Liliana Borcea  
Computational and Applied Mathematics  
Rice University  
borcea@caam.rice.edu

Collaborators:

George Papanicolaou, Mathematics, Stanford University  
Chrysoula Tsogka, Mathematics, University of Chicago

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The array imaging problem

- The source at $x_s$ emits a short pulse $f(t) = e^{-i\omega_0 t} f_B(t)$ and at the array we record the acoustic pressure traces $P(x_r, t; x_s)$, for $r = 1, \ldots, N_r$ and $t \in [t_m, t_M]$.

- Problem: estimate the (support) of reflectivity $r(x)$ from the data traces. The waves propagate in clutter.
Interferometric Imaging

• Uses interferograms $P(x'_r, -t; x_s) \ast_t P(x_r, t; x_s) \rightsquigarrow$ in Fourier domain, we work with

$$\langle \hat{P}(x_r, \omega; x_s) \hat{P}(x'_r, \omega; x_s) \rangle \approx \hat{P}_o(x_r, \omega; x_s) \hat{P}_o(x'_r, \omega; x_s) e^{-\frac{(k\kappa_d)^2|x_r-x'_r|^2}{2}}$$

• The migration to $y^s = (\xi^s, \eta^s)$ is done in the smooth part of the medium, with travel times.

$$\tau(x_r, y^s) - \tau(x'_r, y^s) \approx (x_r - x'_r) \cdot \nabla \tau(x_r, y^s) = \frac{(x_r - x'_r)}{c_0} \cdot \frac{(x_r - \xi^s)}{|(x_r, 0) - y^s|},$$

where $|x_r - x'_r| \leq X_d(\omega)$ and $kX_d(\omega) = \kappa_d^{-1}$. 
Coherent Interferograms

• The interferograms are given by cross-correlation over the whole time axis

\[ P(x_r, t) \ast P(x'_r, -t) = \int ds P(x_r, s)P(x'_r, -(t-s)) \]

\[ = \int \frac{d\omega}{2\pi} \hat{P}(x_r, \omega)\hat{P}(x'_r, \omega)e^{-i\omega t} \]

• Coherent interferograms are computed in a window \( \psi(t; T_d) \), with Fourier transform \( \hat{\psi}(\tilde{\omega}; \Omega_d) \).

\[ \int d\omega \int d\tilde{\omega} \hat{\psi}(\tilde{\omega}; \Omega_d)\hat{P}(x_r, \omega + \tilde{\omega}/2)\hat{P}(x'_r, \omega - \tilde{\omega}/2)e^{-i(\omega + \tilde{\omega}/2)t + i(\omega - \tilde{\omega}/2)t'} \]

\[ = 2\pi \int ds \psi(s + t; T_d)P(x_r, s)P(x'_r, -[(t' - t) - s]) \]

\( T_d = \) delay spread time and \( \Omega_d \sim 1/T_d = \) decoherence frequency.
The phase in the coherent interferograms

• We have the moment formula

\[
\langle \hat{G}(x_r, \omega; y) \hat{G}(x'_r, \omega'; y) \rangle \approx \hat{G}_o(x_r, \omega; y) \hat{G}_o(x'_r, \omega'; y) e^{-\frac{(k \cdot \kappa d)^2 |x_r - x'_r|^2}{2} - \frac{(\omega - \omega')^2}{2 \Omega_d^2}}
\]

where \( \bar{k} = \frac{\omega + \omega'}{2c_o} \).

• The phase in the coherent interferograms is then:

\[
\omega \tau(x_r, y) - \omega' \tau(x'_r, y) = \frac{(\omega + \omega')}{2} \left[ \tau(x_r, y) - \tau(x'_r, y) \right] + (\omega - \omega') \left[ \frac{\tau(x_r, y) + \tau(x'_r, y)}{2} \right]
\]

so we do have travel times, not just differences.

• We let from now on: \( \bar{\omega} = \frac{\omega + \omega'}{2} \) and \( \tilde{\omega} = \omega - \omega' \).
Coherent Interferometric Imaging Function

- Approximate the array by a continuum in planar domain $\mathcal{A}$, where $x_r \sim x = \bar{x} + \frac{\bar{x}}{2}$, $x_r' \sim x' = \bar{x} - \frac{\bar{x}}{2}$.

$$I_{\text{CINT}}(y^s) \sim \int d\tilde{\omega} \int d\bar{x} \int d\hat{\omega} \ \hat{\Psi}(\hat{\omega}; \Omega_d) \int d\tilde{x} \ \hat{\Phi}(k\tilde{x}; \kappa_d^{-1}) \hat{P}\left(\bar{x} + \frac{\bar{x}}{2}, \bar{\omega} + \frac{\bar{\omega}}{2}; x_s\right)$$

$$\exp\left\{ -i \left( \bar{\omega} + \frac{\bar{\omega}}{2} \right) \left[ \tau \left( \bar{x} + \frac{\bar{x}}{2}, y^s \right) + \tau \left( x_s, y^s \right) \right] \right\} \hat{P}\left(\bar{x} - \frac{\bar{x}}{2}, \bar{\omega} - \frac{\bar{\omega}}{2}; x_s\right)$$

$$\exp\left\{ i \left( \bar{\omega} - \frac{\bar{\omega}}{2} \right) \left[ \tau \left( \bar{x} - \frac{\bar{x}}{2}, y^s \right) + \tau \left( x_s, y^s \right) \right] \right\}$$

- The difference frequency is restricted by window $\hat{\Psi}$ to $|\bar{\omega}| \leq \Omega_d$.

- The window $\hat{\Phi}$ of support $O(\kappa_d^{-1})$ ensures $|\bar{x}| \leq X_d(\bar{\omega})$. 
Coherent Interferometry and Statistical Stability

- Expecting small $|\tilde{x}| \leq X_d$, linearize the phases

$$e^{-i\tilde{\omega}}\left[\tau(\tilde{x} + \frac{\tilde{x}}{2}, y) - \tau(\tilde{x} - \frac{\tilde{x}}{2}, y)\right] \approx e^{-i\tilde{\omega} \cdot \nabla_{\tilde{x}} \tau(\tilde{x}, y)} = e^{-i\tilde{k} \cdot \tilde{\kappa}(y)}$$

$$e^{-i\tilde{\omega}}\left[\frac{\tau(\tilde{x} + \frac{\tilde{x}}{2}, y) + \tau(\tilde{x} - \frac{\tilde{x}}{2}, y)}{2} + \tau(x_s, y)\right] \approx e^{-i\tilde{\omega}}[\tau(\tilde{x}, y) + \tau(x_s, y)]$$

- The imaging function becomes

$$I_{\text{CINT}}(y_s) \sim \int d\tilde{x} \int d\kappa \; \Phi(c_0 \nabla_{\tilde{x}} \tau(\tilde{x}, y) - \kappa; \kappa_d) \int dt \Psi(\tau(\tilde{x}, y) + \tau(x_s, y) - t; T_d)$$

$$\int d\tilde{\omega} \; W(\tilde{x}, \kappa, \tilde{\omega}, t),$$

$$W(\tilde{x}, \kappa, \tilde{\omega}, t) = \int d\tilde{\omega} \int d\tilde{x} \; \hat{P}\left(\tilde{x} + \frac{\tilde{x}}{2}, \tilde{\omega} + \frac{\tilde{\omega}}{2}; x_s\right) \hat{P}\left(\tilde{x} - \frac{\tilde{x}}{2}, \tilde{\omega} - \frac{\tilde{\omega}}{2}; x_s\right) e^{-i\tilde{\omega}t - i\tilde{\omega}c_0 \cdot \tilde{\kappa}}$$

- The Wigner distribution $W$ is highly fluctuating, but typically it decorrelates rapidly in $\tilde{\omega}$ and $\kappa \sim$ stability by smoothing.
Smoothing vs. Resolution Trade-off

• Rewriting $I^{\text{CINT}}(y^s)$ once more,

$$I^{\text{CINT}}(y^s) \sim \int d\bar{x} \int d\tilde{x} P\left(\bar{x} + \frac{\tilde{x}}{2}, t + \frac{\kappa \cdot \tilde{x}}{2c_o}; x_s\right) P\left(\bar{x} - \frac{\tilde{x}}{2}, t - \frac{\kappa \cdot \tilde{x}}{2c_o}; x_s\right)$$

$$\ast \kappa \Phi(\kappa; \kappa_d)|_{\kappa = c_o \nabla x \tau(\bar{x}, y^s) \ast t} \Psi(t; T_d)|_{t = \tau(\bar{x}, y^s) + \tau(x_s, y^s)}.$$

• No smoothing ($\Phi$ and $\Psi \sim \delta$ functions) means

$$I^{\text{CINT}}(y^s) \sim \int d\bar{x} \int d\tilde{x} P\left(\bar{x} + \frac{\tilde{x}}{2}, \tau(x_s, y^s) + \tau(\bar{x}, y^s) + \frac{\tilde{x} \cdot \nabla \tilde{x} \tau(\bar{x}, y^s)}{2}; x_s\right) \times$$

$$P\left(\bar{x} - \frac{\tilde{x}}{2}, \tau(x_s, y^s) + \tau(\bar{x}, y^s) - \frac{\tilde{x} \cdot \nabla \tilde{x} \tau(\bar{x}, y^s)}{2}; x_s\right) \approx [I^{\text{KM}}(y^s)]^2.$$

• Smoothing over arrival time by convolution with $\Psi(t; T_d)$ of support $T_d \sim \frac{1}{\Omega_d}$, affects range resolution $\frac{c_o}{\Omega_d}$.

• Smoothing in direction of arrival by convol. with $\Phi(\kappa; \kappa_d)$ supp. in ball of radius $\kappa_d \sim$ cross range resolution $L\kappa_d \sim \lambda_0 L/X_d(\omega_o)$. 
• How can we find $\Omega_d$ and $\kappa_d$?

• We may derive formulae for $\Omega_d$ and $\kappa_d$. But this will be model dependent.

• We can estimate the decoherence parameters using statistical data processing techniques, but this can be tricky.

• We found that a more efficient approach is to do an adaptive estimation of the smoothing parameters, during the image formation process.
- We illuminate from the center of the linear array with 185 elements, of aperture \( a = 92\lambda_o \).

- The reflectors are disks of radius \( \lambda_o/2 \), modeled as acoustic soft scatterers (homogeneous Dirichlet condition on \( P \)).

- The central frequency is 100kHz and the bandwidth at 6dB is 60 – 140kHz. The sound speed is \( c(x) = c_o = 3\text{km/s} \).
Adaptive CINT

- View the imaging function as $I^{\text{CINT}}(y^s; \Omega_d, \kappa_d)$ and seek parameters $\Omega_d$ and $\kappa_d$ by achieving an optimal balance between statistical smoothing and resolution.

Penalize the speckles (left image) by using a norm of the gradient. To obtain a tight image, we should also penalize the blur (see right image) by using a sparsity measure. The “optimal” result is given in the middle.
Adaptive Coherent Interferometry

- The smoothing parameters are determined by minimizing

\[
\|J(y^s; \Omega_d, \kappa_d)\|_{L^1(\mathcal{D})} + \alpha \|\nabla y^s J(y^s; \Omega_d, \kappa_d)\|_{L^1(\mathcal{D})},
\]

where \( J(y^s; \Omega_d, \kappa_d) = \frac{I_{\text{CINT}}(y^s; \Omega_d, \kappa_d)}{\max_{y^s \in \mathcal{D}} I_{\text{CINT}}(y^s; \Omega_d, \kappa_d)}. \)

- This is very different from the usual denoising functionals,

\[
\|N(y^s) - I(y^s)\|_{\text{prox}} + \alpha \|I(y^s)\|_{\text{reg}},
\]

where \( N \) is a given noisy image, \( I \) is the desired denoised image, \( \| \cdot \|_{\text{prox}} \) is a proximity norm, usually \( L^2(\mathcal{D}) \), and \( \| \cdot \|_{\text{reg}} \) is a regularization norm, usually TV.

- We do not have an image such as \( N \) so there is no proximity norm part. We use instead the \( L^1 \) norm of the image which is small, when the image is sparse. We do have however the regularization term.
- The sound speed fluctuates about $c_0 = 3\text{km/s}$, the fluctuations are strong $O(1)$ and the correlation length is $\ell = 30\text{cm}$.

- The carrier wavelength is $\lambda_0 = 3\text{m} = 10\ell$, $B = 0.6-1.3\text{kHz}$ (at 6dB) and the range is $\sim 80\lambda_0$. 
Imaging in layered media: KM vs Adaptive CINT