for α and σ by equating the dt and dB_t terms. This gives us:

$$\alpha = \frac{\alpha_0 - \rho \left(\frac{\pi_t W_t}{X_t S_0} \alpha_0 + r \frac{\pi_t W_t}{X_t S_0} - r \frac{\pi_t^2 W_t}{X_t S_0} \right)}{1 - \rho \frac{\pi_t^2 W_t}{X_t S_0}} \quad (34)$$

$$\sigma = \sigma_0 \frac{1 - \rho \frac{\pi_t W_t}{X_t S_0}}{1 - \rho \frac{\pi_t^2 W_t}{X_t S_0}} \quad (35)$$

This means that we may determine the dynamics of X_t and W_t as soon as we know π_t . The portfolio

allocation decision is determined by the solution to a new Hamilton-Jacobi-Bellman equation:

$$\begin{aligned}
V_t + \sup_{\pi} \left[\pi(\alpha - r)wV_w + \alpha xV_x + \frac{1}{2}\pi^2\sigma^2w^2V_{ww} + \sigma^2\pi xwV_{xw} \right] \\
+ \frac{1}{2}\sigma^2x^2V_{xx} + rwV_w = 0. \\
V(x, w, T) = \frac{w^\lambda}{\lambda}
\end{aligned} \tag{36}$$

5.2 A Numerical Scheme for the Solution of the HJB equation

By solving the nonlinear PDE given in Equation (36), we are able to see how well our small parameter approximations correspond to the actual solution (at least in the vicinity of $\rho = 0$). The numerical solution is also of general interest for ρ away from 0. We use centered finite-difference approximations for the derivatives in Equation (36). We then solve the numerical HJB for π at each time step, using an iterative procedure that repeatedly uses the first order condition for the maximum.

The details are as follows. After using the log transformation ($\tilde{x} = \log(x)$ and $\tilde{w} = \log(w)$), we may write the numerical HJB, which is derived from Equation (36) through simple discretizations of the derivatives, as:

$$\begin{aligned}
\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t} + \sup_{\pi_{i,j}^n} \left[\left(\pi_{i,j}^n(\alpha_{i,j}^n - r) + r \right) \frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2h} \right. \\
+ \frac{1}{2}(\pi_{i,j}^n\sigma_{i,j}^n)^2 \left(\frac{V_{i,j+1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j-1}^{n+1}}{h^2} - \frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2h} \right) \\
+ (\sigma_{i,j}^n)^2\pi_{i,j}^n \left(\frac{V_{i+1,j+1}^{n+1} - V_{i+1,j}^{n+1} - V_{i,j+1}^{n+1} + V_{i,j}^{n+1}}{2h^2} \right) \\
+ \frac{1}{2}(\sigma_{i,j}^n)^2 \left(\frac{V_{i+1,j}^{n+1} - 2V_{i,j}^{n+1} + V_{i-1,j}^{n+1}}{h^2} - \frac{V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1}}{2h} \right) \\
\left. + \alpha_{i,j}^n \frac{V_{i+1,j}^{n+1} - V_{i-1,j}^{n+1}}{2h} \right] = 0
\end{aligned} \tag{37}$$

In the scheme, the first subscript denotes the $\tilde{x}$ co-ordinate in the uniform $(\tilde{x}, \tilde{w})$ -grid, while the second corresponds to the $\tilde{w}$ co-ordinate. The superscript represents the time-step. We note that

we are solving a terminal value problem. Finally, h is the spatial grid step in both the $\tilde{x}$ and $\tilde{w}$ co-ordinates.

In our implementation, we define an extended grid in $\tilde{x}$ and $\tilde{w}$ that is more than 4 standard deviations away from both the initial wealth and the initial stock price. At each time step, we shrink the number of grid points in each of the $\tilde{x}$ and $\tilde{w}$ directions by 2, thereby removing the grid points at the edges and hence minimizing the boundary effects.

The preceding discretization may also be represented as a Markov chain policy iteration scheme. Given some policy, π^n , we solve:

$$\begin{aligned}
V_{i,j}^n = & V_{i,j}^{n+1} \left(1 - \Delta t \frac{(\pi_{i,j}^n \sigma_{i,j}^n)^2 + \Delta t (\sigma_{i,j}^n)^2}{h^2} + \frac{\pi_{i,j}^n (\sigma_{i,j}^n)^2}{2h^2} \right) \\
& + \Delta t V_{i,j+1}^{n+1} \left(\frac{\pi_{i,j}^n (\alpha_{i,j}^n - r) + r - \frac{1}{2} (\pi_{i,j}^n \sigma_{i,j}^n)^2}{2h} + \frac{(\pi_{i,j}^n \sigma_{i,j}^n)^2 - (\sigma_{i,j}^n)^2 \pi_{i,j}^n}{2h^2} \right) \\
& + \Delta t V_{i,j-1}^{n+1} \left(\frac{-(\pi_{i,j}^n (\alpha_{i,j}^n - r) + r) + \frac{1}{2} (\pi_{i,j}^n \sigma_{i,j}^n)^2}{2h} + \frac{(\pi_{i,j}^n \sigma_{i,j}^n)^2}{2h^2} \right) \\
& + \Delta t V_{i+1,j+1}^{n+1} \left(\frac{(\sigma_{i,j}^n)^2 \pi_{i,j}^n}{2h^2} \right) \\
& + \Delta t V_{i+1,j}^{n+1} \left(\frac{\alpha_{i,j}^n - \frac{1}{2} (\sigma_{i,j}^n)^2}{2h} + \frac{(\sigma_{i,j}^n)^2 - \pi_{i,j}^n (\sigma_{i,j}^n)^2}{2h^2} \right) \\
& + \Delta t V_{i-1,j}^{n+1} \left(\frac{-\alpha_{i,j}^n + \frac{1}{2} (\sigma_{i,j}^n)^2}{2h} + \frac{(\sigma_{i,j}^n)^2}{2h^2} \right)
\end{aligned} \tag{38}$$

What remains now is to specify how we calculate the optimal π for each node and each time step. To do this, we perform a policy iteration before proceeding to the next time step. If we have the policy (in this case, the allocation to stock) at time step $n + 1$, then we use that as the seed for the optimal policy at time step n . Once we have an allocation, we use it to find α and σ using Equations (34)-(35). We also use the allocation and the calculated values of α and σ to obtain a value function by substituting this allocation into the discretization discussed in Equation (38). We then calculate an allocation that is consistent with the new value function, by using the first-order

condition for a supremum, namely:

$$\pi_{i,j} = -\frac{1}{\sigma_{i,j}^2 \left(\frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{h^2} \right)} \left((\alpha_{i,j} - r) \frac{V_{i,j+1} - V_{i,j-1}}{2h} + \sigma_{i,j}^2 \frac{V_{i+1,j+1} - V_{i,j+1} - V_{i+1,j} + V_{i,j}}{2h^2} \right).$$

We then repeat until the allocation for a substantial part of the grid does not change much from one iteration to the next. Once this occurs, we proceed to the next time step.

This numerical scheme (and, in particular, the Markov chain) is consistent with the PDE, and we refer the reader to Kushner and Dupuis (1992) for further results about convergence. There is a CFL condition that was satisfied when we used the numerical scheme. Namely, we required at each grid point:

$$\frac{\Delta t}{h^2} \leq \frac{1}{(\pi_{i,j}^n \sigma_{i,j}^n)^2 + (\sigma_{i,j}^n)^2 + \pi_{i,j}^n (\sigma_{i,j}^n)^2}$$

We also note that policy iteration is a common algorithm in dynamic programming (see Howard (1960)), and we use it here as a natural way of solving for the optimal allocation in this nonlinear setting.

5.3 Results for the Numerical Scheme and the Small Parameter Approximation

We compare our small parameter approximation and our numerical solution in two regimes: one with low excess returns relative to variance, and the other with high excess returns relative to variance. If $\alpha_0 - r$ is considered our expected excess return in the absence of the portfolio optimizers, then our two regimes correspond to $\alpha_0 - r = 2\%$ p.a. and $\alpha_0 - r = 4\%$ p.a., respectively. We suppose our volatility, σ_0 , is 20% and that the risk-aversion coefficient, λ , is 0.2. Finally, we take our time horizon to be 30 days. We use a time discretization of 1 day.

We use Equations (28) and (31) to calculate the corrections to the optimal holdings that portfolio optimizers should make given the influence they have on the asset prices. We graph the results over the variable:

$$\xi = \frac{\rho W_0}{X_0 S_0}. \quad (39)$$

Figure 1: Portfolio allocation to stock at time $t = 0$, when stock has low excess returns (left) and high excess returns (right): comparison between approximation ('x') and numerical solution of PDE ('o').

This variable indicates the magnitude of the effect of the portfolio optimizers as it represents the relative amount of wealth held by portfolio optimizers to the total market capitalization of the risky asset. This also means that the relative demand of the portfolio optimizers is small compared to the demand of the reference traders, since with ξ small, $\Phi/S_0 = \pi_0\xi$ is also small (under market conditions where the optimal holding is about 60%). Therefore, after using Equation (3), we know that $D/S_0 = 1 - \Phi/S_0$ accounts for much of the demand for the risky asset.

Figures 1 and 2 are produced by setting $\rho = 0.5$, solving the numerical scheme and then evaluating $\xi = \rho \frac{w}{xS_0}$ at each grid point. In order to make the graphs readable, some grid points were removed in the last graphing stage.

We tested convergence for a variety of ρ by reducing h and Δt , while maintaining the CFL condition. In other words, we changed these discretization parameters but we kept $\frac{\Delta t}{h^2}$ constant. The numerics were robust to these changes.

Figure 2: Local volatility of the stock at time $t = 0$, when stock has low excess returns (left) and high excess returns (right): comparison between approximation ('x') and numerical solution of PDE ('o').

6 Discussion of results

6.1 Optimal Allocations and Volatility of the Underlying Asset

With the analysis and numerical computations for our model we have obtained two main results. First, the volatility of the asset price decreases in the presence of portfolio optimizers in market conditions where $0 < \pi < 1$. This additive increase is proportional to ξ , where ξ was defined in Equation (39). The constant of proportionality is: $(\pi_0 - 1)\sigma_0$. This amounts to as much as a 3% decrease in volatility (3% of 20% volatility in our example) for aggregate wealth equal to 10% of total market capitalization.

Second, we calculate the precise dependence of the optimal allocation to our variable ξ . We find that as ξ increases, the optimal allocation to stock increases in normal trading conditions (i.e. when $\pi_0 < 1$). It amounts to as much as a 2% increase in wealth allocated to the risky asset.

In order to interpret the results, it is helpful to keep in mind the clearing equation (3). Let us suppose that there is a sudden upward movement in the price of X_t when the portfolio optimizers

are holding less than their wealth in stock. Since portfolio optimizers try to keep a constant fraction of their wealth in stocks, this means that their demand for stocks must go down which means that the uptick in X_t is moderated by the portfolio optimizers. Hence it is reasonable that the volatility of the stock decreases in their presence. This effect is in contrast to hedgers who try to hedge a short call option as was the case in the paper by Sircar and Papanicolaou (1998). In that case, hedgers tend to want to buy more stock to cover the decrease in the value of their option position (delta hedging), which will mean that the X_t price movement is not moderated. Instead, demand increases and so it is natural that the price of X_t is further increased because of the hedgers.

However, when portfolio optimizers hold more than their wealth in stock (and have therefore borrowed money to invest in the risky asset), a sudden downward movement in the price of X_t , for example, will result in selling of the risky asset by the portfolio optimizers. They must do this because they have borrowed money to invest in the stocks. This means that the volatility of the stock must increase. In fact, an increase in volatility is expected when there is a lot of leverage in the market. This is seen clearly in our results.

The positive adjustment to the optimal holdings of risky asset in the presence of portfolio optimizers may also be explained with a simple demand argument. As the wealth of the optimizers increases, it is natural that they would demand more risky assets because, to first order, they keep a constant fraction of their wealth in stocks. Since demand increases, it is again natural that X_t should increase, all else being equal. But if the optimizers know this to be the case, it is not surprising that they would want to increase their exposure to the risky asset, as long as they have not borrowed to invest in that risky asset. If they have borrowed, then profit taking turns out to be a better strategy.

6.2 Comparison Between Portfolio Optimizers and Hedgers

In the presence of a small number of hedgers of short positions in options, it was found in Sircar and Papanicolaou (1998) that volatility increased by as much as 10%. Our 3% result shows that

portfolio optimizers tend to have a smaller effect on the underlying security than hedgers. We may do this comparison even when both groups are present because, to first order, their effects on the underlying asset price may be considered to be additive.

6.3 Comparison Between Numerical Solution and Analytical Approximation

From Figures 1 and 2 we see that the deviation of the numerical solution from the small ρ approximation tends to get bigger as ξ increases. In fact, we see that asset allocation to stock, in the case where $\pi_0 < 1$, should be increased even more rapidly than our small ρ estimates indicate, as the ratio of total wealth to the market capitalization increases. This is interesting because it tells us that the second-order ρ effects become substantial and have the same sign as ξ increases.

It is worthwhile noting that in both the numerics and the small ρ estimation, the value function turned out to be independent of time. This follows from the partial differential equation. If we consider α and σ to be some time-independent functions of π , we may separate the value function as:

$$V(x, w, t) = k(t)c(x, w).$$

Substituting this into Equation (36) allows us to pull $k(t)$ outside of the supremum, and so π is independent of time as long as we can find solutions for the resulting PDE in x and w , which is indeed possible, at least for small enough ρ .

6.4 Conclusion

We have shown that portfolio optimizers do influence the price of the underlying asset and do so in such a way as to reduce the asset's volatility, at least in situations where they are not leveraged. Their optimal holdings are also shown to increase as their presence in the market becomes more pronounced. Further research in this area would include hedgers in the market. The exact mechanism through which they may exert their demand is described in Nayak (2006). Also

of theoretical interest are proofs of convergence of the numerical scheme we have described and a general existence theorem for the global Hamilton-Jacobi-Bellman equation, at least for small ρ . We intend to pursue this as part of future work.

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