Lecture 2: Migration imaging as an approximate solution of the linearized inverse problem, in least squares sense.

Outline:

1) Formulation of the least squares problem
2) Migration as an approximate solution (is what worse?)
3) Adjoint calculation details.

1) The least squares problem

- Consider for now a single source at \( x_0 \) and assume that we measure at \( x_R \in \mathcal{A} \), for \( t \in [0,T] \), absorb \( d(x_R,t) \).

- For convenience, let us assume that we have small enough spacing between the receivers so that we can take:

\[
\int_A x_R^\top \sim \int d x_R
\]

- Recall our linearized model for the measurements:

\[
(1) \quad F[c] \phi_c = \{ \delta p(x_R,t) , x_R \in \mathcal{A} , t \in [0,T] \}
\]

where \( \delta p \) is the solution of the linearized seismic eq.

- We seek \( \phi_c \) such that model (1) predicts our measurements:

\[
(2) \quad (F[c] \phi_c)(x_R,t) = \delta p(x_R,t) \approx d(x_R,t) \quad \text{where the approximation is in least squares sense. Explicitly, we minimize}
\]

\[
(3) \quad \mathcal{I}[\phi_c] = \frac{1}{2} \int_0^T \int_A d(x_R) \Phi(x_R) |(F[c] \phi_c)(x_R,t) - d(x_R,t)|^2
\]

we understand dependence on \( c \) & drop it from now on.
The adjoint operator $F^*$

1. Let us define formally, the adjoint $F^*$ of $F$ by

$$
\int_{\mathbb{R}^3} \int_0^T dt \int d\mathbf{x} \psi (\mathbf{x}, t) (F \delta c)(\mathbf{x}, t) = \int_{\mathbb{R}^3} \delta c(x) (F^* \psi)(x)
$$

2. Thus, $F$ acts on $\delta c$ and returns echoes at $\mathbf{x}, t$, whereas operator $F^*$ takes functions $\psi (\mathbf{x}, t)$ (assumed in $L^2(c) \times \mathbb{R}^3$) to $L^2(\mathbb{R}^3)$ functions of $x$.

3. Taking the first variation in (3), we have that $\delta c$ must satisfy:

$$
(F^* \delta \xi) (x) = 0 \quad \forall x \in \mathbb{R}^3,
$$

where

$$
\xi (\mathbf{x}, t) = \left[ (F \delta c)(\mathbf{x}, t) - c \delta (\mathbf{x}) \right] \phi (\mathbf{x}, t)
$$

is the misfit.

The expression of $F^* \delta \xi$ is (we prove this in part 3).

$$
(F^* \delta \xi) (x) = - \frac{2}{c^3(x)} \int_0^T \delta \phi (x, t) \frac{\partial^2 \rho}{\partial t^2} (x, T-t) dt,
$$

where $\rho = \text{pressure field in the background (sound speed } c(x) \text{)}$ and $\delta \phi$ solves the adjoint problem:

$$
\left\{ \begin{array}{l}
\left( \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta \right) \delta \phi (x, t) = - \int d\mathbf{x} \delta c(\mathbf{x}, T-t) \delta (x - \mathbf{x}) \\
\delta \phi (x, t) = 0 \text{ for } t \leq 0
\end{array} \right.
$$
Note: The calculation of the adjoint solution \( \mathbf{F}^\ast (x, t) \) amounts physically to taking the limit \( \mathbf{E} (x, t) \), time reversing it, and sending it back in the medium, to location \( x \).

- We can rewrite (7) in the Fourier domain, and obtain after a straightforward calculation:

\[
\left( \mathbf{F}^\ast \mathbf{E} \right) (x) = - \frac{2}{c_0^3 (x)} \frac{1}{2 \pi} \int \frac{d \omega}{2 \pi} \int \frac{d \omega'}{2 \pi} \left( \mathbf{F}^\ast \mathbf{E} \left( x, \omega, \omega' \right) \right) \mathbf{G} (x, \omega; x_0) \mathbf{G} (x, \omega'; x_0) \mathbf{E} (x_0, \omega') \mathbf{E} (x_0, \omega) \\
\approx 8 (\omega - \omega')^2 \text{ for very long time windows}
\]

\[
\approx \frac{1}{\pi c_0^3 (x)} \int \frac{d \omega}{2 \pi} \int \frac{d \omega'}{2 \pi} \left( \mathbf{F}^\ast \mathbf{E} \left( x, \omega, \omega' \right) \right) \mathbf{G} (x, \omega; x_0) \mathbf{G} (x, \omega'; x_0) \mathbf{E} (x_0, \omega) \\
\]

2) Motion

- The L-S solution satisfies (5) or, equivalently,

\[
\mathbf{F}^\ast \left( \left( \mathbf{F} \mathbf{E} \right) \phi \right) (x) = \mathbf{F}^\ast \left( \chi \phi \right) (x)
\]

which are the normal equations.

- Operator \( \mathbf{F}^\ast \mathbf{F} \) is not invertible in the usual sense but it acts almost as an identity operator on the singularities of \( \mathbf{E} (x) \). That is, it does not move these
Note: \( F^*F \) is the time-reversal operator that will be discussed extensively in this course. This operator has refocusing properties that are very good (depending on aperture and bandwidth) and that get better in non-stationary media.

Conclusion: We can image the singularities of \( \Sigma \) with just the right hand side in (10):

\[
I(x) = F^* \left[ \partial_t \phi \right](x) * \int dx' \phi(x') \int dw \, \omega^2 \frac{\partial^2}{\partial^2 \omega} (w-w_0) \tilde{A}(x_w, \omega)
\]

(11)

\[
\hat{G}(x, \omega; x_0) \hat{G}(x', \omega'; x_0)
\]

Now, we have \( \hat{G}(x, \omega; x_0) \sim \alpha(x, \omega) \hat{\Sigma}(\omega)e^{-i\omega \tau(x, \omega)} \), the high frequency asymptotic formula discussed last time and replace \( x \mapsto y^S = \text{search point mapping to search domain } D \) where we look for the reflector. The result is: Kirchhoff imaging formulae:

\[
I(y^S) = \int \int dx' \phi(x') \alpha(x', y^S) \alpha(y^S, x_0) \int dw \, \omega^2 \hat{A}(x_w, \omega) \frac{i\omega}{[\hat{\Sigma}(\omega)]^2 \hat{A}(x_w, \omega)}
\]

(12)

\[
\exp \left\{ i \omega \left[ \alpha(x, y^S) \alpha(y^S, x_0) - \alpha(x_0, y^S) - \alpha(x, y^S) \right] \right\}
\]

(13)

where \( y \in \text{support of } \Sigma \). Note also that we have a
high frequency regime and therefore, the integral (12) is given approximately by the method of stationary phase. This means that the amplitudes play a negligible role so we can neglect them and obtain a simple imaginary function that we call from now on Kirchhoff Migration (KM):

\[ J^\text{KM}(y^s) = \int d^3 x_0 \phi(x_0) \overline{d(x_0, w)} e^{i w \left[ B(x_0, y^s) + B(x_0, y^s) \right]} \]

\[ = \int \phi(x_0) d(x_0, B(x_0, y^s) + B(x_0, y^s)) d^3 x_0 \]

How does the method of stationary phase give the focus point:

Let us recall the expression of \( d(x_0, t) \), so modeled by Born approximation (see eq. (18) yesterday's notes) and neglect the amplitudes:

\[ J^\text{KM}(y^s) \sim \int d^3 x_0 \phi(x_0) \overline{d(x_0, w)} e^{i w \left[ B(x_0, y^s) + B(x_0, y^s) \right]} \]

\[ \exp \left\{ \frac{i w}{2} \left[ B(x_0, y^s) + B(x_0, y^s) - B(x_0, y^s) - B(x_0, y^s) \right] \right\} \]

where \( n(y) = - \frac{2 \delta c(y) c_0^2}{c^3(y)} \) = reflectivity defined last time.

Now, let us write:

\[ n(y) = \int d^3 \xi \phi(\xi) e^{i \xi \cdot y} \]

\[ n(y) = \int d^3 \xi \phi(\xi) e^{i \xi \cdot y} \]
and obtain:

\[ j^{KM}(y) = \frac{1}{\gamma_0} \int d\theta \hat{P}(\theta) \int d\mathbf{x} \varphi(\mathbf{x}) \sin \omega  \frac{\mathbf{z}^\top}{j_{\gamma_0}} (\omega - \omega_0) \int d\gamma \]

\[(17) \quad \exp \left\{ i \frac{\mathbf{z}^\top \mathbf{y}}{j_{\gamma_0}} + i \omega [\mathbf{Z}(xR, y) + \mathbf{Z}(x_0, y) - \mathbf{Z}(xR, y^S) - \mathbf{Z}(x_0, y^S)] \right\} \]

The method of stationary phase says that when integrating over \( xR \subset A \), \( \omega \in [\omega_0 - \frac{1}{2}, \omega_0 + \frac{1}{2}] \) and \( \gamma_0 \) are such that

because we have high frequency, we obtain the main contribution when the following stationary phase conditions are satisfied:

\[(18.a) \quad \mathbf{Z}(xR, y) + \mathbf{Z}(x_0, y) = \mathbf{Z}(xR, y^S) + \mathbf{Z}(x_0, y^S) \]

\[(18.b) \quad \nabla_{xR} \mathbf{Z}(xR, y) = \nabla_{xR} \mathbf{Z}(xR, y^S) \]

\[(18.c) \quad \frac{\gamma}{j_{\gamma_0}} = -i \omega \nabla_y \left[ \mathbf{Z}(xR, y) + \mathbf{Z}(x_0, y) \right] \]

- Condition (18.a) says that \( y \) & \( y^S \) must lie on the same isochron.
- The gradient in (18.b) is 2-D because \( xR \) sweeps the 2-D array surface. Thus, (18.b) says that \( y \) & \( y^S \) are in the same direction (along the ray) when viewed from \( xR \).

\[ \nabla_{xR} \mathbf{Z}(xR, y) = \frac{\Delta \mathbf{z} - \frac{1}{2} \mathbf{y}}{||\Delta \mathbf{z} - \frac{1}{2} \mathbf{y}||} \quad \text{with} \quad xR = \text{components of} \ xR \]

in plane tongue is A of xR.
Thus, conditions (18.a) & (18.b) determine that \( y^* \) must be the same as some point \( y \in \text{upper} \).

Condition (18.c) tells us what it is that we really image. First of all, note that \( \theta \) points in the normal direction of surfaces, \( \theta(x) e^{i \phi^0 \cdot x} = \text{constant} \) and that \( \theta = \text{along the bisector of the angle} \) made by ray \( x_0 \rightarrow y \) and ray \( y \rightarrow x_0 \).

\[ \Rightarrow \text{surfaces:} \quad y \in \text{upper} \quad \theta \cdot y = \text{constant} \quad \phi^0 \text{act as reflecting surfaces and we have Fermat's law of angle of incidence = angle of reflection.} \]

Second, since \( \theta \) is proportional to \( w \) and since \( w \) is large \( \Rightarrow \) we must have \( 1/\ell \) large too \( \Rightarrow \) we can only determine \( \theta(x) \) for \( 1/\ell \) large.

This is another confirmation that we image just the singularities of \( y(x) \) i.e., its support, in particular.

Note: All this depends on us knowing \( c(x) \), so we can calculate correctly travel times.

Finding out of \( F[\text{I}^{\text{I}}] = \delta t \) both \( \text{I}^{\text{I}} \) and \( c \) is a much harder problem. We linearized in \( \text{I}^{\text{I}} \) but
the dependence on \( c \) is still highly nonlinear. To find \( c \) we need more data.

- A very efficient technique for estimating \( c(x) \) is the differential semblance approach (Symes & collaborators).

The idea behind differential semblance: Uses a lot of data and divides it into smaller data sets (for example according to offsets between source and receivers). Then it constructs images with each subset and compares them. The images will be \( \approx \) identical when the velocity is right.

**Example:**

\[
\begin{align*}
\text{Array model} & : \quad \tilde{x}_R, x_0 \quad \tilde{y}
\end{align*}
\]

The travel time between \( x_R, \tilde{y} \), \( x_0, \tilde{y} \):

\[
\tilde{t}(x_R, \tilde{y}) + \tilde{t}(x_0, \tilde{y}) = \frac{L^2 + \Delta^2}{c_0}
\]

Now, when we have the wrong velocity, \( \hat{c}_0 \), the target is perceived at depth \( \hat{z} \):

\[
2 \sqrt{\hat{z}^2 + \Delta^2} = \frac{2 \sqrt{L^2 + \Delta^2}}{c_0} \Rightarrow \hat{c}_0 = \frac{\sqrt{L^2 + \Delta^2}}{\sqrt{\hat{z}^2 + \Delta^2}}
\]

and this \( \hat{z} = \left( \frac{\hat{c}_0^2 - c_0^2}{c_0^2} \right) \frac{\Delta^2}{4} + \left( \frac{\hat{c}_0}{c_0} \right)^2 L^2 \) changes with the offset. \( \Rightarrow \) The image will change with the offset, \( \hat{c}_0 \neq c_0 \).
3) Details of the adjoint calculation

- We derive the expression of $F^*$, using the linearized version of the wave eq. in 1st order system form.
- Let $u(x,t) = \begin{pmatrix} P(x,t) \\ v(x,t) \end{pmatrix}$ be the four dimensional state variable solving the wave eq. $\frac{\partial u}{\partial t} = H[u, c, t]$ in the medium with speed $c(x)$, where $u(x,0) = 0$ and $H[u, c, t] = \begin{pmatrix} -\frac{1}{\epsilon_0} \nabla \cdot \nabla \cdot u \\ \frac{1}{\epsilon_0} \nabla \cdot u + \frac{E}{\epsilon_0} \end{pmatrix}$.

- The linearization of these equations with respect to $\delta c$ is

$$\frac{2}{\partial t} \delta u = D_u H[u, c, t] \delta u + D_c H[u, c, t] \delta c \quad \text{for } t > 0,$$

$$\delta u = 0 \quad \text{for } t = 0,$$

where:

$$D_u H[u, c, t] \delta u = \begin{pmatrix} -\frac{1}{\epsilon_0} \nabla \cdot \nabla \cdot \delta u \\ \frac{1}{\epsilon_0} \nabla \cdot \delta u \end{pmatrix}; \quad D_c H[u, c, t] \delta c = \begin{pmatrix} -\frac{2}{\epsilon_0} \nabla \cdot \delta u \\ 0 \end{pmatrix}.$$

- The least-squares functional (3) can be rewritten as

$$J[\delta c] = \frac{1}{2} \sum_{k=1}^{T} \int_{A_k} \left[ \int_{A} d[x] \Phi(x_k)(S_\delta u)(x_k, t) - d(x_k, t) \right]^2,$$

where $d$ is the measurement operator

$$(S_\delta u)(x_k, t) = (\Pi \delta u)(x_k, t) \equiv (\Pi S_p)(x_k, t) \equiv S_p(x_k, t)$$

and $\Pi$ = projection of a molar resolved function defined
\[ \forall \mathbf{x} \in \mathbb{R}^3 \text{ to } \mathbf{x} \in \mathcal{A}. \]

- The first variation in (27) is:

\[
\int_{\mathbb{R}^3} \delta c(x) \left[ D^\dagger \right](x) \overset{(27)}{=} \int_{\mathbb{R}^3} \delta c(x) \left[ F^* \mathcal{E} \right](x)
\]

\[
\overset{(27)}{=} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \delta u(x, t) [\mathcal{H}^* (\mathcal{E} \ell^0)] (x),
\]

where \( \mathcal{H}^* \) is the adjoint of the measurement operator \( \mathcal{E} \) is defined by (6).

The adjoint of \( \mathcal{H} \):

For a given \( t \) and arbitrary \( \psi \in L^2(\mathcal{A}) \), we have:

\[
\int_{\mathbb{R}^3} \delta x R(x) (\mathcal{H} \delta u(\cdot, t))(x) = \int_{\mathbb{R}^3} \delta x \delta u(x, t) (\mathcal{H}^* \psi)(x)
\]

(30)

Now, using the definition (28) =>

\[
\int_{\mathbb{R}^3} \delta x \delta u(x, t) (\mathcal{H}^* \psi)(x) = \int_{\mathbb{R}^3} \delta x R(x) \pi [(x, 0) \delta u(\cdot, t)](x)
\]

\[
\overset{(28)}{=} \int_{\mathbb{R}^3} \delta u(x, t) \cdot \left( \begin{array}{c}
\pi\\
0
\end{array} \right)(\pi^* \psi)(x) \quad \Rightarrow
\]

(31) \((\mathcal{H}^* \psi)(x) = \left( \begin{array}{c}
\pi^* \\
0
\end{array} \right)(\pi^* \psi)(x)\)

The adjoint of \( \pi \):

For arbitrary \( \psi \in L^2(\mathcal{A}) \), \( x \in L^2(\mathbb{R}^3) \), we have
\[ \int_A \varphi(x)(\Pi \alpha)(x_R) \, dx_R = \int \alpha(x) (\Pi^* \varphi)(x) \, dx \quad \text{limit, by (28)} \]

\[ \int_A \varphi(x_R)(\Pi \alpha)(x_R) \, dx_R = \int A \varphi(x_R) \, dx_R = \int A \alpha(x) \, dx \]

\[ \int A \varphi(x_R) \delta(x - x_R) = \]

\[ (32) \quad (\Pi^* \varphi)(x) = \int A \varphi(x_R) \delta(x - x_R) \]

\[ \text{Now, using (29), (31), (32)} \Rightarrow \]

\[ (33) \quad \int A \frac{d}{dt} \delta_c(x) \left[ F^*(\mathbf{E}) \right](x) = \int_0^T \int A \left( \frac{d}{dt} \delta_c(x,t) \cdot (\Pi^* \varphi)(x) \right) \, dx_R \mathbf{E}(x_R,t) \delta(x - x_R) \]

\[ \text{The adjoint state equation} \]

\[ \text{Consider the adjoint state: } \psi(t) = \left( \begin{array}{c} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{array} \right) \text{ solving:} \]

\[ \frac{2}{dt} \psi_1(x,t) - \left[ (D_u H)^\star \psi \right](x,t) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \int A \mathbf{E}(x_R,t) \delta(x - x_R) \]

\[ \psi_1(x,T) = 0 \]

\[ \text{Substituting in (33)} \Rightarrow \]

\[ \int A \frac{d}{dt} \delta_c(x) \left[ F^* \mathbf{E} \right](x) = \int_0^T \int A \left( \frac{d}{dt} \delta_c(x,t) \cdot \left[ - \frac{\partial \psi_1(x,t)}{\partial t} - \left[ (D_u H)^\star \psi \right](x,t) \right] \right) \, dx_R \mathbf{E}(x_R,t) \delta(x - x_R) \]

\[ \left[ (D_u H)^\star \psi \right](x,t) = - \left[ \nabla \frac{\partial \psi_1(x,t)}{\partial t} \right] \]
Since \( \delta u = 0 \) at \( t = 0 \) and \( \mathbf{w} = 0 \) at \( t = T \), we have:

\[
\int_0^T \int_{\mathbb{R}^3} dx \cdot \left[ \mathbf{w} \cdot \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \cdot (\nabla \times \mathbf{w}) \right] \quad (25)
\]

\[
= \int_0^T \int_{\mathbb{R}^3} dx \cdot \mathbf{w} \cdot \nabla_c \mathbf{u} \quad (25)
\]

\[
= \int_0^T \int_{\mathbb{R}^3} dx \cdot \left( \nabla_c \right) \mathbf{w} \cdot \nabla_c \mathbf{u} \quad (25)
\]

\[
= \int_0^T \int_{\mathbb{R}^3} dx \cdot \left( \nabla_c \right) \mathbf{w} \cdot \nabla_c \mathbf{u} \quad (25)
\]

Thus,

\[
(F^* \mathbf{E})(x) = (D_c \mathbf{H}^* \mathbf{w})(x) \quad (35)
\]

The adjoint of \( D_c \mathbf{H} \):

Since \( D_c \mathbf{H} \) is a linear operator acting on vector-valued functions in \( L^2(\mathbb{R}^3) \) and returning \( 4 \)-D vector-valued functions in \( L^2(\mathbb{R}^3 \times [0,T]) \rightarrow \mathbb{R}^3 \), we define formally the adjoint

\[
\int_0^T \int_{\mathbb{R}^3} dx \cdot \left( D_c \mathbf{H} \mathbf{w} \right) \cdot \mathbf{u}(x,t) = \int_0^T \int_{\mathbb{R}^3} \left( D_c \mathbf{H}^* \mathbf{w} \right) \cdot \mathbf{u}(x,t) \quad (26)
\]

\[
\Rightarrow \int_0^T \int_{\mathbb{R}^3} dx \cdot \left[ -2 \nabla_c \mathbf{w} \cdot \mathbf{u} \right] = \int_0^T \int_{\mathbb{R}^3} dx \cdot \left[ -2 \nabla_c \mathbf{w} \cdot \mathbf{u} \right] \quad (26)
\]
\[
\begin{align*}
\left( D_u H^* \varphi \right)(x,t) &= - \int_0^T \frac{1}{\xi_0 c(x)} \nabla \cdot S_p(x,t) \, dt = \int_0^T \frac{2}{\xi_0 c(x)} c(x) \nabla \cdot S_p(x,t) \, dt \\
\end{align*}
\]

Now, to obtain the desired formula (17), we must look more carefully at the adjoint \( \varphi \): (34). This requires \( Du H^* \).

The adjoint of \( Du H^* \) is

\( Du H = \text{linear operator taking 4-D vector valued functions with components in } L^2[\mathbb{R}^3 \times [0,1]) \text{ to functions in the same space. We have:} \)

\[
\int_0^T \int_{\mathbb{R}^3} \varphi(x,t) \cdot \mathbf{u}(x,t) \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} \mathbf{u}(x,t) \cdot \left( \nabla \cdot S_p(x,t) \right) \, dx \, dt
\]

\begin{align*}
\int_0^T \int_{\mathbb{R}^3} \varphi(x,t) \cdot \mathbf{u}(x,t) \, dx \, dt &= \int_0^T \int_{\mathbb{R}^3} \mathbf{u}(x,t) \cdot \left( \nabla \cdot S_p(x,t) \right) \, dx \, dt \\
\int_0^T \int_{\mathbb{R}^3} \varphi(x,t) \cdot \mathbf{u}(x,t) \, dx \, dt &= \int_0^T \int_{\mathbb{R}^3} \mathbf{u}(x,t) \cdot \left( \nabla \cdot S_p(x,t) \right) \, dx \, dt \\
\end{align*}

by parts.

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} \left\{ S_p(x,t) \cdot \xi_0 \nabla \left[ \frac{2}{\xi_0 c(x)} c(x) \nabla \cdot S_p(x,t) \right] \right\} \, dx \, dt &= \int_0^T \int_{\mathbb{R}^3} \left\{ S_p(x,t) \cdot \frac{1}{\xi_0} \nabla \cdot S_p(x,t) \right\} \, dx \, dt \\
\end{align*}
\]

\[
\left( D_u H^* \varphi \right)(x,t) = \begin{pmatrix}
\frac{1}{\xi_0} \nabla \cdot S_p(x,t) \\
\xi_0 \nabla \left[ \frac{2}{\xi_0 c(x)} c(x) \nabla \cdot S_p(x,t) \right]
\end{pmatrix}
\]
The adjoint equations (34) are:

\[
\begin{align*}
\frac{\partial^2 \xi}{\partial t^2} - \frac{1}{c_0^2} \nabla \cdot \xi(x, t) &= \int_{\Omega} E(x) \frac{\partial}{\partial t} \delta(x - x_0) \\
- \frac{\partial \xi}{\partial t} - S_0 \nabla \left[ c^2(x) \nabla \psi(x, t) \right] &= 0 \quad \text{for } t < T \\
\xi(x, T) &= 0 ; \quad \psi(x, T) = 0
\end{align*}
\]

Now, eliminate \( \xi \) by taking \( \frac{\partial}{\partial t} \) in the first eqn (38):

\[
\frac{\partial^2 \psi}{\partial t^2} + \Delta \left[ c^2(x) \nabla \psi(x, t) \right] = \int_{\Omega} E(x) \frac{\partial}{\partial t} \delta(x - x_0)
\]

and let \( \psi(x, t) = \frac{\partial}{\partial t} \psi(x, t) \). This defines \( \psi \) up to an additive constant that we fix by \( \psi(x, T) = 0 \).

We also have \( \frac{\partial}{\partial t} \psi(x, t) = \psi(x, T) = 0 \). Using this \( \psi \), eq. (39) becomes:

\[
\frac{\partial}{\partial t} \left[ \frac{\partial^2 \psi}{\partial t^2} + \Delta \left[ c^2(x) \nabla \psi(x, t) \right] - \int_{\Omega} E(x) \frac{\partial}{\partial t} \delta(x - x_0) \right] = 0
\]

for \( t > T \).

(41)

\[
\begin{align*}
- \frac{\partial^2 \psi(x, t)}{\partial t^2} + \Delta \left[ c^2(x) \nabla \psi(x, t) \right] &= \int_{\Omega} E(x) \frac{\partial}{\partial t} \delta(x - x_0) \\
\psi(x, t) &= 0 \quad \text{for } t < T
\end{align*}
\]

Finally, we let:

\[
\psi(x, t) = \frac{1}{c^2(x)} \delta^2(x, T-t)
\]
\[
\begin{align*}
\frac{\partial^2 \delta(x,t)}{\partial t^2} - \Delta \delta(x,t) & = -\int d^2x' \, \xi(x', T-t) \, \delta(x-x') \\
\delta(x,t) & = 0 \quad \text{for } t \leq 0 \\
\delta(x,t) & \neq 0 \quad \text{for } t > 0
\end{align*}
\]

which is precisely eq. (8).

The adjoint formula

Gathering results (35), (36), (42): \( F^* \xi \)

\[
(F^* \xi)(x) = \int_0^T \frac{2}{c^2(x)} \frac{\partial P(x,t)}{\partial t} \frac{\partial}{\partial t} \left[ \frac{1}{c^2(x)} \delta(x, t-t) \right] dt =
\]

\[
= \frac{2}{c^2(x)} \delta(x, T-t) \frac{d}{dt} P(x, t) \bigg|_{t=0}^{T} = \frac{2}{c^2(x)} \int_0^T \frac{\partial^2}{\partial t^2} \delta(x, T-t) dt
\]

\[
= -\frac{2}{c^2(x)} \int_0^T \delta(x, T-t) \frac{\partial^2 P(x, t)}{\partial t^2} dt
\]