Lecture 1: Forward model for array data

Outline:

1) The governing PDE
2) The forward model for passive and active array
3) Linearization (Born approximation)
4) High frequency asymptotics and description of data

1) The PDE:

• Depending on the application, one would like to consider mathematical models that account for:
- material anisotropy
- multiple wave modes (shear, pressure waves)
- polarization effects
- attenuation and frequency dispersion of waves, ...

• We make a big simplification: Acoustic waves ⇝ waves in fluids.

This model allows us to understand key ideas in imaging and it is, in fact, the basis for contemporary radar and seismic imaging.

The acoustic wave equations

• These equations are in fact linearized versions of the equations of conservation of mass and of the momentum balance in a fluid with constitutive relation $p = P(\rho)$, where p = pressure, ρ = mass density

If $p_0, \rho_0, \underline{v}_0$ = pressure, density, particle velocity in the reference state, then

$$c^2 = \frac{dP}{d\rho}(\rho_0) > 0 = \text{change of rate of pressure with density}$$
$$= \text{square of sound speed.}$$

Moreover, $p = p_0 + \delta p$, $\underline{v} = \underline{v}_0 + \delta \underline{v}$ satisfy:

$$(1)\begin{cases} \dfrac{1}{\rho_0 c^2}\dfrac{\partial \delta p}{\partial t} + \nabla \cdot \delta \underline{v} = 0 \\ \rho_0 \dfrac{\partial \delta \underline{v}}{\partial t} + \nabla \delta p = \underline{E} = \text{external force} \quad \text{for } t > 0 \\ \delta \underline{v} = \underline{0}, \quad \delta p = 0 \quad \text{for } t = 0 \end{cases}$$

Note: We assume, for simplicity ρ_0 = constant. However, $c^2 = c^2(\underline{x})$. (It could depend on t, in principle, but we assume a stationary medium).

We also drop the δ in the notation so that: $\delta \underline{v} \rightsquigarrow \underline{v}$ and so on

- In most of the calculations that follow, we work with the 2nd order PDE for p, obtained by eliminating $\underline{v}$ from (1):

$$(2)\begin{cases} \dfrac{1}{c^2(\underline{x})}\dfrac{\partial^2 p(\underline{x},t)}{\partial t^2} - \Delta p(\underline{x},t) = -\nabla \cdot \underline{E} = \mathcal{E}(\underline{x},t) \quad \text{for } t>0 \\ p(\underline{x},t) = 0 \quad \text{for } t \le 0 \end{cases}$$

Depending on the problem, we may have a bounded domain or an unbounded one. For simplicity, we shall work in the entire $\mathbb{R}^3$.

• The solvability (weak sense) of (2) follows from classic studies (Lions book: "Nonhomogeneous boundary value problems and Applications", 1972 or other PDE books)

The force term $\mathcal{E}(x,t)$

• This is usually of the form: $\mathcal{E}(\underline{x},t) = f(t)\,\phi(\underline{x}-\underline{x}_0)$

for ϕ denoting the supp of the source centered at $\underline{x}_0$. We shall idealize this as $\phi(\underline{x}-\underline{x}_0) = \delta(\underline{x}-\underline{x}_0)$

• The pulse is $f(t) = e^{-i\omega_0 t} f_B(t)$, so that in Fourier:

$\hat{f}(\omega) = \int f(t) e^{i\omega t}\, dt = \hat{f}_B(\omega-\omega_0)$. If B is the bandwidth (length of support interval of $\hat{f}_B(\omega)$)

$\Rightarrow$ $\hat{f}(\omega)$ is supported in $\left[\omega_0 - \frac{B}{2},\ \omega_0 + \frac{B}{2}\right]$. We call ω_0 = central (carrier) frequency.

The Green's function

• Given the model $\mathcal{E}(\underline{x},t) = f(t)\,\delta(\underline{x}-\underline{x}_0)$ of the excitation, we shall write the pressure field in terms of the Green's function $G(\underline{x},t;\underline{x}_0)$ solving

$$(3)\quad \left[\frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2} - \Delta_{\underline{x}}\right] G(\underline{x},t;\underline{x}_0) = \delta(\underline{x}-\underline{x}_0)\,\delta(t)$$

• Explicitly, we have:

$$(4)\quad p(\underline{x},t) = f(t) * G(\underline{x},t;\underline{x}_0).$$

- Moreover, due to time convolutions, it is convenient to work in frequency domain:

(5) $\hat{p}(\underline{x}, \omega) = \hat{f}_B(\omega - \omega_0)\, \hat{G}(\underline{x}, \omega; \underline{x}_s)$

where $\hat{G}$ = outgoing solution of

(6) $\Delta_x \hat{G}(\underline{x}, \omega; \underline{x}_s) + k^2 n^2(\underline{x})\, \hat{G}(\underline{x}, \omega; \underline{x}_s) = -\delta(\underline{x} - \underline{x}_s)$,

$k = \frac{\omega}{c_0}$ = wave number ; c_0 = reference sound speed

$n(\underline{x}) = \frac{c_0}{c(\underline{x})}$ = index of refraction

2) The forward model

Passive array and distributed sources:

- When the source at $\underline{y}$ emits signal $f(\underline{y}, t)$, the field recorded at receiver $\underline{x}_R \in \mathcal{A}$, for $t \in [0, T]$ is given by

(7) $p(\underline{x}_R, t) = \int_{\mathcal{Y}} d\underline{y}\;\, f(\underline{y}, t) * G(\underline{x}_R, t; \underline{y})$

where $\mathcal{Y}$ = support of distributed source.

Active array data ⇝ scattered echoes

- Let us write the velocity in the medium as:

(8) $c(\underline{x}) + \delta c(\underline{x})$

where $c(\underline{x})$ = background sound speed and $\delta c(\underline{x})$ = due to reflectors that we wish to image.

Note: $c(\underline{x})$ can consist of a smooth part that we know and can estimate + fluctuations due to inhomogeneities that we don't know and model with random processes.

• We shall neglect in this lecture the fluctuations so for now $c(\underline{x})$ = smooth.

• Let then $p+\delta p$ be the solution of:

$$(9)\quad \begin{cases} \left\{\frac{1}{[c(\underline{x})+\delta c(\underline{x})]^2}\frac{\partial^2}{\partial t^2}-\Delta\right\}\left[p(\underline{x},t)+\delta p(\underline{x},t)\right]=\mathcal{E}(\underline{x},t), & t>0 \\ p(\underline{x},t)+\delta p(\underline{x},t)=0 & \text{for } t\le 0 \end{cases}$$

where $p(\underline{x},t)$ solves (2), the equation in the background, with the reflectors removed.

• The forward model for the array data is:

$$(10)\quad \tilde{\mathcal{F}}[c+\delta c]=\left\{(p+\delta p)(\underline{x}_R,t) \text{ for } \underline{x}_R\in\mathcal{A},\ t\in[0,T]\right\}$$

where we include both the incident & scattered field.

The inverse problem is: find δc given data (10). This is a nonlinear problem.

Note: - There are very few attemps to deal with this nonlinear problem because: - optimization techniques not only lead to local minima but are prohibitively expensive

| in geophysics applications the problems & data sets are huge

| in radar, one needs quick images

etc.

Remedy: linearize (10) w.r.t. δc.

3) Born approximation

- Instead of (10), model the scattered echoes by

(11) $F[c]\delta c = \{\delta p(\underline{x}_R, t), \ \underline{x}_R \in \mathcal{A}, \ t \in [0,T]\}$

where δp solves the linearized equation

(12)
$$\left[\frac{1}{c^2(\underline{x})}\frac{\partial^2}{\partial t^2} - \Delta\right]\delta p(\underline{x},t) = \frac{2\,\delta c(\underline{x})}{c^3(\underline{x})}\,\frac{\partial^2 p(\underline{x},t)}{\partial t^2} \quad \text{for } t>0$$
$$\delta p(\underline{x},t) = 0 \quad \text{for } t\le 0$$

- In the Fourier domain, we have:

(13) $$\left[\Delta + k^2 n^2(\underline{x})\right]\delta\hat{p}(\underline{x},\omega) = -k^2\, r(\underline{x})\,\hat{p}(\underline{x},\omega),$$

where $r(\underline{x})$ = reflectivity,

(14) $$r(\underline{x}) = \frac{c_0^2}{c^2(x)}\left[-\frac{2\,\delta c(x)}{c(x)}\right] \simeq \frac{c_0^2}{[c(x)+\delta c(x)]^2} - \frac{c_0^2}{c^2(x)}.$$

- Using Green's formula and (5), we have:

(15) $$\delta\hat{p}(\underline{x}_R,\omega) = \int d\underline{y}\; r(\underline{y})\; k^2 \hat{f}_B(\omega-\omega_0)\,\hat{G}(\underline{x}_R,\omega;\underline{y})\,\hat{G}(\underline{y},\omega;\underline{x}_s)$$

Note 1: We linearized the forward map in δc => model primary reflections at scatterers we wish to find.

Note 2: The relationship between the data and $c(x)$ is highly nonlinear!

Note 3: How accurate is Born approximation? That is, is it true that $\mathcal{F}[c+\delta c] - \mathcal{F}[c] - F[c]\delta c$ is small?

• There are almost no rigorous results on this question, except in 1-D: Lewis and Symes, Inverse Problems 7, 1991, p. 597-632.

They proved the sharp estimate:

$$\| (p+\delta p) - p - \delta p_{\text{Born}} \|_{L^2[[0,a]\times[0,T]]} \leq K_1 \|\delta c\|_{L^2[0,D]} \|\delta c\|_{H^2[0,D]} + K_2 \|\delta c\|^2_{L^2[0,D]}$$

where K_1, K_2 depend on $\|c\|_{H^2[0,D]}$ and $\|c\|_{H^3[0,D]}$, respectively; D = depth depending on recording time T by causality.

This estimate shows that, for c smooth s.t. K_1, K_2 are bounded, for a sequence $\{\delta c\}$ bounded in $H^2[0,D]$ and yet $\{\delta c\} \to 0$ in $L^2[0,D]$ (that is for δc rough or "oscillatory") the linearization error tends to 0. $\Rightarrow$

Conjecture: Born approximation is OK when we have separation of scales: $c(\underline{x})$ = smooth & $\delta c(\underline{x})$ = rough. (Practice & numerics supports this).

$\Rightarrow$ linearization should give us a good idea about reflectors (rapid changes in δc) which lead to reflected echoes (what we measure).

• We shall accept Born approximation and next, we shall explore further the supposed smoothness of $c(\underline{x})$ to explain how the data looks like $\rightsquigarrow$ high frequency asymptotics.

4) High frequency asymptotics

- In a homogeneous medium, with $c(\underline{x}) = c_0$,

(16) $$G(\underline{x}, t; \underline{x}_0) = \alpha(\underline{x}, \underline{x}_0)\, S(t - \tau(\underline{x}, \underline{x}_0)),$$

where $\tau(\underline{x}, \underline{x}_0) = \dfrac{|\underline{x} - \underline{x}_0|}{c_0}$ = travel time between $\underline{x}, \underline{x}_0$,

$\alpha(\underline{x}, \underline{x}_0)$ = amplitude (smooth) and $S(t)$ = singular at $t = 0$. For example, in 3-D,

(17) $$G(\underline{x}, t; \underline{x}_0) = \frac{\delta[t - \tau(\underline{x}, \underline{x}_0)]}{4\pi |\underline{x} - \underline{x}_0|} \Rightarrow \quad S(t) \rightsquigarrow \delta(t)$$

$$\alpha(\underline{x}, \underline{x}_0) = \frac{1}{4\pi |\underline{x} - \underline{x}_0|}$$

- Using (16) in our forward model (Born approx) (15) $\Rightarrow$

$$\delta\hat{p}(\underline{x}_R, \omega) = \int d\underline{y}\; k^2 \hat{f}_B(\omega - \omega_0) \left[\hat{S}(\omega)\right]^2 \alpha(\underline{x}_R, \underline{y})\, \alpha(\underline{y}, \underline{x}_0)\, r(\underline{y})$$
$$\exp\{i\omega[\tau(\underline{x}_0, \underline{y}) + \tau(\underline{x}_R, \underline{y})]\}$$

and, in time,

(18) $$\delta p(\underline{x}_R, t) = \int d\underline{y}\; r(\underline{y}) \frac{\alpha(\underline{x}_R, \underline{y})\, \alpha(\underline{y}, \underline{x}_0)}{c_0^2} \varphi(t - \tau(\underline{x}_R, \underline{y}) - \tau(\underline{x}_0, \underline{y}))$$

where $\varphi(t)$ has Fourier coefficients:

(19) $$\hat{\varphi}(\omega) = \omega^2 \hat{f}_B(\omega - \omega_0) \left[\hat{S}(\omega)\right]^2 \sim \omega^2 \hat{f}_B(\omega - \omega_0)$$

$\Rightarrow$ if we send a short pulse from $\underline{x}_0$, we expect the scattered echoes (due to reflection of waves at singularities of r) to look as time delayed versions of $\varphi(t)$, by travel times that let us know how far the reflectors are.

• In media with varying $c(\underline{x})$, but smooth, we expect the same conclusion because of the "high frequency" asymptotic result:

(20) $$G(\underline{x}, t; \underline{x}_0) = \alpha(\underline{x}, \underline{x}_0)\, S(t - \tau(\underline{x}, \underline{x}_0)) + R(\underline{x}, t; \underline{x}_0)$$

where α = smooth amplitude to be determined, τ = smooth travel time to be determined, $S(t)$ = singular function at $t=0$ (same singularity as in homogeneous media) and R is a remainder that is smoother that S.

Expression (20) is called the progressing wave expansion (it can be done equivalently in frequency domain $\rightsquigarrow$ geometrical optics or WKB approximation). To derive the eq. for τ and α, substitute (20) in the eq. for G, for time $t > 0$ $\Rightarrow$

$$\left[\frac{1}{c^2(\underline{x})} - |\nabla_{\underline{x}} \tau(\underline{x}, \underline{x}_0)|^2\right] \alpha(\underline{x}, \underline{x}_0)\, S''(t - \tau(\underline{x}, \underline{x}_0)) +$$

(21) $$+ \left[\alpha(\underline{x}, \underline{x}_0)\, \Delta_{\underline{x}} \tau(\underline{x}, \underline{x}_0) + 2 \nabla_{\underline{x}} \tau(\underline{x}, \underline{x}_0) \cdot \nabla_{\underline{x}} \alpha(\underline{x}, \underline{x}_0)\right] S'(t - \tau)$$

$$+ \left\{\left[\frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta_x\right] R(\underline{x}, t; \underline{x}_0) - S(t - \tau(\underline{x}, \underline{x}_0))\, \Delta_x \alpha(\underline{x}, \underline{x}_0)\right\} = 0$$

We have 3 terms with decreasing order of singularity $\Rightarrow$ each term should vanish identically $\Rightarrow$

(22) $$|\nabla_{\underline{x}} \tau(\underline{x}, \underline{x}_0)| = \frac{1}{c(\underline{x})} \quad = \text{eikonal eq for } \tau$$

(23) $\alpha(\underline{x}, \underline{x_0}) \Delta_{\underline{x}} \tau(\underline{x}, \underline{x_0}) + 2 \nabla_{\underline{x}} \tau(\underline{x}, \underline{x_0}) \cdot \nabla_{\underline{x}} \alpha(\underline{x}, \underline{x_0}) = 0$, the transport eq for α and then, we can determine R from the last term in (21).

- One can prove an energy estimate for R:

$$\int d\underline{x} \int_0^T dt \, |R(\underline{x}, t; \underline{x_0})|^2 \leq C \quad \left(\text{constant depends on } c(x)\right)$$

but the key point is that we shall neglect R because: we are in a high frequency regime (this can be seen because wavelengths are $<<$ length scale of change for $c(\underline{x})$ => scaling is implicit here) and because R is smoother that S ~> at high freq ω, $\hat{R}(\underline{x}, \omega; x_0) << \hat{S}(\omega)$ =>

$$f(t) * G(\underline{x}, t; \underline{x_0}) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega) \hat{G}(\underline{x}, \omega; \underline{x_0})$$

(24)
$$\simeq \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega) \alpha(\underline{x}, \underline{x_0}) \hat{S}(\omega) e^{i\omega \tau(\underline{x}, \underline{x_0})}$$

$$= \alpha(\underline{x}, \underline{x_0}) f(t) * S(t - \tau(\underline{x}, \underline{x_0}))$$

$$\left(= \alpha(\underline{x}, \underline{x_0}) f(t - \tau(\underline{x}, \underline{x_0})) \text{ in 3D} \right)$$

- The travel time is given by

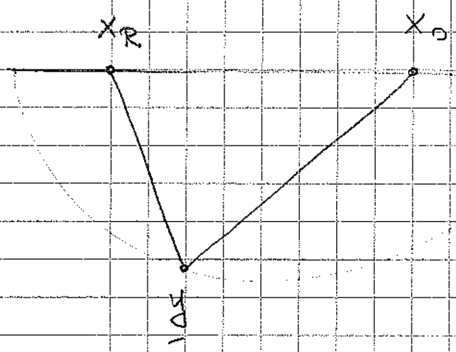
$$\tau(\underline{x}, \underline{x_0}) = \int \frac{dl}{c(\underline{x}(l))} = \text{integral along ray starting at } \underline{x_0}.$$

Ray crossing can occur and this must be addressed properly

Conclusion: In smooth media, the reflection data is given by (18) approximately. In particular, if we had an infinite bandwidth, so that $\varphi(t) \sim \delta(t)$ $\Rightarrow$

$$\sim \int d\underline{y}\ r(\underline{y})\ \alpha(\underline{x}_R, \underline{y})\ \alpha(\underline{y}, \underline{x}_S)\ \delta\left[t - \tau(\underline{x}_R, \underline{y}) - \tau(\underline{x}_S, \underline{y})\right]$$

$\Rightarrow$ the data received at time t is the integral of the reflectivity over the isochron: $\{\underline{y}$ s.t. $t = \tau(\underline{x}_R, \underline{y}) + \tau(\underline{x}_S, \underline{y})\}$



$\leftarrow$ isochron (is an ellipse if $c(x) = c_0$).

$\Rightarrow$ we have a generalized Radon transform, with the isochron's replacing straight lines.