Time reversal, imaging and communications in random media

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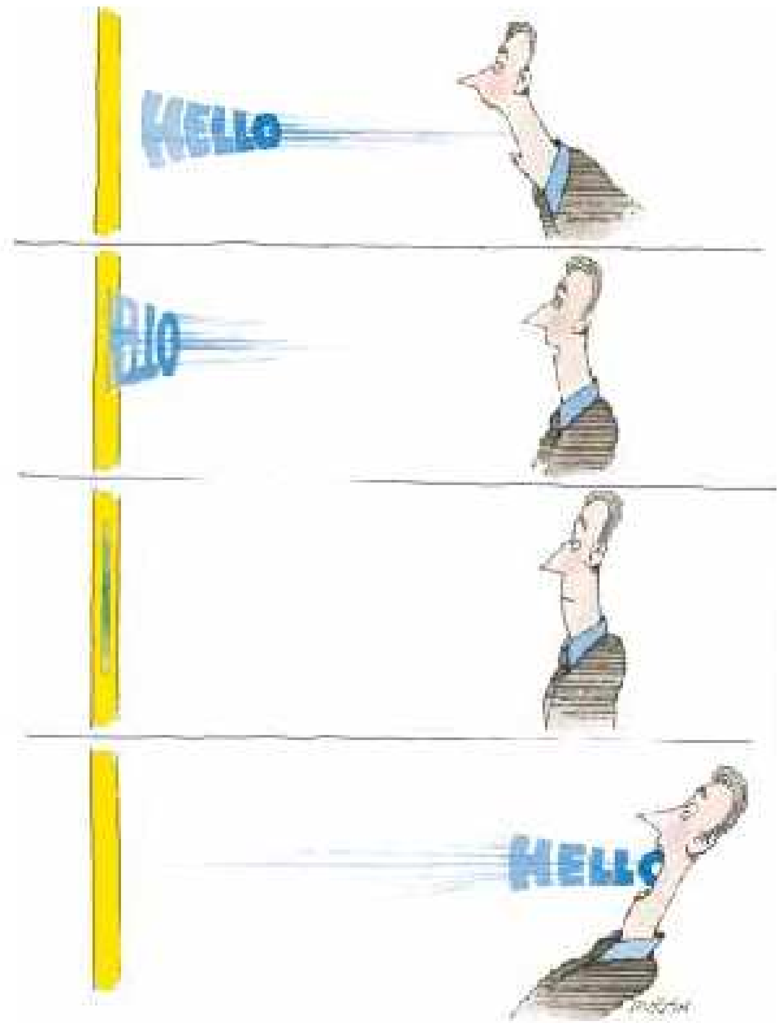
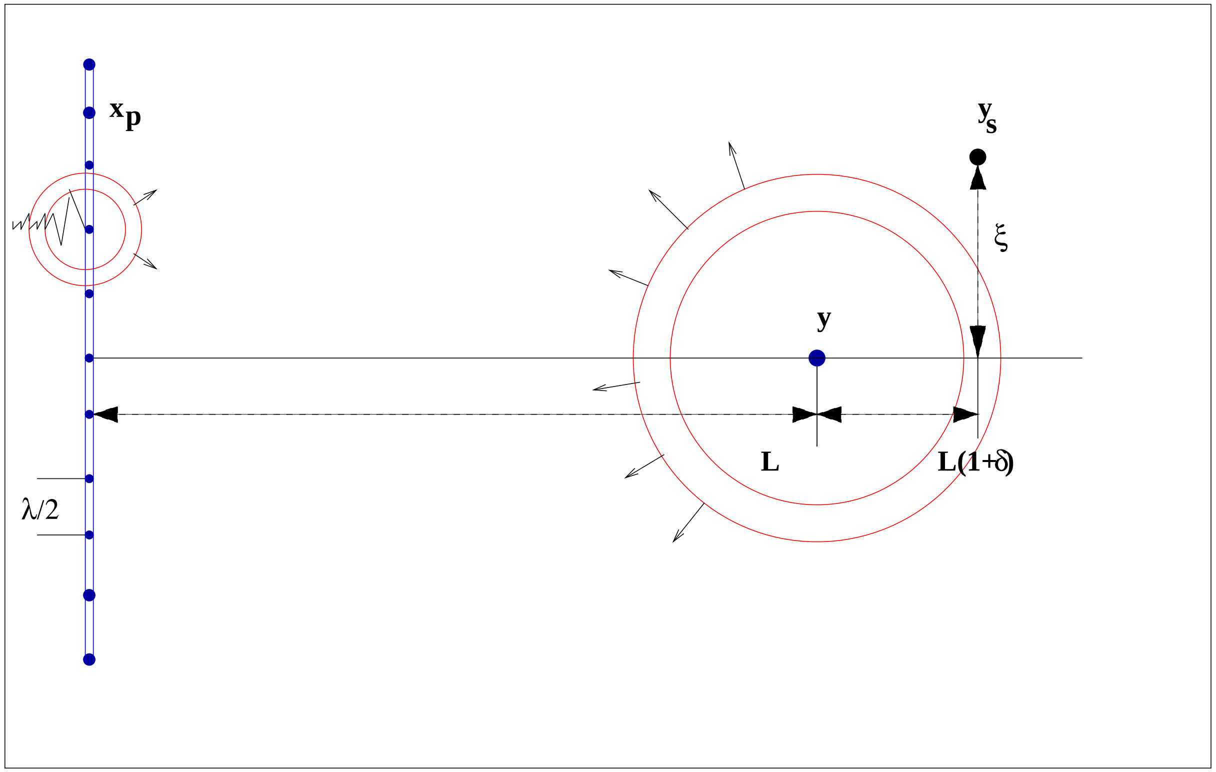
Outline of lectures

- 1. Introduction to the resolution theory of TR
- 2. A more detailed study of resolution theory
- 3. Imaging
- 4. One-dimensional (layered) media
- 5. Communications

Time Reversal Cartoon

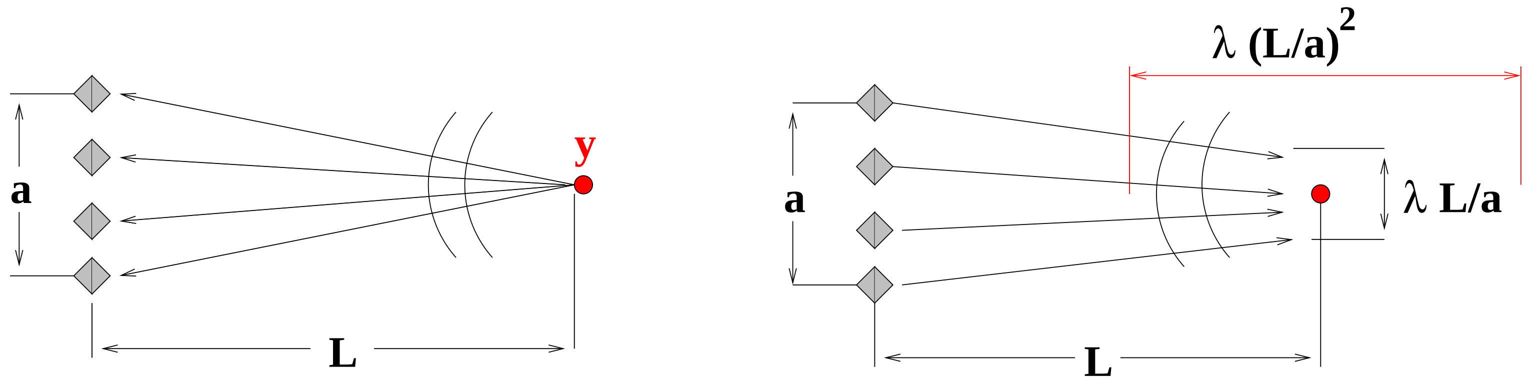


From Scientific American, November 1999 (M. Fink).



Range: L, Carrier wavelength λ, Array size $a = (N-1)\lambda/2$.
Source at y, Search point at y_s, Transducers at x_p.
Remote sensing regime: $\lambda << a << L$.
Random medium: Correlation length $l << L$, fluctuation strength $\sigma << 1$.

Resolution in time reversal



Cross-range (Rayleigh) resolution: $\frac{\lambda L}{a}$

Range resolution: $\frac{\lambda L^2}{a^2}$

Same as for optical instruments.

Wireless: $\lambda = 20cm, \; a = 2m, \; L = 5km$

Cross-range resolution: $500m$

- What happens in a randomly inhomogeneous medium?

SUPER-RESOLUTION

The acoustic wave equation in the time domain

• For numerical simulations of time reversal (and for generating the response matrix $P(t)$ later), we solve in $\mathbb{R}^2$ (mixed finite element)

$$\begin{cases} \rho(\mathbf{x})\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + \nabla p(\mathbf{x},t) = 0, \\ \kappa(\mathbf{x})\frac{\partial p(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{u} = \frac{1}{\rho(\mathbf{x})}\delta_h(\mathbf{x}-\mathbf{y})\int_0^t f(s)ds, \\ \mathbf{u}(\mathbf{x}, t=0) = \mathbf{0}, \\ p(\mathbf{x}, t=0) = 0. \end{cases}$$

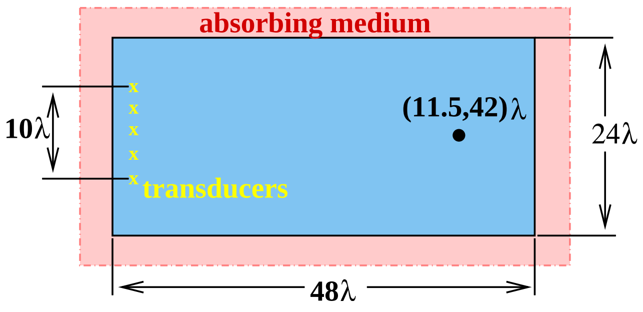
$\mathbf{u}$ is the fluid velocity, p is the acoustic pressure, $f(t)$ is the pulse, ρ is the density, κ is the compressibility and $\delta_h(\cdot)$ is a function of small support (approximate delta function).

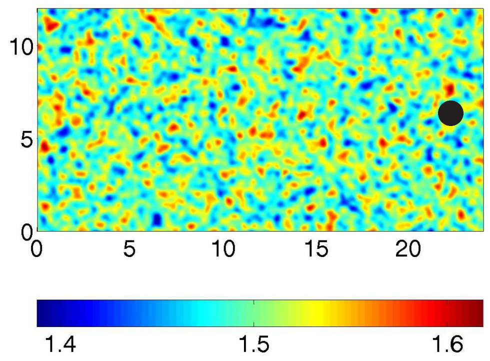
• The field p is recorded at the transducers $\mathbf{x}_j$, time-reversed and re-emitted into the medium.

• The sound speed is the random function $c^2(\mathbf{x}) = \frac{1}{\rho(\mathbf{x})\kappa(\mathbf{x})}$.

• The wave equation for p is

$$\frac{1}{c^2}p_{tt} - \rho\nabla \cdot (\frac{1}{\rho}\nabla p) = \delta_h(\mathbf{x}-\mathbf{y})\, f(t)$$

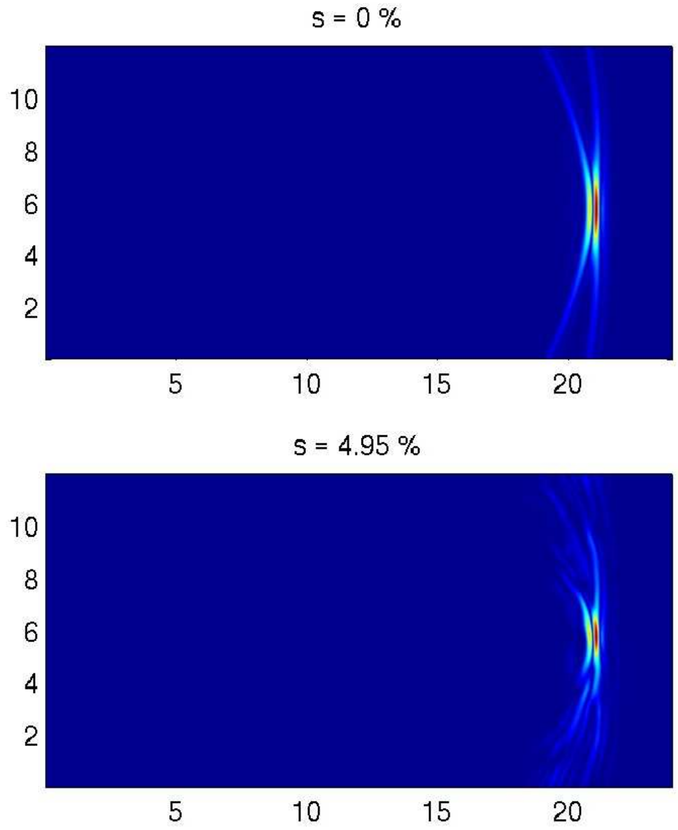




- Solve the wave equation (finite element, time domain) with a randomly fluctuating propagation speed about a constant mean.

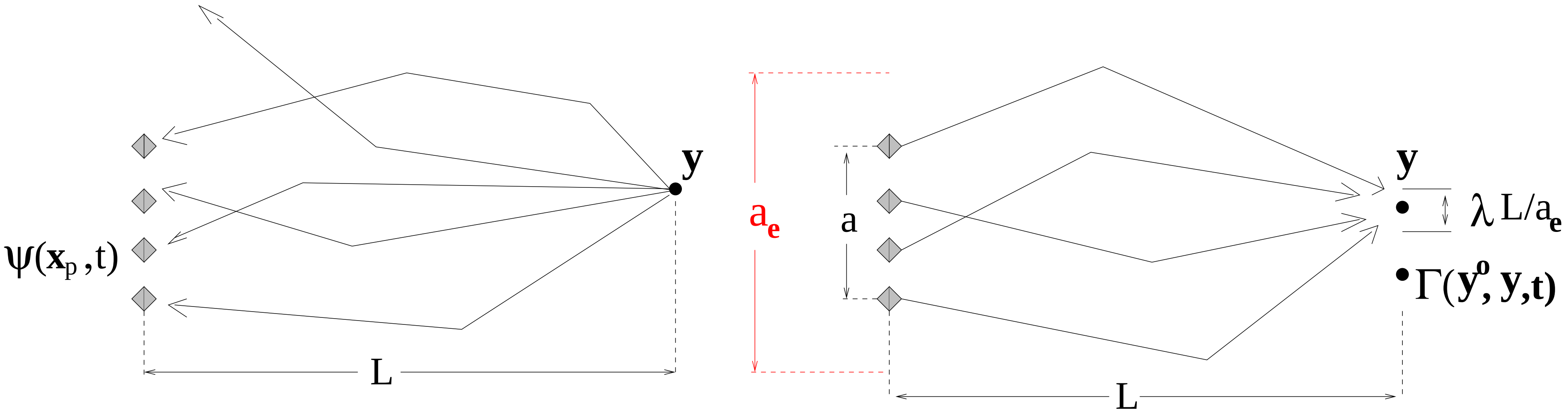
- **Ultrasound probing:** $\lambda = 0.5$mm, $a = 2.5$mm (5mm), $c_0 = 1.5$km/s.

Super-resolution.



Top deterministic. Bottom random. $s =$ standard deviation of the fluctuations of sound speed.

• A source at $\mathbf{y}$ radiates onto the array. The recorded signal is time reversed (conjugated in Fourier domain) and sent into the medium (denoted by $\Gamma(\mathbf{y}, \mathbf{y}^o, t)$)



• **Super-resolution**: In a random medium, because of multi-pathing the **effective** aperture $a_e \gg a$ or, $\frac{\lambda L}{a_e} \ll \frac{\lambda L}{a}$.

• **Self-Averaging**: Nature does the averaging (ergodic theory) in the time domain (broadband).

Applications, history of time-reversal

- **Seismic imaging, migration, '70s**: Physical time reversal not possible. Imaging using back-propagation into an estimated medium. Loss of coherence.

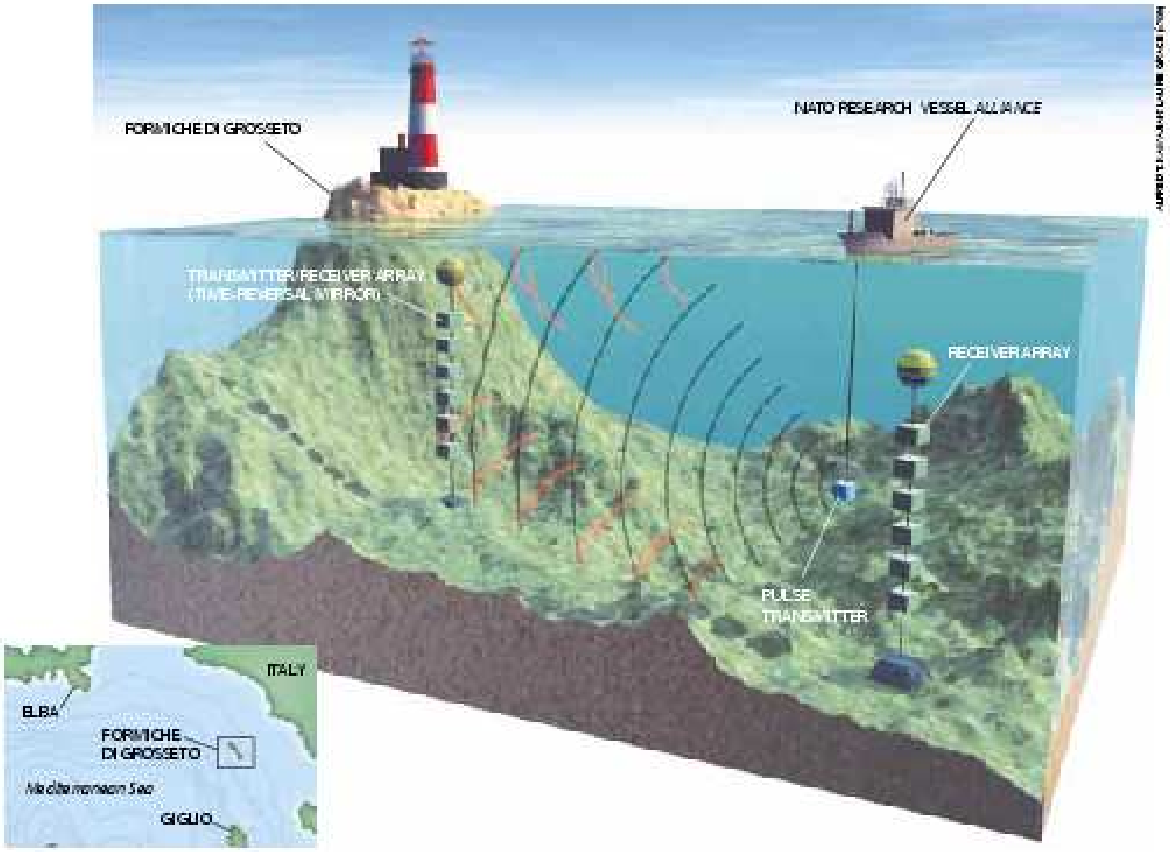
- **Optical phase conjugation, '70s**: Adaptive optics, laser fusion, holography. Single frequency, distributed sources. Super-resolution not (easily) observed.

- **Underwater acoustics, late '80s**: Super-resolution observed, communications, imaging (Kuperman, Jackson, Dowling).

- **Ultrasound, early '90s**: Non-destructive testing, medical applications. Super-resolution observed very clearly (Fink).

- **Wireless**: Use multipathing to increase communication capacity. Radar imaging in cluttered environments (not possible with synthetic aperture radar).

Time Reversal in the Ocean: Experiment.

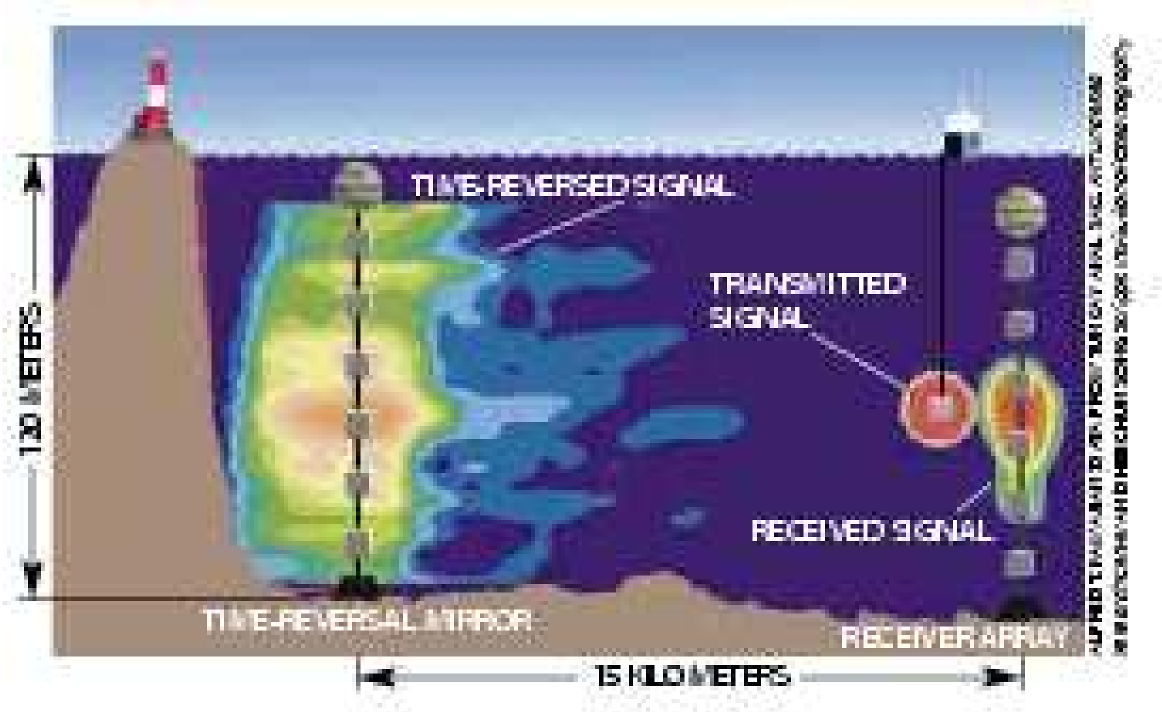


Time reversal experiment with 20-transducer *Time Reversal Mirror* in the Mediterranean Sea (off Italy, at Formiche di Grosseto). 120m depth, 15-30km propagation distance.

Performed by the Scripps Institution of Oceanography (La Jolla, CA) and SACLANT Undersea Research Center (La Spezia, Italy). Published in JASA 1996, 1997 by Kuperman et.al.

From Scientific American, November 1999 (M. Fink).

Time Reversal in the Ocean: Super-Resolution.



Time reversal experiment with 20-transducer *Time Reversal Mirror* in the Mediterranean Sea (off Italy, at Formiche di Grosseto). 120m depth, 15-30km propagation distance.

Note: $L\lambda/a \sim 15000 \times 1/120 \sim 100m$. [No resolution? TR in a waveguide]

From from Scientific American, November 1999 (M. Fink).

The reduced wave equation

The field satisfying the wave equation is written as

$$p(t,x,z) = \int e^{ik(z-c_0 t)}\psi(z,x,k)dk$$

where the complex amplitude ψ solves the "relative" Helmholtz equation

$$\psi_{zz} + 2ik\psi_z + \Delta_x\psi + k^2(n^2-1)\psi = 0,$$

which is a scattering problem. Here, z is the coordinate in the direction of propagation (range), $x \in R^2$ are the transverse coordinates (cross-range), $k = \omega/c_0$ is the wave number, $n(z,x) = c_0/c(z,x)$ is the index of refraction with c_0 a reference speed, and

$$n^2(z,x) - 1 = \sigma\mu(\frac{x}{l},\frac{z}{l})$$

The dimensionless random fluctuations $\mu(x,z)$ are stationary with mean $E\{\mu\} = 0$ and covariance $R(z,x) = E\{\mu(z+z',x+x')\mu(z',x')\}$.

In a deterministic medium $n(z,x) = 1$ and

$$\psi_{zz} + 2ik\psi_z + \Delta_x\psi = 0$$

is the Helmholtz equation (scattering problem).

Scaling

Range: z by L_z, with $L_z = L$ usually
Cross-range: x by L_x, with $L_x = a$ often
Wave number: k by $k_0 = \omega_0/c_0$, with ω_0 the central frequency

Dimensionless parameters

$$\epsilon = \frac{l}{L_z}\ , \quad \delta = \frac{l}{L_x}\ , \quad \gamma = \frac{1}{k_0 l}$$

Scaled and dimensionless Helmholtz equation

$$\frac{\epsilon^2\theta^2}{\delta^2}\psi_{zz} + 2ik\theta\psi_z + \theta^2\Delta_x\psi + \frac{k^2\delta}{\sqrt{\epsilon}}\mu(\frac{x}{\delta},\frac{z}{\epsilon})\psi = 0$$

Here,

$$\theta = \frac{L_z}{k_0 L_x^2} = \frac{\gamma\delta^2}{\epsilon} = \text{Fresnel number}$$

and the STD of the fluctuations is chosen so that $\epsilon = \sigma^{2/3}\delta^{2/3}$.

The Fresnel number is the ration of the spot size to the aperture:

$$\theta = \frac{L_z}{k_0 L_x^2} = \frac{L}{k_0 a} \cdot \frac{1}{a}$$

For $\theta \sim 1$ we have Fresnel diffraction, $\theta << 1$ geometric asymptotics, and for $\theta >> 1$ we have Fraunhoffer diffraction.

Scaling II

- White noise+Paraxial limit: $\epsilon \to 0$. Effectively defines the remote-sensing regime.

- High frequency limit: $\theta \to 0$. Appropriate in the ladar regime and elsewhere. But often $\theta = O(1)$.

- Broad beam limit: $\delta \to 0$. Transverse dimensions of the medium are wide enough for multipathing.

- Many other limits are possible and important in various applications.

White noise + Paraxial limit

The scattering problem for the scaled relative Helmholtz equation defines a stochastic process that, under suitable hypotheses on the fluctuations μ, converges weakly to the process defined by the Ito-Schrödinger equation

$$2ik\theta d_z\psi + \theta^2 \Delta_x \psi dz + \frac{ik^3\delta^2}{4\theta} R_0(0)\psi dz + k^2 \delta \psi d_z B(\frac{x}{\delta}, z) = 0$$

where $B(x,z)$ is a Brownian field, that is, a Guassian process with mean zero and covariance

$$E\{B(x,z_1)B(y,z_2)\} = R_0(x-y)\min\{z_1,z_2\} \ , \qquad R_0(x) = \int_{-\infty}^{\infty} R(s,x)ds$$

The scattering problem becomes an initial-value problem with $\psi(0,x,k) =$ given. If there is no ψ_{zz} term this is the well-known white-noise limit, motivated by the central limit theorem

$$B^\epsilon(x,z) = \frac{1}{\sqrt{\epsilon}} \int_0^z \mu(x, \frac{s}{\epsilon})ds \to B(x,z) \ , \quad \text{weakly as a random field}$$

Compute the covariance.

The time-reversed, back-propagated field

On the plane of the source, at a point with transverse coordinates ξ, the time time harmonic field is

$$\psi^B(L,\xi,k) = \int G_\theta(L,x,\xi;k)\overline{G_\theta(L,\eta,x;k)\psi_0(\eta,k)}\chi_A(x)dxd\eta$$

where G_θ is the (random) Green's function. In the time domain it is

$$\Psi^B(L,\xi,t) = \int e^{-i\omega t}\psi^B(L,\xi,\frac{\omega}{c_0})d\omega$$

Because of the form of this field, and for many other reasons, we introduce and use the Wigner distribution of ψ

$$W_\theta(z,x,p) = \int \frac{dy}{(2\pi)^2}e^{ip\cdot y}\psi(z,x-\frac{\theta y}{2},k)\overline{\psi(z,x+\frac{\theta y}{2},k)}$$

and note that ψ^B can be written entirely in terms of W_θ.

High frequency limit $\theta \to 0$

The Wigner distribution satisfies a linear stochastic equation, the Ito-Wigner equation, that comes from the Ito-Schrödinger equation using the Ito calculus. In the high frequency limit the Wigner process converges weakly to the solution of the Ito-Liouville equation

$$d_z W + \left(\frac{p}{k} \cdot \nabla_x W - \frac{k^2 D}{2} \Delta_p W\right) dz + \frac{k}{2} \nabla_p W \cdot \nabla_x d_z B\left(\frac{x}{\delta}, z\right) = 0$$

where $D = -R_0''(0)/4$ and the wave number scales out: $W = W(z, x, p/k; k = 1)$. The expected value $E\{W\}$ solves the PDE

$$W_z + \frac{p}{k} \cdot \nabla_x W - \frac{k^2 D}{2} \Delta_p W = 0$$

with given initial conditions $W(0, x, p; k)$.

The process W depends on δ but $E\{W\}$ does not.

The mean of the time-reversed, back-propagated field

If we take a source field that is a directed beam

$$e^{ip_0 \cdot x/\theta}\psi_0(\frac{x}{\sigma_s}, k),$$

with σ_s the lateral extent of the source, then in the white-noise ($\epsilon \to 0$) and high-frequency ($\theta \to 0$) limits we have

$$E\{\psi^B(L, \xi, k)\} = e^{-ip_0 \cdot x/\theta}\psi_0(\cdot, -k) * \mathcal{W}(\cdot)(\xi)$$

where $\mathcal{W}$ is the point spread function

$$\mathcal{W}(\eta) = \left(\frac{k}{2\pi L}\right)^2 \widehat{\chi}_A(\frac{\eta k}{L})e^{-\eta^2/(2\sigma_M^2)}$$

and

$$\sigma_M = \frac{L}{ka_e}\ , \quad a_e = \sqrt{\frac{DL^3}{3}}$$

Here $a_e = a_e(L)$ is the effective aperture of the array.

Interpretation of the point spread function

If there is no scattering medium then $D = 0$ and

$$\mathcal{W}(\eta) = \left(\frac{k}{2\pi L}\right)^2 \widehat{\chi}_A(\frac{\eta k}{L})$$

For a square aperture $A = [-\frac{a}{2}, \frac{a}{2}]^2$

$$\mathcal{W}(\eta) = \mathcal{W}(\eta_1, \eta_2) = \frac{1}{\pi^2 \eta_1 \eta_2} \sin(\frac{\eta_1 ka}{2L}) \sin(\frac{\eta_2 ka}{2L})$$

The first zero of the sine function is at

$$\eta_F = \frac{2\pi L}{ka} = \frac{\lambda L}{a} = \text{Rayleigh resolution}$$

If we define $\sigma_F = L/ka$, the Fresnel spot size, then when $\sigma_F << \sigma_M$, or $a >> a_e$, multipathing does not alter the refocused spot size of diffraction theory.

But if $a_e >> a$ then the point spread function is

$$\mathcal{W} \approx \left(\frac{a}{\sqrt{2\pi} a_e}\right)^2 \frac{e^{-\eta^2/(2\sigma_M^2)}}{2\pi\sigma_M^2}$$

Self-averaging

When is the time-reversed, back-propagated field self-averaging? This is a fundumental issue because it determines when super-resolution is observable.

In the present setting there are two results:

- If the source is localized, $\sigma_s \sim \theta$, then, in the limit $\delta \to 0$, the time harmonic field ψ^B is self-averaging

$$\lim_{\delta \to 0} E\{(\psi^B - E\{\psi^B\})^2\} = 0$$

- If the source is distributed, $\sigma_s >> \theta$, then only in the time domain, that is for $\Psi^B(L, \xi, t)$, we have self-averaging in mean square sense as $\delta \to 0$.

- What does $\delta \to 0$ mean? Provides cross-range diversity in multipathing.

The self-averaging is based on the following theorem for the Ito-Liouville process (with $k = 1$) defined by

$$d_z W + (p \cdot \nabla_x W - \frac{D}{2}\Delta_p W)dz + \frac{1}{2}\nabla_p W \cdot \nabla_x d_z B(\frac{x}{\delta}, z) = 0$$

with $W(0, x, p) = \chi_A(x)$:

For any $z > 0$ the integral

$$J_\delta(z, x) = \int W_\delta(z, x, p)dp$$

exists and

$$\lim_{\delta \to 0} E\{(J_\delta - E\{J_\delta\})^2\} = 0$$

where $E\{J_\delta\}$ is independent of δ.

This is proved by using properties of the SDE's (random characteristics) through which the Ito-Liouville equation can be solved.

Time-reversed, back-propagated pulse

In the time domain and for a distributed source, the self-averaging field, in the white-noise and high-frequency limit, is given by

$$\Psi^B(L,\xi,t) = e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\ \chi_A * \left(\frac{e^{-x^2/2a_e^2}}{2\pi a_e^2}\right)\left(\frac{Lc_0p_0}{\omega_0+\omega}\right)$$

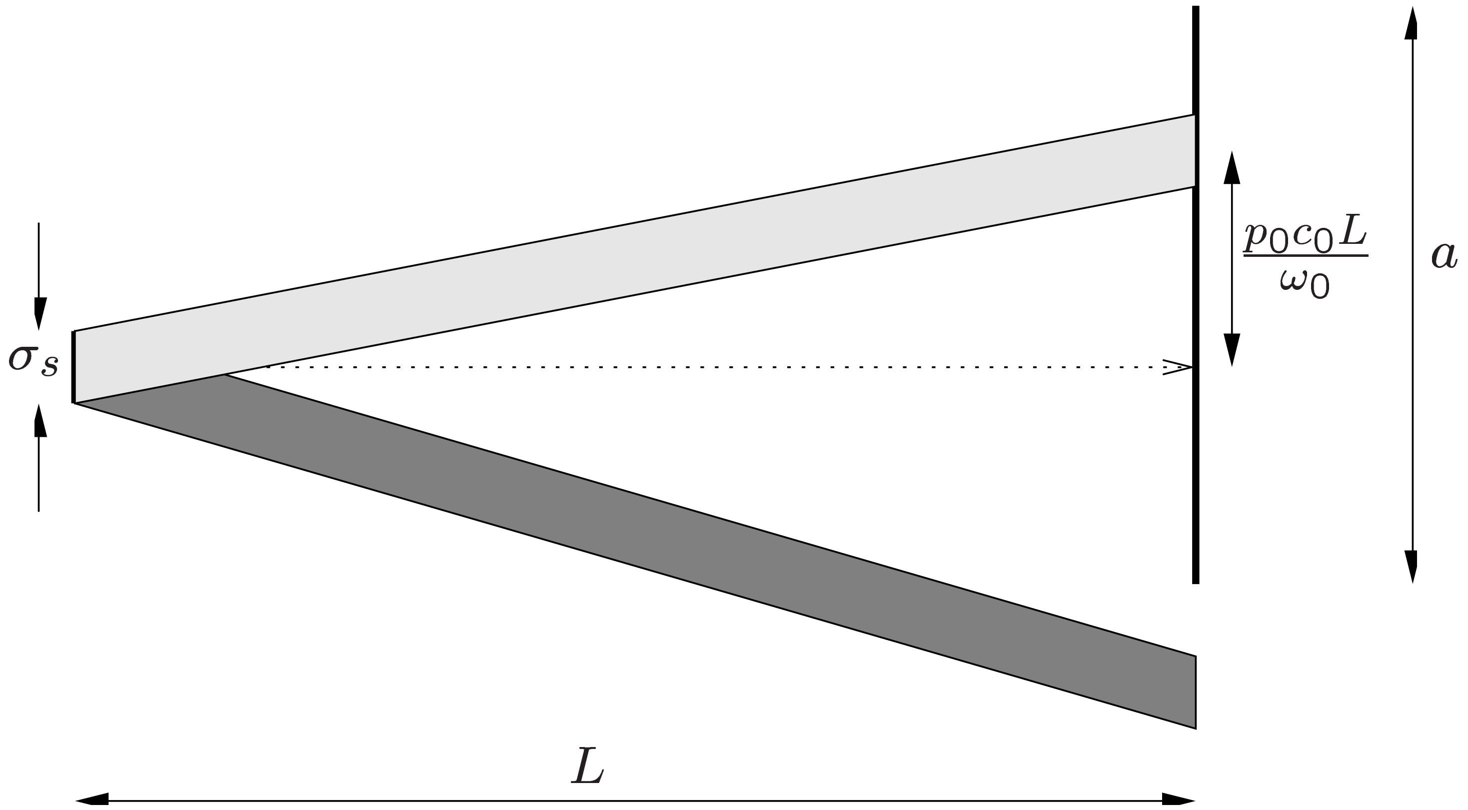
When $a_e << a$, that is, no multipathing, then

$$\Psi^B(L,\xi,t) \sim e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\ \chi_A \left(\frac{Lc_0p_0}{\omega_0+\omega}\right)$$

In this case, if the beam lands entirely withing the TRM then the time-reversed and back-propagated pulse is

$$e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi)g(-t)$$

Time-reversed, back-propagated pulse schematic



A directed field propagates from a distributed source of size σ_s toward the time reversal mirror of size a. The time-reversed, back-propagated field depends on the location of the mirror relative to the direction of the propagating beam.

Time-reversed, back-propagated pulse with multipathing

When multipathing is strong, $a_e >> a$, then the self-averaging time-reversed and back-propagated pulse is given by

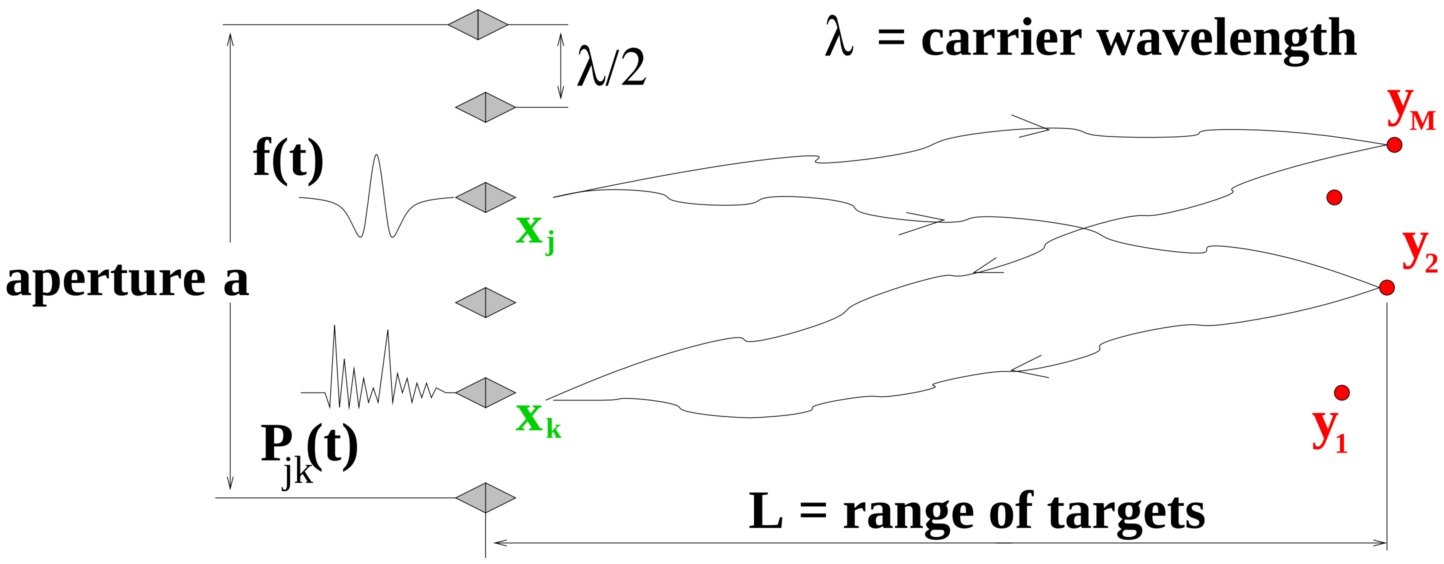
$$\Psi^B(L,\xi,t) \sim e^{-i(p_0\cdot\xi+\omega_o t)}\psi_0(\xi) \cdot \left(\frac{a}{\sqrt{2\pi}a_e}\right)^2 \int_{\{|\omega|<\Omega\}} \frac{d\omega}{2\pi} e^{-i\omega t}\widehat{g}(-\omega)\; e^{-\frac{1}{2}\left(\frac{Lc_0p_0}{a_e(\omega_0+\omega)}\right)^2}$$

Note that, remarkably, this expression **is almost independent of the time reversal mirror**!

Use this formula to estimate the most important quantity in time reversal with strong multipathing: the effective aperture a_e.

Point the beam in different directions toward the TRM, measure the time reversed pulse and estimate a_e by fitting to the formula.

Active array imaging.



An array of N transducers probes a medium with M small scatterers with a pulse $f(t)$ emitted from transducer j, located at $\mathbf{x}_j$, and the echos $P_{jk}(t)$ are recorded at $\mathbf{x}_k$, for $k = 1, \ldots N$.

Imaging: From the $N \times N$ response matrix $\big(P_{jk}(t)\big)$ find the number $M \leq N$ of scatterers and their location $\mathbf{y}_1, \mathbf{y}_2 \ldots \mathbf{y}_M$.

Random media: In the regime $\lambda \leq l \ll a = N\lambda/2 \ll L$, where there is significant **multipathing**, how is imaging affected?.

How to do imaging in random media

- We use the statistical stability and the super-resolution of time reversal in random media, for imaging.
- Identify the unknown targets as minima or maxima of objective functions that are self-averaging.

Motivate echo mode passive target array imaging with active target array imaging (locate an active source).

To find the location $\mathbf{y}^*$ of that source we use the matched field method in the time domain, which works well in random media.

This means: time reverse and then back propagate in a **homogeneous** medium, numerically.

For a single active source

$$\mathbf{y}^* = \text{argmin}_{\mathbf{z}} F(\mathbf{z}, 0)$$

where

$$F(\mathbf{z}, t) = \mathcal{F}^{-1}\left\{ |(\widehat{f}(\omega)\widehat{\mathbf{g}}(\mathbf{y}, \omega))^H \frac{\widehat{\mathbf{g}}_0(\mathbf{z}, \omega)}{|\widehat{\mathbf{g}}_0(\mathbf{z}, \omega)|}|^2 \right\}$$

where

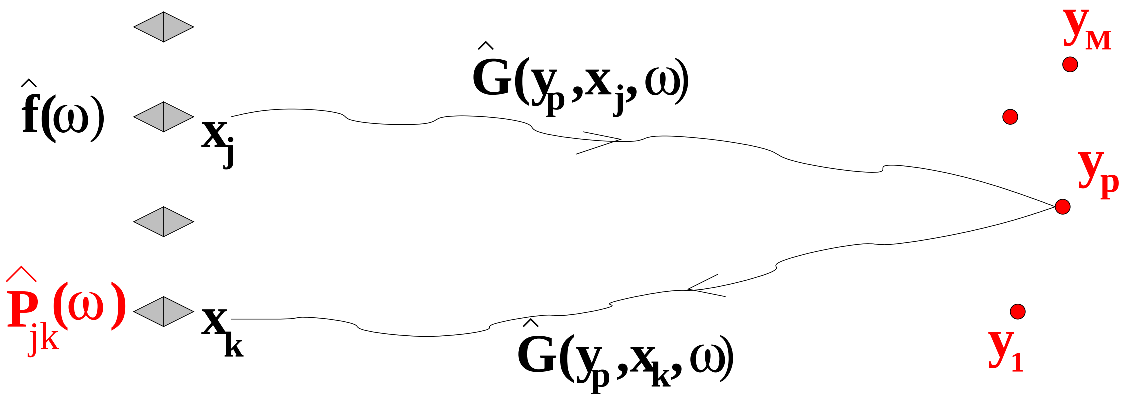
$$\widehat{\mathbf{g}}(\mathbf{y}, \omega) = \begin{pmatrix} \widehat{G}(\mathbf{y}, \mathbf{x}_1; \omega) \\ \vdots \\ \widehat{G}(\mathbf{y}, \mathbf{x}_N; \omega) \end{pmatrix}$$

where $\widehat{G}(\mathbf{y}, \mathbf{x}; \omega)$ is the (random) Green's function. The vector $\widehat{\mathbf{g}}(\mathbf{y}, \omega)$ is the radiation vector.

The subscript zero indicates free space (deterministic medium).

The matched field functional is statistically stable.

Echo mode array imaging. Forward model of response matrix P, at frequency ω.



$\widehat{P}_{jk}(\omega) =$ sum of fields scattered from all targets $\mathbf{y}_p$, $p = 1, \ldots M$.

We neglect multiple scattering.

- $\widehat{P}(\omega) = \widehat{f}(\omega) \sum_{p=1}^{M} \widehat{\tau}_p(\omega) \widehat{\mathbf{g}}(\mathbf{y}_p, \omega) \widehat{\mathbf{g}}^T(\mathbf{y}_p, \omega)$, $\widehat{\mathbf{g}}(\mathbf{y}_p, \omega) = \begin{pmatrix} \widehat{G}(\mathbf{y}_p, \mathbf{x}_1; \omega) \\ \vdots \\ \widehat{G}(\mathbf{y}_p, \mathbf{x}_N; \omega) \end{pmatrix}$

$\widehat{\tau}_p(\omega) =$scattering coefficient of target p.

- rank $\left[\widehat{P}(\omega)\right] = M \leq N$, except for very special configurations of targets and transducers.

SVD of $\widehat{P}(\omega)$. Statistically stable imaging.

- Why SVD?
- M = number of significant singular values $\sigma_j(\omega)$.
- The left singular vectors $\widehat{\mathbf{V}}_j$, for $j = 1, \ldots M$, are linear combinations of $\widehat{\mathbf{g}}(\mathbf{y}_p, \omega)$, the random, radiating Green's vectors.

Imaging M targets by projection (MUSIC)

For search point $\mathbf{y}$, if projection onto the null space of $\widehat{P}(\omega)$, $\mathcal{P}_N \widehat{\mathbf{g}}(\mathbf{y}, \omega) = 0$, then $\mathbf{y} = \mathbf{y}_p$, for some $p = 1, \ldots M$.

In imaging, $\widehat{\mathbf{g}}(\mathbf{y}, \omega)$ is unknown. Use the deterministic $\widehat{\mathbf{g}}_0(\mathbf{y}, \omega)$ instead with the objective functional (single target):

$$\mathbf{F}(\mathbf{y}, t) = \int e^{-i\omega t} \sigma(\omega) \mathcal{P}_N \widehat{\mathbf{g}}_0(\mathbf{y}, \omega) d\omega.$$

- Minima of $|\, \mathbf{F}(\mathbf{y}, t) \,| \Longrightarrow$ information about location of targets.

It works because $\mathbf{F}(\mathbf{y}, t)$ is self-averaging.

Connection between imaging and time reversal

For simplicity of explanation, take one target. It extends to multiple targets, with modifications.

- For a search point $\mathbf{y}$, we calculate

$$\mathbf{F}(\mathbf{y},t) = \int e^{-i\omega t}\sigma(\omega)\left\{\left[\widehat{\mathbf{V}}_1^H(\omega)\widehat{\mathbf{g}}_0(\mathbf{y},\omega)\right]\widehat{\mathbf{V}}_1(\omega) - \mid \widehat{\mathbf{V}}_1(\omega) \mid^2 \widehat{\mathbf{g}}_0(\mathbf{y},\omega)\right\},$$

  with normalized $\widehat{\mathbf{V}}_1(\omega)$, corresponding to the singular value $\sigma(\omega)$.

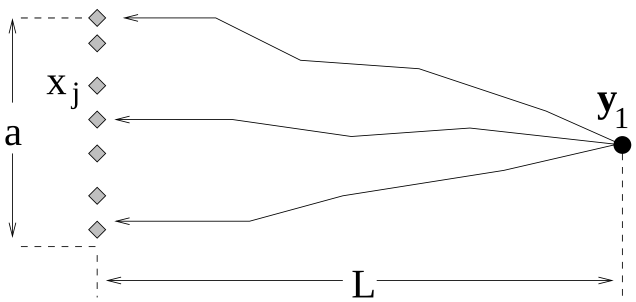
Each term in $\mathbf{F}(\mathbf{y},t)$ has a time reversal interpretation and is self-averaging.

At what time do we evaluate this vector? Try the deterministic arrival times $t_j(\mathbf{y})$. Plot the reciprocal of the norm of this vector as function of $\mathbf{y}$.

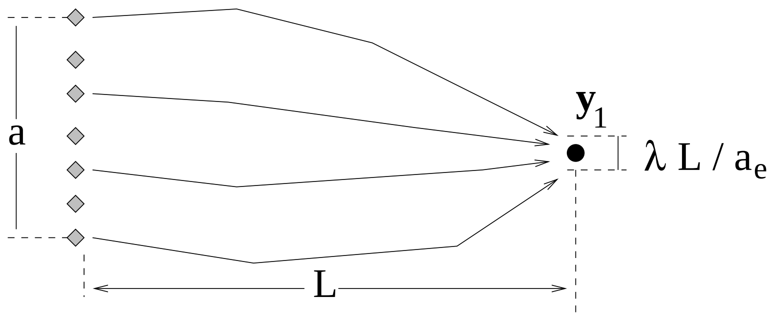
$$\mathbf{F}^{(1)}(\mathbf{y},t) = \int e^{-i\omega t} \mid \widehat{V}_1(\omega) \mid^2 \widehat{\mathbf{g}}_0(\mathbf{y},\omega) d\omega =$$

$$\int e^{-i\omega t} \mid \widehat{f}(\omega) \mid \widehat{\mathbf{g}}_0(\mathbf{y},\omega) \sum_{j=1}^{N} \overline{\widehat{G}}(\mathbf{y}_1,\mathbf{x}_j,\omega) \widehat{G}(\mathbf{y}_1,\mathbf{x}_j,\omega) d\omega.$$

Echos received from unknown target $\mathbf{y}_1$ in random medium



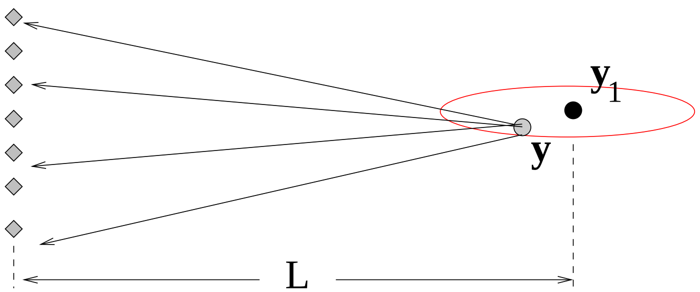
TR and resend into random medium



Search point y illuminates the array in fictitious homogeneous medium

Eval. $F^{(1)}$ at deterministic time. (self−averaging).

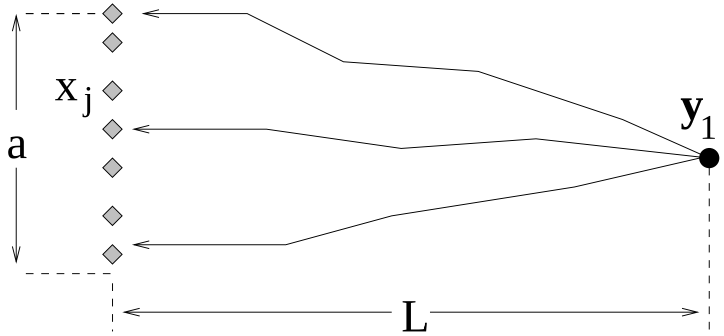
$\mathbf{F}_j^{(1)}(\mathbf{y},t_j^0(\mathbf{y}))$



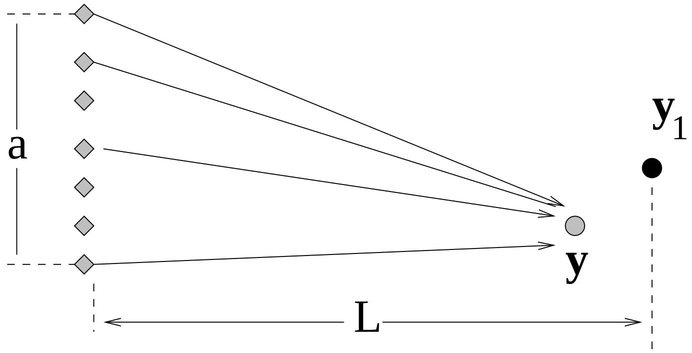
$$\mathbf{F}^{(2)}(\mathbf{y},t) = \int e^{-i\omega t}\left[\widehat{\mathbf{V}}_1^H(\omega)\widehat{\mathbf{g}}_0(\mathbf{y},\omega)\right]\widehat{\mathbf{V}}_1(\omega)d\omega =$$

$$\int e^{-i\omega t} \mid \widehat{f}(\omega) \mid \widehat{\mathbf{g}}(\mathbf{y}_1,\omega)\sum_{j=1}^{N}\overline{\widehat{G}}(\mathbf{y}_1,\mathbf{x}_j,\omega)\widehat{\mathbf{G}}_0(\mathbf{y},\mathbf{x}_j,\omega)d\omega.$$

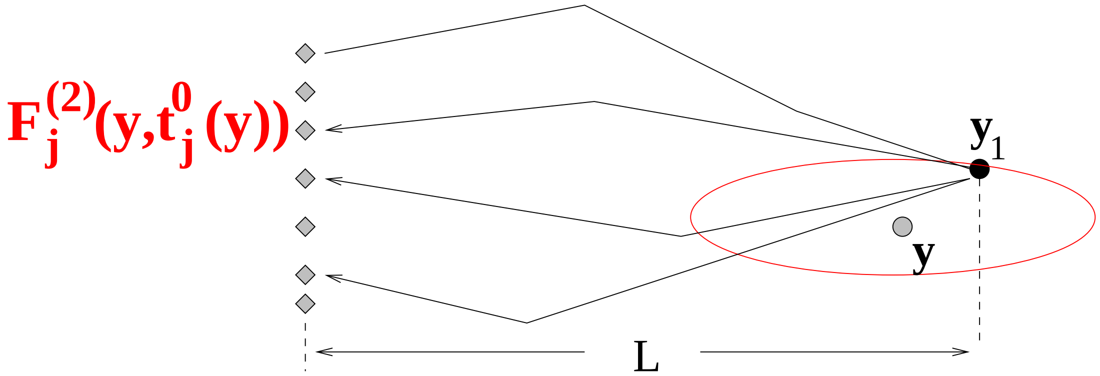
Echos received from unknown target y_1 in random medium.



TR and propagate to search point y in fictitious homog. med.



Illuminate the array from unknown y_1 in the random medium



Extension to imaging multiple targets

- For a search point $\mathbf{y}$, we project the deterministic $\widehat{\mathbf{g}}_0(\mathbf{y},\omega)$ onto $\text{Null}\left(\widehat{P}(\omega)\right)$, for all ω in the band.

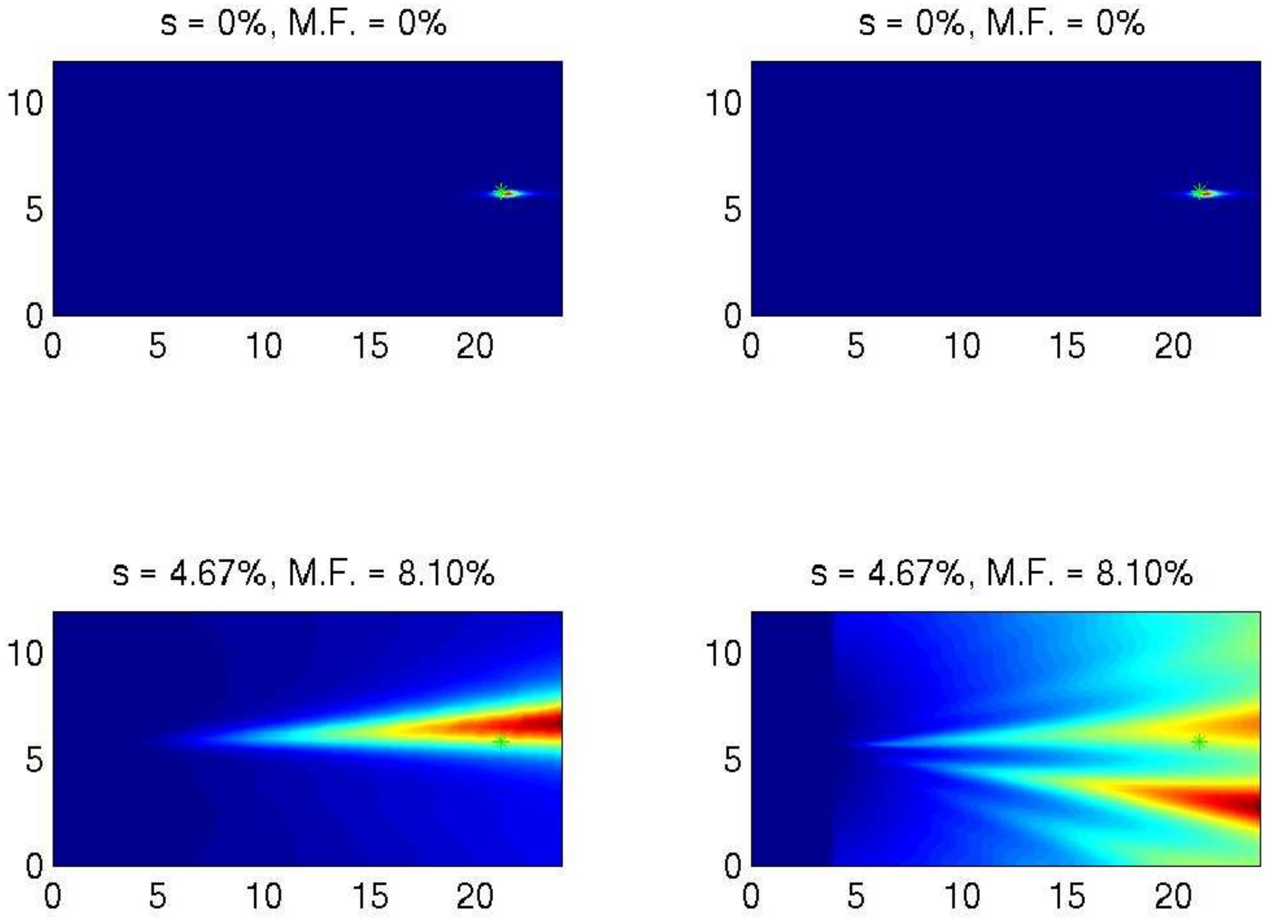
  Up to a normalization, $\mathcal{F}(\mathbf{y},t)$ is the inverse Fourier transform of this projection.

- We evaluate the j^{th} component of $\mathcal{F}(\mathbf{y},t)$ at $t = t_j^0(\mathbf{y})$, the deterministic arrival time from test point $\mathbf{y}$ to receiver $\mathbf{x}_j$.

  This is justified by the self-averaging property of $\mathcal{F}(\mathbf{y},t)$.

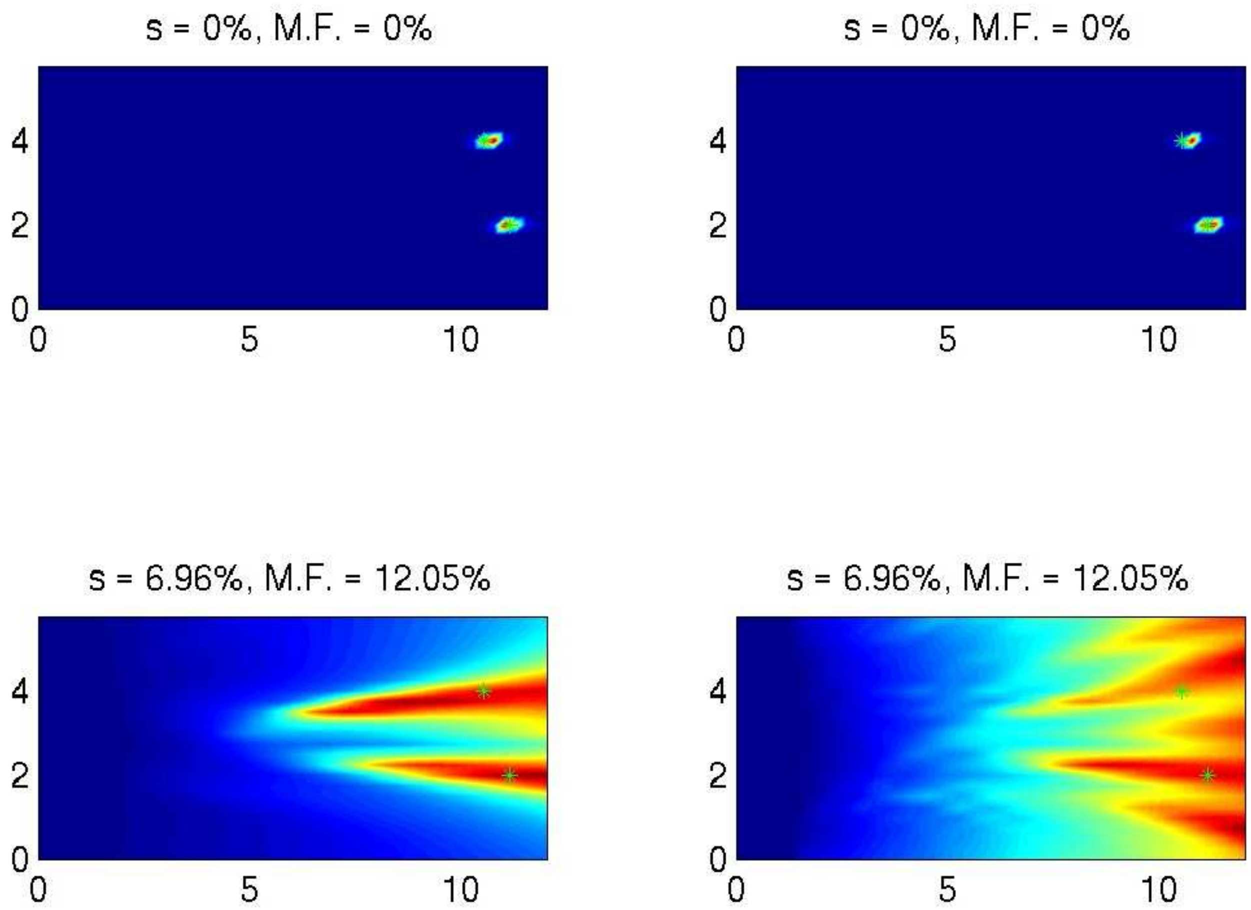
- For display, we use $\mathcal{R}(\mathbf{y}) = 1/\sum_{j=1}^{N}\left[\mathcal{F}_j(\mathbf{y},t_j^0(\mathbf{y}))\right]^2$.

  Estimates of target locations are the maxima of $\mathcal{R}(\mathbf{y})$.



On the left time domain estimation, on the right fixed frequency MUSIC.

Direction of arrival estimation II



On the left time domain estimation, on the right fixed frequency MUSIC.

Sensitivity of time domain direction of arrival estimation to arrival times used.

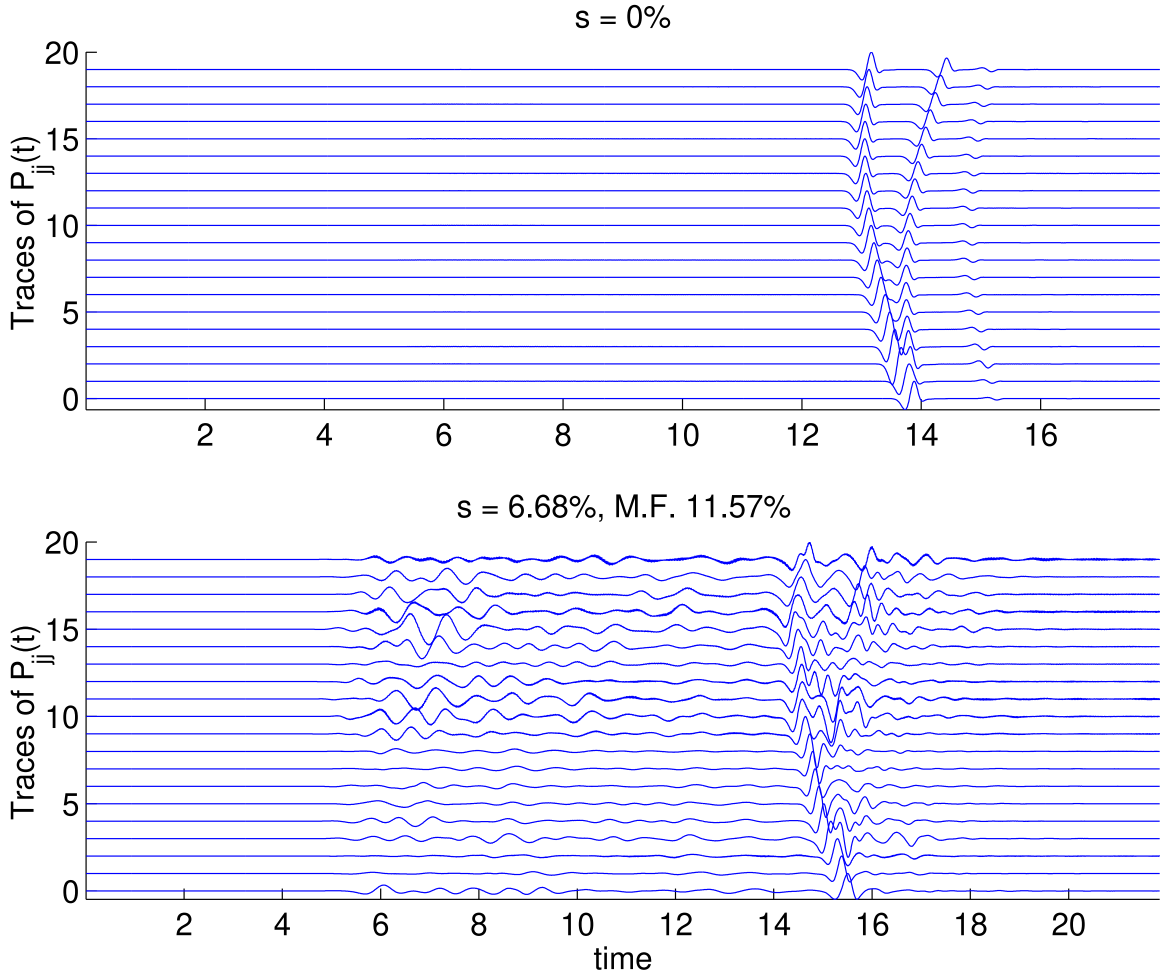
- Get very bad range estimation when using deterministic arrival times
- Using estimated arrival times improves cross range estimation greatly
- Need for good arrival time estimation

Sensitivity of estimation to arrival time (the conical region) is related to super-resolution in time-reversal.

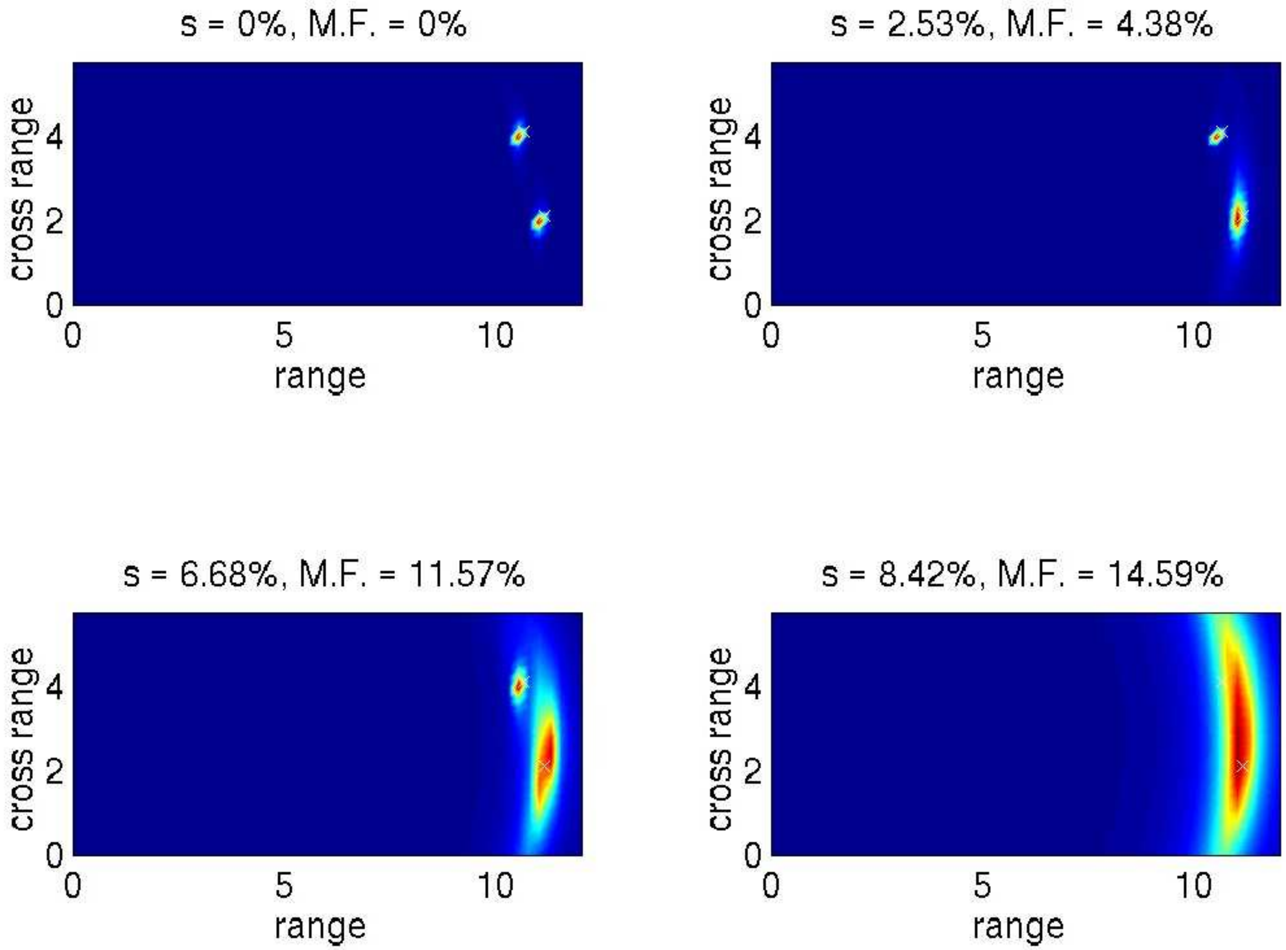
New phenomenon when imaging in random media: It is NOT possible to separate cross-range and range estimation. They interact.

The phase (arrival time) in the data gives the range of targets.

Traditional arrival time analysis. SAR



ATA imaging of two targets in a random medium.



Using **only** arrival time information and LSQ fitting of hyperbolas (common in seismic imaging).

• Singular vectors are also used to get better range estimates. (Illuminate one target and shade the other).

- SVD gives complex, singular vectors $\widehat{\mathbf{V}}_j(\omega)$ up to arbitrary, $e^{i\phi(\omega)} \Longrightarrow \int e^{-i\omega t}\widehat{\mathbf{V}}_j(\omega)d\omega$ is not coherent in time.

• Calculate N coherent in time vectors:

$$\mathbf{v}_j^{(m)}(t) = \int e^{-i\omega t}\left[\widehat{\mathbf{V}}_j^H(\omega)\cdot\widehat{P}(\omega)\mathbf{e}_m\right]\widehat{\mathbf{V}}_j(\omega)d\omega.$$

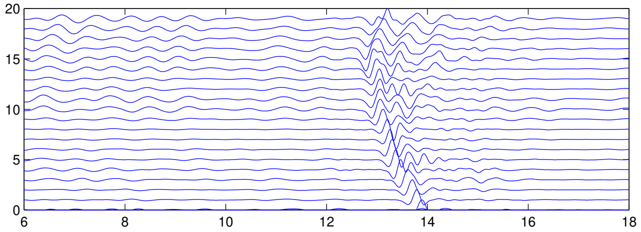
Equivalent to taking m^{th} column of $P(t)$, time reverse it and send it into the medium. Repeat this an infinite number of times.

• Stack the $\mathbf{v}_j^{(m)}(t)$ to get better estimates of arrival times.

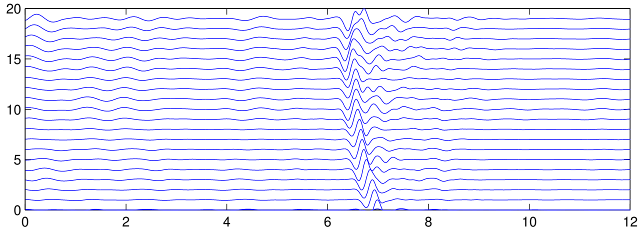
$$\left\langle \mathbf{v}_j(t)\right\rangle = \frac{1}{N}\sum_{j=1}^{N}\mathbf{v}_j^{(m)}(t-\tau_m), \qquad \min_{\tau_m}\int_0^T\sum_{m=1}^{N}\left[\mathbf{v}_j^{(m)}(t-\tau_m)-\left\langle \mathbf{v}_j(t)\right\rangle\right]^2.$$

Traces for random medium with st. dev. $s = 6.68\%$

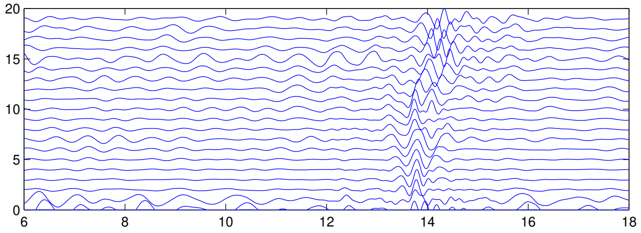
(a1)



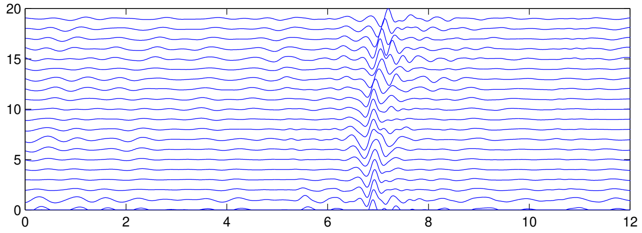
(b1)



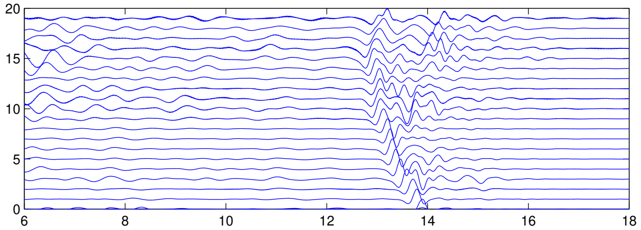
(a2)



(b2)



(c)



(a1) traces of m^{th} component of $\mathbf{v}_1^{(m)}(t)$, $m = 1, \ldots N$.

(a2) traces of m^{th} component of $\mathbf{v}_2^{(m)}(t)$, $m = 1, \ldots N$.

(b1) and (b2): traces of $\langle \mathbf{v}_1(t) \rangle$ and $\langle \mathbf{v}_2(t) \rangle$.

(c) traces of $P_{jj}(t)$.

The SAT estimation

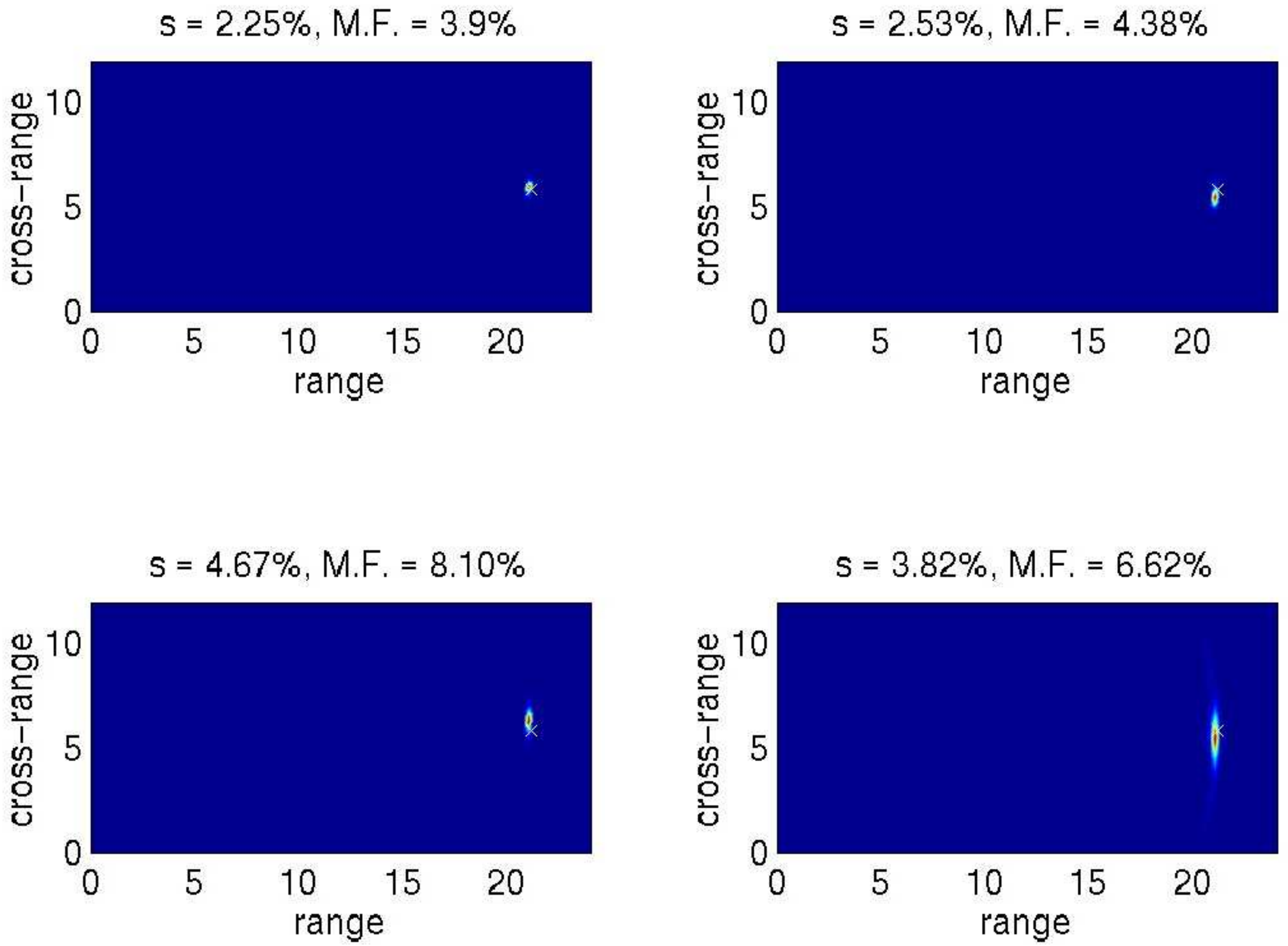
- The SAT objective function is basically an intersection between SVD direction of arrival estimation and subspace arrival time analysis.

- The D-o-A estimation gives good cross-range resolution of the targets.

- The subspace arrival time analysis gives good range resolution of each target.

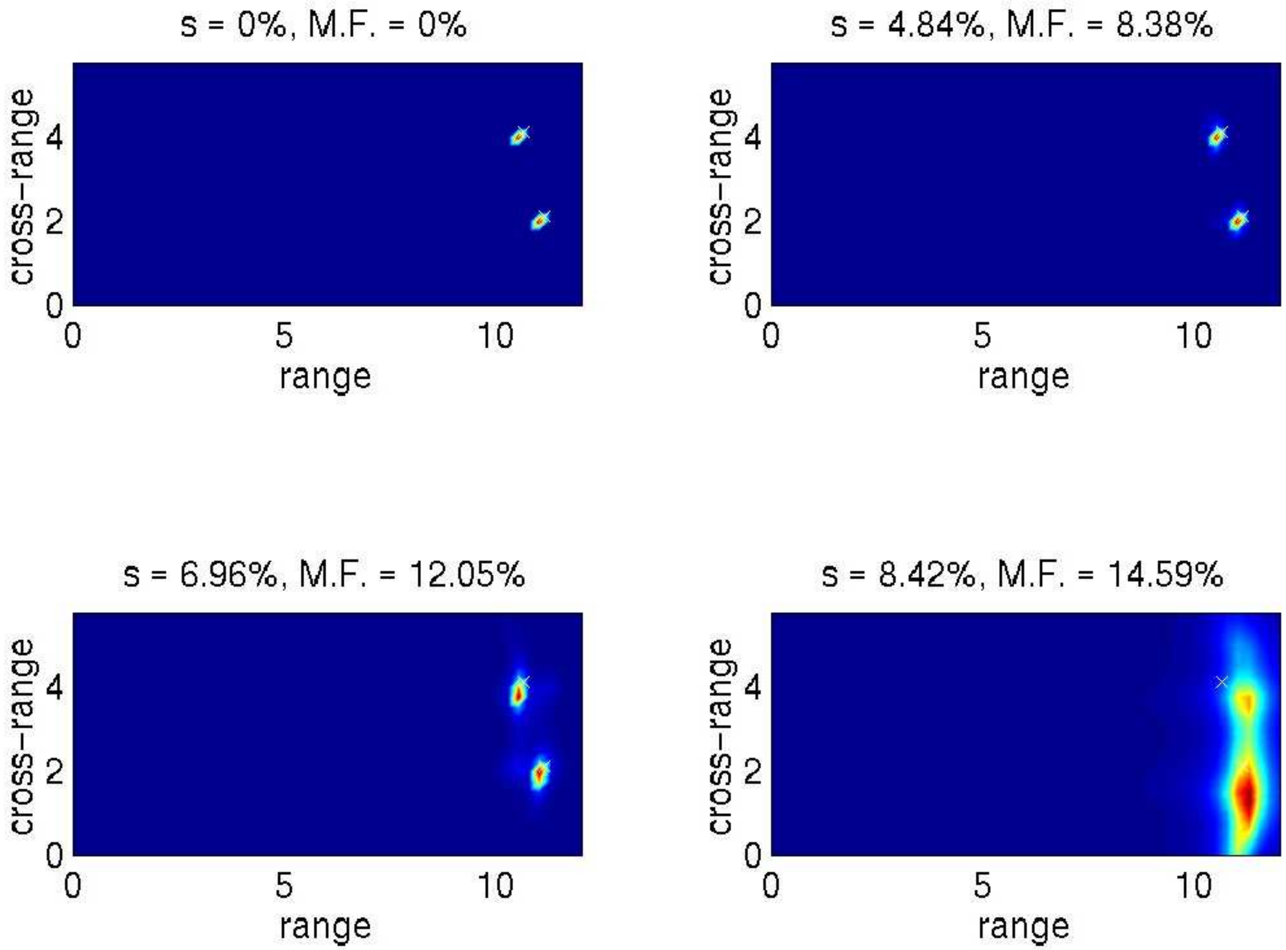
- The intersection gives good cross-range and range resolution.

SAT imaging of one target in a random medium



s = standard deviation, M.F. = max. fluctuations of sound speed

SAT imaging of two targets in a random medium



s = standard deviation, M.F. = max. fluctuations of sound speed

Why SAR does not work in a random medium

- SAR is when only the **diagonal** $P_{jj}(t)$ of the response matrix is measured.

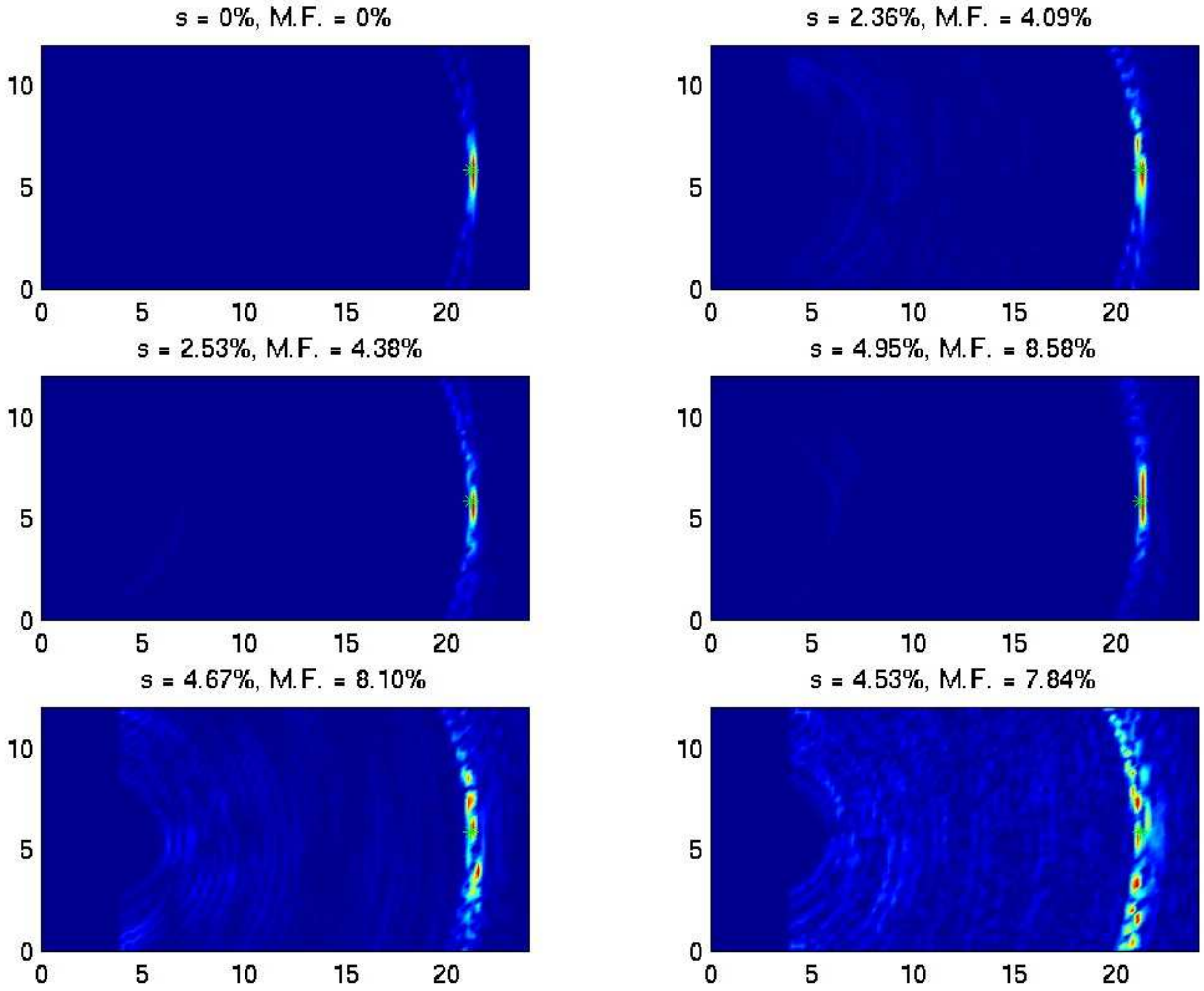
- Maxima of

$$\mathcal{R}(\mathbf{y}^s) = \sum_{j=1}^{N} P_{jj}\left(\frac{2|\mathbf{x}_j - \mathbf{y}^s|}{c_0}\right)$$

  give target estimates.

- No projection, no ‘time reversal’, no statistical stability.

One target SAR in a random medium



Summary and conclusions

- Time reversal in a random medium is important because of super-resolution and self-averaging, which are phenomena that are difficult to analyze and understand quantitatively, and require new and interesting mathematics.

- Applications abound, are enormously exciting and limited only by the **hardware**, our imagination, and also our analytical understanding: Direct TR applications, Imaging, Communications.

- Current research: the ergodic (asymptotic) theory of SPDE (like the $\delta \to 0$ limit), array imaging that uses self-averaging to minimize the effect of clutter, and communications in rich multipathing environments using TR protocols.