Lecture 2: Migration imaging as an approximate solution of the linearized inverse problem, in least squares sense.

Outline:

1) Formulation of the least squares problem
2) Migration as an approximate solution (in what sense?)
3) Adjoint calculation details.

1) The least squares problem

- Consider for now a single source at $\underline{x}_s$ and assume that we measure at $\underline{x}_R \in \mathcal{A}$, for $t \in [0,T]$, echoes $d(\underline{x}_R, t)$.
- For convenience, let us assume that we have small enough spacing between the receivers so that we can take: $\sum x_R \sim \int_{\mathcal{A}} dx_R$
- Recall our linearized model for the measurements:

(1) $F[c]\delta c = \{ \delta p(\underline{x}_R, t), \ \underline{x}_R \in \mathcal{A}, \ t \in [0,T] \}$,

where δp is the solution of the linearized acoustic eq.

- We seek δc such that model (1) predicts our measurements:

(2) $(F[c]\delta c)(\underline{x}_R, t) = \delta p(\underline{x}_R, t) \simeq d(\underline{x}_R, t)$ where the approximation is in least squares sense. Explicitly, we minimize

(3) $J[\delta c] = \frac{1}{2}\int_0^T dt \int_{\mathcal{A}} d\underline{x}_R \, \phi(\underline{x}_R) \left| (F[c]\delta c)(\underline{x}_R, t) - d(\underline{x}_R, t) \right|^2$,

we understand dependence on c & drop it from now on where Φ = smooth cutoff (mute) function.

The adjoint operator F^*

- Let us define formally, the adjoint F^* of F by

$$(4)\quad \int_0^T dt \int_{\mathcal{A}} d\underline{x}_R \, \psi(\underline{x}_R, t)\, (F\delta c)(\underline{x}_R, t) = \int_{\mathbb{R}^3} d\underline{x}\, \delta c(\underline{x}) \, (F^*\psi)(\underline{x})$$

Thus: F acts on δc and returns echoes at $\underline{x}_R, t$, whereas operator F^* takes functions $\psi(\underline{x}_R, t)$ (assumed in $L^2(\mathcal{A}\times[0,T])$) to $L^2(\mathbb{R}^3)$ functions of $\underline{x}$.

- Taking the first variation in (3), we have that δc must satisfy:

$$(5)\quad (F^*\mathcal{E})(\underline{x}) = 0 \quad \forall\, \underline{x} \in \mathbb{R}^3,$$

where

$$(6)\quad \mathcal{E}(\underline{x}_R, t) = \left[(F\delta c)(\underline{x}_R, t) - d(\underline{x}_R, t)\right] \Phi(\underline{x}_R, t)$$

is the misfit.

The expression of $F^*\mathcal{E}$ is: (we prove this in part 3)).

$$(7)\quad (F^*\mathcal{E})(\underline{x}) = -\frac{2}{c^3(\underline{x})} \int_0^T \vartheta(\underline{x}, t)\, \frac{\partial^2 p}{\partial t^2}(\underline{x}, T-t)\, dt,$$

where p = pressure field in the background (sound speed is $c(\underline{x})$) and ϑ solves the adjoint problem:

$$(8)\quad \begin{cases} \left(\dfrac{1}{c^2(x)}\dfrac{\partial^2}{\partial t^2} - \Delta\right) \vartheta(\underline{x}, t) = -\displaystyle\int_{\mathcal{A}} d\underline{x}_R\, \mathcal{E}(\underline{x}_R, T-t)\, \delta(\underline{x} - \underline{x}_R) & \text{for } t>0 \\ \vartheta(\underline{x}, t) = 0 \quad \text{for } t \le 0 \end{cases}$$

Note: The calculation of the adjoint solution $\delta(\underline{x}, t)$ amounts physically to taking the misfit $\varepsilon(\underline{x}_R, t)$, time reversing it and sending it back in the medium, to location $\underline{x}$.

• We can rewrite (7) in the Fourier domain, and obtain after a straightforward calculation:

$$(9)\quad (F^*\varepsilon)(\underline{x}) = -\frac{2}{c_0^3(\underline{x})}\frac{1}{2\pi}\int d\omega \int d\omega'\, \omega^2 \hat{f}_B(\omega-\omega_0)\, \hat{G}(\underline{x}, \omega; x_s) \int_{\mathcal{A}} dx_R\, \hat{G}(\underline{x}, \omega'; \underline{x}_R)\, \overline{\hat{\varepsilon}(\underline{x}_R, \omega')}\, \underbrace{\int_0^T \frac{dt}{2\pi} e^{i(\omega'-\omega)(T-t)}}_{\approx\, \delta(\omega'-\omega) \text{ for very long time windows}}$$

$$\sim \frac{1}{\pi c_0^3(\underline{x})}\int_{\mathcal{A}} dx_R \int d\omega\, \omega^2 \hat{f}_B(\omega-\omega_0)\, \hat{G}(\underline{x}, \omega; \underline{x}_s)\, \hat{G}(\underline{x}, \omega; x_R)\, \overline{\hat{\varepsilon}(\underline{x}_R, \omega)}$$

2) Migration

• The L-S solution satisfies (5) or, equivalently,

$$(10)\quad F^*\left[(F\delta c)\phi\right](\underline{x}) = F^*\left[d\,\phi\right](\underline{x})$$

which are the normal equations.

• Operator F^*F is not invertible in the usual sense but it acts almost as an identity operator on the singularities of $\delta c(x)$. That is, it does not move these singularities and it does not create new ones.

Note: F^*F = the time reversal operator that will be discussed extensively in this course. This operator has refocusing properties that are very good (depending on aperture and bandwidth) and that get better in random media.

Conclusion: We can image the singularities of δc with just the right hand side in (10):

$$\mathcal{I}(\underline{x}) = F^*[d\,\phi](\underline{x}) \sim \int_{\mathcal{A}} d\underline{x}_R\, \phi(\underline{x}_R) \int d\omega\, \omega^2 \hat{f}_B(\omega-\omega_0)\, \overline{\hat{d}(\underline{x}_R,\omega)}\, \hat{G}(\underline{x},\omega;\underline{x}_0)\, \hat{G}(\underline{x}_R,\omega;\underline{x}) \tag{11}$$

• Now, use $\hat{G}(\underline{x},\omega;\underline{x}_0) \simeq \alpha(\underline{x},\underline{x}_0)\hat{S}(\omega) e^{i\omega \tau(\underline{x},\underline{x}_0)}$, the high frequency asymptotic formula discussed last time and replace $\underline{x} \rightsquigarrow \underline{y}^S$ = search point sweeping a search domain $\mathcal{D}$ where we look for the reflectors. The result is: Kirchhoff imaging formulae:

$$\mathcal{I}(\underline{y}^S) \sim \int_{\mathcal{A}} d\underline{x}_R\, \phi(\underline{x}_R)\, \alpha(\underline{x}_R,\underline{y}^S)\, \alpha(\underline{y}^S,\underline{x}_0) \int d\omega\, \omega^2 \hat{f}_B(\omega-\omega_0)\, [\hat{S}(\omega)]^2\, \overline{\hat{d}(\underline{x}_R,\omega)}\, e^{i\omega[\tau(\underline{x}_R,\underline{y}^S)+\tau(\underline{x}_0,\underline{y}^S)]} \tag{12}$$

• Recall the Born approximation model for $\hat{d}(\underline{x}_R,\omega)$ and note that (12) contains the oscillatory term:

$$\exp\left\{i\omega\left[\tau(\underline{x}_R,\underline{y}^S)+\tau(\underline{x}_0,\underline{y}^S)-\tau(\underline{x}_R,\underline{y})-\tau(\underline{x}_0,\underline{y})\right]\right\}, \tag{13}$$

where $\underline{y} \in$ support of δc. Note also that we have a high frequency regime and therefore, the integral (12) is given approximately by the method of stationary phase. This means that the amplitudes play a negligible role so we can neglect them and obtain a simple imaging function that we call from now on Kirchhoff Migration (KM):

$$J^{KM}(\underline{y}^S) = \int_A d\underline{x}_R\, \phi(\underline{x}_R) \int d\omega\, \hat{d}(\underline{x}_R, \omega)\, e^{\overline{i\omega[\tau(\underline{x}_R, \underline{y}^S) + \tau(\underline{x}_0, \underline{y}^S)]}}$$

(14)

$$= \int_A \phi(\underline{x}_R)\, d(\underline{x}_R, \tau(\underline{x}_R, \underline{y}^S) + \tau(\underline{x}_0, \underline{y}^S))\, d\underline{x}_R$$

How does the method of stationary phase give the focus points:

Let us recall the expression of $d(\underline{x}_R, t)$, as modeled by Born approximation (see eq. (18) yesterday's notes) and neglect the amplitudes:

$$J^{KM}(\underline{y}^S) \sim \int_A d\underline{x}_R \phi(\underline{x}_R) \int d\omega\, \omega^2 \vec{f}_B(\omega-\omega_0) \int d\underline{y}\, r(\underline{y}) \,.$$

(15)

$$\exp\{i\omega[\tau(\underline{x}_R, \underline{y}) + \tau(\underline{x}_0, \underline{y}) - \tau(\underline{x}_R, \underline{y}^S) - \tau(\underline{x}_0, \underline{y}^S)]\}$$

where $r(\underline{y}) = -\dfrac{2\,\delta c(\underline{y})}{c^3(\underline{y})} c_0^{\,2} \equiv$ reflectivity defined last time.

Now, let us write:

(16) $$r(\underline{y}) = \int d\underline{\gamma}\, \rho(\underline{\gamma})\, e^{i\underline{\gamma}\cdot\underline{y}}$$

and obtain:

$$I^{KM}(\underline{y}^S) \sim \int d\underline{\gamma}\, \rho(\underline{\gamma}) \int_{\mathcal{A}} d\underline{x}_R\, \phi(\underline{x}_R) \int d\omega\, \omega^2 \hat{f}_B(\omega-\omega_0) \int d\underline{y}$$

(17) $$\exp\left\{ i\,\underline{\gamma}\cdot\underline{y} + i\omega\left[\tau(\underline{x}_R,\underline{y}) + \tau(\underline{x}_0,\underline{y}) - \tau(\underline{x}_R,\underline{y}^S) - \tau(\underline{x}_0,\underline{y}^S)\right]\right\}$$

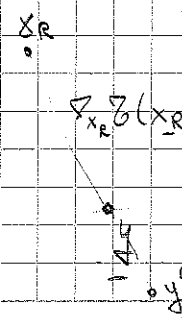
The method of stationary phase says that when integrating over $\underline{x}_R \in \mathcal{A}$, $\omega \in [\omega_0 - \frac{B}{2}, \omega_0 + \frac{B}{2}]$ and $\underline{y} \in \mathrm{supp}\,\rho$, because we have high frequencies, we obtain the main contribution where the following stationarity conditions are satisfied:

(18.a) $$\tau(\underline{x}_R,\underline{y}) + \tau(\underline{x}_0,\underline{y}) = \tau(\underline{x}_R,\underline{y}^S) + \tau(\underline{x}_0,\underline{y}^S)$$

(18.b) $$\nabla_{x_R} \tau(\underline{x}_R,\underline{y}) = \nabla_{x_R} \tau(\underline{x}_R,\underline{y}^S)$$

(18.c) $$\underline{\gamma} = -\omega\, \nabla_{\underline{y}}\left[\tau(\underline{x}_R,\underline{y}) + \tau(\underline{x}_0,\underline{y})\right]$$

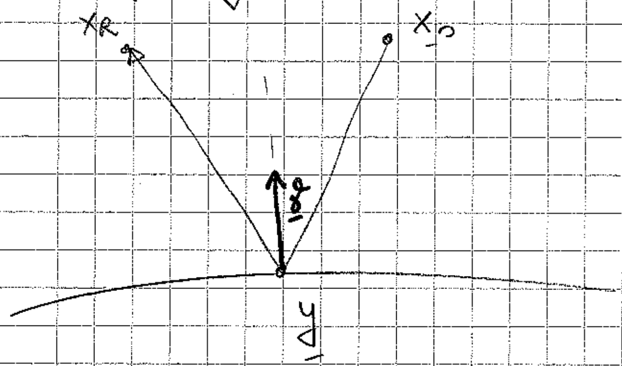
- Condition (18.a) says that $\underline{y}$ & $\underline{y}^S$ must lie on the same isochron.
- The gradients in (18.b) are 2-D because $\underline{x}_R$ sweeps the 2-D array surface. Thus, (18.b) says that $\underline{y}$ & $\underline{y}^S$ are in the same direction (along the ray) when viewed from $\underline{x}_R$.



$$\nabla_{x_R}\tau(\underline{x}_R,\underline{y}) = \frac{\underline{x}_R^{\perp} - \underline{y}^{\perp}}{|\underline{x}_R - \underline{y}|}$$, with $\underline{x}_R^{\perp}$ = components of $\underline{x}_R$ in plane tangent to $\mathcal{A}$ at $\underline{x}_R$

Thus, conditions (18.a) & (18.b) determine that $\underline{y}^s$ must be the same as some point $\underline{y} \in \text{supp}\, r$.

• Condition (18.c) tells us what it is that we really image. First of all, note that $\underline{\delta}$ points in the normal direction of surfaces: $\rho(\underline{\delta})\, e^{i\underline{\delta}\cdot\underline{y}} = \text{constant}$ and that $-\underline{\delta}$ = along the bisector of the angle made by ray $\underline{x}_s \rightsquigarrow \underline{y}$ and ray $\underline{y} \rightsquigarrow \underline{x}_s$



$\Rightarrow$ surfaces $\{\underline{y} \in \text{supp}\, r$ s.t. $\underline{\delta}\cdot\underline{y} = \text{constant}\}$ act as reflecting surfaces and we have Fermat's law of angle of incidence = angle of reflection.

Second, since $\underline{\delta}$ is proportional to ω and since ω is large $\Rightarrow$ we must have $|\underline{\delta}|$ large too so, we can only determine $\rho(\underline{\delta})$ for $|\underline{\delta}|$ large. This is another confirmation that we image just the singularities of $r(\underline{y})$ i.e., its support, in particular.

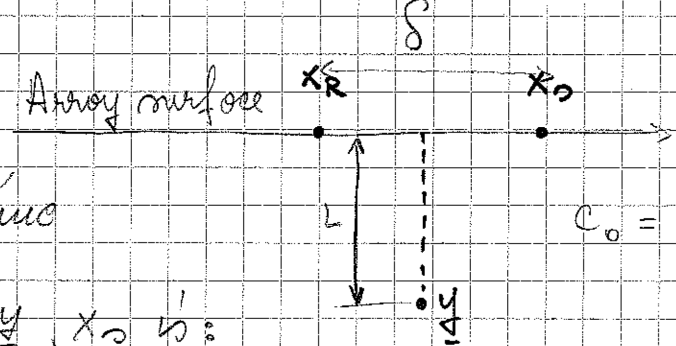
Note: All this depends on us knowing $c(\underline{x})$, so we can calculate correctly traveltimes.

• Finding out of $F[c]\delta c \cong \text{data}$ both δc and c is a much harder problem. We linearized in δc but the dependence on c is still highly nonlinear. To find c we need more data. (8)

• A very efficient technique for estimating $c(\underline{x})$ is the differential semblance approach (Symes & collaborators)

The idea behind differential semblance: Uses a lot of data and divides it into smaller data sets (for example according to offsets between source and receivers). Then, it constructs images with each subset and compares them. The images will be $\simeq$ identical when the velocity is right.

Example:



The travel time between x_R, $\underline{y}$, x_s is: $\quad c_0$ = correct speed

$$\tau(x_R, \underline{y}) + \tau(\underline{x}_s, \underline{y}) = \frac{2\sqrt{L^2 + \frac{\delta^2}{4}}}{c_0}$$

• Now, when we have the wrong velocity, $\tilde{c}_0$, the target is perceived at depth $\tilde{L}$:

$$\frac{2\sqrt{\tilde{L}^2 + \frac{\delta^2}{4}}}{\tilde{c}_0} = \frac{2\sqrt{L^2 + \frac{\delta^2}{4}}}{c_0}$$

and this $\tilde{L}^2 = \left(\frac{\tilde{c}_0^2 - c_0^2}{c_0^2}\right)\frac{\delta^2}{4} + \left(\frac{\tilde{c}_0}{c_0}\right)^2 L^2$ changes with the offset. $\Rightarrow$ The image will change with the offset when $\tilde{c}_0 \neq c_0$

3) Details of the adjoint calculation

- We derive the expression of F^*, using the linearized version of the wave eq. in 1st order system form:
- Let $\underline{u}(\underline{x},t) = \begin{pmatrix} p(\underline{x},t) \\ \underline{v}(\underline{x},t) \end{pmatrix}$ be the four dimensional state

variable solving the wave eq: $\frac{\partial \underline{u}}{\partial t} = \underline{H}[\underline{u}, c, t]$ in the medium with sound speed $c(\underline{x})$, where $\underline{u}(\underline{x},0) = \underline{0}$ and

$$\underline{H}[\underline{u}, c, t] = \begin{pmatrix} -\rho_0 c^2 \, \nabla\cdot\underline{v} \\ -\frac{1}{\rho_0}\nabla p + \frac{\underline{F}}{\rho_0} \end{pmatrix}.$$

- The linearization of these equations with respect to δc is

$$(25) \quad \frac{\partial \delta\underline{u}}{\partial t} = D_{\underline{u}}\underline{H}[\underline{u}, c, t)\,\delta\underline{u} + D_c\underline{H}[\underline{u}, c, t]\,\delta c \quad \text{for } t>0$$

$$\delta\underline{u} = 0 \quad \text{for } t = 0,$$

where:

$$(26) \quad D_{\underline{u}} H[\underline{u}, c, t]\,\delta\underline{u} = \begin{pmatrix} -\rho_0 c^2 \, \nabla\cdot\delta\underline{v} \\ -\frac{1}{\rho_0}\nabla\delta p \end{pmatrix}; \quad D_c H[u,c,t]\,\delta c = \begin{pmatrix} -2\rho_0 c\,\delta c\, \nabla\cdot\underline{v} \\ \underline{0} \end{pmatrix}$$

- The least-squares functional (3) can be rewritten as

$$(27) \quad J[\delta c] = \frac{1}{2}\int_0^T dt \int_{\mathcal{A}} d\underline{x}_R \, \Phi(\underline{x}_R) \left| (\mathcal{M}\,\delta\underline{u})(\underline{x}_R, t) - d(\underline{x}_R, t) \right|^2,$$

where $\mathcal{M}$ is the measurement operator

$$(28) \quad (\mathcal{M}\,\delta\underline{u})(\underline{x}_R, t) = \Big(\Pi\,[(1,\underline{0})\,\delta\underline{u}]\Big)(\underline{x}_R, t) \equiv (\Pi\,\delta p)(\underline{x}_R, t) = \delta p(\underline{x}_R, t)$$

and Π = projection of a scalar valued function defined of $\underline{x} \in \mathbb{R}^3$ to $\underline{x}_R \in \mathcal{A}$.

• The first variation in (27) is:

(29)
$$\int_{\mathbb{R}^3} d\underline{x}\ \delta c(\underline{x})\,(D\,]\,)(\underline{x}) \overset{(3)}{=} \int_{\mathbb{R}^3} d\underline{x}\ \delta c(\underline{x})\,[F^*\,\varepsilon\,](\underline{x}) \overset{(27)}{=}$$

$$= \int_0^T dt \int_{\mathbb{R}^3} d\underline{x}\ \delta\underline{u}(\underline{x},t)\cdot[\mathcal{M}^*(\,\varepsilon(\cdot,t))](\underline{x}) \,,$$

where $\mathcal{M}^*$ = adjoint of measurement operator & ε is def. by (6).

The adjoint of $\mathcal{M}$:

For a given t and arbitrary $\varphi \in L^2(\mathcal{A})$, we have

(30)
$$\int_{\mathcal{A}} d\underline{x}_R\ \varphi(\underline{x}_R)\,(\mathcal{M}\,\delta\underline{u}(\cdot,t))(\underline{x}_R) = \int_{\mathbb{R}^3} d\underline{x}\ \delta\underline{u}(\underline{x},t)\cdot(\mathcal{M}^*\varphi)(\underline{x})$$

Now, using the definition (28) =>

$$\int_{\mathbb{R}^3} d\underline{x}\ \delta\underline{u}(\underline{x},t)\cdot(\mathcal{M}^*\varphi)(\underline{x}) = \int_{\mathcal{A}} d\underline{x}_R\ \varphi(\underline{x}_R)\ \Pi[(1,\underline{0})\,\delta\underline{u}(\cdot,t)](\underline{x}_R)$$

$$= \int_{\mathbb{R}^3} \delta u(\underline{x},t)\cdot\begin{pmatrix}1\\ \underline{0}\end{pmatrix}(\Pi^*\varphi)(\underline{x}) \quad =>$$

(31)
$$(\mathcal{M}^*\varphi)(\underline{x}) = \begin{pmatrix}1\\ \underline{0}\end{pmatrix}(\Pi^*\varphi)(\underline{x})$$

The adjoint of Π:

For arbitrary $\varphi \in L^2(\mathcal{A})$, $\alpha \in L^2(\mathbb{R}^3)$, we have

$$\int_{\mathcal{A}} \varphi(\underline{x}_R)(\Pi\alpha)(\underline{x}_R)\,d\underline{x}_R = \int_{\mathbb{R}^3} \alpha(\underline{x})\,(\Pi^*\varphi)(\underline{x})\,d\underline{x}$$ but, by (28),

$$\int_{\mathcal{A}} \varphi(\underline{x}_R)(\Pi\alpha)(\underline{x}_R)\,d\underline{x}_R = \int_{\mathcal{A}} \varphi(\underline{x}_R)\,\alpha(\underline{x}_R)\,d\underline{x}_R = \int_{\mathbb{R}^3} d\underline{x}\;\alpha(\underline{x}) \int_{\mathcal{A}} d\underline{x}_R\;\varphi(\underline{x}_R)\,\delta(\underline{x}-\underline{x}_R) \Rightarrow$$

$$(32)\quad (\Pi^*\varphi)(\underline{x}) = \int_{\mathcal{A}} d\underline{x}_R\;\varphi(\underline{x}_R)\,\delta(\underline{x}-\underline{x}_R)$$

- Now, using (29), (31), (32) $\Rightarrow$

$$(33)\quad \int_{\mathbb{R}^3} d\underline{x}\;\delta c(\underline{x})\left[F^*(\mathcal{E})\right](\underline{x}) = \int_0^T dt \int_{\mathbb{R}^3} d\underline{x}\;\delta\underline{u}(\underline{x},t)\cdot\begin{pmatrix}1\\ \underline{0}\end{pmatrix}\int_{\mathcal{A}} d\underline{x}_R\;\mathcal{E}(\underline{x}_R,t)\cdot\delta(\underline{x}-\underline{x}_R)$$

The adjoint state equation

Consider the adjoint state: $\underline{w} = \begin{pmatrix}\eta\\ \underline{\xi}\end{pmatrix}$ solving:

$$(34)\quad \begin{cases} -\dfrac{\partial}{\partial t}\underline{w}(\underline{x},t) - \left[(D_u H)^*\underline{w}\right](\underline{x},t) = \begin{pmatrix}1\\ \underline{0}\end{pmatrix}\displaystyle\int_{\mathcal{A}} d\underline{x}_R\;\mathcal{E}(\underline{x}_R,t)\,\delta(\underline{x}-\underline{x}_R) & \text{for } t<T \\ \underline{w}(\underline{x},T)=0 \end{cases}$$

Substituting in (33) $\Rightarrow$

$$\int_{\mathbb{R}^3} d\underline{x}\;\delta c(\underline{x})\left[F^*\mathcal{E}\right](\underline{x}) = \int_0^T dt\int_{\mathbb{R}^3} d\underline{x}\;\delta\underline{u}(\underline{x},t)\cdot\left\{-\frac{\partial\underline{w}}{\partial t}(\underline{x},t) - \left[(D_u H)^*\underline{w}\right](\underline{x},t)\right\} = -\int_{\mathbb{R}^3} d\underline{x}\;\delta\underline{u}\cdot\underline{w}\Big|_0^T +$$

$$+ \int_0^T dt \int_{\mathbb{R}^3} d\underline{x} \left\{ \underline{w} \cdot \frac{\partial \delta \underline{u}}{\partial t} - \delta \underline{u} \cdot (D_u H)^* \underline{w} \right\} .$$

Since $\delta \underline{u} = \underline{0}$ at $t=0$ and $\underline{w} = \underline{0}$ at $t = T$, we have:

$$\int_{\mathbb{R}^3} d\underline{x}\ \delta c(\underline{x}) \left[F^* \mathcal{E}\right](\underline{x}) = \int_0^T dt \int_{\mathbb{R}^3} d\underline{x}\ \underline{w} \cdot \left\{ \frac{\partial \delta \underline{u}}{\partial t} - D_u H \delta \underline{u} \right\}$$

$$\overset{(25)}{=} \int_0^T dt \int_{\mathbb{R}^3} d\underline{x}\ \underline{w} \cdot D_c H \delta c =$$

$$= \int_{\mathbb{R}^3} d\underline{x}\ \delta c(\underline{x}) \left(D_c H^* \underline{w}\right)(\underline{x}).$$

Thus,

$$(35) \qquad (F^* \mathcal{E})(\underline{x}) = (D_c H^* \underline{w})(\underline{x})$$

The adjoint of $D_c H$:

Since $D_c H$ = linear operator acting on scalar valued functions in $L^2(\mathbb{R}^3)$ and returning 4-D vector valued functions in $L^2(\mathbb{R}^3 \times [0,T])$ => we define formally the adjoint:

$$\int_0^T dt \int_{\mathbb{R}^3} d\underline{x}\ (D_c H \delta c)(\underline{x},t) \cdot \underbrace{\underline{w}(\underline{x},t)}_{= \binom{\eta}{\underline{w}}} = \int_{\mathbb{R}^3} \delta c(\underline{x}) \left[D_c H^* \underline{w}\right](\underline{x})$$

$$\overset{(26)}{=} \int_0^T dt \int_{\mathbb{R}^3} d\underline{x} \left[-2 \rho_0 c\, \delta c\, \nabla \cdot \underline{v}\right] \eta \quad =>$$

$$(36)\quad (D_c H^* \underline{w})(\underline{x}) = -\int_0^T 2\, \rho_0\, c(\underline{x})\, \eta(\underline{x},t)\, \underbrace{\nabla\cdot\underline{v}(\underline{x},t)}_{= \, -\frac{1}{\rho_0 c^2(\underline{x})}\frac{\partial p(\underline{x},t)}{\partial t}}\, dt =$$

$$= \int_0^T \frac{2}{c(\underline{x})}\, \eta(\underline{x},t)\, \frac{\partial p(\underline{x},t)}{\partial t}.$$

Now, to obtain the desired formula (7), we must look more carefully at the adjoint eq. (34). This requires $D_u H^*$

The adjoint of $D_u H$:

$D_u H$ = linear operator taking 4-D vector valued functions with components in $L^2[\mathbb{R}^3 \times [0,T]]$ to functions in the same space. We have:

$$\int_0^T dt \int_{\mathbb{R}^3} d\underline{x}\, (D_u H\, \delta \underline{u})(\underline{x},t) \cdot \underbrace{\underline{w}(\underline{x},t)}_{= \binom{\eta}{\underline{\xi}}} = \int_0^T \int_{\mathbb{R}^3} d\underline{x}\, \delta\underline{u}(\underline{x},t) \cdot (D_u H^* \underline{w})(\underline{x},t)$$

$$\overset{(26)}{=} \int_0^T dt \int_{\mathbb{R}^3} d\underline{x} \left[-\eta(\underline{x},t)\, \rho_0\, c^2(\underline{x})\, \nabla\cdot \delta\underline{v}(\underline{x},t) - \frac{\underline{\xi}(\underline{x},t)}{\rho_0} \nabla \delta p(\underline{x},t) \right]$$

$$\overset{\text{int by parts}}{=} \int_0^T dt \int_{\mathbb{R}^3} d\underline{x} \left\{ \delta\underline{v}(\underline{x},t) \cdot \rho_0 \nabla[\eta(\underline{x},t)\, c^2(\underline{x})] + \delta p(\underline{x},t)\, \frac{1}{\rho_0}\, \nabla\cdot \underline{\xi}(\underline{x},t) \right\} \Rightarrow$$

$$(37)\quad (D_u H^* \underline{w})(\underline{x},t) = \begin{pmatrix} \frac{1}{\rho_0} \nabla\cdot \underline{\xi}(\underline{x},t) \\ \rho_0 \nabla[\eta(\underline{x},t)\, c^2(\underline{x})] \end{pmatrix}$$

The adjoint equations (34) are:

$$(38)\quad \begin{cases} -\dfrac{\partial \eta}{\partial t} - \dfrac{1}{\rho_0}\nabla\cdot\underline{\xi}(\underline{x},t) = \displaystyle\int_{A} d\underline{x}_R\ \varepsilon(x_R,t)\,\delta(\underline{x}-x_R) \\ -\dfrac{\partial \underline{\xi}}{\partial t} - \rho_0\nabla\left[c^2(\underline{x})\,\eta(\underline{x},t)\right] = \underline{0} \qquad \text{for } t<T \\ \underline{\xi}(\underline{x},T)=\underline{0}\ ;\quad \eta(\underline{x},T)=0 \end{cases}$$

- Now, eliminate $\underline{\xi}$ by taking $\dfrac{\partial}{\partial t}$ in the first eq in (38):

$$(39)\quad -\frac{\partial^2\eta}{\partial t^2} + \Delta\left[c^2(\underline{x})\,\eta(\underline{x},t)\right] = \int_{A} d\underline{x}_R\ \frac{\partial\varepsilon}{\partial t}(\underline{x}_R,t)\,\delta(\underline{x}-\underline{x}_R)$$

and let $\eta(\underline{x},t)=\dfrac{\partial}{\partial t}\varphi(\underline{x},t)$. This defines φ up to an additive constant that we fix by $\varphi(\underline{x},T)=0$.

- We also have $\dfrac{\partial\varphi}{\partial t}(\underline{x},T)=\eta(\underline{x},T)=0$. Using this φ, eq. (39) becomes:

$$(40)\quad \frac{\partial}{\partial t}\left\{-\frac{\partial^2\varphi}{\partial t^2} + \Delta\left[c^2(\underline{x})\,\varphi(\underline{x},t)\right] - \int_{A} d\underline{x}_R\ \varepsilon(\underline{x}_R,t)\,\delta(\underline{x}-\underline{x}_R)\right\}=0$$

$\forall t \Rightarrow$

since at $t\geq T$ we have everything quiet $\Rightarrow$

$$(41)\quad \begin{cases} -\dfrac{\partial^2\varphi}{\partial t^2}(\underline{x},t) + \Delta\left[c^2(\underline{x})\,\varphi(\underline{x},t)\right] = \displaystyle\int_{A} d\underline{x}_R\ \varepsilon(\underline{x}_R,t)\,\delta(\underline{x}-\underline{x}_R) \qquad \text{for } t<T \\ \varphi(\underline{x},t)=0 \quad \text{for } t\geq T \end{cases}$$

- Finally, we let:

$$(42)\quad \varphi(\underline{x},t) = \frac{1}{c^2(\underline{x})}\,\gamma(\underline{x},T-t)$$

and have that:

$$(43)\begin{cases} \dfrac{\partial^2 \gamma}{\partial t^2}(\underline{x},t) - \Delta \gamma(\underline{x},t) = -\displaystyle\int_{\mathcal{A}} d\underline{x}_R\, \mathcal{E}(\underline{x}_R, T-t)\, \delta(\underline{x}-\underline{x}_R) & \text{for } t>0 \\ \gamma(\underline{x},t) = 0 \quad \text{for } t \le 0 \end{cases}$$

which is precisely eq. (8).

- The adjoint formula

Gathering results (35), (36), (42) =>

$$(F^* \mathcal{E})(\underline{x}) = \int_0^T \frac{2}{c(\underline{x})} \frac{\partial p(\underline{x},t)}{\partial t} \frac{\partial}{\partial t}\left[\frac{1}{c^2(\underline{x})}\gamma(\underline{x},T-t)\right] dt =$$

$$= \frac{2}{c^3(\underline{x})} \gamma(\underline{x},T-t) \frac{\partial p}{\partial t}(\underline{x},t)\Big|_{t=0}^{T} - \frac{2}{c^3(\underline{x})}\int_0^T \frac{\partial^2 p}{\partial t^2}(\underline{x},t)\, \gamma(\underline{x},T-t)\, dt$$

$$= \frac{-2}{c^3(\underline{x})}\int_0^T \gamma(\underline{x},T-t) \frac{\partial^2 p}{\partial t^2}(\underline{x},t) \equiv \text{formula (7).}$$