Oberwolfach Lecture 5: Adaptive Coherent Interferometric Imaging

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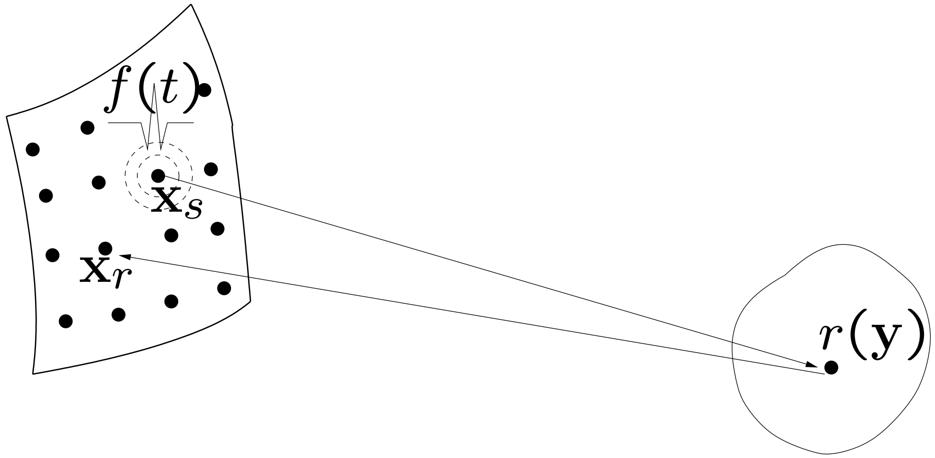
The array imaging problem

- The source at $\mathbf{x}_s$ emits a short pulse $f(t) = e^{-i\omega_o t} f_B(t)$ and at the array we record the acoustic pressure traces $P(\mathbf{x}_r, t; \mathbf{x}_s)$, for $r = 1, \ldots, N_r$ and $t \in [t_m, t_M]$.



- Problem: estimate the (support) of reflectivity $r(\mathbf{x})$ from the data traces. The waves propagate in clutter.

Interferometric Imaging

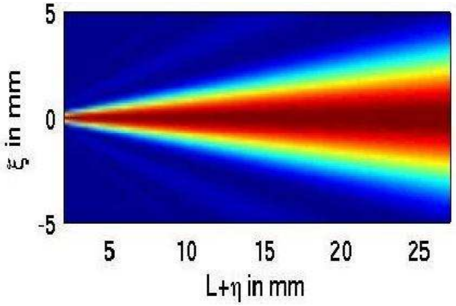
- Uses interferograms $P(\mathbf{x}_r', -t; \mathbf{x}_s) \star_t P(\mathbf{x}_r, t; \mathbf{x}_s) \rightsquigarrow$ in Fourier domain, we work with

$$\left\langle \widehat{P}(\mathbf{x}_r, \omega; \mathbf{x}_s) \overline{\widehat{P}(\mathbf{x}_r', \omega; \mathbf{x}_s)} \right\rangle \approx \widehat{P}_o(\mathbf{x}_r, \omega; \mathbf{x}_s) \overline{\widehat{P}_o(\mathbf{x}_r', \omega; \mathbf{x}_s)} e^{-\frac{(k\kappa_d)^2 |\mathbf{x}_r - \mathbf{x}_r'|^2}{2}}$$

- The migration to $\mathbf{y}^s = (\xi^s, \eta^s)$ is done in the smooth part of the medium, with travel times.

$$\tau(\mathbf{x}_r, \mathbf{y}^s) - \tau(\mathbf{x}_r', \mathbf{y}^s) \approx (\mathbf{x}_r - \mathbf{x}_r') \cdot \nabla \tau(\mathbf{x}_r, \mathbf{y}^s) = \frac{(\mathbf{x}_r - \mathbf{x}_r')}{c_o} \cdot \frac{(\mathbf{x}_r - \xi^s)}{|(\mathbf{x}_r, 0) - \mathbf{y}^s|},$$

where $|\mathbf{x}_r - \mathbf{x}_r'| \leq X_d(\omega)$ and $k X_d(\omega) = \kappa_d^{-1}$.



Coherent Interferograms

- The interferograms are given by cross-correlation over the whole time axis

$$P(\mathbf{x}_r,t)\star P(\mathbf{x}_r',-t)=\int ds P(\mathbf{x}_r,s)P(\mathbf{x}_r',-(t-s))$$

$$=\int\frac{d\omega}{2\pi}\widehat{P}(\mathbf{x}_r,\omega)\overline{\widehat{P}(\mathbf{x}_r',\omega)}e^{-i\omega t}$$

- Coherent interferograms are computed in a window $\psi(t;T_d)$, with Fourier transform $\widehat{\psi}(\tilde{\omega};\Omega_d)$.

$$\int d\varpi\int d\tilde{\omega}\,\widehat{\psi}(\tilde{\omega};\Omega_d)\widehat{P}\left(\mathbf{x}_r,\varpi+\frac{\tilde{\omega}}{2}\right)\overline{\widehat{P}\left(\mathbf{x}_r',\varpi-\frac{\tilde{\omega}}{2}\right)}e^{-i\left(\varpi+\frac{\tilde{\omega}}{2}\right)t+i\left(\varpi-\frac{\tilde{\omega}}{2}\right)t'}$$

$$=2\pi\int ds\,\psi(s+t;T_d)P(\mathbf{x}_r,s)P(\mathbf{x}_r',-[(t'-t)-s])$$

$T_d=$ delay spread time and $\Omega_d\sim 1/T_d=$ decoherence frequency.

The phase in the coherent interferograms

- We have the moment formula

$$\left\langle \widehat{G}(\mathbf{x}_r,\omega;\mathbf{y})\overline{\widehat{G}(\mathbf{x}_r',\omega';\mathbf{y})}\right\rangle \approx \widehat{G}_o(\mathbf{x}_r,\omega;\mathbf{y})\overline{\widehat{G}_o(\mathbf{x}_r',\omega';\mathbf{y})}e^{-\frac{(\overline{k}\,\kappa_d)^2|\mathbf{x}_r-\mathbf{x}_r'|^2}{2}-\frac{(\omega-\omega')^2}{2\Omega_d^2}}$$

where $\overline{k}=\frac{\omega+\omega'}{2c_o}$.

- The phase in the coherent interferograms is then:

$$\omega\tau(\mathbf{x}_r,\mathbf{y})-\omega'\tau(\mathbf{x}_r',\mathbf{y})=\frac{(\omega+\omega')}{2}\left[\tau(\mathbf{x}_r,\mathbf{y})-\tau(\mathbf{x}_r',\mathbf{y})\right]$$

$$+(\omega-\omega')\left[\frac{\tau(\mathbf{x}_r,\mathbf{y})+\tau(\mathbf{x}_r',\mathbf{y})}{2}\right]$$

so we do have travel times, not just differences.

- We let from now on: $\overline{\omega}=\frac{\omega+\omega'}{2}$ and $\tilde{\omega}=\omega-\omega'$.

Coherent Interferometric Imaging Function

- Approximate the array by a continuum in planar domain $\mathcal{A}$, where $\mathbf{x}_r \rightsquigarrow \mathbf{x} = \overline{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \quad \mathbf{x}_r' \rightsquigarrow \mathbf{x}' = \overline{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}$.

$$\mathcal{I}^{\text{CINT}}(\mathbf{y}^s) \sim \int d\overline{\omega} \int d\overline{\mathbf{x}} \int d\tilde{\omega}\ \widehat{\Psi}(\tilde{\omega};\Omega_d) \int d\tilde{\mathbf{x}}\ \widehat{\Phi}\left(\overline{k}\tilde{\mathbf{x}};\kappa_d^{-1}\right) \widehat{P}\left(\overline{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \overline{\omega} + \frac{\tilde{\omega}}{2}; \mathbf{x}_s\right)$$
$$\exp\left\{-i\left(\overline{\omega} + \tfrac{\tilde{\omega}}{2}\right)\left[\tau\left(\overline{\mathbf{x}} + \tfrac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s\right) + \tau\left(\mathbf{x}_s, \mathbf{y}^s\right)\right]\right\} \overline{\widehat{P}\left(\overline{\mathbf{x}} - \tfrac{\tilde{\mathbf{x}}}{2}, \overline{\omega} - \tfrac{\tilde{\omega}}{2}; \mathbf{x}_s\right)}$$
$$\exp\left\{i\left(\overline{\omega} - \tfrac{\tilde{\omega}}{2}\right)\left[\tau\left(\overline{\mathbf{x}} - \tfrac{\tilde{\mathbf{x}}}{2}, \mathbf{y}^s\right) + \tau\left(\mathbf{x}_s, \mathbf{y}^s\right)\right]\right\}$$

- The difference frequency is restricted by window $\widehat{\Psi}$ to $|\tilde{\omega}| \le \Omega_d$.

- The window $\widehat{\Phi}$ of support $O(\kappa_d^{-1})$ ensures $|\tilde{\mathbf{x}}| \le X_d(\overline{\omega})$.

Coherent Interferometry and Statistical Stability

- Expecting small $|\tilde{\mathbf{x}}| \leq X_d$, linearize the phases

$$e^{-i\overline{\omega}\left[\tau(\overline{\mathbf{x}}+\frac{\tilde{\mathbf{x}}}{2},\mathbf{y}^s)-\tau(\overline{\mathbf{x}}-\frac{\tilde{\mathbf{x}}}{2},\mathbf{y}^s)\right]} \approx e^{-i\overline{\omega}\tilde{\mathbf{x}}\cdot\nabla_{\overline{\mathbf{x}}}\tau(\overline{\mathbf{x}},\mathbf{y}^s)} = e^{-i\overline{k}\tilde{\mathbf{x}}\cdot\boldsymbol{\kappa}(\mathbf{y}^s)}$$

$$e^{-i\tilde{\omega}\left[\frac{\tau(\overline{\mathbf{x}}+\frac{\tilde{\mathbf{x}}}{2},\mathbf{y}^s)+\tau(\overline{\mathbf{x}}-\frac{\tilde{\mathbf{x}}}{2},\mathbf{y}^s)}{2}+\tau(\mathbf{x}_s,\mathbf{y}^s)\right]} \approx e^{-i\tilde{\omega}[\tau(\overline{\mathbf{x}},\mathbf{y}^s)+\tau(\mathbf{x}_s,\mathbf{y}^s)]}$$

- The imaging function becomes

$$\mathcal{I}^{\mathsf{CINT}}(\mathbf{y}^s) \sim \int d\overline{\mathbf{x}} \int d\boldsymbol{\kappa}\ \Phi\left(c_o\nabla_{\overline{\mathbf{x}}}\tau(\overline{\mathbf{x}},\mathbf{y}^s)-\boldsymbol{\kappa};\kappa_d\right) \int dt\Psi\left(\tau(\overline{\mathbf{x}},\mathbf{y}^s)+\tau(\mathbf{x}_s,\mathbf{y}^s)-t;T_d\right)$$
$$\int d\overline{\omega}\ W\left(\overline{\mathbf{x}},\boldsymbol{\kappa},\overline{\omega},t\right),$$

$$W\left(\overline{\mathbf{x}},\boldsymbol{\kappa},\overline{\omega},t\right) = \int d\tilde{\omega} \int d\tilde{\mathbf{x}}\ \widehat{P}\left(\overline{\mathbf{x}}+\frac{\tilde{\mathbf{x}}}{2},\overline{\omega}+\frac{\tilde{\omega}}{2};\mathbf{x}_s\right)\overline{\widehat{P}\left(\overline{\mathbf{x}}-\frac{\tilde{\mathbf{x}}}{2},\overline{\omega}-\frac{\tilde{\omega}}{2};\mathbf{x}_s\right)}e^{-i\tilde{\omega}t-i\frac{\overline{\omega}}{c_o}\tilde{\mathbf{x}}\cdot\boldsymbol{\kappa}}$$

- The Wigner distribution W is highly fluctuating, but typically it decorrelates rapidly in $\overline{\omega}$ and $\boldsymbol{\kappa}$ ⇝ stability by smoothing.

Smoothing vs. Resolution Trade-off

- Rewriting $\mathcal{I}^{\mathsf{CINT}}(\mathbf{y}^s)$ once more,

$$\mathcal{I}^{\mathsf{CINT}}(\mathbf{y}^s) \sim \int d\overline{\mathbf{x}} \int d\tilde{\mathbf{x}}\, P\left(\overline{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, t + \frac{\boldsymbol{\kappa}\cdot\tilde{\mathbf{x}}}{2c_o}; \mathbf{x}_s\right) P\left(\overline{\mathbf{x}} - \frac{\tilde{\mathbf{x}}}{2}, t - \frac{\boldsymbol{\kappa}\cdot\tilde{\mathbf{x}}}{2c_o}; \mathbf{x}_s\right)$$

$$\star_{\boldsymbol{\kappa}} \Phi(\boldsymbol{\kappa}; \kappa_d)\big|_{\boldsymbol{\kappa}=c_o\nabla_{\overline{\mathbf{x}}}\tau(\overline{\mathbf{x}},\mathbf{y}^s)} \star_t \Psi(t; T_d)\big|_{t=\tau(\overline{\mathbf{x}},\mathbf{y}^s)+\tau(\mathbf{x}_s,\mathbf{y}^s)} \cdot$$

- No smoothing (Φ and $\Psi \sim \delta$ functions) means

$$\mathcal{I}^{\mathsf{CINT}}(\mathbf{y}^s) \rightsquigarrow \int d\overline{\mathbf{x}} \int d\tilde{\mathbf{x}}\, P\left(\overline{\mathbf{x}} + \frac{\tilde{\mathbf{x}}}{2}, \tau(\mathbf{x}_s, \mathbf{y}^s) + \tau(\overline{\mathbf{x}}, \mathbf{y}^s) + \frac{\tilde{\mathbf{x}}\cdot\nabla_{\tilde{\mathbf{x}}}\tau(\overline{\mathbf{x}}, \mathbf{y}^s)}{2}; \mathbf{x}_s\right) \times$$

$$P\left(\overline{\mathbf{x}} - \tfrac{\tilde{\mathbf{x}}}{2}, \tau(\mathbf{x}_s, \mathbf{y}^s) + \tau(\overline{\mathbf{x}}, \mathbf{y}^s) - \tfrac{\tilde{\mathbf{x}}\cdot\nabla_{\tilde{\mathbf{x}}}\tau(\overline{\mathbf{x}},\mathbf{y}^s)}{2}; \mathbf{x}_s\right) \approx [\mathcal{I}^{\mathsf{KM}}(\mathbf{y}^s)]^2.$$

- Smoothing over arrival time by convolution with $\Psi(t; T_d)$ of support $T_d \sim \frac{1}{\Omega_d}$, affects range resolution $\frac{c_o}{\Omega_d}$.

- Smoothing in direction of arrival by convol. with $\Phi(\boldsymbol{\kappa}; \kappa_d)$ supp. in ball of radius $\kappa_d \rightsquigarrow$ cross range resolution $L\kappa_d \sim \lambda_o L / X_d(\omega_o)$.

Adaptive Coherent Interferometry

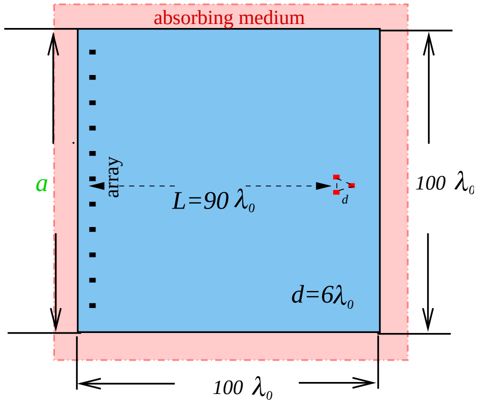
- How can we find Ω_d and κ_d?

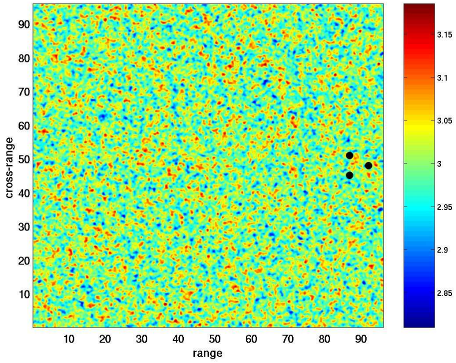
- We may derive formulae for Ω_d and κ_d. But this will be model dependent.

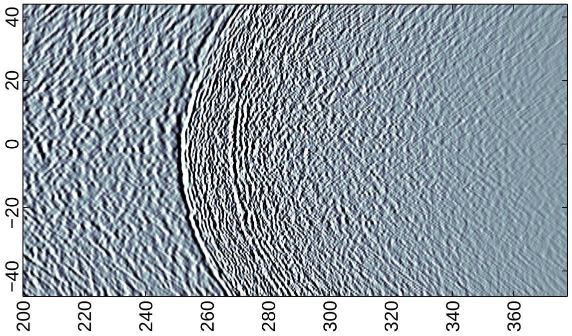
- We can estimate the decoherence parameters using statistical data processing techniques, but this can be tricky.

- We found that a more efficient approach is to do an adaptive estimation of the smoothing parameters, during the image formation process.

Computation Setup





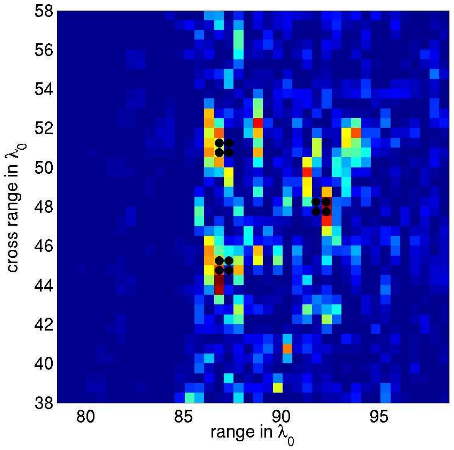


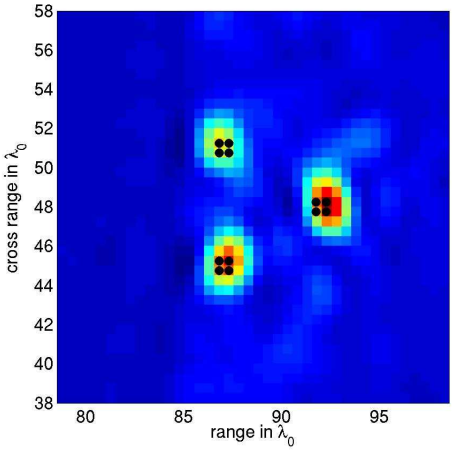
- We illuminate from the center of the linear array with 185 elements, of aperture $a = 92\lambda_o$.
- The reflectors are disks of radius $\lambda_o/2$, modeled as acoustic soft scatterers (homogeneous Dirichlet condition on P).
- The central frequency is 100kHz and the bandwidth at 6dB is $60 - 140$kHz. The sound speed is $c(\mathbf{x}) = c_o = 3$km/s.

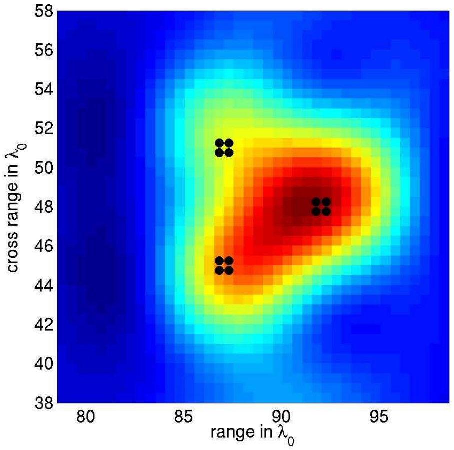
Adaptive CINT

- View the imaging function as $\mathcal{I}^{\text{CINT}}(\mathbf{y}^s; \Omega_d, \kappa_d)$ and seek parameters Ω_d and κ_d by achieving an optimal balance between statistical smoothing and resolution.

Penalize the speckles (left image) by using a norm of the gradient. To obtain a tight image, we should also penalize the blur (see right image) by using a sparsity measure. The “optimal” result is given in the middle.







Adaptive Coherent Interferometry

- The smoothing parameters are determined by minimizing

$$\|\mathcal{J}(\mathbf{y}^s;\Omega_d,\kappa_d)\|_{L^1(\mathcal{D})}+\alpha\|\nabla_{\mathbf{y}^s}\mathcal{J}(\mathbf{y}^s;\Omega_d,\kappa_d)\|_{L^1(\mathcal{D})},$$

where $\mathcal{J}(\mathbf{y}^s;\Omega_d,\kappa_d)=\frac{\mathcal{I}^{\mathrm{CINT}}(\mathbf{y}^s;\Omega_d,\kappa_d)}{\max_{\mathbf{y}^s\in\mathcal{D}}\mathcal{I}^{\mathrm{CINT}}(\mathbf{y}^s;\Omega_d,\kappa_d)}$.

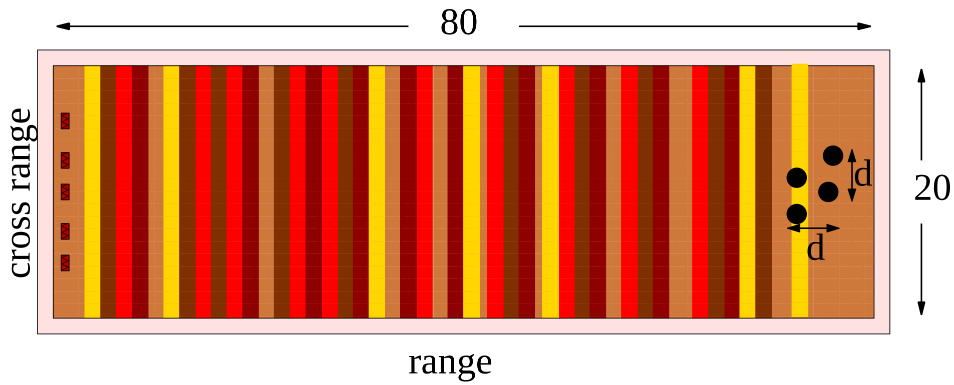
- This is very different from the usual denoising functionals,

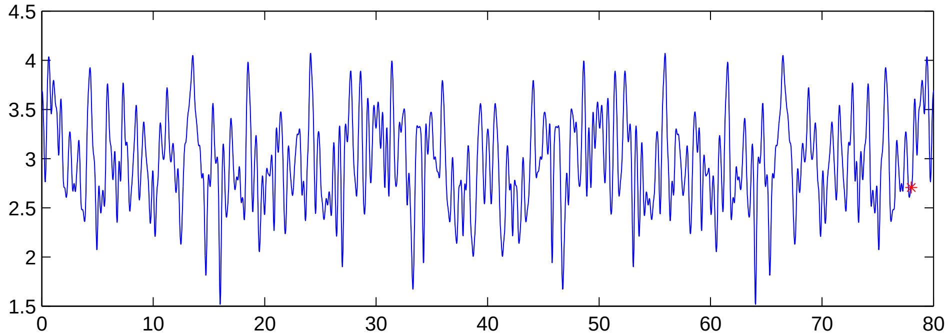
$$\|\mathcal{N}(\mathbf{y}^s)-\mathcal{I}(\mathbf{y}^s)\|_{\mathrm{prox}}+\alpha\|\mathcal{I}(\mathbf{y}^s)\|_{\mathrm{reg}},$$

where $\mathcal{N}$ is a given noisy image, $\mathcal{I}$ is the desired denoised image, $\|\cdot\|_{\mathrm{prox}}$ is a proximity norm, usually $L^2(\mathcal{D})$, and $\|\cdot\|_{\mathrm{reg}}$ is a regularization norm, usually TV.

- We do not have an image such as $\mathcal{N}$ so there is no proximity norm part. We use instead the L^1 norm of the image which is small, when the image is sparse. We do have however the regularization term.

Anisotropic clutter: Finely layered media





- The sound speed fluctuates about $c_0 = 3$km/s, the fluctuations are strong $O(1)$ and the correlation length is $\ell = 30$cm.

- The carrier wavelength is $\lambda_0 = 3\text{m} = 10\ell$, $B = 0.6 - 1.3$kHz (at 6dB) and the range is $\sim 80\lambda_0$.

Imaging in layered media: KM vs Adaptive CINT