Lecture 3: Optimal illumination for imaging ⇝ smooth media.

Outline

1) Contrast between the detection and migration imaging
2) Algorithm for optimal waveform design
*3) Analysis for a point scatterer ⇝ Point Spread Function

1) Detection vs. Migration imaging

• As it was explained in the previous lecture, frequency by frequency, it is best for detection to send signals:

$$(1) \quad \hat{g}(\underline{x}_s, \omega) = \frac{1}{\sqrt{B}} \hat{g}_s^{(M)}(\omega) \quad , \quad |\omega - \omega_0| \le \frac{B}{2} \, ,$$

where $\underline{\hat{g}}^{(M)}(\omega) = \begin{pmatrix} \hat{g}_1^{(M)}(\omega) \\ \vdots \\ \hat{g}_{N_s}^{(M)}(\omega) \end{pmatrix}$ = eigenvector of time reversal matrix $\hat{\Pi}^*(\omega)\hat{\Pi}(\omega)$, corresponding to the largest eigenvalue.

• Let us consider a simple example with a point target at $\underline{y}$. The impulse response matrix is:

$$(2) \quad \hat{\Pi}_{Rs}(\omega) = \hat{\xi}(\omega)\, \hat{G}(\underline{x}_R, \omega; \underline{y})\, \hat{G}(\underline{y}, \omega, \underline{x}_s) \, ,$$

where $\hat{\xi}(\omega)$ = reflectivity of the target (recall the Born approximation model from lecture 1, eq. (15)).

In compact form: (3) $\hat{\Pi}(\omega) = \hat{\xi}(\omega)\, \underline{\hat{G}}(:, \omega; \underline{y})\, \underline{\hat{G}}^T(:, \omega; \underline{y})$

The TR matrix is:

$$\hat{\Pi}^*(\omega)\hat{\Pi}(\omega) = |\hat{\xi}(\omega)|^2 \, \|\underline{\hat{G}}(:,\omega;\underline{y})\|^2 \, \overline{\underline{\hat{G}}(:,\omega;\underline{y})} \, \underline{\hat{G}}^T(:,\omega;\underline{y})$$

so, $$\underline{\hat{g}}^M(\omega) = \frac{\overline{\underline{\hat{G}}(:,\omega;\underline{y})}}{\|\underline{\hat{G}}(:,\omega;\underline{y})\|} \quad \text{and} \quad \lambda^{(M)}(\omega) = |\hat{\xi}(\omega)|^2 \|\underline{\hat{G}}(:,\omega,\underline{y})\|^4$$

$\Rightarrow$ The signal received at receiver $\underline{x}_R$, when sending signals (1) simultaneously, from all N_s sources is:

$$\hat{p}(\underline{x}_R,\omega) = \sum_{s=1}^{N_s} \hat{g}(\underline{x}_s,\omega)\,\hat{\Pi}_{Rs}(\omega) \sim$$

$$(3) \qquad \sim \sum_{s=1}^{N_s} \hat{\Pi}_{Rs}(\omega)\, e^{-i\omega\tau(\underline{x}_s,\underline{y})}$$

where I used the approximation:

$$(4) \qquad \underline{\hat{G}}(x_s,\omega;\underline{y}) \sim e^{-i\omega\tau(\underline{x}_s,\underline{y})}.$$

This is OK as explained in lecture 1, for smooth media and for remote targets such that the amplitudes of the Green's function don't vary much over the array aperture.

Conclusion: In detection, we send signals (1) simultaneously from all sources and deliver efficiently energy on the target at $\underline{y}$ by beamforming. Mathematically, this is achieved with the phases $\omega\tau(\underline{x}_s,\underline{y})$ in (3).

- In imaging, we do not send signals simultaneously from all $\underline{x}_s$. Migration imaging requires measurements of traces for one source at a time, so that we can migrate these traces and achieve the cancellation of phases at the targets.

• To image with N_s sources, we imagine that we sent signal $\hat{f}(\underline{x}_s, \omega)$ from the source at $\underline{x}_s$ and measure the echoes

(5) $\hat{f}(\underline{x}_s, \omega)\, \hat{\Pi}_{R,s}(\omega)$, $\quad |\omega - \omega_0| \le \frac{B}{2}$.

The image is obtained as:

(6) $\mathcal{J}^{KM}(\underline{y}^s; \hat{f}) \sim \int d\omega \sum_{R=1}^{N_R} \left[\sum_{s=1}^{N_s} \hat{f}(\underline{x}_s, \omega)\, \hat{\Pi}_{R,s}(\omega)\, e^{-i\omega \tau(\underline{x}_s, \underline{y}^s)} \right] e^{-i\omega \tau(\underline{x}_R, \underline{y}^s)}$

Note: In imaging, we have "illumination" functions

(7) $\hat{g}(\underline{x}_s, \omega) = \hat{f}(\underline{x}_s, \omega)\, e^{-i\omega \tau(\underline{x}_s, \underline{y}^s)}$

that allow us to beamform on $\underline{y}^s$, where we look for a target.

Question is: How to choose $\hat{f}(\underline{x}_s, \omega)$ so that we get the best possible image?

How do we measure the image quality? First, we normalize the image by its maximum

(8) $\bar{\mathcal{J}}(\underline{y}^s; \hat{f}) = \dfrac{\mathcal{J}^{KM}(\underline{y}^s; \hat{f})}{\max_{\underline{y}^s \in \mathcal{D}} |\mathcal{J}^{KM}(\underline{y}^s; \hat{f})|}$,

because we are interested in the geometrical features of the

image and not the magnitude. Obviously, $|\mathcal{J}(\underline{y}^s;\hat{f})| \in [0,1]$.
Now, to get a tight image, i.e. a small support of $\mathcal{J}$, we seek to minimize:

$$(9) \quad \mathcal{O}(\hat{f}) = \|\mathcal{J}(\underline{y}^s;\hat{f})\|^2_{L^2(\mathcal{D})}$$

Note: Slightly better results are obtained using the L^1 norm or L^p, for $0<p<1$ or other sparsity promoting measures. We choose (9) because it is convenient for analysis and computations. Expression (9) is also justified through its resemblance to the $L^1(\mathcal{D})$ norm of the coherent interferometric imaging function that we describe later in the course.

2) Algorithm for optimal waveform design

- Our goal is to minimize (9) over a class of illuminations $\hat{f}(\underline{x}_s,\omega)$.
- To reduce the size of the problem, we begin with the assumption that:

$$(10) \quad \hat{f}(\underline{x}_s,\omega) = w_s \hat{f}_B(\omega-\omega_0) \qquad \forall s=1,\dots N_s$$
$$|\omega-\omega_0| \le \frac{B}{2}$$

- This corresponds to sending the same pulse, given by $f(t) = e^{-i\omega_0 t} f_B(t)$, from all sources and recording $f(t) * \Pi_{RS}(t)$.

- The w_s are then weights assigned to the sources and so, they satisfy:

$$(11) \quad w_s \ge 0 \quad ; \quad \sum_{s=1}^{N_s} w_s = 1.$$

- The pulse $\hat{f}_B(\omega-\omega_0)$ is restricted to the frequency band $[\omega_0 - \frac{B}{2}, \omega_0 + \frac{B}{2}]$ and for the sake of simplicity, we ask that

$$(12) \qquad \hat{f}_B(\omega-\omega_0) \geq 0 \quad \& \quad \int d\omega\, \hat{f}_B(\omega-\omega_0) = 1.$$

Note 1: - Choice $\hat{f}_B \geq 0$ has a good justification for imaging in random media (we shall see this later)

- Even in smooth media, we shall see that $\|\mathcal{J}^{KM}\|^2_{L^2(\mathcal{D})}$ depends just on $|\hat{f}_B(\omega-\omega_0)|$ and not $\hat{f}_B$ itself! (This is however a consequence of assumption (10)).

Note 2: - All the choices made above turn out to be optimal for a true point target (Point Spread Function (PSF)). They may not be optimal in general but they lead to stable optimization that can be easily done.

Note 3: When noise at the receivers is an issue, we must take into account that what we measure is:

$$f(t) * \Pi_{Rs}(t) + \text{noise}, \quad \forall s = 1, \dots N_s$$

so we ask that:

$$(13) \quad w_s^2 \Big\{ \sum_{R=1}^{N_R} \underbrace{\int d\omega\, |\hat{f}_B(\omega-\omega_0)|^2 |\hat{\Pi}_{Rs}(\omega)|^2}_{\overset{\text{(Parseval)}}{=} \int dt\, |f(t) * \Pi_{Rs}(t)|^2} - \mathcal{N} \Big\} \geq 0, \quad \forall s = 1, \dots N_s$$

$\mathcal{N}$ reflects the noise level

If we let $\mathcal{P}$ = maximal power achievable by sending a pulse from any source:

$$(14)\quad \mathcal{P} = \max_{s=1,\dots N_s} \max_{\hat{f}_s \text{ satisfying } (12)} \sum_{R=1}^{N_R} \int d\omega \, |\hat{f}_s(\omega-\omega_0)|^2 \, |\hat{\Pi}_{Rs}(\omega)|^2,$$

we can define a proxy SNR:

$$(15)\quad SNR = \frac{\mathcal{P}}{\mathcal{N}}.$$

- The higher the SNR, the less important is the constraint (13) and the better the image (you will see this in the numerical results). At low SNR, we get waveforms $\hat{f}_s$ that are narrow band, similar to what we saw in detection ⇒ images deteriorate.

3) Analysis of the optimal illumination problem for a point scatterer (PSF study) in homogeneous medium

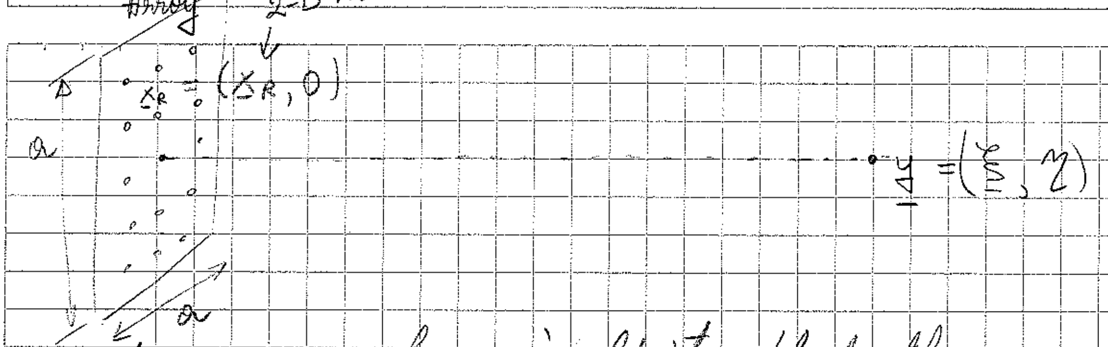
- Let us begin by computing the image: given that for a true point scatterer at location $\underline{y}$,

$$(16)\quad \hat{\Pi}(\omega) = \hat{\underline{G}}(:,\omega,\underline{y})\, \hat{\underline{G}}^T(:,\omega;\underline{y}).$$

The imaging function (6) is:

$$(17)\quad \mathcal{I}^{KM}(\underline{y}^S, \hat{\underline{f}}) \sim \int d\omega \sum_{R=1}^{N_R} \sum_{s=1}^{N_s} \hat{f}(\underline{x}_s,\omega)\, \hat{G}(\underline{x}_R,\omega;\underline{y})\, \hat{G}(\underline{x}_s,\omega;\underline{y}) \exp\{-i\omega[\tau(\underline{x}_R,\underline{y}^S)+\tau(\underline{x}_s,\underline{y}^S)]\}$$

• The set-up



• We suppose for simplicity that the array is planar, of square shape and aperture a. The orientation of the array is orthogonal to the direction of propagation.

• We distinguish between range and cross-range coordinates in the notation.

• Each element in the array plays the double role of a source and receiver so $N_s = N_R = N$ = number of transducers.

• We suppose a dense sampling of the array aperture s.t.

$$(18) \quad \sum_{R=1}^{N} \varphi(\underline{x}_R) \sim \int_{A} d\underline{x}_R \, \varphi(\underline{x}_R), \qquad \forall \varphi \text{ defined on } A$$

here $\underline{x}_R$ varies continuously in A. Index R reminds us that these are receivers

• We assume a remote target, which means $\eta \gg |\underline{x}_R - \underline{\xi}|$ and

$$(19) \quad \hat{G}(\underline{x}_R, \omega; \underline{y}) \simeq \frac{1}{4\pi\eta} \exp\left\{ i \frac{\omega}{c_0} \sqrt{\eta^2 + |\underline{x}_R - \underline{\xi}|^2} \right\}$$

• The imaging function becomes:

$$(20) \quad I^{KM}(\underline{y}^S; \hat{f}) \sim \frac{1}{\eta^2} \int d\omega \int_{A} d\underline{x}_s \int_{A} d\underline{x}_R \, \hat{f}(x_s, \omega) \exp\Big\{ i\omega \Big[\sqrt{\eta^2 + |\underline{x}_R - \underline{\xi}|^2} - \sqrt{(\eta^S)^2 + |\underline{x}_R - \underline{\xi}^S|^2} + \sqrt{\eta^2 + |\underline{x}_s - \underline{\xi}|^2} - \sqrt{(\eta^S)^2 + |\underline{x}_s - \underline{\xi}^S|^2} \Big] \Big\}$$

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- Linearization of the phase: We know from past lectures that $\mathcal{J}^{KM}$ is large for $\underline{y}^s$ near $\underline{y}$, so we linearize the phase in (20) as:

$$(21)\quad \sqrt{\eta^2 + |\underline{x}_R - \underline{\xi}|^2} - \sqrt{(\eta^s)^2 + |\underline{x}_R - \underline{\xi}^s|^2} \simeq \left(\frac{\underline{\xi} - \underline{x}_R}{\sqrt{\eta^2 + |\underline{x}_R - \underline{\xi}|^2}}, \frac{\eta}{\sqrt{\eta^2 + |\underline{x}_R - \underline{\xi}|^2}} \right) \cdot (\underline{\xi} - \underline{\xi}^s, \eta - \eta^s)$$

$$\simeq (\underline{\xi} - \underline{\xi}^s) \cdot \frac{(\underline{\xi} - \underline{x}_R)}{\eta} + \eta - \eta^s. \implies$$

$$(22)\quad \mathcal{J}^{KM}(\underline{y}^s; \hat{f}) \sim \frac{1}{\eta^2} \int_{\mathcal{A}} d\underline{x}_s \int d\omega\, \hat{f}(\underline{x}_s, \omega)\, e^{2i\frac{\omega}{c_0}(\eta - \eta^s) + \frac{2i\omega}{c_0}\frac{(\underline{\xi} - \underline{\xi}^s)}{\eta} \cdot \left(\underline{\xi} - \frac{\underline{x}_s}{2}\right)}$$

$$\underbrace{\int_{\mathcal{A}} d\underline{x}_R \exp\left[-\frac{\omega}{c_0} \frac{(\underline{\xi} - \underline{\xi}^s)}{\eta} \cdot \underline{x}_R \right]}_{\simeq \prod_{j=1}^{2} 2\sin\left[\frac{\omega}{c_0} \frac{a}{2\eta} e_j \cdot (\underline{\xi} - \underline{\xi}^s) \right] \Big/ \frac{\omega}{c_0} \frac{e_j \cdot (\underline{\xi} - \underline{\xi}^s)}{\eta}}$$

- Normalization of the image: Obviously, the maximum of $|\mathcal{J}^{KM}(\underline{y}^s; \hat{f})|$ occurs at $\underline{y}^s = \underline{y}$, where the phases cancel out precisely $\implies$

$$(23)\quad \max_{y^s} |\mathcal{J}^{KM}(\underline{y}^s; \hat{f})| \sim \frac{|\mathcal{A}|}{\eta^2} \left| \int_{\mathcal{A}} d\underline{x}_s \int d\omega\, \hat{f}(\underline{x}_s, \omega) \right|,$$

where $|\mathcal{A}| = a^2$.

This suggests that we can keep the $\max |\mathcal{J}^{KM}|$ constant by simply asking that:

$$(24)\quad \int_{\mathcal{A}} d\underline{x}_s \int d\omega\, \hat{f}(\underline{x}_s, \omega) = \text{constant} = 1.$$

This is a normalization of the illuminations, as we see next.

Now, the optimization of (9) is equivalent to:

(25) minimize $\| \mathcal{I}^{KM}(\underline{y}^s; \hat{f}) \|^2_{L^2(\mathcal{D})}$ over $\hat{f}$, subject to (24).

From (22), we have:

$$\| \mathcal{I}^{K,M}(\underline{y}^s; \hat{f}) \|^2_{L^2(\mathcal{D})} \sim \int_{\mathcal{A}} d\underline{x}_s \int_{\mathcal{A}} d\underline{x}'_s \int d\omega \int d\omega' \, \hat{f}(\underline{x}_s, \omega) \overline{\hat{f}(\underline{x}'_s, \omega')}$$

(26)
$$\int d\underline{\xi}^s \int d\eta^s \prod_{j=1}^{2} \frac{\sin\left[\frac{\omega a}{2c_0}\left(\frac{\underline{\xi} - \underline{\xi}^s}{\eta}\right)\cdot \underline{e}_j\right] \sin\left[\frac{\omega' a}{2c_0}\left(\frac{\underline{\xi} - \underline{\xi}^s}{\eta}\right)\cdot \underline{e}_j\right]}{\left[\frac{\omega}{c_0}\left(\frac{\underline{\xi} - \underline{\xi}^s}{\eta}\right)\cdot \underline{e}_j\right]\left[\frac{\omega'}{c_0}\left(\frac{\underline{\xi} - \underline{\xi}^s}{\eta}\right)\cdot \underline{e}_j\right]} \cdot$$

$$\exp\left\{ 2i\frac{(\omega-\omega')}{c_0}(\eta - \eta^s) + 2i\frac{\omega}{c_0}\frac{(\underline{\xi}-\underline{\xi}^s)\cdot\left(\underline{\xi} - \frac{\underline{x}_s}{2}\right)}{\eta} - 2i\frac{\omega'}{c_0}\frac{(\underline{\xi}-\underline{\xi}^s)\cdot\left(\underline{\xi} - \frac{\underline{x}'_s}{2}\right)}{\eta} \right\}$$

Now,

(27) $\int d\eta^s \, e^{2i\frac{(\omega-\omega')}{c_0}(\eta-\eta^s)} \sim \delta(\omega - \omega')$ if we have a large domain, centered at the target. Using (27) in (26),

$$\| \mathcal{I}^{KM}(\underline{y}^s; \hat{f}) \|^2_{L^2(\mathcal{D})} \sim \int_{\mathcal{A}} d\underline{x}_s \int_{\mathcal{A}} d\underline{x}'_s \int d\omega \, \hat{f}(\underline{x}_s, \omega) \overline{\hat{f}(\underline{x}'_s, \omega)} \cdot$$

(28)
$$\int d\underline{\xi}^s \prod_{j=1}^{2} \frac{\sin^2\left[\frac{\omega a}{2c_0}\left(\frac{\underline{\xi}-\underline{\xi}^s}{\eta}\right)\cdot \underline{e}_j\right]}{\left[\frac{\omega}{c_0}\left(\frac{\underline{\xi}-\underline{\xi}^s}{\eta}\right)\cdot \underline{e}_j\right]} e^{i\frac{\omega}{c_0}\frac{(\underline{\xi}-\underline{\xi}^s)}{\eta}\cdot(\underline{x}'_s - \underline{x}_s)}$$

. It turns out that we can integrate over $\underline{\xi}^s$ explicitly (again assuming a sufficiently large domain) and obtain: $\prod_{j=1}^{2} \frac{\eta}{\omega}\left[1 - \frac{|\underline{e}_j^T\cdot(\underline{x}'_s - \underline{x}_s)|}{a}\right]$ (29)

Thus, our objective function is:

$$\|\mathcal{J}^{KM}(y^s, \hat{f})\|^2_{L^2(\mathcal{D})} \sim \int_{\mathcal{A}} d\underline{x}_s \int_{\mathcal{A}} d\underline{x}_s' \int d\omega \; \hat{f}(\underline{x}_s, \omega) \overline{\hat{f}(\underline{x}_s', \omega)} \tag{30}$$

$$\prod_{j=1}^{2} \frac{1}{\omega}\left[1 - \frac{|e_j^T \cdot (x_s' - x_s)|}{a}\right]$$

The minimizer of (30), subject to constraint (24) satisfies:

$$\int_{\mathcal{A}} d\underline{x}_s' \; \hat{f}(\underline{x}_s', \omega) \prod_{j=1}^{2} \frac{1}{\omega}\left[1 - \frac{|\underline{e}_j^T \cdot (\underline{x}_s' - \underline{x}_s)|}{a}\right] = \tag{31}$$

$$= \lambda \quad \forall \quad \omega \in \left[\omega_0 - \frac{B}{2}, \omega_0 + \frac{B}{2}\right] \text{ and } \forall\, \underline{x}_s \in \mathcal{A}$$

λ ↳ Lagrange multiplier

Because the right hand side is constant => solution is:

$$\hat{f}(\underline{x}_s, \omega) = \frac{\omega^2}{\int d\omega\, \omega^2} \prod_{j=1}^{2} \left\{\delta\left[\underline{e}_j \cdot (\underline{x}_s - \underline{\xi}) - \frac{a}{2}\right] + \delta\left[\underline{e}_j \cdot (\underline{x}_s - \underline{\xi}) + \frac{a}{2}\right]\right\} \tag{32}$$

which satisfies the assumptions made in the algorithm.

Conclusion: At infinite SNR, the optimal PSF (as measured by L^2 norm) is given by a pulse that increases quadratically in ω (in 2-D it will increase linearly) and weights placed at the corner of the array.

This result changes for finite SNR, due to constraints (13).