Lecture 1: Forward model for array data

Outline:
1) The governing PDE
2) The forward model for passive and active array
3) Linearization (Born approximation)
4) High frequency asymptotics and description of data

1) The PDE:

- Depending on the application, one would like to consider mathematical models that account for:
  - material anisotropy
  - multiple wave modes (shear, pressure waves)
  - polarization effects
  - attenuation and frequency dispersion of waves...

- We make a big simplification: Acoustic waves $\rightarrow$ waves in fluids.

   This model allows us to understand key ideas in imaging and it is, in fact, the basis for contemporary medical and seismic imaging.

The acoustic wave equations

- These equations are in fact linearized versions of the equations of conservation of mass and of the momentum balance in a fluid with constitutive relation $p = P(s)$, where $p =$ pressure, $s =$ mass density
If $p_0, s_0, v_0$ = pressure, density, particle velocity in the reference state, then

$$c^2 = \frac{dP(s)}{ds} > 0 = \text{change of rate of pressure with density}$$

$$= \text{speed of sound speed}$$

Moreover, $p = p_0 + \delta p$, $v = v_0 + \delta v$ satisfy:

$$\begin{cases}
\frac{1}{s_0 c^2} \frac{\partial \delta p}{\partial t} + \nabla \delta v = 0 \\
\frac{s_0}{2} \frac{\partial \delta v}{\partial t} + \nabla \delta p = E = \text{external force} \quad \text{for } t > 0 \\
\delta v = 0, \quad \delta p = 0 \quad \text{for } t = 0
\end{cases}$$

Note: We assume, for simplicity $s_0 = \text{constant}$. However, $c^2 = c^2(x)$. (It could depend on $t$, in principle, but we assume a stationary medium).

We also drop the $\delta$ in the notation so that $\delta v \rightarrow v$ and $\delta p \rightarrow p$.

In most of the calculations that follow, we work with the 2nd order PDE for $p$, obtained by eliminating $v$ from (1):

$$\begin{cases}
\frac{1}{c^2(x)} \frac{\partial^2 p(x,t)}{\partial t^2} - \Delta p(x,t) = -\nabla \cdot E = E(x,t) \quad \text{for } t > 0 \\
p(x,t) = 0 \quad \text{for } t \leq 0
\end{cases}$$

Depending on the problem, we may have a bounded domain
or in unbounded one. For simplicity, we shall work in the entire $\mathbb{R}^3$.

- The solvability (weak sense) of (2) follows from classic studies (Lions' book: "Nonhomogeneous boundary value problems and Applications", 1972 or other PDE books).

The force term $\mathcal{E}(x,t)$

- This is usually of the form: $\mathcal{E}(x,t) = f(t) \Phi(x-x_0)$ for $\Phi$ denoting the supp of the source centered at $x_0$.
- We shall idealize this as $\Phi(x-x_0) = \delta(x-x_0)$.
- The pulse is $f(t) = e^{-i\omega t} f_0(t)$, so that in Fourier,
- $\hat{f}(\omega) = \int f(t) e^{-i\omega t} dt = \hat{f}_0(\omega - \omega_0)$. If $\delta_0$ is the bandwidth (length of support) interval of $\hat{f}_0(\omega)$, $\Rightarrow \hat{f}(\omega)$ is supported in $[\omega_0 - \delta_0, \omega_0 + \delta_0]$. We call $\omega_0 = \text{central (carrier) frequency}$.

The Green's function

- Given the model $\mathcal{E}(x,t) = f(t) \delta(x-x_0)$ of the excitation, we shall write the pressure field in terms of the Green's function $G(x,t;x_0)$, solving
- \[ \left[ \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta_x \right] G(x,t;x_0) = \delta(x-x_0) \delta(t) \]

- Explicitly, we have:
- \[ p(x,t) = f(t) \ast G(x,t;x_0). \]
Moreover, due to time convolutions, it is convenient to work in frequency domain:

\[
\hat{p}(x, \omega) = \tilde{f}_b(\omega - w_0) \hat{G}(x, \omega; \omega_0)
\]

where \( \hat{G} \) = outgoing solution of

\[
\Delta \hat{G}(x, \omega; \omega_0) + k^2 m^2(x) \hat{G}(x, \omega; \omega_0) = -\delta(x - x_0)
\]

\[ k = \frac{\omega}{c_0} = \text{wave number} \quad ; \quad c_0 = \text{reference sound speed} \]

\[ m(x) = \frac{c_0}{c(x)} = \text{index of refraction} \]

2) The forward model

Passive array and distributed sources.

- When the source of a unit signal \( f(y, t) \), the field recorded at receiver \( x_R \in A \), for \( t \in [0, T] \) is given by

\[
p(x_R, t) = \int dy f(y, t) \ast G(x_R, t; y)
\]

where \( S \) = support of distributed source.

Active array data = scattered echoes

- Let us write the velocity in the medium as:

\[
c(x) + \delta c(x)
\]

where \( c(x) = \text{background sound speed} \) and \( \delta c(x) = \text{due to reflectors that we wish to image} \).
Note: \( c(x) \) can consist of a smooth part that we know and can estimate + fluctuations due to inhomogeneities that we don’t know and model with random processes.

- We shall neglect in this lecture the fluctuations so for now \( c(x) = \text{smooth} \).
- Let then \( p + \delta p \) be the solution of:

\[
\left\{ \begin{array}{l}
\frac{1}{[c(x) + 8c_0]^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \left[ p(x,t) + \delta p(x,t) \right] = \varepsilon(x,t), \quad t > 0 \\
p(x,t) + \delta p(x,t) = 0 \quad \text{for } t \leq 0
\end{array} \right.
\]

where \( p(x,t) \) solves (2), the equation in the background with the reflectors removed.

The forward model for the array data is:

\[
\mathcal{F}[c + \delta c] = \left\{ (p + \delta p)(x, t) \text{ for } x \in A, \ t \in [0,T] \right\}
\]

where we include both the incident & reflected field.

The inverse problem is: find \( \delta c \) given data (10). This is a nonlinear problem.

Note: There are very few attempts to deal with this nonlinear problem because: optimization techniques not only lead to local minimums but are prohibitively expensive in geophysics applications the problems & data sets are huge

Remedy: linearize (10) w.r.t. \( \delta c \).
3) Born approximation

- Instead of (10), model the scattered echoes by

\[ F[c, \delta c] = \{ \delta p(x_R, t) \mid x_R \in \mathbb{A}, \ t \in [0, T] \} \]

where \( \delta p \) solves the linearized equation

\[ \left[ \frac{1}{c^2(x)} \frac{\delta^2}{\delta t^2} - \Delta \right] \delta p(x, t) = \frac{2 \delta c(x)}{c^3(x)} \frac{\partial^2 p(x, t)}{\partial t^2} \quad \text{for } t > 0 \]

\[ \delta p(x, t) = 0 \quad \text{for } t < 0 \]

- In the Fourier domain, we have:

\[ \left[ \Delta + \kappa^2 n^2(x) \right] \hat{\delta p}(x, \omega) = -\kappa^2 n(x) \hat{p}(x, \omega), \]

where \( n(x) \) = reflectivity,

\[ n(x) = \frac{c_0^2}{c^2(x)} \left[ -2 \frac{\delta c(x)}{c(x)} \right] \approx \frac{c_0^2}{[c(x) + \delta c(x)]^2} - \frac{c_0^2}{c^2(x)} \]

- Using Green's formula and (5), we have:

\[ \hat{\delta p}(x_R, \omega) = \int d\omega' n(y) \kappa^2 \hat{f}_0(\omega - \omega') \hat{G}(x_R, y; \omega') \hat{G}(y, \omega'; \omega) \]

Note 1: We linearized the forward map in \( \delta c \rightarrow \) model primary reflections of scatterers we wish to find.

Note 2: The relationship between the data and \( c(x) \) is highly nonlinear!

Note 3: How accurate is Born approximation? That is, is it true that \( F[c + \delta c] - F[c] \approx F[c] \delta c \)?
There are almost no rigorous results on this question, except in 1-D: Leuen and Synnev, Inverse Problems 7, 1991, p. 597-632.

They proved the sharp estimate:

\[ \| (p + \delta p) - p - \delta p \|_{L^2([0,T] \times \mathbb{R}^3)} \leq K_1 \| \delta c \|_{L^2([0,T] \times \mathbb{R}^3)} \leq K_2 \| \delta c \|_{L^2([0,T])} + K_2 \| \delta c \|_{L^2([0,T])} \]

where \( K_1, K_2 \) depend on \( \| c \|_{H^2_\mathcal{O}} \) and \( \| c \|_{H^3_\mathcal{O}} \), respectively; \( D \) = depth depending on recording time \( T \) by causality.

This estimate shows that, for a smooth \( c \), \( K_1, K_2 \) are bounded, for a sequence \( \{ \delta c \} \) bounded in \( H^2_\mathcal{O} \) and yet \( \{ \delta c \} \rightarrow 0 \) in \( L^2_\mathcal{O} \), that is for \( \delta c \) rough or oscillatory, the linearization error tends to 0.

Conjecture: Born approximation is OK when we have separation of scales: \( c(x) = \) smooth & \( \delta c(x) = \) rough. (Practice & numerics support this).

=> linearization should give us a good idea about reflectors (rapid changes in \( \delta c \)) which lead to reflected waves (what we measure).

- We shall accept Born approximation and next, we shall explore further the supposed smoothness of \( c(x) \) to explain how the data looks like high frequency asymptotics.
4) High frequency asymptotics

- In a homogeneous medium, with \( c(x) = c_0 \),

\[
G(x, t; x_0) = \alpha(x, x_0) S(t - \tau(x, x_0)),
\]

where \( \tau(x, x_0) = \frac{|x - x_0|}{c_0} = \) travel time between \( x, x_0 \),
\( \alpha(x, x_0) = \) amplitude (smooth) and \( S(t) = \) singularity at \( t = 0 \). For example, in 3-D,

\[
G(x, t, x_0) = \frac{S\left[ \frac{t - \tau(x, x_0)}{4 \pi c_0 (x - x_0)} \right]}{4 \pi c_0 (x - x_0)} \Rightarrow \frac{S(t)}{\alpha(x, x_0)} = \frac{1}{4 \pi c_0 (x - x_0)}
\]

- Using (16) in our forward model (Born approx) (15):

\[
\delta \phi(x, \omega) = \int \frac{dy}{k^2} \delta \hat{\Phi}((\omega - \omega_0)[\hat{S}(\omega)])^2 \alpha(x, y) \alpha(y, x_0) \hat{\Phi}(y)
\]

\[
\exp \left\{ i \omega \left[ \tau(x, y) + \tau(x_0, y) \right] \right\}
\]

and, in time,

\[
\delta \phi(x, t) = \int \frac{dy}{k^2} \hat{\Phi}(y) \alpha(x, y) \alpha(y, x_0) \frac{S(t - \tau(x, y) - \tau(x_0, y))}{c_0^2}
\]

where \( \gamma(t) \) has Fourier coefficients.

\[
\hat{\gamma}(\omega) = \omega^2 \frac{1}{k^2} \delta \hat{\Phi}((\omega - \omega_0)[\hat{S}(\omega)])^2 \sim \omega^2 \frac{1}{k^2} \delta \hat{\Phi}((\omega - \omega_0))
\]

\[
\Rightarrow \text{if we send a short pulse from } x_0, \text{ we expect the scattered echoes (due to reflection of waves at singularities of } \alpha) \text{ to look as time delayed versions of } \gamma(t), \text{ by travel times that let us know how far the reflectors are.}
In media with varying \( c(x) \), but smooth, we expect the same conclusion because of the "high frequency"
asympotic result:

\[
G(x, t; x_0) = \alpha(x, x_0) \mathcal{S}(t - \mathcal{T}(x, x_0)) + R(x, t, x_0)
\]

where \( \alpha = \) smooth amplitude to be determined, \( \mathcal{T} = \) smooth travel time to be determined, \( \mathcal{S} = \) singular function at \( t=0 \) (same singularity as in homogeneous media) and \( R \) is a remainder that is smoother than \( \mathcal{S} \).

Expression (20) is called the progressing wave expansion (it can be done equivalently in frequency domain with geometrical optics or WKB approximations). To derive the eq. for \( \mathcal{T} \) and \( \alpha \), substitute (20) in the eq. for \( G \), for time \( t > 0 \) =>

\[
\left[ \frac{1}{c^2(x)} \alpha(x, x_0) \right] \alpha''(t - \mathcal{T}(x, x_0)) + \mathcal{S}''(t - \mathcal{T}(x, x_0)) + \mathcal{S}(t - \mathcal{T}(x, x_0)) \mathcal{T}'(t - \mathcal{T}(x, x_0)) = 0
\]

\[
\left[ \alpha(x, x_0) \Delta x \mathcal{T}(x, x_0) + 2 \nabla_x \mathcal{T}(x, x_0) \cdot \nabla_x \alpha(x, x_0) \right] \mathcal{S}(t - \mathcal{T}(x, x_0)) + \left[ \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta x \right] R(x, t; x_0) - \mathcal{S}(t - \mathcal{T}(x, x_0)) \Delta x \alpha(x, x_0) = 0
\]

We have 3 terms with decreasing order of singularity => each term should vanish identically =>

\[
\left| \nabla_x \mathcal{T}(x, x_0) \right| = \frac{1}{c(x)} = \text{constant eq. for } \mathcal{T}
\]
\( \alpha(x, x_0) \Delta x \beta(x, x_0) + 2 \nabla_{x} \beta(x, x_0) \cdot \nabla_{x} \alpha(x, x_0) = 0, \) the transport eq for \( \alpha \) and then, we can determine \( R \) from the last term in (21).

- One can prove an energy estimate for \( R_i \):
  \[
  \int_0^1 \int_0^1 |R_i(x, t; x_0)|^2 \, dt \, dx \leq C \quad \text{(constant depends on \( c(t) \))}
  \]

  but the key point is that we shall neglect \( R \), because:

  - we are in a high frequency regime (this can be seen because
    wavelengths are \( \ll \) length scale of change for \( c(x) \))
  - neglecting \( \hat{R}(x, \omega; x_0) \ll \hat{\beta}(\omega) \)

  \[ f(t) \ast G(x, t; x_0) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{\beta}(\omega) \hat{G}(x, \omega; x_0) \]

(24)

\[
= \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{\beta}(\omega) \alpha(x, x_0) \hat{\delta}(\omega) e^{i\omega \beta(x, x_0)}
\]

\[
= \alpha(x, x_0) f(t) \ast \delta(t - \beta(x, x_0))
\]

\[
= \alpha(x, x_0) f(t - \beta(x, x_0)) \quad \text{in 3D}
\]

- The travel time is given by

\[ t \beta(x, x_0) = \int \frac{dt}{c(t(x))} \quad \text{- integral along ray starting at } x_0. \]

Ray crossing can occur and this must be addressed properly.
Conclusion: In smooth media, the reflection data is given approximately by \((18)\). In particular, if we had an infinite bandwidth, so that \(\chi(t) \sim \delta(t) \Rightarrow\)

\[
\sim \int \delta(y) \chi(x, y) \chi(y, x_0) \delta(t - z(x, y) - z(x_0, y)) 
\]

\(\Rightarrow\) the data received at time \(t\) is the integral of the reflectivity over the isochronus \(\{ y \text{ s.t. } t = z(x, y) + z(x_0, y) \}\), an ellipse \(y(x) = c_0\).

\(\Rightarrow\) we have a generalized Radon transform, with the isochron's replacing straight lines.