Pulse Stabilization in a Strongly Heterogeneous Layered Medium

Pawel Lewicki, Robert Burridge
Schlumberger-Doll Research
Old Quarry Road
Ridgefield, CT 06877

George Papanicolaou
Department of Mathematics
Stanford University
Stanford, CA 94305

ABSTRACT

A strongly heterogeneous stratified elastic medium occupies a slab of finite thickness. The medium parameters vary randomly with correlation length of $O(\epsilon^2)$. A pulse with $O(\epsilon)$ wavelength enters the slab at one end. It travels coherently through the slab with a slowness near to that predicted by effective medium theory. Relative to a certain realization-dependent co-moving frame of reference the pulse stabilizes to its mean as ϵ tends to zero. It broadens as it travels as in the O'Doherty-Anstey theory, but here the variations in medium properties are not assumed to be small as in that theory, and the pulse travels with a travel time differing significantly from the characteristic first arrival time.

Contents

1 Introduction

In this paper we study the evolution of an initially pulse-like acoustic wavefield as it travels through a rapidly varying medium where the variations are finite in amplitude, i.e. not assumed to be weak. The ratio of the length scale of the medium variations to the wavelength is the same as the ratio of the wavelength to the distance traveled, and is small. This ratio is used as a small parameter for an asymptotic analysis which shows that if a reference frame is suitably chosen, moving at a slowness close to that predicted by the effective medium theory, the waveform consists of a stabilized pulse (head) plus a fluctuating coda, the latter decaying with the small parameter. This situation is illustrated in Figure 1. Here, for numerical illustration the medium consist of a stack of many uniform layers where the properties change by finite amounts from one layer to the next. A Gaussian pulse enters the stack at one end and the forward propagating wavefield is observed after traversing 10000 layers. The waveform is again a Gaussian pulse accompanied by a fluctuating coda. The time of arrival is close to the time predicted by effective medium theory, 1.45 as opposed to 1.41.

A similar phenomenon was observed for weak variations by O'Doherty and Anstey [18] and has been extensively studied since [2,4,5,6,9,16,17]. On the other hand the coda of the wave has been studied thoroughly even when the variations are finite [1,7,8,15,19] and that work already contained some numerical examples showing the pulse-like head of the wave.

In our analysis we regard our rapidly varying medium as a single (typical) realization of a stochastic process, whose correlation length is on the same scale as the medium variations. We then find an exact formula for the space-time trajectory of the centroid of the pulse, which travels at close to the effective slowness, and for the shape of the stabilized pulse, which appears to diffuse about its centroid as it travels, the rate of diffusion being proportional to the integral of the autocorrelation function of the medium parameters.

The theory starts from first principles, namely the momentum equation and the linearly elastic constitutive law. We study the limit as the small ratio of scales tends to zero and our results are true almost surely with respect to the underlying probability measure. The analysis is based on invariant imbedding equations for the time-harmonic reflection and transmission coefficients. The recognition of the importance of the invariant imbedding technique to the analysis of multiple scattering came with the series of papers mentioned above [1,7,8,15,19], devoted to the behavior of the coda which follows the main pulse. Here, a new insight into the versatility of the invariant imbedding technique enabled us to find a moving reference frame relative to which the pulse stabilizes to a deterministic waveform. This co-moving frame moves at the deterministic effective slowness with a realization-dependent correction. It is analogous to the stabilization observed by Stanke and Burridge in the O'Doherty-Anstey regime [10].

In Section 2 we state the hyperbolic system of one-dimensional linear acoustics and introduce the small parameter ϵ, the common ratio among the three length scales. Section 3 describes an effective medium approximation based upon homogenization. It is valid for wave propagation over distances on the same order as the wavelength. In Section 4 we consider wave propagation over a much longer distance. We define the up-going and down-going amplitudes A and B and also the random travel time relative to which the pulse is stabilized. Our main result is stated as Theorem 1 where a precise formula for the stable pulse shape is given. When the incoming pulse is Gaussian then it is a Gaussian in time which appears to diffuse as the distance traveled increases. Here time and space have roles which are the reverse of those usual in diffusion. In Section 5 we introduce a time-harmonic reflection coefficient , and a time-harmonic transmission coefficient Θ which together satisfy a non-linear first order system of ODE's. We then obtain the integral representation for the quantity of interest, the transmitted pressure, in terms of the time-harmonic transmission coefficient. The stabilization of the pulse as ϵ converges to zero is seen from the evaluation of the variance of the transmitted pressure, i.e the difference between the second moment and the square of the mean. In Section 6 we present the analysis of the moment equations, which is based on the theory of mixing stochastic differential equations as described in the Appendix. The formula for the mean of the transmitted pressure yields the waveform of the stable pulse. This proves our main result that the waveform is stabilized when it is observed in the correct reference frame. In Section 7 we consider the evolution of the pulse relative to a frame of reference moving with the effective slowness. Here the pulse *does not* stabilize and its mean is considerably broader than the stable pulse, because of the random time-differences between the two co-moving frames. The results of this section were also obtained by Clouet and Fouque [11] by similar methods. In Section 8 are displayed some numerical illustrations which illustrate how the pulse tends to the stable waveform as the small parameter converges to zero, and also how it varies as the depth of observation increases, or as the variance of the medium properties changes. In the Appendix we state the main theorem about mixing stochastic differential equations that we need and we derive from it the moment equations which we use in Section 6.

2 Formulation of the problem

We consider propagation of acoustic pulses in a layered medium. The linearized equation of momentum and the constitutive relation for the component of particle velocity perpendicular to the layering $u(z,t)$ and the pressure $p(z,t)$ are

$$\begin{aligned} \rho u_t + p_z &= 0, \\ K^{-1} p_t + u_z + \alpha p &= 0. \end{aligned} \tag{1}$$

where z is a cartesian coordinate, $-z$ is thought of as depth into the earth, t is time, and $\rho = \rho(z)$, $K = K(z)$ and $\alpha = \alpha(z)$ are the density, bulk modulus and absorption of the medium, respectively. This particular form of absorption[1] arises for a special linear viscoelastic material known as a Maxwell solid (see chapter 5, [14]). The bulk compliance of the medium is $K^{-1}(z)$. The local wave slowness is given by

$$\gamma(z) = \left(\frac{\rho(z)}{K(z)}\right)^{1/2}. \tag{2}$$

The medium consists of a homogeneous part, occupying the half space $z > 0$, where the parameters $\rho,\ K,\ \alpha$ are constant and equal to $\rho_1,\ K_1,\ \alpha_1$; a layered part, occupying the slab $-L < z < 0$, where the parameters $\rho,\ K,\ \alpha$ change with z; and another homogeneous part, occupying the half space $z < -L$, where the parameters $\rho,\ K,\ \alpha$ are equal again to $\rho_1,\ K_1,\ \alpha_1$. The fluctuations of $\rho,\ K$ and α in the layered medium are rapid and strong: for a small parameter ϵ they are random functions of a fast variable z/ϵ^2. In non-dimensional variables we have

$$\rho(z) = \begin{cases} \rho_1(z)\left(1 + \eta(z, z/\epsilon^2)\right) & z \in [-L, 0], \\ \rho_1 & \text{otherwise}, \end{cases} \tag{3}$$

$$K^{-1}(z) = \begin{cases} K_1^{-1}(z)\left(1 + \nu(z, z/\epsilon^2)\right) & z \in [-L, 0], \\ K_1^{-1} & \text{otherwise}, \end{cases} \tag{4}$$

[1] In [1] absorption is modeled by viscous damping: $\rho u_t + \alpha u + p_z = 0$, $K^{-1} p_t + u_z = 0$. The modifications of our theory required in this case are minor and are given in Theorem 1 below.

$$\alpha(z) = \begin{cases} \alpha_1(z)\,(1+\chi(z, z/\epsilon^2)) & z \in [-L, 0], \\ \alpha_1 = 0 & \text{otherwise}, \end{cases} \tag{5}$$

where the means $\rho_1(z),\ K_1^{-1}(z),\ \alpha_1(z)$ are smooth and positive functions and the fluctuations $\eta(z,\cdot),\ \nu(z,\cdot),\ \chi(z,\cdot)$ are mean-zero, stochastic processes that are stationary for each z fixed, take values bounded by a constant less than 1 and have correlation functions that decay rapidly. The mean parameters are matched at the interfaces: $\rho_1(0) = \rho_1 = \rho(-L)$, $K_1(0) = K_1 = K_1(-L)$ and $\alpha_1(0) = \alpha_1 = \alpha_1(-L)$. Occasionally we suppress the dependence on the slow variable z.

The wave is incident on the layered random medium $-L < z < 0$ from the homogeneous half space $z > 0$ in the form of an impulse of short time duration. We assume the incident pulse to be

$$\begin{aligned} u_I(z,t) &= \zeta_1^{-1/2} f\left(\tfrac{t+\gamma_1 z}{\epsilon}\right), \\ p_I(z,t) &= -\zeta_1^{1/2} f\left(\tfrac{t+\gamma_1 z}{\epsilon}\right), \end{aligned} \tag{6}$$

for $t < 0$, where $\zeta_1 = \sqrt{\rho_1 K_1}$ is the effective impedance, $\gamma_1 = \sqrt{\rho_1/K_1}$ is the effective slowness and the amplitude $f(\cdot)$ has support in $[0,1]$.

The width of the incident pulse is intermediate between the macroscopic scale $O(1)$ and the microscopic scale $O(\epsilon^2)$ and it is taken to be $O(\epsilon)$. The numerical evidence shows that in this scaling the leading part of the propagating pulse appears to retain its initial amplitude, regardless of the strong reflection coefficients that result from large inhomogeneities in the structure of the material.

The main interest of our work is the long time evolution of the pulse for the intermediate-scale incident wave in the rapidly oscillating medium. Starting from first principles i.e. the wave equation (1), we investigate the travel time of the pulse and its shape.

3 Effective medium theory

The simplest approximation theory for the propagation of long waves in a homogeneous medium is the effective medium theory obtained by homogenization, which states that waves propagate as in a homogeneous medium whose density and compliance are constants related to the original parameters. Note that in our case, the wavelength being of order $O(\epsilon)$ and the size of inhomogeneities being of order $O(\epsilon^2)$, we deal in fact with the problem of long wave propagation. However, the effective medium theory is restricted to propagation over distances of the order of a wavelength and, therefore, is not appropriate for our problem (where the distance of propagation is on the order of $1/\epsilon$ wavelengths). Nevertheless, the parameters of the effective medium model are the first approximation to the parameters of the long time evolution model we develop in next section, as the latter must reduce to the former for short times. Therefore, we show here briefly how to find the effective parameters for the one-dimensional acoustic equation (1) with a long incident pulse and rapid fluctuations in the medium parameters.

To study the effective medium approximation we introduce the intermediate space and time variables $\tilde{z} = z/\epsilon, \tilde{t} = t/\epsilon$. Neglecting absorption, equations (1) become

$$\begin{aligned} \rho u_{\tilde{t}} + p_{\tilde{z}} &= 0, \\ K^{-1} p_{\tilde{t}} + u_{\tilde{z}} &= 0. \end{aligned} \tag{7}$$

with ϵ-independent initial conditions for $p(\tilde{z}, \tilde{t} = 0), u(\tilde{z}, \tilde{t} = 0)$. In the case of constant profiles $\rho_1(\cdot) \equiv \rho_1, K_1(\cdot) \equiv K_1$ the coefficients $\rho,\ K$ in equation (7) are functions of the fast variable $\tilde{z}/\epsilon$.

To simplify the analysis we assume ρ, K to be periodic. The random case yields the same effective parameters after a more complicated derivation.

The multiple scale expansion method yields the effective equations as follows: let

$$\begin{aligned} p &= p_0(\tilde{t}, \tilde{z}, \frac{\tilde{z}}{\epsilon}) + \epsilon p_1(\tilde{t}, \tilde{z}, \frac{\tilde{z}}{\epsilon}) + \cdots , \\ u &= u_0(\tilde{t}, \tilde{z}, \frac{\tilde{z}}{\epsilon}) + \epsilon u_1(\tilde{t}, \tilde{z}, \frac{\tilde{z}}{\epsilon}) + \cdots , \end{aligned} \tag{8}$$

where the dependence on $y = \tilde{z}/\epsilon$ is assumed to be periodic. Writing $\frac{\partial}{\partial \tilde{z}} + \frac{1}{\epsilon}\frac{\partial}{\partial y}$ instead of $\frac{\partial}{\partial \tilde{z}}$ in (7), substituting (8) into (7) and equating the coefficients of consecutive powers of ϵ to 0 we obtain:

- the ϵ^{-1}-equations imply that p_0, u_0 are y-independent,
- the solvability conditions for the ϵ^0-equations require that

$$\begin{aligned} \langle \rho \rangle u_{0,\tilde{t}} + p_{0,\tilde{z}} &= 0, \\ \langle K^{-1} \rangle p_{0,\tilde{t}} + u_{0,\tilde{z}} &= 0, \end{aligned} \tag{9}$$

  where $\langle \cdot \rangle$ denotes the average of the quantity over a period cell.

Equations (9) constitute the effective equations for wave propagation over a distance comparable with a wavelength. The effective slowness, i.e. the reciprocal of the characteristic speed of propagation for the hyperbolic system (9), is given by

$$\gamma_1 = \sqrt{\rho_1 / K_1}. \tag{10}$$

The equation (9) has constant coefficients implying that pulses travel unchanged through the medium over distances on the order of a wavelength. If we try to use this result over larger distances, on the order of $1/\epsilon$ times the wavelength, it would again predict pulses propagating unchanged. However, we know from simulations, see Figure 1, and earlier related studies [1] that the pulse is significantly attenuated and broadened when it propagates over distances on this scale. Therefore, we need a new approach for this problem.

4 Long time evolution theory

Let A, B be defined by

$$\begin{aligned} p(z,t) &= \zeta_1^{1/2} \left[A(z, t - \tau_0(z) + 2\tau_0(-L)) - B(z, t + \tau_0(z)) \right], \\ u(z,t) &= \zeta_1^{-1/2} \left[A(z, t - \tau_0(z) + 2\tau_0(-L)) + B(z, t + \tau_0(z)) \right], \end{aligned} \tag{11}$$

where

$$\zeta_1(z) = \sqrt{\rho_1(z) K_1(z)}, \quad \gamma_1(z) = \sqrt{\rho_1(z)/K_1(z)} \tag{12}$$

are the effective impedance and slowness, and

$$\tau_\xi(z) = \int_\xi^z ds\, \gamma_1(s)[1 + \mu(s, \frac{s}{\epsilon^2})] \tag{13}$$

is a realization-dependent travel time. We do not specify the correction μ appearing in (13) at this moment. We assume only that $\mu = \mu(z, \frac{z}{\epsilon^2})$ is a rapidly varying, mean zero quantity which can be expressed in terms of the medium defining parameters $\rho_1(z), K_1(z), \alpha_1(z), \eta(z, \frac{z}{\epsilon^2}), \nu(z, \frac{z}{\epsilon^2}), \chi(z, \frac{z}{\epsilon^2})$.

The travel time according to which the pulse stabilizes is determined by a unique μ, which is defined in the Theorem below.

Note that A, B, the amplitudes of the up-going and down-going waves, are centered in time relative to the system of "characteristics" $t + \tau_0(z) = \text{const}, t - \tau_0(z) = \text{const}$ defined by the long-distance travel time τ_0, with the time coordinate of B measured relative to the down-going characteristic $t+\tau_0(z) = 0$ starting from the interface $z = 0$ and the time coordinate of A measured relative to the up-going characteristic $t - \tau_0(z) = -2\tau_0(-L)$ (see Figure 2).

The quantity of interest is the transmitted pressure, which is represented in (11) by the down-going wave B, for times t near its arrival time $-\tau_0(z)$ at the depth z:

$$p_{\mathrm{TR}}(z, \epsilon s - \tau_0(z)) = \zeta_1^{1/2}(z) B(z, \epsilon s) \tag{14}$$

Theorem 1 *There exists a unique travel time according to which the pulse stabilizes. It is defined by taking*

$$\mu = \mu^{\mathrm{s}}(z, \frac{z}{\epsilon^2}) \equiv \frac{1}{2}(\eta(z, \frac{z}{\epsilon^2}) + \nu(z, \frac{z}{\epsilon^2})) \tag{15}$$

in the definition (13). The limiting deterministic transmitted pulse shape as ϵ tends to zero is with probability one

$$p_{\mathrm{TR}}(z, \epsilon s - \tau_0(z)) \simeq \frac{\zeta_1^{1/2}(z)}{2\pi} e^{-U(z)} \int e^{-i\omega s} \hat{f}(\omega) e^{-\frac{1}{2}V(z)\omega^2}\, d\omega \tag{16}$$

where[2]

$$V(z) = \int_z^0 \frac{1}{2}\gamma_1^2(\xi) \int_0^\infty E\{(\eta - \nu)(\xi, 0)(\eta - \nu)(\xi, r)\}\, dr\, d\xi, \tag{17}$$

$$U(z) = \int_z^0 \alpha_1(\xi)\zeta_1(\xi) d\xi \tag{18}$$

and $\hat{f}(\omega)$ is the standard Fourier transform of $f(\cdot)$.

Equation (16) for the limiting pulse shape states that the pulse at the depth of observation z is just the initial pulse convolved with a Gaussian kernel whose variance $V(z)$ depends on the medium traversed. The variance $V(z)$ is a positive function for $z < 0$, as the integral of the covariance functions of $(\mu - \nu)$ is positive. In the simple case when the incident pulse is Gaussian with squared width v_I, i.e.

$$f(s) = e^{-\frac{1}{2}s^2/v_I},$$

the limiting waveform (16) reduces to

$$B(z, \epsilon s) = e^{-U(z)} \sqrt{\frac{v_I}{v_I + V(z)}} e^{-\frac{1}{2}s^2/(v_I + V(z))}.$$

Thus, we observe both attenuation, since the height of the pulse decreases according to $e^{-U(z)}(1 + V(z)/v_I)^{-1/2}$ as $z \downarrow -\infty$, and broadening, since the pulse width increases as $(v_I + V(z))^{1/2}$ as $z \downarrow -\infty$.

Note that the attenuation of the pulse is due to both absorption, by the factor $e^{-U(z)}$, and multiple scattering, by the factor $(1 + V(z)/v_I)^{-1/2}$. Pulse broadening is due to multiple scattering only, however. This allows us to discriminate between absorption (intrinsic attenuation) and

[2] If we model absorption by viscous damping as in the footnote following (1) then $U(z) = \int_z^0 \frac{\alpha_1(\xi)}{\zeta_1(\xi)} d\xi$.

attenuation due to multiple scattering very accurately. We measure the peak of the pulse that passes by observation points at different depths z and estimate the slope as a function of z of the logarithm of this amplitude. A more precise way to estimate $U(z)$ is to first estimate $V(z)$ from the broadening of the pulse at different observation points z and then to estimate $U(z)$ from the logarithm of the peak amplitude

$$U(z) = -\log B(z,0) + \log\sqrt{\frac{v_I}{v_I + V(z)}}.$$

These estimations are robust because they do not depend on the unknown random time at which the pulse passes by different observation points, which is precisely the situation covered by our theory.

The center of the pulse travels with travel time $\tau_0(\cdot)$. Note that this is not the effective travel time

$$\tau^e(z) = \int_0^z \gamma_1(s)\,ds \tag{19}$$

but rather the effective travel time "corrected" for the particular realization of the medium. The size of this correction, as $\epsilon \to 0$, is of order $O(\epsilon)$ since

$$\frac{1}{\epsilon}\left(\tau_0(-L) - \int_0^{-L}\gamma_1(s)\,ds\right) \to \mathcal{N},$$

where $\to$ denotes convergence in distribution and $\mathcal{N}$ is a mean-zero, Gaussian, random variable.

It is of interest to compare the travel time τ_0 with the characteristic travel time τ^c i.e. the first arrival time:

$$\tau^c_\xi(z) = \int_\xi^z \gamma(s)\,ds, \tag{20}$$

where $\gamma(\cdot)$ is the characteristic slowness (2). A simple application of the Schwartz inequality yields:

$$\tau^c_0(z) \simeq \int_0^z \gamma_1(s)\,ds \mathrm{E}\sqrt{(1+\eta)(1+\nu)} < \int_0^z \gamma_1(s)\,ds \simeq \tau_0(z), \tag{21}$$

where $\simeq$ denotes the equality up to the error of order $O(\epsilon)$. Therefore, unless the medium is homogeneous: $\nu \equiv 0, \eta \equiv 0$, there is *always* a discrepancy between the first arrival time $\tau^c(z)$, when the first non-zero disturbance reaches depth z, and the true long-distance travel time $\tau(z)$, when the main pulse reaches depth z.

We note that in order to find the limiting waveform (16) at the depth z we need only the information about the medium-related parameters between source and receiver, i.e. $\eta(z',\cdot), \nu(z',\cdot)$ for $z' \in [z, 0]$. This exemplifies the fact that the wavefront, as it travels, feels only the traversed part $[z, 0]$ of the medium, although, as we noted above in (21), the hyperbolic domain of dependence at point $z, \tau_0(z)$ reaches much deeper into the medium than z (as the characteristic speed is grater than the effective speed). This property confirms our intuition that the pulse, although it consists of multiple reflections, behaves as a single entity.

Finally, we note that the transmitted field p_{TR} that emerges from the slab at $z = -L$ constitutes the whole pressure field at this depth, as the homogeneous half space $z < -L$ does not support any up-going waves. Thus, in spite of the fact that we observe only the down-going, transmitted, part of the pulse in its passage through the medium, eventually, at $z = -L$, the pulse-shape formula (16) contains the whole (total) pulse.

In the rest of the paper we shall assume for simplicity that absorption is absent, $\alpha \equiv 0$. It can be easily included in the calculations to yield the result stated in our theorem.

5 Integral representation of transmitted field

The analysis of the transmitted pressure is based on the invariant imbedding representation of the time harmonic reflection and transmission coefficients. They are defined as follows: let $\hat{p}, \hat{u}, \hat{A}, \hat{B}$ be the scaled Fourier transforms

$$
\begin{aligned}
\hat{p}(z,\omega) &= \int e^{i\omega t/\epsilon}\, p(z,t)\, dt, \quad \hat{u}(z,\omega) = \int e^{i\omega t/\epsilon}\, u(z,t)\, dt, \\
\hat{A}(z,\omega) &= \int e^{i\omega t/\epsilon}\, A(z,t)\, dt, \quad \hat{B}(z,\omega) = \int e^{i\omega t/\epsilon}\, B(z,t)\, dt.
\end{aligned} \tag{22}
$$

Then, the relations (11) among p, u and A, B translate in the frequency domain to

$$
\begin{aligned}
\hat{p} &= \zeta^{1/2}\left(\hat{A}e^{i\omega(\tau_0 - 2\tau_0(-L))/\epsilon} - \hat{B}e^{-i\omega\tau_0/\epsilon}\right), \\
\hat{u} &= \zeta^{-1/2}\left(\hat{A}e^{i\omega(\tau_0 - 2\tau_0(-L))/\epsilon} + \hat{B}e^{-i\omega\tau_0/\epsilon}\right),
\end{aligned} \tag{23}
$$

where τ_0 is the travel time (13).
The amplitudes $\hat{A}, \hat{B}$ satisfy a system of ordinary differential equations in $-L < z < 0$:

$$
\begin{aligned}
\frac{d\hat{A}}{dz} &= \frac{i\omega}{\epsilon}[n\hat{B}e^{-2i\omega\tau_{-L}/\epsilon} + m\hat{A}] + \frac{1}{2\zeta_1}\frac{d\zeta_1}{dz}\hat{B}e^{-2i\omega\tau_{-L}/\epsilon}, \\
\frac{d\hat{B}}{dz} &= -\frac{i\omega}{\epsilon}[n\hat{A}e^{2i\omega\tau_{-L}/\epsilon} + m\hat{B}] + \frac{1}{2\zeta_1}\frac{d\zeta_1}{dz}\hat{A}e^{2i\omega\tau_{-L}/\epsilon},
\end{aligned} \tag{24}
$$

where

$$
n(z,\tfrac{z}{\epsilon^2}) = \tfrac{1}{2}\gamma_1(z)\left(\eta(z,\tfrac{z}{\epsilon^2}) - \nu(z,\tfrac{z}{\epsilon^2})\right), \tag{25}
$$

$$
m(z,\tfrac{z}{\epsilon^2}) = \tfrac{1}{2}\gamma_1(z)\left(\eta(z,\tfrac{z}{\epsilon^2}) + \nu(z,\tfrac{z}{\epsilon^2}) - 2\mu(z,\tfrac{z}{\epsilon^2})\right). \tag{26}
$$

As mentioned at the end of the previous section, we assume for simplicity that there is no absorption so that $\alpha \equiv 0$.

The time harmonic reflection coefficient Γ is defined by

$$
\Gamma(z) = A(z)/B(z), \quad -L < z < 0. \tag{27}
$$

It satisfies the stochastic Riccati equation

$$
\begin{aligned}
\frac{d\Gamma}{dz} = \frac{i\omega}{\epsilon}&\left[ne^{-2i\omega\tau_{-L}(z)/\epsilon} + 2m\Gamma + n\Gamma^2 e^{2i\omega\tau_{-L}(z)/\epsilon}\right] \\
&+\frac{1}{2\zeta_1}\frac{d\zeta_1}{dz}\left[e^{-2i\omega\tau_{-L}(z)/\epsilon} - \Gamma^2 e^{2i\omega\tau_{-L}(z)/\epsilon}\right]
\end{aligned} \tag{28}
$$

$$
\Gamma|_{z=-L} = 0 \tag{29}
$$

where the initial condition (29) is implied by the continuity of $\hat{p}, \hat{u}$ across the interface $z = -L$ and the zero value of the up-going amplitude $\hat{A}$ in $z < -L$.

Let $-L_1$, with $-L < -L_1 < 0$, be a fixed depth where we observe the transmitted pressure. The time harmonic transmission coefficient Θ is defined by

$$
\Theta(z) = \hat{B}(-L_1)/\hat{B}(z), \quad -L_1 < z < 0. \tag{30}
$$

It satisfies a nonlinear stochastic equation coupled with the , -equation (28):

$$\frac{d\Theta(z)}{dz} = \frac{i\omega}{\epsilon}[n\Theta,\ e^{2i\omega\tau_{-L}/\epsilon} + m\Theta] - \frac{1}{2\zeta_1}\frac{d\zeta_1}{dz}\Theta,\ e^{2i\omega\tau_{-L}/\epsilon}, \tag{31}$$

which is complemented by the initial condition

$$\Theta(-L_1) = 1, \tag{32}$$

from (30).

Note that the , -equation has a natural initial condition at $z = -L < -L_1$. Therefore, we artificially extend $\Theta(\cdot)$ to the interval $z \in [-L, -L_1]$ by putting its value there equal to $\Theta(-L_1)$. This extension can be realized by assuming that $\Theta(\cdot)$ satisfies a modified equation

$$\frac{d\Theta(z)}{dz} = \frac{i\omega}{\epsilon}[n^\star\Theta,\ e^{2i\omega\tau_{-L}/\epsilon} + m^\star\Theta] - \frac{1^\star}{2\zeta_1}\frac{d\zeta_1}{dz}\Theta,\ e^{2i\omega\tau_{-L}/\epsilon} \tag{33}$$

in $-L < z < 0$, where a superscript * on a function $f(\cdot)$ denotes that it is multiplied by the characteristic function of the interval $[-L_1, 0]$. The modified initial condition for Θ is

$$\Theta(-L) = 1. \tag{34}$$

At this point it is useful to realize the meaning of the position of the initial conditions for , , Θ in the context of the time domain causality. We already noticed in Section 4 that the characteristic travel time $|\tau^c|$ (20) is less than the travel time $|\tau|$ (13). Consequently, when $-L_1$ is so near to $-L$ that $|\tau_0^c(-L)| + |\tau_{-L}^c(-L_1)| < |\tau_0(-L_1)|$, then the transmitted wave B feels the interface at $z = -L$ because the hyperbolic domain of dependence at the space-time point $(-L_1, -\tau_0(-L_1))$, i.e. where we claim the pulse is found at the depth $-L_1$, includes points deeper than $-L$. Nevertheless, as we show in the next section, the limiting waveform depends only on the properties of the medium in the traversed material $[-L_1, 0]$.

The integral representation of the transmitted pressure p_{TR} is now obtained from the inverse Fourier transform of $\hat{B}(-L_1)$ as follows. We have

$$\hat{B}(-L_1, \omega) = \hat{B}(0, \omega)\Theta(0, \omega) = \epsilon\hat{f}(\omega)\Theta(0, \omega) \tag{35}$$

where we used (30), (6) and (23); $\hat{f}$ denotes the standard Fourier transform of f. Therefore

$$p_{\mathrm{TR}}(-L_1, t - \tau_0(-L_1)) = \frac{\zeta_1^{1/2}(-L_1)}{2\pi}\int e^{-i\omega t/\epsilon}\Theta(0, \omega)\hat{f}(\omega)\, d\omega. \tag{36}$$

Formula (36) expresses the quantity of interest $p_{\mathrm{TR}}(\cdot, t - \tau(\cdot))$, t small, in terms of the solution of the stochastic system of equations (28), (33) with the initial values (29), (34). The randomness enters into p_{TR} only through the transmission coefficient Θ.

The stabilization of the transmitted pressure p_{TR} near the arrival time $-\tau_0(z)$ to the deterministic waveform (16) can be easily expressed in terms of the moments of p_{TR}. Let $\langle\cdot\rangle$ denote the mean of the quantity with respect to the underlying probability measure. It is enough to prove that

- for small t

$$\lim_{\epsilon\to 0}\left[\langle p_{\mathrm{TR}}(z, t - \tau_0(z))^2\rangle - \langle p_{\mathrm{TR}}(z, t - \tau_0(z))\rangle^2\right]_{z=-L_1} = 0, \tag{37}$$

  which states that for times near the arrival time $-\tau_0(z)$ the transmitted pressure $p_{\mathrm{TR}}(\cdot)$ is almost surely equal to its mean $\langle p_{\mathrm{TR}}(\cdot)\rangle$, and

- its mean $\langle p_{\rm TR}(\cdot)\rangle$ is given by formula (16).

if and only if μ is as stated in Theorem 1.

Note that in formula (37) we first center $p_{\rm TR}$ on the random, i.e. realization dependent, travel time $\tau_0(z)$ and only then do we calculate its variance. This sequence is essential for the theory to hold: the variance of the transmitted pressure centered around the effective travel time τ^e is non-zero as a result of the random fluctuations in the difference between μ and μ^s, even when $\mu = 0$. (see equation (13), and Section 8 for a more detailed discussion of this fact).

In view of our interest in the transmitted pressure for times t near the arrival time $-\tau_0(z)$ we introduce a time-rescaled version of this quantity

$$p^{\epsilon}_{\rm TR}(z,s) \equiv p_{\rm TR}(z, \epsilon s - \tau_0(z)) = \frac{\zeta_1^{1/2}(z)}{2\pi}\int e^{-i\omega s}\Theta(0,\omega)\hat{f}(\omega)\,d\omega. \tag{38}$$

We note that the dependence of $p^{\epsilon}_{\rm TR}$ on ϵ is only through Θ. Thus, to find the first and second moments of $p^{\epsilon}_{\rm TR}$ we have to analyze the , , Θ-system of equations (28),(33).

6 Moment equations

The variance of the time-scaled transmitted pressure $p^{\epsilon}_{\rm TR}$ may be found from the , , Θ-equations (28), (22) and the integral representation formula (38) in two steps:

- find the equation for the limit of the mean of the transmission coefficient at a single frequency $\lim_{\epsilon\to 0}\langle\Theta(\cdot,\omega)\rangle$. The solution of this equation evaluated at $z = 0$ yields the mean of the transmitted pressure $\lim_{\epsilon\to 0}\langle p^{\epsilon}_{\rm TR}(-L_1,s)\rangle$,
- find the equation for the limit of the mean of the product of the transmission coefficient evaluated at two different frequencies $\lim_{\epsilon\to 0}\langle\Theta(\cdot,\omega_1)\Theta(\cdot,\omega_2)\rangle$. The solution of this equation evaluated at $z = 0$ yields the second moment of the transmitted pressure $\lim_{\epsilon\to 0}\langle p^{\epsilon}_{\rm TR}(-L_1,s)^2\rangle$,

Both steps are similar. To find the equation for $\lim_{\epsilon\to 0}\langle\Theta(\cdot\omega)\rangle$ we analyze the , , Θ-system of equations (28), (33), which we rewrite here in a compact form:

$$\frac{dX^{\epsilon}}{dz} = \frac{1}{\epsilon}F_X\left(z, q(\frac{z}{\epsilon^2}), \frac{\bar{\tau}(z)}{\epsilon}, X^{\epsilon}, T^{\epsilon}\right) + G_X\left(z, q(\frac{z}{\epsilon^2}), \frac{\bar{\tau}(z)}{\epsilon}, X^{\epsilon}, T^{\epsilon}\right), \tag{39}$$

where X^{ϵ} is the vector $[, , \Theta]$. The operators F_X, G_X are defined to be

$$F_X(z,q,\bar{\tau},X,T) = i\begin{pmatrix} \omega[ne(-\omega) + 2m, \ + ne(\omega), \ ^2] \\ \\ \omega[n^{\star}e(\omega)\Theta, \ + m^{\star}\Theta] \end{pmatrix} \tag{40}$$

and

$$G_X(z,q,\bar{\tau},X,T) = \frac{1}{2\zeta_1}\frac{d\zeta_1}{dz}\begin{pmatrix} e(-\omega) - e(\omega), \ ^2 \\ \\ -e(\omega)\Theta, \end{pmatrix}, \tag{41}$$

where

$$e(\omega) = e^{2i\omega(\bar{\tau}+T)}, \tag{42}$$

$q = [\mu, \nu, \eta]$ is a three-dimensional random process defining the medium and the correction to the travel time (see (3),(4),(5) and (13)), $\bar{\tau}$ is the deterministic part τ^e of the travel time τ, and T^{ϵ} is the scaled random part of the travel time τ defined by (13)

$$T^{\epsilon}(z) = \frac{1}{\epsilon}\int_{-L}^{z}\gamma_1(s)\mu(s,\frac{s}{\epsilon^2})\,ds. \tag{43}$$

The subscript X on F_X, G_X derives from the significance of F_X, G_X in the larger system (64), (65). The operators F_X, G_X have special properties: F_X has mean zero with respect to its second, random, rapidly varying, argument $q = q(z/\epsilon^2) = [\mu(z/\epsilon^2), \nu(z/\epsilon^2), \eta(z/\epsilon^2)]$ and both F_X and G_X depend on the deterministic rapidly varying argument $\bar{\tau}(z)/\epsilon$ through a periodic function. The asymptotic behavior of this type of equation is well understood [13]. In the Appendix we summarize the relevant theory that yields the following result:

$$M_\Theta(z) = \lim_{\epsilon \to 0} \langle \Theta(z, \omega) \rangle \tag{44}$$

exists and satisfies the following ordinary differential equation:

$$\frac{dM_\Theta(z)}{dz} = -\omega^2[\overline{mm}(z) + \overline{nn}(z)]1^\star \, M_\Theta(z),$$

$$M_\Theta(-L) = 1, \tag{45}$$

where

$$\overline{mm}(z) = \int_0^\infty \mathrm{E}\{m(z,0)m(z,r)\}\, dr,$$

$$\overline{nn}(z) = \int_0^\infty \mathrm{E}\{n(z,0)n(z,r)\}\, dr. \tag{46}$$

Therefore, we obtain immediately that the limit of the mean of the transmission coefficient at a single frequency evaluated at $z = 0$ is given by

$$M_\Theta(0) = \exp\Big\{-\int_{-L_1}^{0} \omega^2[\overline{mm}(z) + \overline{nn}(z)]\, dz\Big\}. \tag{47}$$

This concludes Step 1.

To find the equation for $\lim_{\epsilon\to 0}\langle \Theta(\cdot, \omega_1)\Theta(\cdot, \omega_2)\rangle$ we analyze the 3-dimensional system of equations for $,_1 = ,(\omega_1)$, $,_2 = ,(\omega_2)$, and $\Phi = \Theta(\omega_1)\Theta(\omega_2)$. This system may be written in a form analogous to (39), where now X^ϵ is a vector $[,_1, ,_2, \Phi]$. The new operators F_X, G_X are defined to be

$$F_X(z, q, \bar{\tau}, X, T) = i \begin{pmatrix} \omega_1[ne(-\omega_1) + 2m,_1 + ne(\omega_1),_1^2] \\ \\ \omega_2[ne(-\omega_2) + 2m,_2 + ne(\omega_2),_2^2] \\ \\ \omega_1[n^\star,_1 e(\omega_1) + m^\star]\Phi + \omega_2[n^\star,_2 e(\omega_2) + m^\star]\Phi \end{pmatrix}, \tag{48}$$

and

$$G_X(z, q, \bar{\tau}, X, T) = \frac{1}{2\zeta_1}\frac{d\zeta_1}{dz} \begin{pmatrix} e(-\omega_1) + e(\omega_1),_1^2 \\ \\ e(-\omega_2) + e(\omega_2),_2^2 \\ \\ -[,_1 e(\omega_1) - ,_2 e(\omega_2)]\Phi \end{pmatrix}, \tag{49}$$

The operators defined in (48) and (49) have the same properties as the ones defined in (40) and (41). Applying once again the theory summarized in the Appendix we obtain:

$$M_\Phi(z) = \lim_{\epsilon\to 0}\langle \Theta(z, \omega_1)\Theta(z, \omega_2)\rangle \tag{50}$$

exists and satisfies the following ordinary differential equation

$$\frac{dM_\Phi(z)}{dz} = -[(\omega_1^2+\omega_2^2)\overline{nn}\,1^\star + (\omega_1+\omega_2)^2\overline{mm}\,1^\star]M_\Phi(z),$$

$$M_\Phi(-L) = 1, \tag{51}$$

where $\overline{mm}$, $\overline{nn}$ are defined in (46). Thus, we obtain a simple formula for the limit of the mean of the product of transmission coefficient evaluated at two different frequencies, $\lim_{\epsilon\to 0}\langle \Theta(\cdot,\omega_1)\Theta(\cdot,\omega_2)\rangle$:

$$M_\Phi(0) = \exp\Big\{-\int_{-L_1}^{0}[(\omega_1^2+\omega_2^2)\overline{nn} + (\omega_1+\omega_2)^2\overline{mm}]\,dr\Big\} \tag{52}$$

Combining (47) with (52) we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0}\langle p^\epsilon_{\mathrm{TR}}(-L_1,s)^2\rangle - \langle p^\epsilon_{\mathrm{TR}}(-L_1,s)\rangle^2 = \\
&\quad = \frac{\zeta_1(-L_1)}{(2\pi)^2}\int\int e^{-i(\omega_1+\omega_2)s}\hat{f}(\omega_1)\hat{f}(\omega_2)\times \\
&\qquad \times\exp\Big\{-\int_{-L_1}^{0}(\omega_1^2+\omega_2^2)(\overline{nn}(z)+\overline{mm}(z))\,dz\Big\}\times \\
&\qquad \times\Big[\exp\Big\{-2\omega_1\omega_2\int_{-L_1}^{0}\overline{mm}(z)\Big\}-1\Big]\,d\omega_1 d\omega_2.
\end{aligned} \tag{53}$$

Now, if μ is taken to be

$$\mu^{\mathrm{s}}(z,\frac{z}{\epsilon^2}) = \frac{1}{2}\gamma_1(z)[\eta(z,\frac{z}{\epsilon^2})+\nu(z,\frac{z}{\epsilon^2})], \tag{54}$$

then function $\overline{mm}(z)$ is identically zero, and therefore we infer from (53) that the variance of the transmitted pressure p^ϵ_{TR} is zero. In that case formula (47) yields the following expression for the stable waveform

$$\frac{\zeta_1^{1/2}(-L_1)}{2\pi}\int e^{-i\omega s}\hat{f}(\omega)\exp\Big\{-\int_{-L_1}^{0}\omega^2\overline{nn}(z)\,dz\Big\}\,d\omega, \tag{55}$$

which is exactly the same as the waveform (16) of the Theorem. Moreover, changing variables $\omega_1\to\omega_1$, $\omega_2\to-\omega_2$ in (53) and expanding

$$\exp\Big\{2\omega_1\omega_2\int_{-L_1}^{0}\overline{mm}(z)\,dz\Big\}$$

in an exponential series, we obtain that if $\int_{-L_1}^{0}\overline{mm}(z)\,dz$ is different from zero each term in the series is strictly positive, and consequently the variance of the transmitted pressure (53) is strictly positive. However, the time lag between the centroids of the transmitted pressure observed according to the travel time τ, defined respectively by an arbitrary μ and μ^{s} of the Theorem, converges to a Gaussian random variable with variance $\int_{-L_1}^{0}\overline{mm}(z)\,dz$. Thus, the pulse does not stabilize in any frame significantly different from the one defined by μ^{s}, i.e. when $\int_{-L_1}^{0}\overline{mm}(z)\,dz>0$ and the time lag between frames is almost surely non zero; But it stabilizes in any frame effectively identical to the one defined by μ^{s}, i.e. when $\int_{-L_1}^{0}\overline{mm}(z)\,dz=0$ and the time lag between frames is almost surely 0.

7 Reference frame moving with effective slowness

The stabilization of the wavefront is visible only in the coordinate frame F^r moving according to the random travel time τ (13). In a frame F' moving according to a travel time different from τ, the fluctuations in the time lag between frames F^r and F' disguise the phenomenon of stabilization to the deterministic waveform (16). It is particularly interesting to observe this problem for the frame F^e which moves with the effective travel time τ^e (19). We rewrite the limiting waveform formula (16) as

$$p_{\mathrm{TR}}(-L_1, t+\gamma_1 L_1) \simeq \frac{\zeta_1^{1/2}}{2\pi} \int e^{-i\omega t/\epsilon} \hat{f}(\omega) e^{-i\omega T^\epsilon} e^{-\frac{1}{2}V^-(-L_1)\omega^2}\, d\omega \tag{56}$$

where (cf (13))

$$T^\epsilon = \frac{1}{\epsilon}\int_0^{-L_1} \frac{1}{2}\gamma_1(s)(\eta(s/\epsilon^2)+\nu(s/\epsilon^2))\, ds \tag{57}$$

and the limiting distribution of T^ϵ is known to be Gaussian with mean zero and variance $V^+(-L_1)$ where

$$V^\pm(z) = \int_z^0 \frac{1}{2}\gamma_1^2(\xi) \int_0^\infty \mathrm{E}\{(\eta\pm\nu)(\xi,0)(\eta\pm\nu)(\xi,r)\}\, dr\, d\xi. \tag{58}$$

Now, using formula (56) and the expression for the characteristic function of a Gaussian random variable $\mathcal{N}$ with mean zero and variance v, $\mathrm{E}e^{i\omega\mathcal{N}} = e^{-\omega^2 v/2}$, we calculate the mean of the transmitted field when centered around the effective travel time $-\tau^e(-L_1) = \gamma_1 L_1$. It is

$$\langle p_{\mathrm{TR}}(-L_1, t+\gamma_1 L_1)\rangle \simeq \frac{\zeta_1^{1/2}}{2\pi} \int e^{-i\omega t/\epsilon} \hat{f}(\omega) e^{-\frac{1}{2}(V^+(-L_1)+V^-(-L_1))\omega^2}\, d\omega. \tag{59}$$

Comparing the coefficients of ω^2 in the exponents in of formulas (59) and (16) we see a discrepancy of $\frac{1}{2}V^+$. It is the factor $e^{-i\omega T^\epsilon}$ in (56) which increases the variance of the modifying Gaussian kernel. This factor corresponds to fluctuations in the time lag between coordinate frames F^r and F^e. In fact the pulse shape (59) is the convolution in time of the stable pulse shape (16) with the probability density of the travel time difference (57) between the effective travel time τ^e and the stabilizing travel time. This shows again that waveform stabilization is not observed in any frame other than F^r.

We note also that in the special case of constant density, i.e. $\eta \equiv 0$, and fluctuating bulk compliance, for constant mean parameters $\rho_1(z) \equiv \rho_1, K_1(z) \equiv K_1$, formula (59) reduces to an expression originally found in [1] (formula (3.183)).

8 Numerical experiments

We conducted a series of numerical experiments to illustrate the approximate pulse-evolution theory. We considered a Goupillaud medium, without absorption, where the density is the reciprocal of the bulk compliance, so that the characteristic slowness (2) is unity, and both are stepwise constant functions with discontinuities equally spaced along the depth axis. The evolution equations (1) simplify in that case to a system of algebraic relations for the amplitudes of the up- and down-going waves at the interfaces between consecutive layers (see for details [3]). The exact solution of

(1) generated according to this scheme was compared with the limiting wavefront formula (16). In this experiment we take

$$K^{-1}(z) = \rho^{-1}(z) = e^{\alpha u_i} \quad \text{when } z \in [i\epsilon^2, (i+1)\epsilon^2], \tag{60}$$

where α is a positive constant and $\{u_i\}_{i=0}^{\lfloor \epsilon^{-2} \rfloor}$ is a sequence of independent, random numbers generated according to the uniform distribution in $[-1, 1]$. The constant α controls the size of fluctuations in the properties of the medium. The typical value for α was 0.5 which allows reflection coefficients to take values in $[-0.46, 0.46]$, where $0.46 = \tanh(\frac{1}{2})$. The incident pulse was a scaled Gaussian given by

$$f(t) = e^{-\frac{1}{2}(t/\epsilon)^2/v_I},$$

where the constant v_I is the squared width of the incident pulse in units of ϵ.

In Figures 3,4,5 we illustrate different aspects of our theory. In each figure there are two curves: the observed exact down-going pulse (the wriggling line) and the limiting waveform (the stable line). The time axis is centered around the long-distance travel time, i.e. 0 corresponds to the arrival time $-\tau_0(z)$ and it is scaled in units of length $O(\epsilon)$, i.e. according to the pulse width scale.

Figure 3 illustrates the convergence of the observed pulse to the limiting waveform. We clearly see that as ϵ decreases to 0, the discrepancy between the two curves decays to 0. This picture is typical of all experiments conducted. Figure 4 illustrates the influence of the depth of observations on the shape of the pulse: as depth $-z$ increases the pulse broadens and attenuates. In Figure 5 we observe that a similar effect appears when the magnitude of the fluctuations increases, i.e α increases. This is understandable in view of formula (17): the variance of the Gaussian kernel that modifies the initial pulse increases with both the depth and with the magnitude of the fluctuations in medium parameters.

Finally, in Figure 6 we observe the down-going pulse not just for times near the travel time τ but for much later times as well. The time axis in this figure is scaled according to the pulse width scale as in the previous figures, but now the axis is centered around the first arrival time τ^c. We see that there is a rapidly and randomly fluctuating coda that follows the main pulse. The behavior of the coda, which is clearly not described by our deterministic limiting waveform (16), is studied by the W-equation method in [1].

Appendix

We review briefly the elementary theory that deals with equations of the form

$$\frac{dU^\epsilon}{dz} = \frac{1}{\epsilon} F\left(z, q(\frac{z}{\epsilon^2}), \frac{\bar{\tau}(z)}{\epsilon}, U^\epsilon\right) + G\left(z, q(\frac{z}{\epsilon^2}), \frac{\bar{\tau}(z)}{\epsilon}, U^\epsilon\right), \tag{61}$$

where U^ϵ takes values in some finite dimensional space R^d; $q(z/\epsilon^2)$ represents random, and $\bar{\tau}(z)/\epsilon$ deterministic, rapidly varying coefficients. The stochastic process $q(\cdot)$ is assumed to be stationary, mean zero, with rapidly decaying correlation function. The dependence on $\bar{\tau}$ is through a periodic function and we assume

$$\frac{d\bar{\tau}(z)}{dz} \neq 0.$$

The operator F is assumed to have mean zero with respect to the underlying probability measure:

$$\mathrm{E}\{F(z, q(\xi), \bar{\tau}, U)\} = 0$$

for any $z, \xi, \bar{\tau}, U$.
Then, the solution process U^ϵ converges weakly as $\epsilon \to 0$ to a diffusion process whose infinitesimal generator is given by

$$
\begin{aligned}
(\mathcal{L}_z \phi)(U) = \lim_{S\to\infty} \frac{1}{S} \int_0^S ds \int_0^\infty & dr \mathrm{E}\{F(z, q(0), s, U) \cdot \\
& \nabla (F(z, q(r), s, U) \cdot \nabla \phi(U))\} \\
+ \lim_{S\to\infty} \frac{1}{S} \int_0^S & ds \mathrm{E}\{G(z, q(0), s, U) \cdot \nabla \phi(U)\}.
\end{aligned}
\tag{62}
$$

where $\cdot$ denotes the scalar product. Let $A(z, U)$ and $B(z, U)$ be defined by

$$
\mathcal{L}_z \phi(U) = \frac{1}{2} A(z, U) \cdot \nabla^2 \phi(U) + B(z, U) \cdot \nabla \phi(U).
$$

In the special case when the the drift vector $B(z, U)$ is linear in U: $B(z, U) = B(z)U$, $B(z)$ a linear operator on R^d, the limit of the means $\overline{U} = \lim_{\epsilon\to 0} \mathrm{E} U^\epsilon$ satisfies a linear equation

$$
\frac{d\overline{U}}{dz} = B(z)\overline{U}. \tag{63}
$$

The quantities of interest, M_Θ, M_Φ of (44), (50) are defined to be the limits of the means $\lim_{\epsilon\to 0}\langle \Theta(\omega)\rangle$, $\lim_{\epsilon\to 0}\langle \Theta(\omega_1)\Theta(\omega_2)\rangle$. Therefore, we must find the exact form of the drift linear operator B in (63) corresponding to the equation (39) for X^ϵ, T^ϵ with F_X, G_X defined by (40),(41) and (48),(49), which we repeat here:

$$
\frac{dX^\epsilon}{dz} = \frac{1}{\epsilon} F_X \left(z, q(\frac{z}{\epsilon^2}), \frac{\bar{\tau}(z)}{\epsilon}, X^\epsilon, T^\epsilon\right) + G_X \left(z, q(\frac{z}{\epsilon^2}), \frac{\bar{\tau}(z)}{\epsilon}, X^\epsilon, T^\epsilon\right), \tag{64}
$$

$$
\frac{dT^\epsilon}{dz} = \frac{1}{\epsilon} F_T(z, q(\frac{z}{\epsilon^2})), \tag{65}
$$

where $F_T(z, q) = \gamma_1(z)\mu$.
Equations (64), (65) form a closed system of the form (61) for the vector $[X^\epsilon, T^\epsilon]$. Therefore, if we denote the X_j-coordinate of the drift vector B by $B^j(z, X, T)$ we find

$$
\begin{aligned}
B^j(z, X, T) = & \\
& \lim_{S\to\infty} \frac{1}{S} \int_0^S ds \int_0^\infty dr \mathrm{E}\{F_X(z, q(0), s, X, T) \cdot \nabla_X F_X^j(z, q(r), s, X, T)\} \\
& \qquad + \mathrm{E}\{F_T(z, q(0)) \cdot \nabla_T F_X^j(z, q(r), s, X, T)\} \\
& + \lim_{S\to\infty} \frac{1}{S} \int_0^S ds \mathrm{E}\{G_X^j(z, q(0), s, X, T)\}.
\end{aligned}
\tag{66}
$$

We note here that in the expression (66) all terms with deterministic, rapidly oscillating factors disappear, since

$$
\lim_{S\to\infty} \frac{1}{S} \int_0^S ds e^{\pm 2i\omega(s+T)} = 0.
$$

Consequently, there is no contribution from $F_T \cdot \nabla_T F_X^j$ and G_X^j in (66), for either of the systems defined by (40),(41) or by (48),(49). After a simple calculation we find that the Θ-coordinate of B for the system (40),(41) is given by

$$
B^\Theta(z) = -[\omega^2(\overline{mm}(z) + \overline{nn}(z))]1^\star \Theta, \tag{67}
$$

and the Φ-coordinate of B for the system (48),(49) is given by

$$B^{\Phi}(z) = -[(\omega_1^2 + \omega_2^2)(\overline{mm}(z) + \overline{nn}(z)) + 2\omega_1\omega_2\overline{mm}(z)]1^{\star}\,\Phi. \tag{68}$$

Note that both expression are linear in Θ, Φ, respectively, and they are independent of T. Therefore the ordinary differential equations for the limits of the moments $\lim_{\epsilon\to 0}\langle\Theta(\omega)\rangle, \lim_{\epsilon\to 0}\langle\Theta(\omega_1)\Theta(\omega_2)\rangle$ form closed equations (45),(51), decoupled from the other equations implied by (63).

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