Imaging and time reversal in random media

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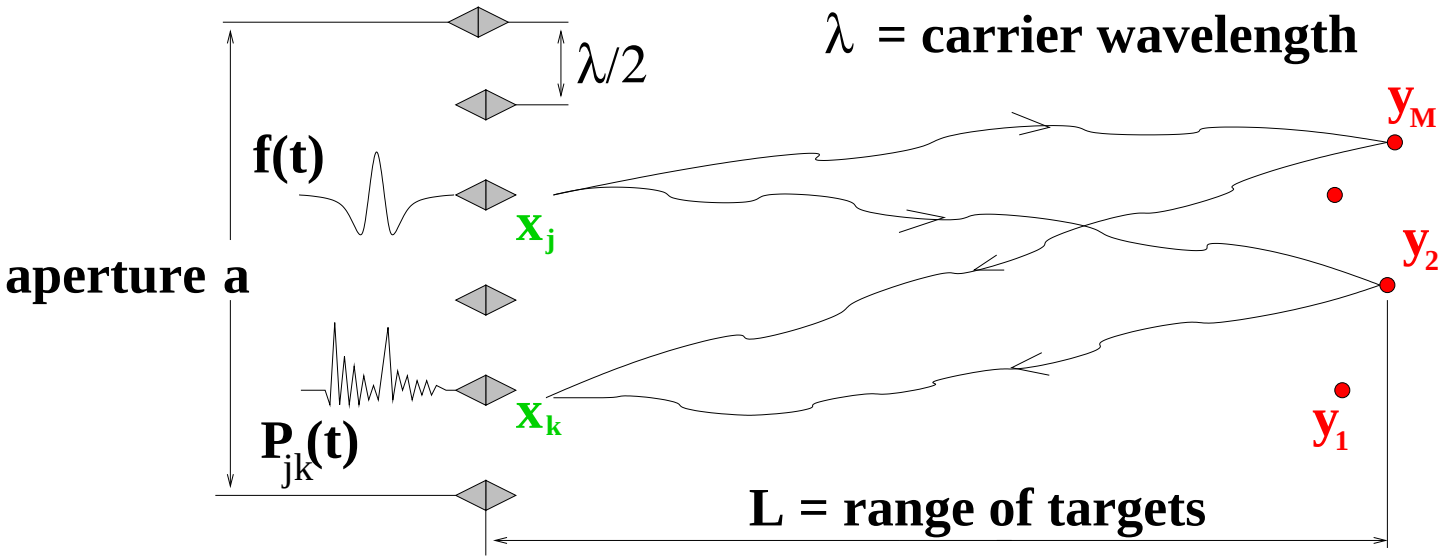
Active array imaging.



An array of N transducers probes a medium with M small scatterers with a pulse $f(t)$ emitted from transducer j, located at $\mathbf{x}_j$, and the echos $P_{jk}(t)$ are recorded at $\mathbf{x}_k$, for $k = 1, \ldots N$.

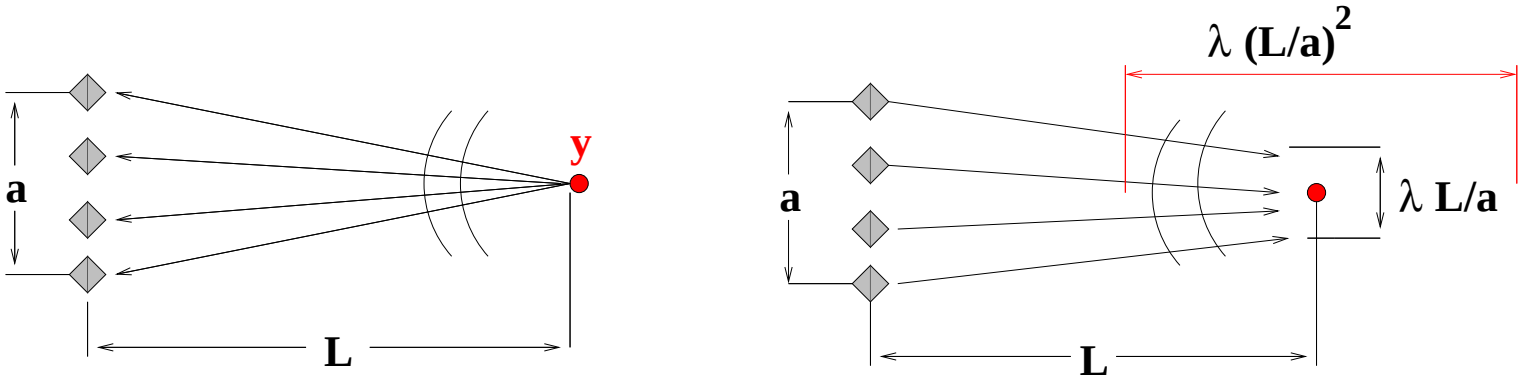
Imaging: From the $N \times N$ response matrix $\left(P_{jk}(t)\right)$ find the number $M \leq N$ of scatterers and their location $\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_M$.

Random media: In the regime $\lambda \leq l \ll a = N\lambda/2 \ll L$, where there is significant **multipathing**, how is imaging affected?.

Imaging strategy

• **Imaging**: Requires some back propagation or **time reversal** of the measured fields into a medium (real or fictitious).

• Need to understand time reversal in random media.



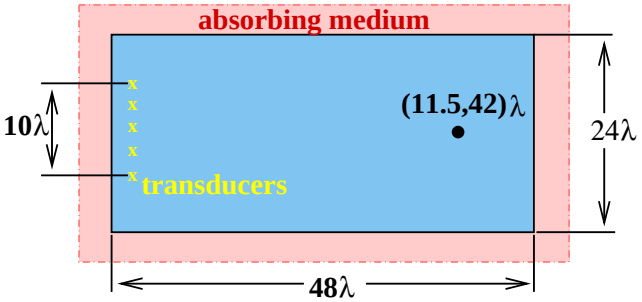
We know that significant **multipathing** in the random medium results in **super-resolution** in time reversal, which is **statistically stabile**.

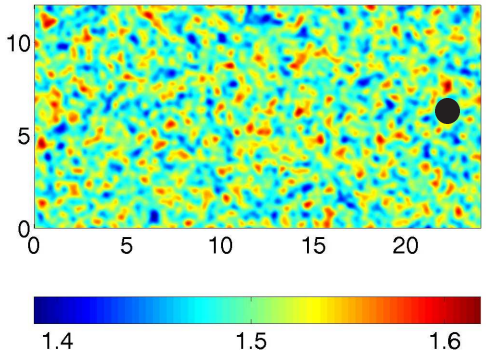
• **Objective in imaging**:
- Exploit super-resolution & statistical stability of time reversal.
- Use the **coherent** (time-resolved) nature of the response matrix.

Applications of imaging in random media

- We are interested in imaging in random media in which there is **significant multipathing**. Some applications are:
  - Foliage penetrating and ground penetrating radar.
  - Identification of underground or underwater mines and structures.
  - Non destructive testing for locating defects in concrete, airplane structures, etc.
  - Secure communication in cluttered environments (not discussed here).

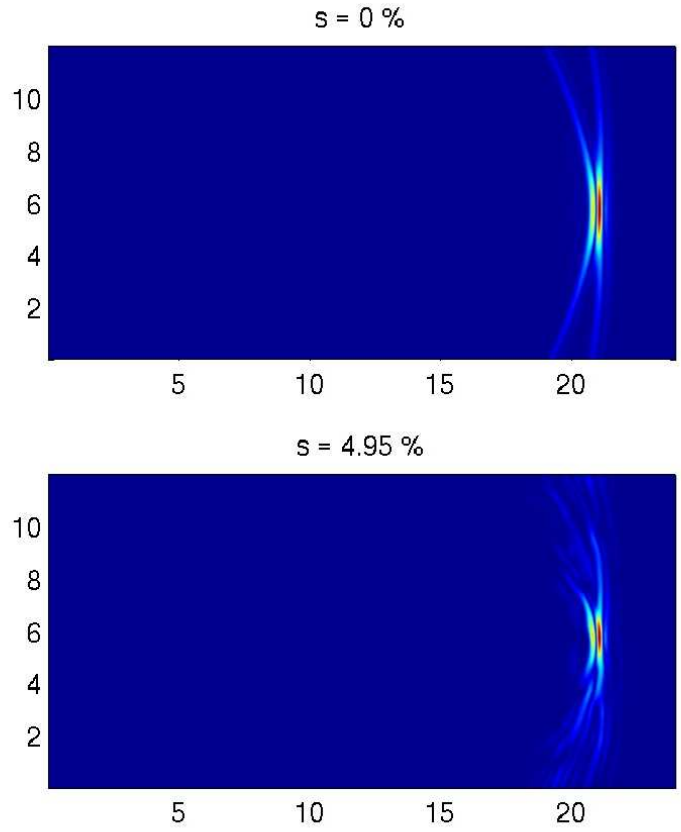
Numerical simulations (FE-TD)





- Targets (one or two) are simulated as soft scatterers (Dirichelt BC at their surface).
- **Ultrasound probing:** $\lambda = 0.5$mm, $a = 2.5$mm (5mm), $c_0 = 1.5$km/s.
- **Underwater sound:** scale lengths by $\sim 10^3$. $\lambda \sim 1$m, $L = 1 - 50$km, $a = 50 - 100$m.
- **Radar** microwaves: scale lengths by $\sim 10^2$. $\lambda = 10 - 30$cm, $L \sim 1$km, $a = 1 - 10$ m.

Simulation of super-resolution.

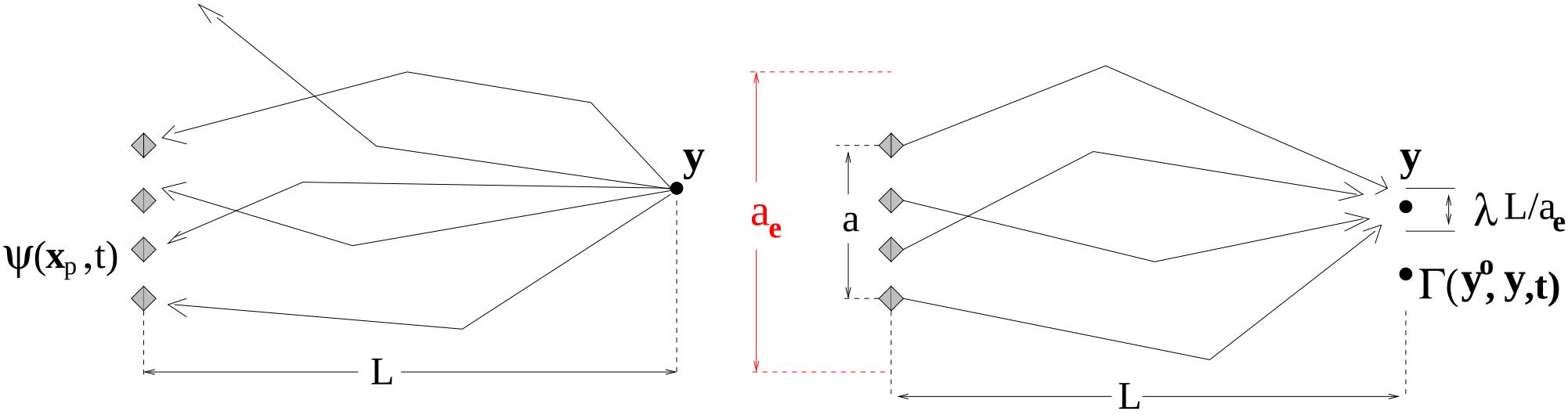


s = standard deviation of the fluctuations of sound speed.

Time reversal in random media. Multipathing.

• A point target at $\mathbf{y}_1$ radiates onto the array. The recorded signal is time reversed (conjugated in Fourier domain) and sent into medium. The back propagated field at the observation point $\mathbf{y}$ is

$$\Gamma(\mathbf{y}, \mathbf{y}_1, t) = \int e^{-i\omega t} \sum_{j=1}^{N} \widehat{G}(\mathbf{y}, \mathbf{x}_j, \omega) \overline{\widehat{f}(\omega) \widehat{G}(\mathbf{y}_1, \mathbf{x}_j, \omega)} d\omega$$



• **Super-resolution**: In a random medium, because of multipathing the **effective** aperture $a_e \gg a$ or, $\frac{\lambda L}{a_e} \ll \frac{\lambda L}{a}$.

Self averaging of back propagated field $\Gamma(\mathbf{y}, \mathbf{y}_1, t)$

- In a suitable asymptotic regime, decorrelation of $\widehat{\Gamma}(\mathbf{y}, \mathbf{y}_1, \omega) = \overline{\widehat{f}(\omega)\widehat{G}(\mathbf{y}, \mathbf{x}_j, \omega)\widehat{G}(\mathbf{y}_1, \mathbf{x}_j, \omega)}$ for frequencies $\omega_1 \neq \omega_2$ gives

$$E\left\{\Gamma^2(\mathbf{y}, \mathbf{y}_1, t)\right\} \approx E\left\{\Gamma(\mathbf{y}, \mathbf{y}_1, t)\right\}^2 .$$

- Back propagated field is **self averaging**: $\Gamma(\mathbf{y}, \mathbf{y}_1, t) \approx E\left\{\Gamma(\mathbf{y}, \mathbf{y}_1, t)\right\}$.

- The analytic expression for $\Gamma(\mathbf{y}, \mathbf{y}_1, t)$ is close to the expression for the back propagated field in a homogeneous medium for an array of **effective aperture** $\mathbf{a_e} \gg \mathbf{a}$.

- Self-averaging and super-resolution are observed **in time** only, not for monochromatic fields. Broadband is essential (BPZ-02).

How to do imaging in random media

- We use the statistical stability and the super-resolution of time reversal in random media, for imaging.
- Identify the unknown targets as minima or maxima of objective functions that are self-averaging.

Motivate echo mode passive target array imaging with active target array imaging.

Let $\widehat{\mathbf{p}}(\omega)$ be the N-vector signal received by the array issuing from a single source. To find the location $\mathbf{y}^*$ of that source we use beamforming (frequency domain; not good in random media) or the matched field method (in the time domain; works well in random media).

$$\mathbf{y}^* = \text{argmin}_{\mathbf{y}} F(\mathbf{y}, \tau_{\text{max}}(\mathbf{y}))$$

where

$$F(\mathbf{y}, t) = \mathcal{F}^{-1}\left\{|\widehat{\mathbf{p}}(\omega)^H \widehat{\mathbf{g}}_0(\mathbf{y}, \omega)|^2\right\}$$

Active target imaging; matched field

For a single active source

$$\widehat{\mathbf{p}}(\omega) = \widehat{f}(\omega)\widehat{\mathbf{g}}(\mathbf{y},\omega) \quad , \qquad \widehat{\mathbf{g}}(\mathbf{y},\omega) = \begin{pmatrix} \widehat{G}(\mathbf{y},\mathbf{x}_1;\omega) \\ \vdots \\ \widehat{G}(\mathbf{y},\mathbf{x}_N;\omega) \end{pmatrix}$$

where $\widehat{G}(\mathbf{y},\mathbf{x};\omega)$ is the (random) Green's function. The vector $\widehat{\mathbf{g}}(\mathbf{y},\omega)$ is the radiation vector.

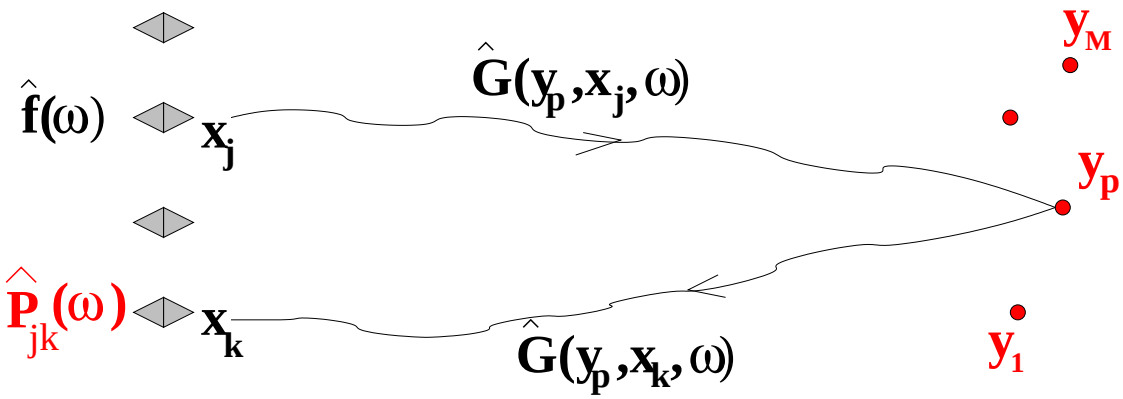
The subscript zero indicates free space (deterministic medium).

Why is the matched field functional statistically stable? It has a time reversal interpretation: Time-autocorrelation of time-reversed, backpropagated field. No arrival time estimation needed for a single active target.

What is the difference between physical Time-Reversal and Imaging? (The subscript zero and the autocorrelation.)

Other methods, like projection, work better in echo mode but need arrival time estimation.

Echo mode array imaging. Forward model of response matrix P, at frequency ω.



$\widehat{P}_{jk}(\omega) =$ sum of fields scattered from all targets $\mathbf{y}_p,\ p = 1, \ldots M$.

We neglect multiple scattering.

- $\widehat{P}(\omega) = \widehat{f}(\omega) \sum_{p=1}^{M} \widehat{\tau}_p(\omega) \widehat{\mathbf{g}}(\mathbf{y}_p, \omega) \widehat{\mathbf{g}}^T(\mathbf{y}_p, \omega)$, $\widehat{\mathbf{g}}(\mathbf{y}_p, \omega) = \begin{pmatrix} \widehat{G}(\mathbf{y}_p, \mathbf{x}_1; \omega) \\ \vdots \\ \widehat{G}(\mathbf{y}_p, \mathbf{x}_N; \omega) \end{pmatrix}$

$\widehat{\tau}_p(\omega) =$scattering coefficient of target p.

- rank $\left[\widehat{P}(\omega)\right] = M \leq N$, except for very special configurations of targets and transducers.

SVD of $\widehat{P}(\omega)$. Statistically stable imaging.

- Why SVD?
- M = number of significant singular values $\sigma_j(\omega)$.
- The left singular vectors $\widehat{\mathbf{V}}_j$, for $j = 1, \ldots M$, are linear combinations of $\widehat{\mathbf{g}}(\mathbf{y}_p, \omega)$, the random, radiating Green's vectors.

Imaging M targets by projection (MUSIC)

For search point $\mathbf{y}$, if projection onto the null space of $\widehat{P}(\omega)$, $\mathcal{P}_N \widehat{\mathbf{g}}(\mathbf{y}, \omega) = 0$, then $\mathbf{y} = \mathbf{y}_p$, for some $p = 1, \ldots M$.

In imaging, $\widehat{\mathbf{g}}(\mathbf{y}, \omega)$ is unknown. Use the deterministic $\widehat{\mathbf{g}}_0(\mathbf{y}, \omega)$ instead with the objective functional (single target):

$$\mathbf{F}(\mathbf{y}, t) = \int e^{-i\omega t} \sigma(\omega) \mathcal{P}_N \widehat{\mathbf{g}}_0(\mathbf{y}, \omega) d\omega.$$

• Minima of $| \mathbf{F}(\mathbf{y}, t) | \Longrightarrow$ information about location of targets.
It works because $\mathbf{F}(\mathbf{y}, t)$ is self-averaging.

Connection between imaging and time reversal

For simplicity of explanation, take one target. It extends to multiple targets, with modifications.

- For a search point $\mathbf{y}$, we calculate

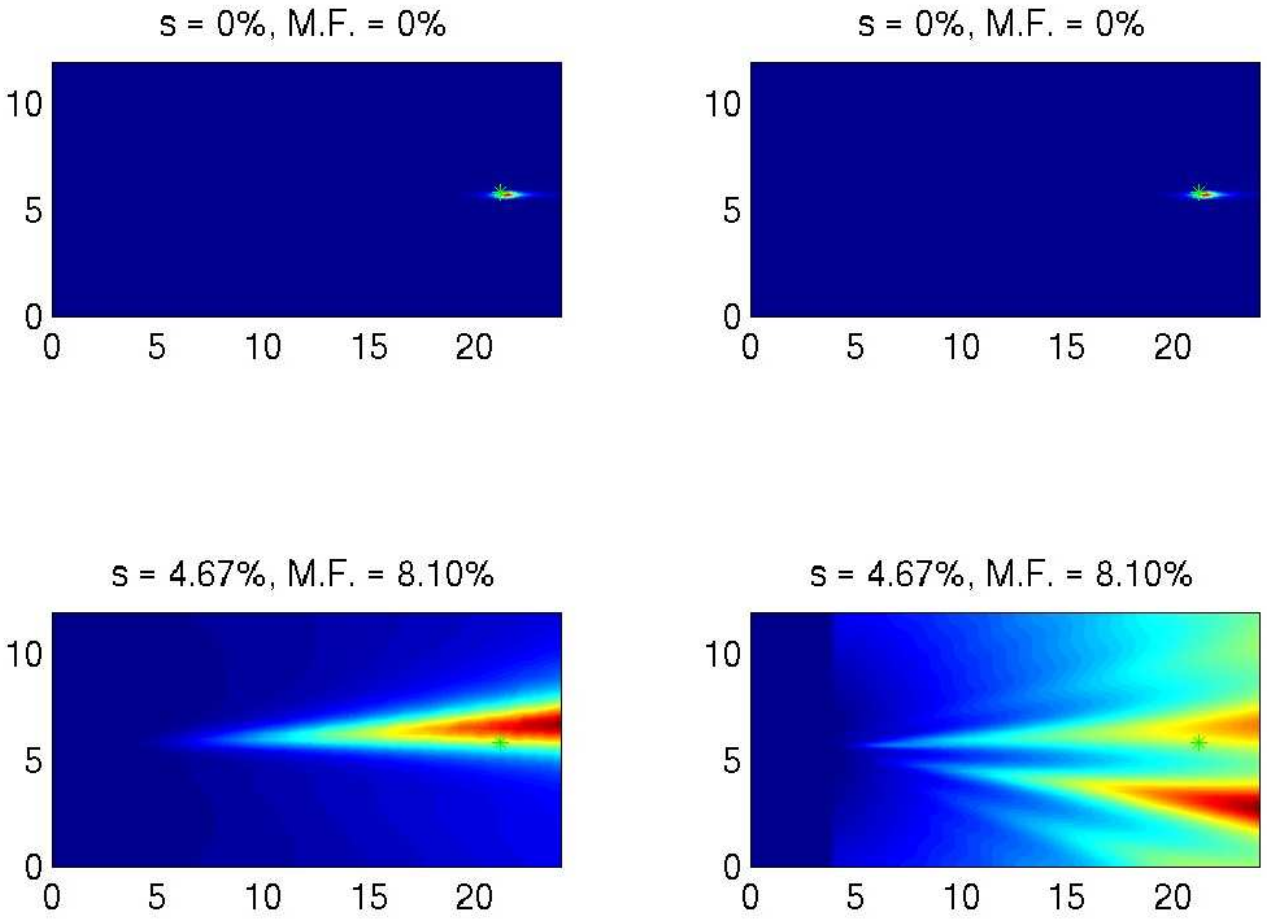
$$\mathbf{F}(\mathbf{y},t) = \int e^{-i\omega t}\sigma(\omega)\left\{\left[\widehat{\mathbf{V}}_1^H(\omega)\hat{\mathbf{g}}_0(\mathbf{y},\omega)\right]\widehat{\mathbf{V}}_1(\omega) - \mid \widehat{\mathbf{V}}_1(\omega)\mid^2 \hat{\mathbf{g}}_0(\mathbf{y},\omega)\right\},$$

  with normalized $\widehat{\mathbf{V}}_1(\omega)$, corresponding to the singular value $\sigma(\omega)$.

Each term in $\mathbf{F}(\mathbf{y},t)$ has a time reversal interpretation and is self-averaging.

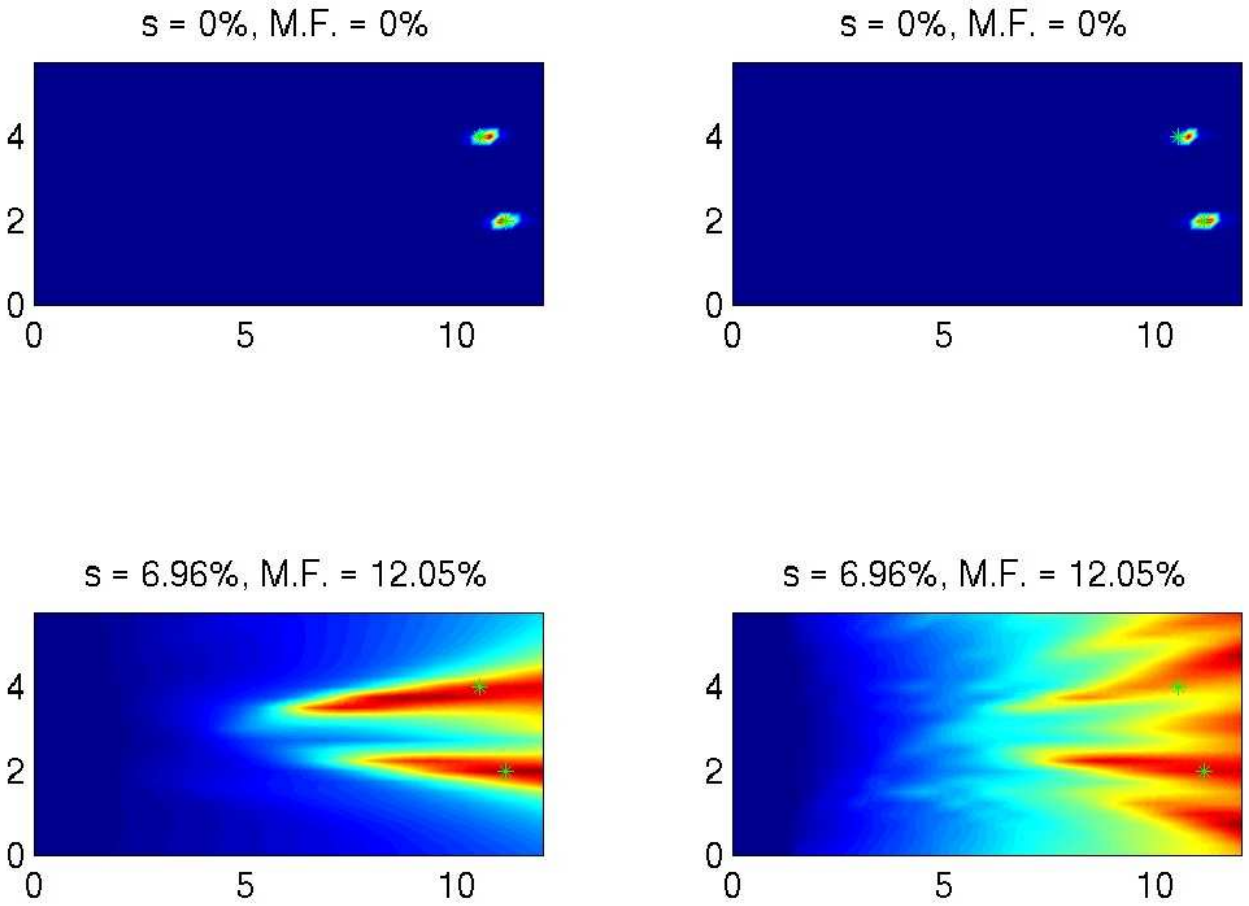
At what time do we evaluate this vector? Try the deterministic arrival times $t_j(\mathbf{y})$. Plot the reciprocal of the norm of this vector as function of $\mathbf{y}$.

Direction of arrival estimation I.



On the left time domain estimation, on the right fixed frequency MUSIC.

Direction of arrival estimation II



On the left time domain estimation, on the right fixed frequency MUSIC.

Sensitivity of time domain direction of arrival estimation to arrival times used.

- Get very bad range estimation when using deterministic arrival times
- Using estimated arrival times improves cross range estimation greatly
- Need for good arrival time estimation

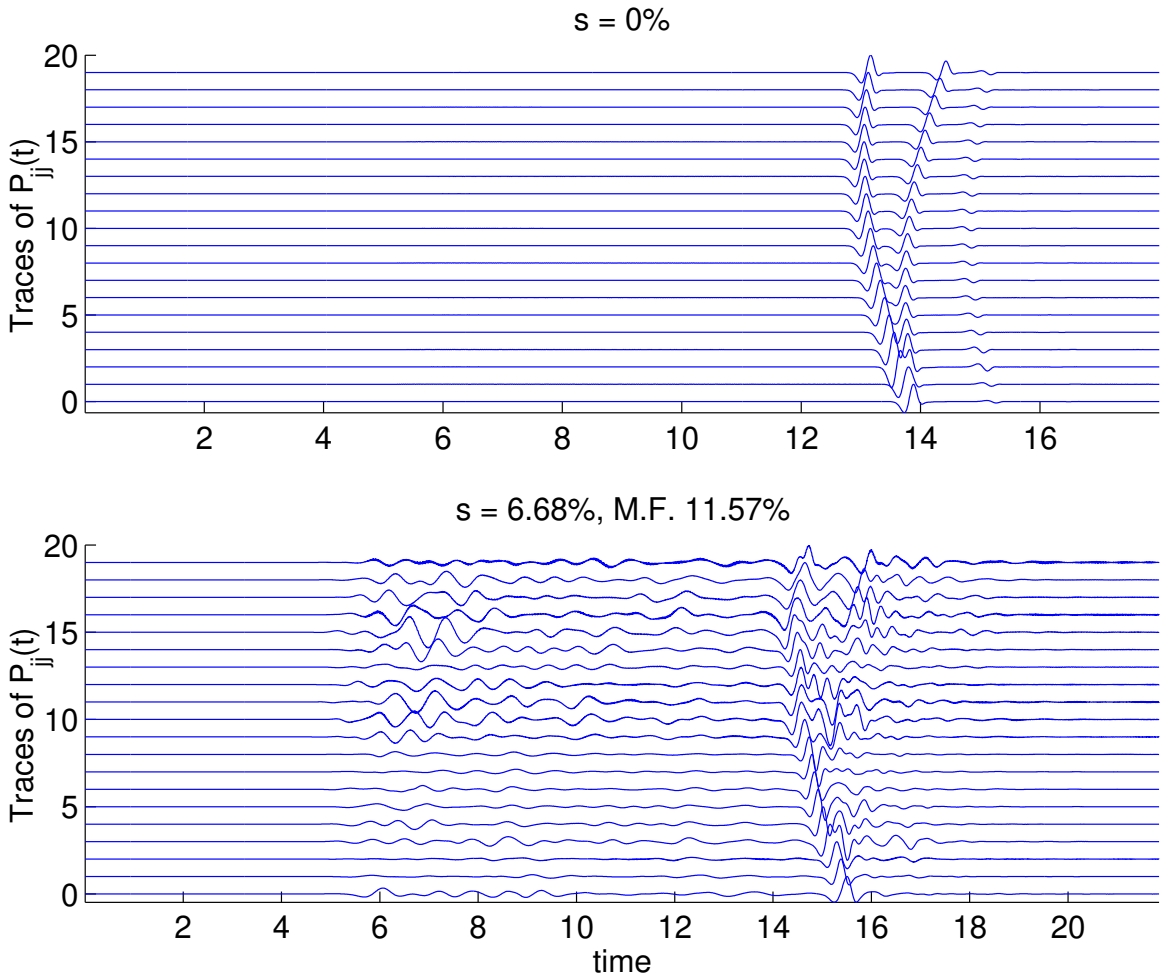
Sensitivity of estimation to arrival time (the conical region) is related to super-resolution in time-reversal.

New phenomenon when imaging in random media: It is NOT possible to separate cross-range and range estimation. They interact.

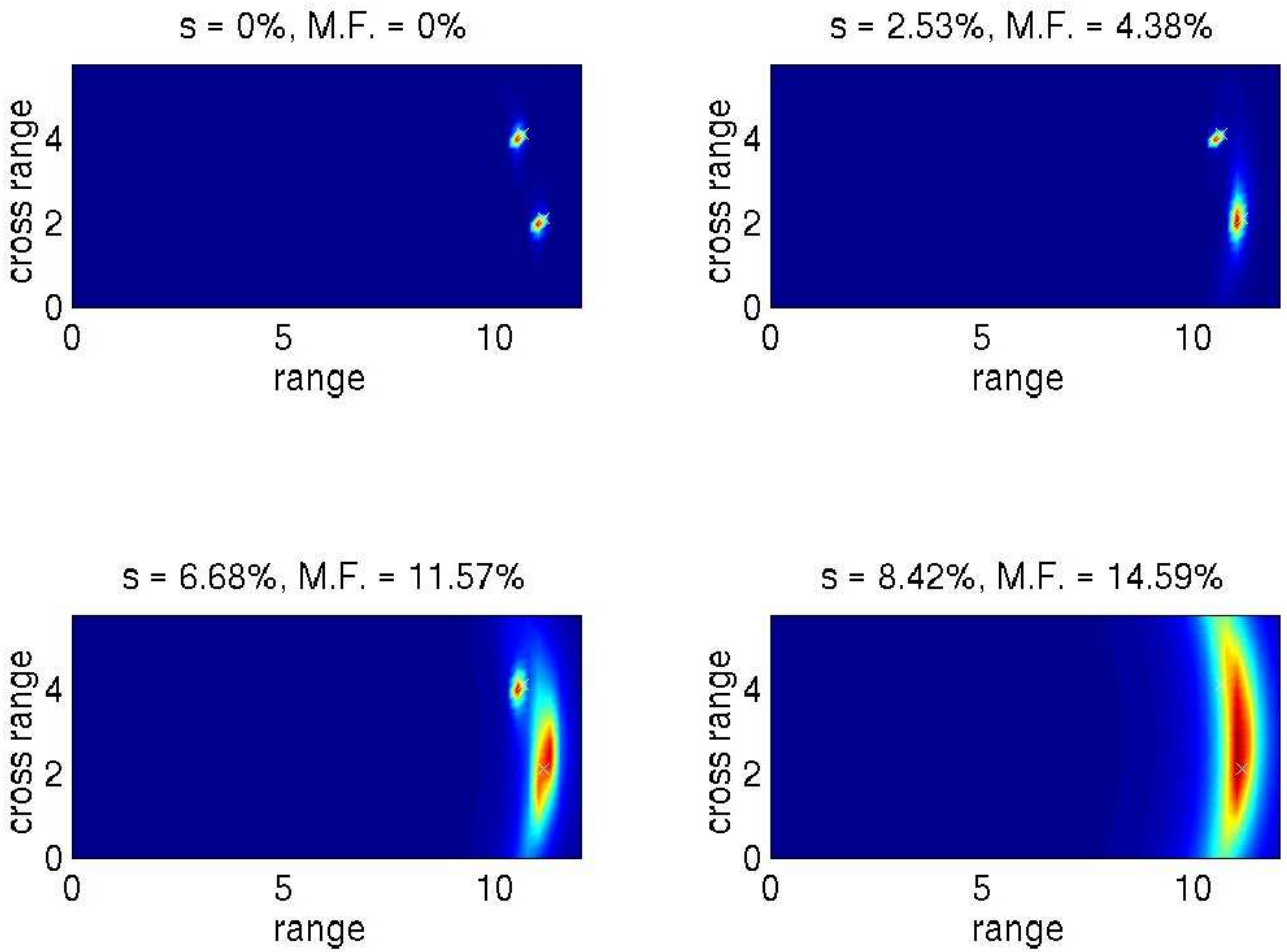
Coherent imaging: target range estimation

The phase (arrival time) in the data gives the range of targets.

Traditional arrival time analysis. SAR



ATA imaging of two targets in a random medium.



Using **only** arrival time information and LSQ fitting of hyperbolas (common in seismic imaging).

Subspace - Arrival Time (SAT) analysis

• Singular vectors are also used to get better range estimates. (Illuminate one target and shade the other).

- SVD gives complex, singular vectors $\widehat{\mathbf{V}}_j(\omega)$ up to arbitrary, $e^{i\phi(\omega)} \Longrightarrow \int e^{-i\omega t}\widehat{\mathbf{V}}_j(\omega)d\omega$ is not coherent in time.

• Calculate N coherent in time vectors:

$$\mathbf{v}_j^{(m)}(t) = \int e^{-i\omega t}\left[\widehat{\mathbf{V}}_j^H(\omega)\cdot\hat{P}(\omega)\mathbf{e}_m\right]\widehat{\mathbf{V}}_j(\omega)d\omega.$$

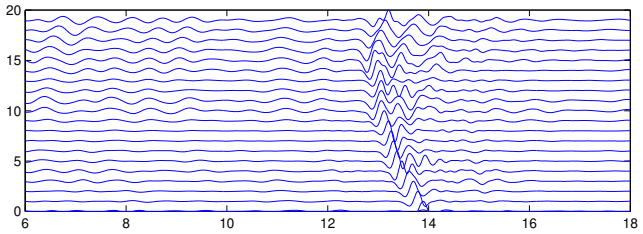
Equivalent to taking m^{th} column of $P(t)$, time reverse it and send it into the medium. Repeat this an infinite number of times.

• Stack the $\mathbf{v}_j^{(m)}(t)$ to get better estimates of arrival times.

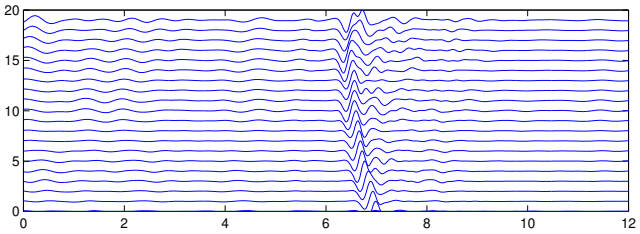
$$\left\langle \mathbf{v}_j(t)\right\rangle = \tfrac{1}{N}\sum_{j=1}^{N}\mathbf{v}_j^{(m)}(t-\tau_m), \qquad \min_{\tau_m}\int_0^T\sum_{m=1}^{N}\left[\mathbf{v}_j^{(m)}(t-\tau_m)-\left\langle \mathbf{v}_j(t)\right\rangle\right]^2.$$

Traces for random medium with st. dev. $s = 6.68\%$

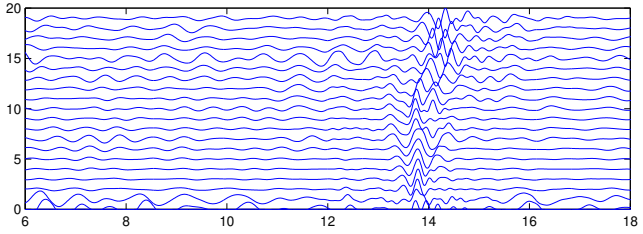
(a1)



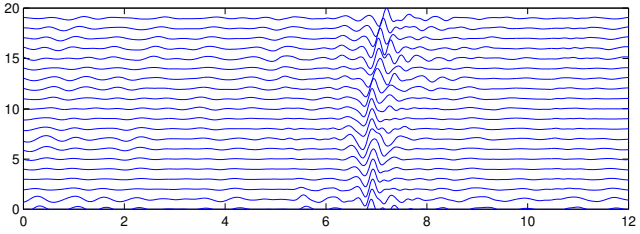
(b1)



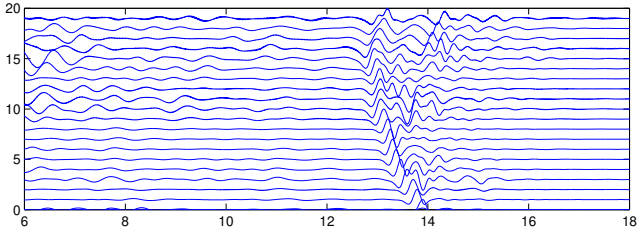
(a2)



(b2)



(c)



(a1) traces of m^{th} component of $\mathbf{v}_1^{(m)}(t)$, $m = 1, \ldots N$.

(a2) traces of m^{th} component of $\mathbf{v}_2^{(m)}(t)$, $m = 1, \ldots N$.

(b1) and (b2): traces of $\langle \mathbf{v}_1(t) \rangle$ and $\langle \mathbf{v}_2(t) \rangle$.

(c) traces of $P_{jj}(t)$.

The SAT estimation

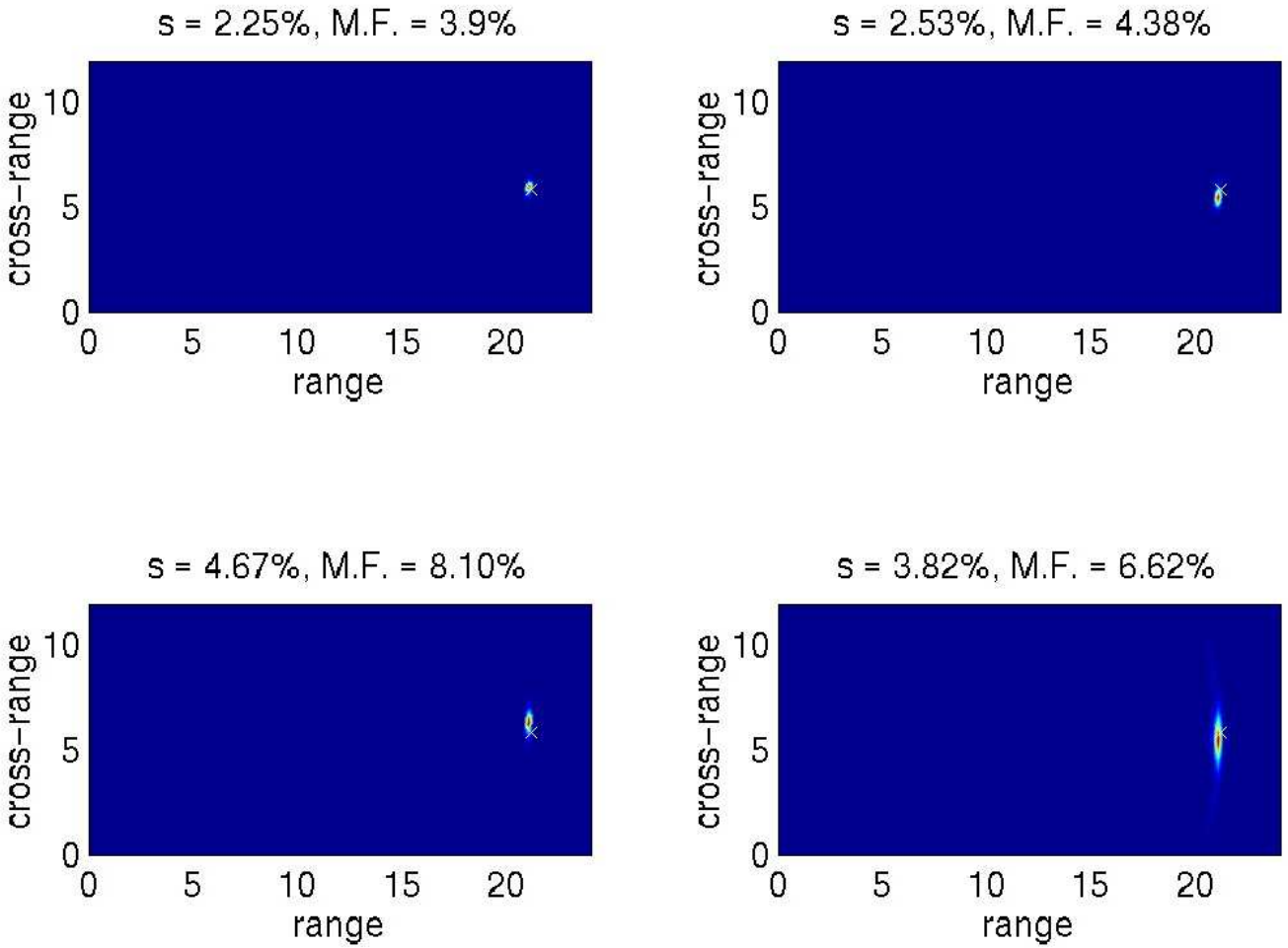
- The SAT objective function is basically an intersection between SVD direction of arrival estimation and subspace arrival time analysis.

- The D-o-A estimation gives good cross-range resolution of the targets.

- The subspace arrival time analysis gives good range resolution of each target.

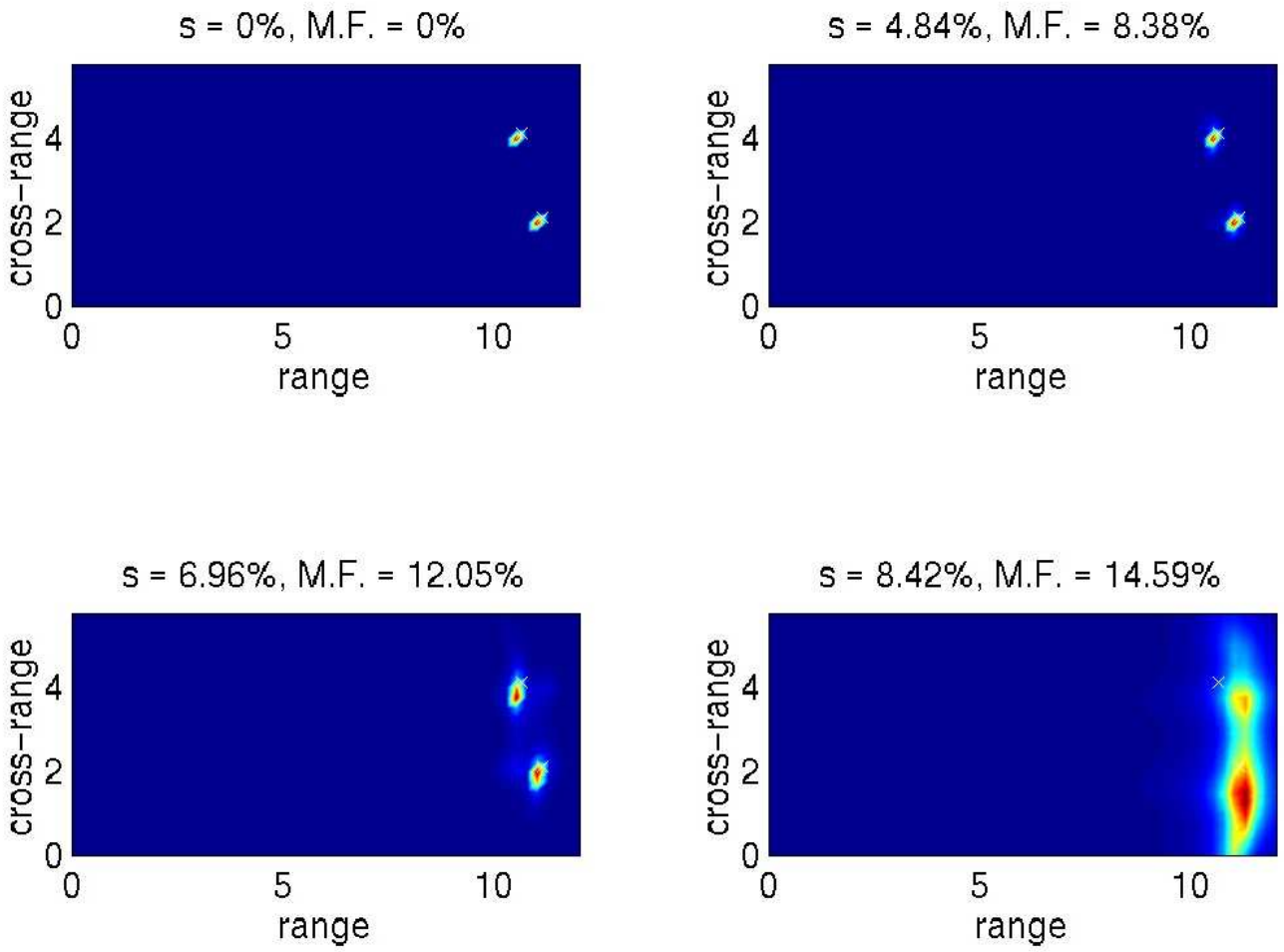
- The intersection gives good cross-range and range resolution.

SAT imaging of one target in a random medium



s = standard deviation, M.F. = max. fluctuations of sound speed

SAT imaging of two targets in a random medium



s = standard deviation, M.F. = max. fluctuations of sound speed

Why SAR does not work in a random medium

- SAR is when only the **diagonal** $P_{jj}(t)$ of the response matrix is measured.

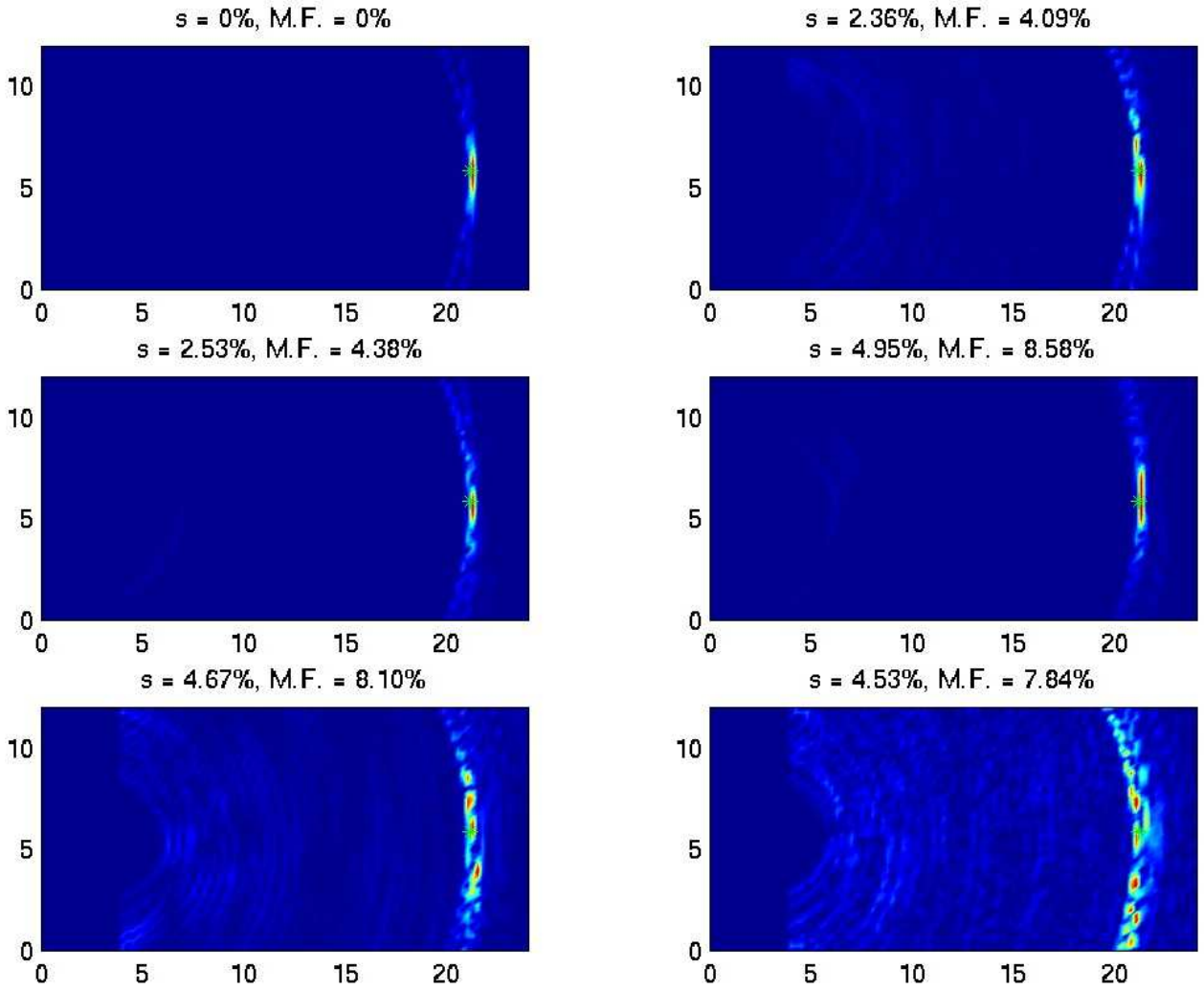
- Maxima of

$$\mathcal{R}(\mathbf{y}^s) = \sum_{j=1}^{N} P_{jj}(\frac{2|\mathbf{x}_j - \mathbf{y}^s|}{c_0})$$

  give target estimates.

- No projection, no 'time reversal', no statistical stability.

One target SAR in a random medium

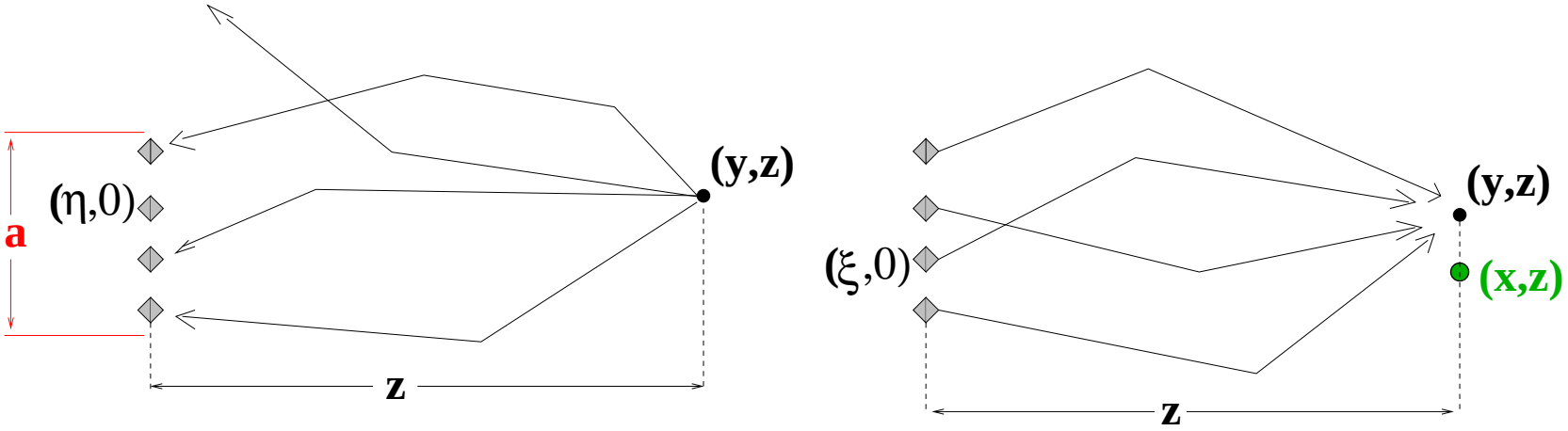


Analysis of statistical stability

• Our success in imaging in random media is due to the **self-averaging** property of the objective functionals.

• The random part of the functionals is of the form

$$\Gamma(z;\mathbf{x},\mathbf{y};\xi,\eta,t)=\int e^{-i\omega t}\widehat{f}(\omega)\widehat{G}(z,0;\mathbf{x},\xi;\omega)\overline{\widehat{G}}(z,0;\mathbf{y},\eta;\omega)d\omega$$



$|\,\mathbf{x}\,|$, $|\,\mathbf{y}\,|$ are $O(a)$, $|\,\mathbf{x}-\mathbf{y}\,|$ is small and $z\sim L$, the propagation distance.

Analysis of statistical stability

• For long propagation distances L and weak sound velocity fluctuations (forward scattering), use the parabolic Green's function:

$$\begin{cases} 2ik\frac{\partial \widehat{G}}{\partial z} + \Delta_{\mathbf{x}}\widehat{G} + k^2\sigma\mu\left(\frac{\mathbf{x}}{l}, \frac{z}{l}\right)\widehat{G} = 0, \text{ for } z > 0, \\ \\ \widehat{G} = \delta(\mathbf{x} - \xi) \text{ at } z = 0. \end{cases}$$

σ is the standard deviation of the fluctuations $\mu(\mathbf{x}, z) = \frac{c_0^2}{c(\mathbf{x},z)^2} - 1$.

• In a regime $L \gg a \gg l$ and weak fluctuations (σ small) $\rightarrow$ the white noise limit (convergence in law):

$$d\widehat{G} = \left[\frac{i}{2k}\Delta_{\mathbf{x}}\widehat{G} - \frac{k^2}{8}R_0(0)\widehat{G}\right]dz + \frac{ik}{2}\widehat{G}\; dB(\frac{\mathbf{x}}{l}, z), \text{ for } z > 0.$$

• $B(\mathbf{x}, z)$ is a field of Brownian motions in z, with covariance

$$E\left\{B(\mathbf{x}_1, z_1)B(\mathbf{x_2}, z_2)\right\} = R_0(\mathbf{x}_1 - \mathbf{x}_2)z_1 \bigwedge z_2,$$

$$R_0(r) = \int R(r, z)dz, \; R(r, z) = E\left\{\mu(r + r', z + z')\mu(r, z)\right\}.$$

The white noise limit

$$\begin{cases} d\widehat{\Gamma} &= \left\{ \frac{i}{2k} \left(\Delta_{\mathbf{x}} - \Delta_{\mathbf{y}} \right) \widehat{\Gamma} + \frac{k^2}{4} \left[R_0(\frac{\mathbf{x}}{l} - \frac{\mathbf{y}}{l}) - R_0(0) \right] \widehat{\Gamma} \right\} dz + \\ & \\ & \frac{ik}{2} \widehat{\Gamma} \left[dB(\frac{\mathbf{x}}{l}, z) - dB(\frac{\mathbf{y}}{l}, z) \right], \text{ for } z > 0, \\ & \\ \widehat{\Gamma} &= \delta(\mathbf{x} - \xi)\delta(\mathbf{y} - \eta), \quad \text{at } z = 0. \end{cases}$$

• We are interested in points $\mathbf{x}$ and $\mathbf{y}$ that are close together, such that $\mathbf{Y} = \mathbf{y} - \mathbf{x}$ satisfies $\mid \mathbf{Y} \mid \ll l$.

• Also, if $\mathbf{X} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$ is the center point, then $\mid \mathbf{X} \mid \gg l$. In particular, $\frac{l}{a} = \epsilon \ll 1$.

• Take the Fourier transform with respect to Y and define the Wigner distribution:

$$W = \int e^{i\mathbf{P}\cdot\mathbf{Y}} \widehat{\Gamma}(z; \mathbf{X}, \mathbf{Y}; \xi, \eta, k) d\mathbf{Y},$$

Statistical stability

- After a rescaling, we obtain the Ito Liouville equation

$$\begin{cases} dW = \{-\mathbf{P}\cdot\nabla_{\mathbf{X}}W + D\Delta_{\mathbf{P}}W\}\,dz - \frac{1}{2}\nabla_{\mathbf{P}}W\cdot d\nabla_{\mathbf{X}}B(\frac{\mathbf{X}}{k^3\epsilon}, z), \\ \\ W = e^{i\mathbf{P}\cdot(\eta-\xi)}\ \delta\left(\mathbf{X} - \frac{\xi+\eta}{2}\right) \quad \text{at } z = 0; \quad D = -\Delta R_0(\mathbf{X})\mid_{\mathbf{X}=\mathbf{0}}. \end{cases}$$

- Decorrelation of W or $\widehat{\Gamma}$, at different frequencies.

The coupling term in the equation for $W(X,P,z;k_1)W(X,P,z;k_2)$ is in terms of

$$R_0\left(\frac{\mathbf{X}}{k_1^3\epsilon} - \frac{\mathbf{X}}{k_2^3\epsilon}\right) \approx 0, \quad \text{since } \epsilon \ll 1.$$

- Self-averaging in time of $\Gamma = \int d\omega \int d\mathbf{P} e^{-i(\omega t + \mathbf{P}\cdot\mathbf{Y})} W$ follows.

Conclusions and future research

• We have introduced and explored with simulations a theory for super-resolution and statistical stability of time reversal in random media (BPZ, JASA Jan 02).

• We have introduced a coherent array imaging method for target identification in random media, which exploits the super-resolution and statistical stability of time reversal (BBPT, IP submitted 01).

Research plans

- Apply to foliage and ground penetrating radar
- Imaging with very large, moving (synthetic) arrays. Hierarchical, multiscale methods for analyzing very large response matrices.
- Multipathing in wireless for secure communications in a cluttered environment.