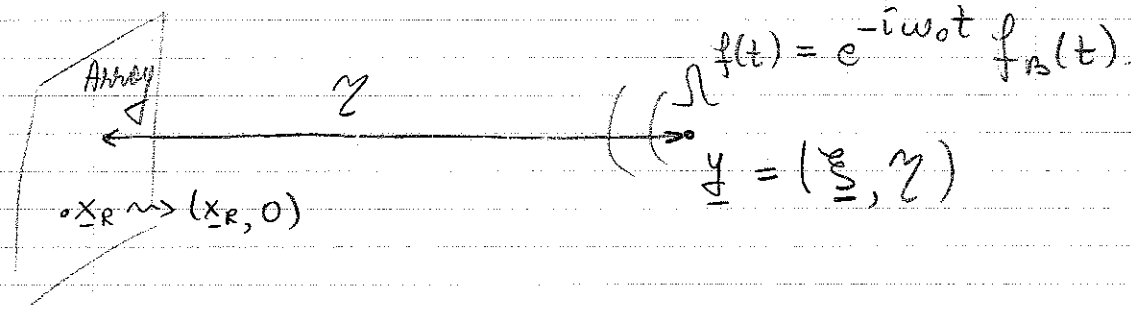
Lecture 4: Interferometry as a stable imaging approach in random media.

Outline:

1) Interferometric (Matched Field) imaging and comparison with TR
2) Resolution analysis.
3) Coherent interferometric imaging

1) Interferometric imaging

Take the case of a point source and a passive array



• In Kirchhoff migration, we migrate the recorded traces

$$(1)\quad P(\underline{x}_R, t) = f(t) * G(\underline{x}_R, t ; \underline{y}) = \int \frac{d\omega}{2\pi} \hat{f}_B(\omega-\omega_0)\, \hat{G}(\underline{x}_R, t; \underline{y})\, e^{-i\omega t}$$

to $\underline{y}^s \in \mathcal{D}$, using travel times $\tau(\underline{x}_R, \underline{y}^s)$ computed in the smooth part of the medium:

$$(2)\quad \mathcal{I}^{KM}(\underline{y}^s) = \int d\omega \sum_{\underline{x}_R \in \mathcal{A}} \hat{P}(\underline{x}_R, \omega)\, e^{-i\omega \tau(\underline{x}_R, \underline{y}^s)}$$

• Problem: We do not deal with the clutter effect on the traces. The only chance for removing the "clutter noise" in (2) is by summation over $\underline{x}_R$ and $\int d\omega$. But this

would work only for additive, uncorrelated noise and (2)
clutter effects are not like that.

TR = ideal imaging and it is what the Least-Squares solution (recall Lecture 2) tells us to do:

$$(3)\quad \mathcal{J}^{TR}(\underline{y}^S) = \int d\omega \sum_{\underline{x}_R \in \mathcal{A}} \hat{P}(\underline{x}_R,\omega)\, \overline{\hat{G}(\underline{x}_R,\omega;\underline{y}^S)} \sim \Gamma^{TR}(\underline{y}^S, t=0)$$

where $\Gamma^{TR}(\underline{y}^S, t)$ = time reversal field observed at $\underline{y}^S$:

$$(4)\quad \Gamma^{TR}(\underline{y}^S, t) = \int \frac{d\omega}{2\pi} \sum_{x_R \in \mathcal{A}} \hat{P}(\underline{x}_R,\omega)\, \overline{\hat{G}(\underline{x}_R,\omega;y^S)}\, e^{-i\omega t}$$

$$\simeq \langle \Gamma^{TR}(\underline{y}^S, t)\rangle .$$

We call $\mathcal{J}^{TR}$ ideal because $\rightsquigarrow$ it is statistically stable
$\rightsquigarrow$ it focuses better in random media

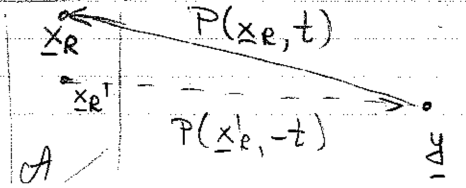
- We cannot achieve (3) because we do not know the fluctuations in the medium, i.e. $\hat{G}(\underline{x}_R,\omega;\underline{y}^S)$ $\forall\, \underline{y}^S \in \mathcal{D}$.

Interferometry: we suppress the clutter effects and obtain statistically stable results by working with interferograms:

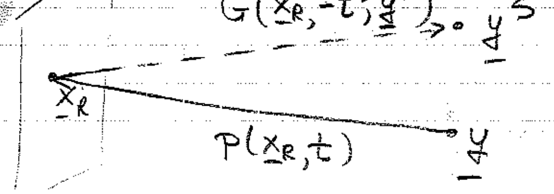
$$(5)\quad P(\underline{x}_R, t) * P(\underline{x}_R', -t) = \int \frac{d\omega}{2\pi}\, \hat{P}(\underline{x}_R,\omega)\, \overline{\hat{P}(\underline{x}_R',\omega)}\, e^{-i\omega t} .$$

Note: In light of (1), we see a strong similarity between the mathematical expression of (5) and (4).

Interferometry



TR:



- The statistical stability of (5) follows just as that of (4), due to the cancellation of random phases and the integration over frequency.

Note: $\langle \hat{P}(\underline{x}_R, \omega) \overline{\hat{P}(\underline{x}'_R, \omega)} \rangle \simeq 0$ for $|\underline{x}_R - \underline{x}_R'| \geq X_d(\omega)$, which means that $\hat{P}(\underline{x}_R, \omega)$ and $\hat{P}(\underline{x}_{R'}, \omega)$ are statistically uncorrelated if the receivers are at distance greater than decoherence length $X_d(\omega)$.

- Because of similarity of (5), (4), we see that $X_d(\omega)$ is the same as radius of support of $\langle \hat{\Gamma}^{TR}(\underline{y}^S, \omega) \rangle$ for a very small array ($\mathcal{A}$ shrinked to ~ point).
- Recalling the previous lecture,

$$(6) \quad X_d(\omega) = \frac{\eta}{k a_e} \sim \frac{\lambda \eta}{a_e}$$

with a_e = "effective aperture" due to the random medium at distance of propagation η.

The interferometric imaging function is:

$$\mathcal{J}^{INT}(\underline{y}^S) = \int d\omega \left| \sum_{x_R \in \mathcal{A}} \hat{P}(\underline{x}_R, \omega) e^{-i\omega \tau(\underline{x}_R, \underline{y}^S)} \right|^2 \cong$$

$$(7) \qquad \cong \int d\omega \sum_{x_R \in \mathcal{A}} \sum_{x'_R \in \mathcal{A}} \langle \hat{P}(\underline{x}_R, \omega) \overline{\hat{P}(\underline{x}'_R, \omega)} \rangle \exp\{-i\omega \tau(\underline{x}_R, \underline{y}^S) + i\omega \tau(\underline{x}'_R, \underline{y}^S)\},$$

where the double sum extends, in fact, over $x_R, x_R' \in \mathcal{A}$ s.t. $|x_R - x_R'| \leq X_d(\omega)$.

• Let us rewrite (7) using the approximation $\sum_{x_r \in \mathcal{A}} \sim \int_{\mathcal{A}} d\underline{x}$ and the change of variables: $\frac{x+x'}{2} = \bar{x}$; $x - x' = \tilde{x}$: (4)

$$J^{INT}(\underline{y}^s) \sim \int_{\mathcal{A}} d\bar{x} \int d\tilde{x} \int d\omega \; \hat{\phi}(k\tilde{x}; X_d^{-1}) \langle \hat{P}(\bar{x}+\tfrac{\tilde{x}}{2}, \omega) \cdot \overline{\hat{P}(\bar{x}-\tfrac{\tilde{x}}{2}, \omega)} \rangle e^{-i\omega\left[\tau(\bar{x}+\frac{\tilde{x}}{2}, y^s) - \tau(\bar{x}-\frac{\tilde{x}}{2}, y^s)\right]} \quad (8)$$

where $\hat{\phi}$ is a window function of support in the disk of radius X_d^{-1}. This window ensures $|\tilde{x}| \leq X_d(\omega)$ so, by (6), $K_d = \frac{a_e}{\eta}$.

• Because usually $X_d(\omega)$ is small, we can linearize the phase in (8):

$$(9) \quad \tau(\bar{x}+\tfrac{\tilde{x}}{2}, y^s) - \tau(\bar{x}-\tfrac{\tilde{x}}{2}, y^s) \simeq \tilde{x} \cdot \nabla_x \tau(\bar{x}, y^s) = \frac{1}{c_0} \tilde{x} \cdot \frac{(\bar{x} - \xi^s)}{|(\bar{x},0) - y^s|}$$
$$= \frac{1}{c_0} \tilde{x} \cdot \underline{K}(\underline{y}^s),$$

where $\underline{K}(y^s) = \frac{\bar{x} - \xi^s}{|(\bar{x},0) - y^s|}$ is the direction of arrival from $\underline{y}^s$, to $(\bar{x}, 0)$, in a homogeneous medium.

• Substitute (9) in (8) and observe that:

$$(10) \quad \hat{\phi}(k\tilde{x}; K_d^{-1}) \, e^{-ik\tilde{x}\cdot\underline{K}(y^s)} = \int \frac{d\underline{K}}{(2\pi)^2} \phi(\underline{K}(y^s) - \underline{K}; K_d) \cdot e^{-ik\tilde{x}\cdot\underline{K}},$$

where $\phi(\underline{K}, K_d) = \int \hat{\phi}(k\tilde{x}; K_d^{-1}) \, e^{-ik\tilde{x}\cdot\underline{K}} \, d(k\tilde{x})$ = window of support $O(K_d)$. The arguments of ϕ are direction vectors $\underline{K}$.

- The imaging function becomes:

$$\mathcal{I}^{INT}(\underline{y}^s) \sim \int d\bar{x} \int d\tilde{x} \int d\underline{K}\; \phi(\underline{K}(\underline{y}^s) - \underline{K}; K_d) \cdot$$

$$\left\langle \int d\omega \underbrace{\left[\hat{P}(\bar{x}+\tfrac{\tilde{x}}{2},\omega) e^{-i\frac{\omega}{c_0}\frac{\tilde{x}}{2}\cdot\underline{K}}\right]}_{\substack{\text{Fourier coeff. of} \\ P(\bar{x}+\frac{\tilde{x}}{2},\, t+\frac{\underline{K}\cdot\tilde{x}}{2c_0})}} \overline{\underbrace{\left[\hat{P}(\bar{x}-\tfrac{\tilde{x}}{2},\omega) e^{i\frac{\omega}{c_0}\frac{\tilde{x}\cdot\underline{K}}{2}}\right]}_{\substack{\text{Fourier coeff. of} \\ P(\bar{x}-\frac{\tilde{x}}{2},\, t-\frac{\underline{K}\cdot\tilde{x}}{2c_0})}}} \right\rangle$$

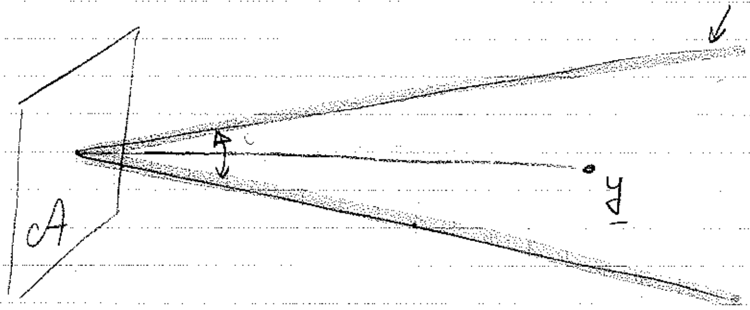
therefore,

$$\mathcal{I}^{INT}(\underline{y}^s) \sim \int d\bar{x} \int d\tilde{x} \int d\underline{K}\; \phi(\underline{K}(\underline{y}^s) - \underline{K}; K_d) \cdot$$

(11)
$$\left\langle \left[P\left(\bar{x}+\frac{\tilde{x}}{2}, t+\frac{\underline{K}\cdot\tilde{x}}{2c_0}\right) * P\left(\bar{x}-\frac{\tilde{x}}{2}, -t-\frac{\underline{K}\cdot\tilde{x}}{2c_0}\right)\right]\Bigg|_{t=0} \right\rangle$$

2) Resolution analysis

- It is clear from (11) that in $\mathcal{I}^{INT}(y^s)$ we have lost all the range information.
- We only have direction of arrival information, due to $\underline{K}(\underline{y}^s)$. But, our uncertainty of the medium is translated into our uncertainty in the direction of arrival, quantified by $K_d = \frac{a_e}{\eta}$. This appears in the form of smoothing, by convolution with ϕ, over the arrival directions $\Rightarrow$ we loose resolution in cross-range.



cone of uncertainity has opening angle: $2\arcsin(Kd)$

Note: We could reach the same conclusion through an explicit calculation, that uses a model for the random medium and a particular wave scattering regime (scaling). In that case, we have the moment formula:

$$\left\langle \hat{P}(\bar{x}+\tfrac{\tilde{x}}{2},\omega)\,\overline{\hat{P}(\bar{x}-\tfrac{\tilde{x}}{2},\omega)} \right\rangle \simeq \hat{P}_0(\bar{x}+\tfrac{\tilde{x}}{2},\omega)\,\overline{\hat{P}_0(\bar{x}-\tfrac{\tilde{x}}{2},\omega)}\;\exp\left\{-\frac{k^2 a_e^2}{2\eta^2}|\tilde{x}|^2\right\} \tag{12}$$

Substituting (12) in (8) and forgetting the window, since the Gaussian in (12) ensures that $|\tilde{x}| \leq X_d(\omega)$, we have:

$$\mathcal{J}^{INT}(\underline{y}^s) \sim \int d\omega\, |\hat{f}_B(\omega-\omega_0)|^2 \int d\bar{x}\int d\tilde{x}\; e^{-\frac{k^2 a_e^2}{2\eta^2}|\tilde{x}|^2 + ik\tilde{x}\cdot[\underline{K}(y)-\underline{K}(y^s)]} \tag{13}$$

where we used that

$$\hat{P}_0(\bar{x}\pm\tfrac{\tilde{x}}{2},\omega) = \hat{f}_B(\omega-\omega_0)\,\frac{e^{i\omega\tau(\bar{x}\pm\frac{\tilde{x}}{2},y)}}{|(\bar{x}\pm\frac{\tilde{x}}{2},0)-y|} \simeq \hat{f}_B(\omega-\omega_0)\,\frac{e^{i\omega\tau(\bar{x},y)\pm i\frac{k\tilde{x}}{2}\cdot K(\underline{y})}}{|(\bar{x},0)-y|}, \tag{14}$$

$\underline{K}(y) = \dfrac{\bar{x}-\xi}{|(\bar{x},0)-y|}$ and where we neglected the amplitudes

An explicit calculation yields:

$$(15)\quad \mathcal{J}^{INT}(\underline{y}^s) \sim \exp\left\{-\frac{|\underline{K}(\underline{y}^s)-\underline{K}(\underline{y})|^2}{2\; K_d^2}\right\} \;;\quad K_d = \frac{a_e}{\eta}$$

Conclusion: Although we began with expressions that looked similar to those in TR, $\mathcal{J}^{INT}(\underline{y}^s)$ has a VERY POOR resolution in comparison to $\mathcal{J}^{TR}(\underline{y}^s)$.

- In $\mathcal{J}^{INT}$, we exploited the coherence in the direction of arrival in order to get statistical stability. We get back an estimate of the direction of the target, but with uncertainty K_d.
- Next, we exploit the coherence in frequency, to get back a much better image.

3) Coherent Interferometric Imaging