Reflection and Transmission of Acoustic Waves by a Locally-layered Random Slab

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Abstract

We consider acoustic propagation through a slab in which rapidly-varying random plane layering has been subjected to smooth small amplitude undulations. The reflection and transmission properties of such a locally-layered slab are studied. Of principal interest is the question of the robustness of the plane layered theory. Do phenomena such as localization and the results of O'Doherty-Anstey theory persist under perturbations of plane layering? We establish some results affirming the robustness of these phenomena.

Contents

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1 Introduction

The study of wave propagation in disordered or random media has been extensively pursued, being motivated by applications in many of the physical sciences. References [1], [2] and [3] provide an overview of some recent activity. A sizable portion of this work has been focussed upon the study of one-dimensional models. Such models have relevance to geophysical applications; they are also the most tractable mathematically.

One-dimensional random wave propagation exhibits some striking features. Coherent multiple scattering, induced by the fluctuations in the underlying random medium, suppresses transmission and effectively localizes the wave (*c.f.* [4], [5]). If the random medium fluctuations are weak, the corresponding transmission length scale, or localization length, is large. O'Doherty and Anstey [6] discovered that in this regime, a pulse, observed in a Lagrangian frame moving with the appropriate random velocity, will appear to retain its shape up to a slow spreading; references [7, 8, 9, 14] provide further insight into this O'Doherty-Anstey phenomenon.

The question of robustness naturally arises. Do these phenomena persist under modest geometric perturbations of the underlying random layering? The difficulty inherent in providing an answer to this question lies in the fact that such geometric perturbations, no matter how modest, destroys transverse homogeneity and forces one to deal with stochastic partial, rather than ordinary, differential equations.

In [15] we studied extensively the problem of acoustic wave propagation in a rapidly-varying plane layered slab. A small parameter ε was used to delineate three relevant length scales. The largest scale or macroscale (*e.g.* the slab thickness L) was assumed to be $O(1)$. The acoustic wavelength defined an intermediate $O(\varepsilon)$ length scale while the correlation length of the random layering was assumed to vary on the smallest $O(\varepsilon^2)$ scale. On the one hand, therefore, the wavelength of the acoustic radiation was assumed to be small relative to the scale on which deterministic variations in the slab constitutive parameters occurred, so geometric acoustics might reasonably be expected to play an important role. On the other hand, the acoustic wavelength was assumed to span many correlation lengths and one might further expect this scaling to produce a meaningful probabilistic limit asymptotically as $\varepsilon \to 0$. These expectations were indeed realized in [15]. The multiple scattering induced by the random layering created localization phenomena which profoundly affected the reflection and transmission properties of the slab. The O'Doherty-Anstey phenomenon was also shown to follow from a weak-fluctuation variant of this model. In [17] we studied the reflection and transmission of elastic, time harmonic plane waves by randomly layered media, especially mode conversion by the inhomogeneities. In [18] we extended the statistical inverses theory for randomly layered media of [15] to the case of acoustic wave pulses generated by a point source.

In this paper our goal is to test the robustness of this theory with respect to perturbations in the layering geometry. In other words, will the essential features of the theory be preserved if we introduce gentle deformations (undulations) or other reasonably small perturbations into the plane layering structure? We shall refer to this deformed geometry as a locally layered medium. The O'Doherty-Anstey phenomenon is analyzed for a different class of locally layered random media by [14]. The deterministic background fluctuations are not small in amplitude but they are slowly varying.

In what follows, we extend much of the development of [15] to the more complicated locally layered setting, adhering closely to the setup and notation of that paper. In Section 2 the acoustic problem in a locally layered geometry is formulated, and scales are delineated in terms of the small parameter ε as described above. We assume a generalization of the plane-layered model that allows for small spatially-varying deviations in the direction of the random layering and also allows for some small, but fully three-dimensional nonrandom variations in the background. As will be shown, this generalization of the model does produce interesting leading order, three-dimensional spatial dependences in the final results.

In Sections 3 and 4 the quantities of interest, *i.e.* the reflected and transmitted pressure waves, are expressed in terms of reflection and transmission operators. In Section 5 the basic conservation relations for these operators are derived, and in Section 6, a limit for the operators is obtained, valid as $\varepsilon \downarrow 0$. This stochastic limit is conveniently expressed in terms of Ito "white noise" equations, and it is from these governing Ito equations that all the relevant statistics are derived in the remainder of the paper. We check, in Section 7, that this stochastic limit for the operators does indeed reduce to the strictly plane layered theory of [15] when the appropriate expressions are set equal to zero.

In Sections 8, 9 and 10 we investigate, both for reflected and transmitted waves, the coherent field, the space and time correlation functions, and the localization length. A generalized O'Doherty-Anstey theory is presented in Section 11, also for both transmitted and reflected waves. For all these cases, the main qualitative features of the strictly plane-layered theory survive the three-dimensional perturbations, although some $O(1)$ three-dimensional effects can be observed.

2 Problem Formulation and Scaling

As in [15], let (x, y) be coordinates transverse to an initially unperturbed plane layering with the scattering region in the interval $-L \leq z \leq 0$. To define the locally layered geometry, we introduce the coordinate transformation

$$x' = x, \quad y' = y, \quad z' = z + \varepsilon\phi(x, y, z), \tag{2.1}$$

where ϕ is a smooth and bounded function. The undulations are defined by the surfaces z' = constant and exhibit an $O(\varepsilon)$ (wavelength scale) amplitude variation over $O(1)$ (macroscale) transverse distances.

We are primarily interested in the multiple scattering properties of the bulk locally layered material. Therefore, we shall model the problem so as to eliminate interface complications. We shall flatten the undulations at both ends of the slab by assuming

$$\phi(x, y, -L) = \phi(x, y, 0) = 0 \tag{2.2}$$

so that $z' = z$ at $-L$ and 0. The half-spaces $z < -L$ and $z > 0$ will be assumed to be constant and homogeneous acoustic media. In $z > 0$, the density, bulk modulus and sound speed will be denoted by ρ_0, K_0 and c_0, respectively (with $K_0 = \rho_0 c_0^2$). The subscript 2 will be used to index their constant counterparts in $z < -L$. In the randomly undulating slab region $-L \leq z \leq 0$, the subscript 1 will be used. As in [15], the density will be assumed to be constant, *i.e.* $\rho = \rho_1$. The bulk modulus will be modeled as

$$K^{-1} = K_1^{-1}(1 + \nu(z'/\varepsilon^2)) + \varepsilon K_{11}^{-1}(x', y', z') \ , \qquad -L \leq z' \leq 0, \tag{2.3}$$

where K_1 is a constant, K_{11} is a deterministic spatially-varying perturbation and ν is a zero mean, stationary stochastic process bounded in modulus by a constant less than one. Since it is a function of z'/ε^2, the process ν decorrelates on the smallest $O(\varepsilon^2)$ spatial scale in a direction perpendicular to the undulations.

We shall assume that acoustic energy is incident upon the random slab from above. For the present discussion, we shall assume that this radiation is emitted by a point source located at $(0, 0, z_s)$ and having orientation defined by the unit vector $\mathbf{e}$. Then the governing acoustic equations for pressure p and particle velocity $\mathbf{u}$ are (*c.f.* equations (2.17)–(2.19) in [15])

$$\rho \partial_t \mathbf{u} + \nabla p = \varepsilon^{1/2} f(t/\varepsilon)\delta(x)\delta(y)\delta(z - z_s)\mathbf{e} \tag{2.4a}$$

$$K^{-1}\partial_t p + \nabla \cdot \mathbf{u} = 0 \tag{2.4b}$$

where

$$\rho = \begin{cases} \rho_0, & z > 0 \\ \rho_1, & -L < z < 0 \\ \rho_2, & z < -L \end{cases} \tag{2.5}$$

and

$$K^{-1} = \begin{cases} K_0^{-1}, & z > 0 \\ K_1^{-1}(1 + \nu(z'/\varepsilon^2)) + \varepsilon K_{11}^{-1}(x', y', z'), & -L < z < 0 \\ K_2^{-1}, & z < -L \end{cases} \tag{2.6}$$

with x', y' and z' defined by (2.1). The function f in (2.4a) defines the pulsed temporal signal emitted by the point source; the $\varepsilon^{1/2}$ factor makes the total energy released by the pulsed point source independent of ε.

We begin by rewriting equations (2.4), within the (source-free) slab region, in terms of the primed variables. Let $u^{(j)}, j = 1, 2, 3$, denote the three components of particle velocity in the fixed, unprimed Cartesian system. Let a function ψ be defined in terms of the inverse map as follows

$$x = x' \ , \ y = y' \ , \ z = z' - \varepsilon\psi(x', y', z', \varepsilon) \ . \tag{2.7}$$

Note from (2.1) that ψ satisfies the functional equation:

$$\psi(x',y',z',\varepsilon) = \phi(x',y',z'-\varepsilon\psi(x',y',z',\varepsilon)) \ . \tag{2.8}$$

In the slab region $-L \le z' \le 0$, the component acoustic equations become

$$\begin{aligned}
\rho_1 u_t^{(1)} + p_{x'} + \varepsilon\psi_{x'}(1-\varepsilon\psi_{z'})^{-1}p_{z'} &= 0 \\
\rho_1 u_t^{(2)} + p_{y'} + \varepsilon\psi_{y'}(1-\varepsilon\psi_{z'})^{-1}p_{z'} &= 0 \\
\rho_1 u_t^{(3)} + p_{z'} + \varepsilon\psi_{z'}(1-\varepsilon\psi_{z'})^{-1}p_{z'} &= 0 \\
\left[K_1^{-1}(1+\nu)+\varepsilon K_{11}^{-1}\right]p_t + u_{x'}^{(1)} + u_{y'}^{(2)} + u_{z'}^{(3)} & \\
+\varepsilon(1-\varepsilon\psi_{z'})^{-1}\left[\psi_{x'}u_{z'}^{(1)} + \psi_{y'}u_{z'}^{(2)} + \psi_{z'}u_{z'}^{(3)}\right] &= 0.
\end{aligned} \tag{2.9}$$

Equations (2.9), in turn, can be used to obtain the following system of equations for p and $u^{(3)}$

$$\begin{aligned}
p_{z'} &= -\rho_1(1-\varepsilon\psi_{z'})u_t^{(3)} \\
\left[1+\varepsilon^2(\psi_{x'}^2+\psi_{y'}^2)\right]u_{z't}^{(3)} &= (1-\varepsilon\psi_{z'})\left[\rho_1^{-1}(p_{x'x'}+p_{y'y'}) - [K_1^{-1}(1+\nu)+\varepsilon K_{11}^{-1}]\,p_{tt}\right. \\
&\quad \left. -\varepsilon\left[2\partial_{x'}(\psi_{x'}u_t^{(3)}) + 2\partial_{y'}(\psi_{y'}u_t^{(3)}) - (\psi_{x'x'}+\psi_{y'y'})u_t^{(3)}\right]\right], \\
&\quad -L \le z' \le 0 \ .
\end{aligned} \tag{2.10}$$

For simplicity of notation, we shall now drop the primes and the (3) - superscript. As in equation (2.21) of [15], we introduce the following Fourier transforms

$$\begin{aligned}
\hat{p}(\boldsymbol{\kappa},\omega,z) &= \iiint p e^{i\omega(t-\boldsymbol{\kappa}\cdot\mathbf{x})/\varepsilon}dtd\mathbf{x} \\
\hat{u}(\boldsymbol{\kappa},\omega,z) &= \iiint u e^{i\omega(t-\boldsymbol{\kappa}\cdot\mathbf{x})/\varepsilon}dtd\mathbf{x}\ ,\ \boldsymbol{\kappa}=(\kappa_1,\kappa_2),\ \mathbf{x}=(x,y)\ .
\end{aligned} \tag{2.11}$$

From (2.8) we infer that

$$\psi = \phi - \varepsilon\phi_z\phi + O(\varepsilon^2) \ . \tag{2.12}$$

It then follows that the transformed pressure and particle velocity satisfy the following system of integro-differential equations

$$\partial_z\hat{p} = i\frac{\omega}{\varepsilon}\rho_1\hat{u} - i\omega\rho_1\hat{\phi}_z * \hat{u} + \ldots$$

$$
\begin{aligned}
\partial_z \hat{u} &= i\frac{\omega}{\varepsilon}\left[K_1^{-1} - \rho_1^{-1}\kappa^2\right]\hat{p} + i\frac{\omega}{\varepsilon}K_1^{-1}\nu\hat{p} - i\omega\left[[K_1^{-1}\hat{\phi}_z - (\widehat{K_{11}^{-1}})] * \hat{p}\right] + \\
&\quad + i\omega\rho_1^{-1}\hat{\phi}_z * [\kappa^2\hat{p}] + 2\omega^2\left[\kappa_1[(\kappa_1\hat{\phi}) * \hat{u}] + \kappa_2[(\kappa_2\hat{\phi}) * \hat{u}]\right] + \dots \\
&\quad -L \le z \le 0,
\end{aligned} \tag{2.13}
$$

where $\kappa^2 = \boldsymbol{\kappa}\cdot\boldsymbol{\kappa}$ and

$$
\begin{aligned}
\hat{\phi}(\omega\boldsymbol{\kappa}, z) &= \iint \phi e^{-i\omega\boldsymbol{\kappa}\cdot\mathbf{x}} d\mathbf{x} \\
(\hat{\phi} * \hat{u})(\boldsymbol{\kappa}, \omega, z) &= \left(\frac{\omega}{2\pi}\right)^2 \iint \hat{\phi}(\omega\boldsymbol{\lambda}, z)\hat{u}(\boldsymbol{\kappa} - \varepsilon\boldsymbol{\lambda}, \omega, z)d\boldsymbol{\lambda},
\end{aligned} \tag{2.14}
$$

and the ellipses in equations (2.13) indicate negligible terms which shall, in subsequent equations, simply be dropped. An examination of equations (2.13) reveals that the deformation function ϕ and the deterministic bulk modulus perturbation K_{11}^{-1} introduce basically the same level of complexity, *i.e.* $O(1)$ convolution terms.

We are ultimately interested in recasting equations (2.13) into equations for the amplitudes of upgoing and downgoing waves. Sound speeds c_j and acoustic impedances ζ_j for each of the three regions are defined as

$$
c_j = (K_j/\rho_j)^{1/2}, \quad \zeta_j = \rho_j c_j/(1 - \kappa^2 c_j^2)^{1/2}, \quad j = 0, 1, 2. \tag{2.15}
$$

A travel time τ relative to $z = 0$ is then defined as

$$
\tau(z,\kappa) = \begin{cases} (1-\kappa^2 c_0^2)^{1/2} c_0^{-1} z, & z > 0 \\ (1-\kappa^2 c_1^2)^{1/2} c_1^{-1} z, & -L \le z \le 0 \\ -(1-\kappa^2 c_1^2)^{1/2} c_1^{-1} L + (1-\kappa^2 c_2^2)^{1/2} c_2^{-1}(z+L), & z < -L \ . \end{cases} \tag{2.16}
$$

(Recall that the depth variable in (2.16) is actually the primed variable. At points outside the slab, we simply equate primed and unprimed depths.) As in equation (2.26) of [15] , we introduce amplitudes A and B by the relations

$$
\begin{aligned}
\hat{p} &= \zeta^{1/2}[Ae^{i\frac{\omega}{\varepsilon}\tau} - Be^{-i\frac{\omega}{\varepsilon}\tau}] \\
\hat{u} &= \zeta^{-1/2}[Ae^{i\frac{\omega}{\varepsilon}\tau} + Be^{-i\frac{\omega}{\varepsilon}\tau}],
\end{aligned} \tag{2.17}
$$

where ζ is the piecewise-constant acoustic impedance taking on the values defined by (2.15). When equations (2.17) are substituted into (2.13), we obtain the following equations for A and B:

$$\begin{aligned} \frac{d}{dz}A &= i\frac{\omega}{\varepsilon}n\left[A - Be^{-i2\frac{\omega\tau}{\varepsilon}}\right] - i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2 \iint \partial_z\left[\hat{\phi}e^{-i\omega\tau_{\boldsymbol{\kappa}}\cdot\boldsymbol{\lambda}}\right]A(\boldsymbol{\kappa}-\varepsilon\boldsymbol{\lambda},\omega,z)d\boldsymbol{\lambda} \\ &\quad + i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2 \iint (\widehat{K_{11}^{-1}})e^{-i\omega\tau_{\boldsymbol{\kappa}}\cdot\boldsymbol{\lambda}}A(\boldsymbol{\kappa}-\varepsilon\boldsymbol{\lambda},\omega,z)d\boldsymbol{\lambda} \\ \frac{d}{dz}B &= i\frac{\omega}{\varepsilon}n\left[Ae^{i2\frac{\omega}{\varepsilon}\tau} - B\right] + i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2 \iint \partial_z\left[\hat{\phi}e^{i\omega\tau_{\boldsymbol{\kappa}}\cdot\boldsymbol{\lambda}}\right]B(\boldsymbol{\kappa}-\varepsilon\boldsymbol{\lambda},\omega,z)d\boldsymbol{\lambda} \\ &\quad - i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2 \iint (\widehat{K_{11}^{-1}})\ e^{i\omega\tau_{\boldsymbol{\kappa}}\cdot\boldsymbol{\lambda}}B(\boldsymbol{\kappa}-\varepsilon\boldsymbol{\lambda},\omega,z)d\boldsymbol{\lambda}, \qquad -L < z < 0, \end{aligned} \tag{2.18}$$

where (*c.f.* equation (2.28) of [15]):

$$\begin{aligned} n &= \frac{1}{2}K_1^{-1}\zeta_1\nu = \frac{\nu}{2c_1(1-\kappa^2c_1^2)^{1/2}} \\ \tau_{\boldsymbol{\kappa}} &= \nabla_{\boldsymbol{\kappa}}\tau = \frac{-c_1\boldsymbol{\kappa}z}{(1-\kappa^2c_1^2)^{1/2}}. \end{aligned} \tag{2.19}$$

The boundary conditions accompanying equations (2.18) follow from the requirement that pressure and the normal component of particle velocity be continuous across the slab interfaces. Because we have assumed that the undulations flatten out at these extremities (*c.f.* (2.1), (2.2)), it follows that $\hat{p}$ and $\hat{u}_3 = \hat{u}$ must be continuous at $z = -L$ and 0. Equations (2.17), in turn, transform these relations into corresponding interface relations for A and B.

In the homogeneous half-space below the slab, *i.e.* $z < -L$, the radiation condition requiring the waves to be downward-propagating leads to $A = 0$. We shall assume that the slab effective medium is matched to the upper homogeneous half-space, *i.e.* that $\zeta_1 = \zeta_0$. This condition suppresses deterministic coherent reflections from the upper interface, $z = 0$. All energy returning to the upper half-space from the slab will do so because of multiple scattering by the undulating layers and/or reflection by the impedance mismatch at $z = -L$. For the point source excitation of (2.4a) with $\mathbf{e} = \mathbf{z}_0$, we obtain the boundary conditions (equations (2.30) - (2.32) and (A.1) of [15])

$$\begin{aligned} A(\boldsymbol{\kappa},\omega,-L^+) &= \,,_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}B(\boldsymbol{\kappa},\omega,-L^+) \\ B(\boldsymbol{\kappa},\omega,0^-) &= B(\boldsymbol{\kappa},\omega,0^+) = \varepsilon^{3/2}\frac{\hat{f}(\omega)}{2}\zeta_0^{-1/2}e^{i\frac{\omega}{\varepsilon}\tau(z_s)}, \end{aligned} \tag{2.20}$$

where

$$
\begin{aligned}
, _I(-L) &= \frac{\zeta_1 - \zeta_2}{\zeta_1 + \zeta_2} \\
\hat{f}(\omega) &= \int f(t) e^{i\omega t} dt \ .
\end{aligned}
\tag{2.21}
$$

In some of the subsequent discussions, we shall consider plane wave excitation. For those cases, the second of boundary conditions (2.20) will be appropriately changed.

The integral terms in (2.18) couple the amplitudes A and B for all values of slowness $\boldsymbol{\lambda}$. To emphasize this point, as well as to simultaneously recenter the equations, we make the change of dependent variables

$$
\begin{aligned}
\widetilde{A}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z) &= A(\boldsymbol{\kappa} + \varepsilon\boldsymbol{\lambda}, \omega, z) e^{i\omega\tau\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}} \\
\widetilde{B}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z) &= B(\boldsymbol{\kappa} + \varepsilon\boldsymbol{\lambda}, \omega, z) e^{-i\omega\tau\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}} \ .
\end{aligned}
\tag{2.22}
$$

Equations (2.18) transform into

$$
\begin{aligned}
\frac{d}{dz}\widetilde{A} + i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}\widetilde{A} &= i\frac{\omega}{\varepsilon}n\left[\widetilde{A} - \widetilde{B}e^{-i2\frac{\omega\tau}{\varepsilon}}\right] - i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2 \iint \left[\hat{\phi}_z(\omega\boldsymbol{\lambda}', z)\right. \\
&\quad \left. + i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}'\hat{\phi}(w\boldsymbol{\lambda}', z)\ \right] \widetilde{A}(\boldsymbol{\kappa}, \boldsymbol{\lambda} - \boldsymbol{\lambda}', \omega, z) d\boldsymbol{\lambda}' \\
&\quad + i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2 \iint (\widehat{K_{11}^{-1}})(\omega\boldsymbol{\lambda}', z)\widetilde{A}(\boldsymbol{\kappa}, \boldsymbol{\lambda} - \boldsymbol{\lambda}', \omega, z) d\boldsymbol{\lambda}' \\
\frac{d}{dz}\widetilde{B} - i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}\widetilde{B} &= i\frac{\omega}{\varepsilon}n\left[\widetilde{A}e^{i2\frac{\omega\tau}{\varepsilon}} - \widetilde{B}\right] + i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2 \iint \left[\hat{\phi}_z(\omega\boldsymbol{\lambda}', z)\right. \\
&\quad \left. - i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}'\hat{\phi}(\omega\boldsymbol{\lambda}', z)\ \right] \widetilde{B}(\boldsymbol{\kappa}, \boldsymbol{\lambda} - \boldsymbol{\lambda}', \omega, z) d\boldsymbol{\lambda}' \\
&\quad - i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2 \iint (\widehat{K_{11}^{-1}})(\omega\boldsymbol{\lambda}', z)\widetilde{B}(\boldsymbol{\kappa}, \boldsymbol{\lambda} - \boldsymbol{\lambda}', \omega, z) d\boldsymbol{\lambda}',
\end{aligned}
\tag{2.23}
$$

where all $\boldsymbol{\lambda}$ dependence in the coefficients, unless explicitly noted, has been collapsed.

The structure of equations (2.23) suggests that we view $\boldsymbol{\kappa}$ and $\boldsymbol{\lambda}$ as independent variables and introduce Fourier transforms with respect to slowness $\boldsymbol{\lambda}$. The correct conjugate spatial variable will, in fact, turn out to be the macroscopic transverse variable $\mathbf{x}$. Anticipating this fact, we define

$$
\overline{A}(\boldsymbol{\kappa}, \mathbf{x}, \omega, z) = \left(\frac{\omega}{2\pi}\right)^2 \iint \widetilde{A}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z) e^{i\omega\boldsymbol{\lambda}\cdot\mathbf{x}} d\boldsymbol{\lambda}
$$

$$\overline{B}(\boldsymbol{\kappa},\mathbf{x},\omega,z) \;=\; \left(\frac{\omega}{2\pi}\right)^2 \iint \widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda},\omega,z)e^{i\omega\boldsymbol{\lambda}\cdot\mathbf{x}}d\boldsymbol{\lambda} \;. \tag{2.24}$$

Equations (2.23) transform into the following transport equations for $\overline{A}$ and $\overline{B}$

$$\begin{aligned}
\frac{d}{dz}\overline{A}+\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\overline{A} &= i\frac{\omega}{\varepsilon}n\left[\overline{A}-\overline{B}e^{-i2\frac{\omega\tau}{\varepsilon}}\right]-i\omega\frac{\rho_1}{\zeta_1}\left[\phi_z(\mathbf{x},z)+\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{x},z)\right]\overline{A} \\
&\quad +i\frac{\omega}{2}\zeta_1 K_{11}^{-1}(\mathbf{x},z)\overline{A} \\
\frac{d}{dz}\overline{B}-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\overline{B} &= i\frac{\omega}{\varepsilon}n\left[\overline{A}e^{i2\frac{\omega\tau}{\varepsilon}}-\overline{B}\right]+i\omega\frac{\rho_1}{\zeta_1}\left[\phi_z(\mathbf{x},z)-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{x},z)\right]\overline{B} \\
&\quad -i\frac{\omega}{2}\zeta_1 K_{11}^{-1}(\mathbf{x},z)\overline{B} \;.
\end{aligned} \tag{2.25}$$

Equations (2.23) and (2.25), along with corresponding boundary conditions, will form the basis for our discussion. The boundary conditions accompanying (2.25) are obvious adaptations of (2.20). The variables $\tilde{A},\tilde{B},\overline{A}$ and $\overline{B}$ have natural physical interpretations. Recall that the primes were dropped and that z actually corresponds to z'. Thus, $\overline{A}(\boldsymbol{\kappa},\mathbf{x},\omega,z')$ represents an upgoing wave amplitude corresponding to slowness $\boldsymbol{\kappa}$ and frequency ω at spatial position (x',y',z'); upgoing is interpreted locally in the sense of increasing z'.

3 Reflection and Transmission Operators

In the case of a plane layered slab, the scattering process is local in the slowness variable $\boldsymbol{\kappa}$. The upgoing amplitude can be equated to the product of the downgoing amplitude and a reflection coefficient – all evaluated at the same value of $\boldsymbol{\kappa}$ (equation (2.31) of [15]). The presence of undulations or spatially varying perturbations delocalizes the scattering process. Energy incident at one value of $\boldsymbol{\lambda}$ will be scattered into other values of $\boldsymbol{\lambda}$. Therefore, we formulate the following reflection relation

$$\widetilde{A}(\boldsymbol{\kappa},\boldsymbol{\lambda},\omega,z)=\iint\widetilde{\Gamma}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,z)\widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}' \;. \tag{3.1}$$

The upgoing wave at slowness $\boldsymbol{\lambda}$ is thus the superposition of contributions from downgoing waves incident at all values of slowness $\boldsymbol{\lambda}'$. The kernel of the integral operator, $\widetilde{\Gamma}$, determines how this slowness conversion occurs through scattering.

We can also define an analogous scattering relation in the spatial domain

$$\overline{A}(\boldsymbol{\kappa},\mathbf{x},\omega,z)=\iint\overline{\Gamma}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z)\overline{B}(\boldsymbol{\kappa},\mathbf{x}',\omega,z)d\mathbf{x}' \;. \tag{3.2}$$

The corresponding physical interpretation of (3.2) is that energy scattered upward at depth z and transverse location $\mathbf{x}$ is a linear superposition of multiple scattering contributions from energy incident at depth z and all transverse locations $\mathbf{x}'$.

From (2.24), it follows that the reflection kernels $\tilde{,}$ and $\bar{,}$ are related as Fourier transforms.

$$\bar{,}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z) = \left(\frac{\omega}{2\pi}\right)^2 \int\cdot\cdot\int \tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,z)e^{i\omega(\boldsymbol{\lambda}\cdot\mathbf{x}-\boldsymbol{\lambda}'\cdot\mathbf{x}')}d\boldsymbol{\lambda}d\boldsymbol{\lambda}' \ . \tag{3.3}$$

The defining relations (3.1), (3.2), when substituted into (2.23) and (2.25), respectively, lead to Riccati equations for the scattering kernels $\tilde{,}$ and $\bar{,}$. The initial conditions for these equations follow from the first of boundary conditions (2.20). We obtain two related initial value problems.

$$\begin{aligned}
\partial_z \tilde{,} \quad = \quad & i\frac{\omega}{\varepsilon}n\left[2\tilde{,} - e^{-i2\frac{\omega}{\varepsilon}\tau}\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}') - e^{i2\frac{\omega}{\varepsilon}\tau}\iint \tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}'',\omega,z)\tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda}'',\boldsymbol{\lambda}',w,z)d\boldsymbol{\lambda}''\right] \\
& -i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot(\boldsymbol{\lambda}+\boldsymbol{\lambda}')\tilde{,} - i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2\iint \Big[\hat{\phi}_z(\omega(\boldsymbol{\lambda}-\boldsymbol{\lambda}''),z) \\
& +i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot(\boldsymbol{\lambda}-\boldsymbol{\lambda}'')\hat{\phi}(\omega(\boldsymbol{\lambda}-\boldsymbol{\lambda}''),z)\ \Big]\tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda}'',\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}'' \\
& +i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2\iint (\widehat{K_{11}^{-1}})(\omega(\boldsymbol{\lambda}-\boldsymbol{\lambda}''),z)\tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda}'',\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}'' \\
& -i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2\iint \tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}'',\omega,z)\Big[\hat{\phi}_z(\omega(\boldsymbol{\lambda}''-\boldsymbol{\lambda}'),z) \\
& -i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot(\boldsymbol{\lambda}''-\boldsymbol{\lambda}')\hat{\phi}(\omega(\boldsymbol{\lambda}''-\boldsymbol{\lambda}'),z)\ \Big]d\boldsymbol{\lambda}'' \\
& +i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2\iint \tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}'',\omega,z)(\widehat{K_{11}^{-1}})(\omega(\boldsymbol{\lambda}''-\boldsymbol{\lambda}'),z)d\boldsymbol{\lambda}'',
\end{aligned} \tag{3.4}$$

with initial condition

$$\tilde{,}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,-L^+) = \,,_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}')$$

, and where (equation (2.30) of [15])

$$,_I(-L) = \frac{\zeta_1-\zeta_2}{\zeta_1+\zeta_2} \tag{3.5}$$

is an interface reflection coefficient. We also have

$$\begin{aligned}
\partial_z\bar{,} + \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot(\nabla_{\mathbf{x}}-\nabla_{\mathbf{x}'})\bar{,} \quad = \quad & i\frac{\omega}{\varepsilon}n\Big[2\bar{,} - e^{-i2\frac{\omega}{\varepsilon}\tau}\delta(\mathbf{x}-\mathbf{x}') - e^{i2\frac{\omega}{\varepsilon}\tau}\cdot \\
& \cdot\iint \bar{,}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}'',\omega,z)\bar{,}(\boldsymbol{\kappa},\mathbf{x}'',\mathbf{x}',\omega,z)d\mathbf{x}''\Big]
\end{aligned}$$

$$
\begin{aligned}
&-i\omega\frac{\rho_1}{\zeta_1}\left[\phi_z(\mathbf{x},z)+\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{x},z)+\phi_z(\mathbf{x}',z)\right.\\
&\left.-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'}\phi(\mathbf{x}',z)\right]\overline{\Gamma}\\
&+i\frac{\omega}{2}\zeta_1\left(K_{11}^{-1}(\mathbf{x},z)+K_{11}^{-1}(\mathbf{x}',z)\right)\overline{\Gamma},
\end{aligned}
\tag{3.6}
$$

with initial condition

$$
\overline{\Gamma}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,-L^+)=\Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\delta(\mathbf{x}-\mathbf{x}')\ .
$$

Transmission operators are introduced as linear integral operators relating the downgoing wave at the slab bottom to the downgoing wave at the variable depth z, *i.e.*

$$
\widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda},\omega,-L^-)=\iint\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,z)\widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}',
\tag{3.7}
$$

and

$$
\overline{B}(\boldsymbol{\kappa},\mathbf{x},\omega,-L^-)=\iint\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z)\overline{B}(\boldsymbol{\kappa},\mathbf{x}',\omega,z)d\mathbf{x}'\ .
\tag{3.8}
$$

Equations for the transmission kernels are obtained by differentiating (3.7) and (3.8) with respect to z and using (2.23) – (3.2). We obtain the following initial value problems

$$
\begin{aligned}
\partial_z\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,z)+i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}'\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,z)&=-i\frac{\omega}{\varepsilon}n\left[e^{i2\frac{\omega}{\varepsilon}\tau}\iint\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}'',\omega,z)\cdot\right.\\
&\left.\cdot\widetilde{\Gamma}(\boldsymbol{\kappa},\boldsymbol{\lambda}'',\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}''-\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,z)\right]\\
&-i\omega\frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2\iint\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}'',\omega,z)\left[\hat{\phi}_z(\omega(\boldsymbol{\lambda}''-\boldsymbol{\lambda}'),z)\right.\\
&\left.-i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot(\boldsymbol{\lambda}''-\boldsymbol{\lambda}')\hat{\phi}(\omega(\boldsymbol{\lambda}''-\boldsymbol{\lambda}'),z)\right]d\boldsymbol{\lambda}''\\
&+i\frac{\omega}{2}\zeta_1\left(\frac{\omega}{2\pi}\right)^2\iint\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}'',\omega,z)(\widehat{K_{11}^{-1}})(\omega(\boldsymbol{\lambda}''-\boldsymbol{\lambda}'),z)d\boldsymbol{\lambda}''\\
\widetilde{T}(\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\lambda}',\omega,-L^+)&=\frac{2(\zeta_1\zeta_2)^{1/2}}{\zeta_1+\zeta_2}\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}'),
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\partial_z\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z)-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'}\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z)&=-i\frac{\omega}{\varepsilon}n\left[e^{i2\frac{\omega}{\varepsilon}\tau}\iint\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}'',\omega,z)\right.\\
&\overline{\Gamma}(\boldsymbol{\kappa},\mathbf{x}'',\mathbf{x}',\omega,z)d\mathbf{x}''\\
&\left.-\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,x)\right]-\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z)\left[i\omega\frac{\rho_1}{\zeta_1}\left[\phi_z(\mathbf{x}',z)\right.\right.
\end{aligned}
$$

$$-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'}\phi(\mathbf{x}',z)\;]-i\frac{\omega}{2}\zeta_1 K_{11}^{-1}(\mathbf{x}',z)\Big]$$

$$\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,-L^+)=\frac{2(\zeta_1\zeta_2)^{1/2}}{\zeta_1+\zeta_2}\delta(\mathbf{x}-\mathbf{x}')\ . \tag{3.10}$$

Note that the two transmission kernels $\widetilde{T}$ and $\overline{T}$ satisfy transform relation (3.3).

4 Quantities of Interest

The reflected pressure at $z=0$ and the transmitted pressure, *i.e.* the total pressure at $z=-L$, are the quantities of interest. Consider first the reflected pressure. Recall that we have assumed that the slab effective medium is matched to the upper half-space ($\zeta_1=\zeta_0$). From (2.11), (2.17) and (2.22), noting that $\tau_{\boldsymbol{\kappa}}(0)=\mathbf{0}$, we obtain

$$p_{\text{refl}}(t,\mathbf{x},0)=(2\pi\varepsilon)^{-3}\iiint e^{i\omega(\boldsymbol{\kappa}\cdot\mathbf{x}-t)/\varepsilon}\,\zeta_0^{1/2}\omega^2\Big[\iint\widetilde{,}(\boldsymbol{\kappa},\mathbf{0},\boldsymbol{\lambda}',\omega,0)\cdot$$
$$\cdot B(\boldsymbol{\kappa}+\varepsilon\boldsymbol{\lambda}',\omega,0)d\boldsymbol{\lambda}'\Big]d\omega d\boldsymbol{\kappa}\ . \tag{4.1}$$

Equation (4.1) can be recast into a form that makes its physical content more transparent. We make the change of variables $\overline{\boldsymbol{\kappa}}=\boldsymbol{\kappa}+\varepsilon\boldsymbol{\lambda}',\overline{\boldsymbol{\lambda}}=\boldsymbol{\lambda}'$ (and subsequently drop the overbar); we use the fact that $\widetilde{,}(\boldsymbol{\kappa}-\varepsilon\boldsymbol{\lambda},\mathbf{0},\boldsymbol{\lambda},\omega,0)=\widetilde{,}(\boldsymbol{\kappa},-\boldsymbol{\lambda},\mathbf{0},\omega,0)$. With these steps and the use of transform relation (3.3), (4.1) can be rewritten as

$$p_{\text{refl}}(t,\mathbf{x},0)=(2\pi\varepsilon)^{-3}\iiint e^{i\omega(\boldsymbol{\kappa}\cdot\mathbf{x}-t)/\varepsilon}\Big[\iint\overline{,}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\Big]\zeta_0^{1/2}\cdot$$
$$\cdot B(\boldsymbol{\kappa},\omega,0)\omega^2 d\omega d\boldsymbol{\kappa}\ . \tag{4.2}$$

The physical content of (4.2) is clear. The spectral synthesis involves a product of two terms. The first term, $\overline{\overline{,}}(\boldsymbol{\kappa},\mathbf{x},\omega)\equiv\iint\overline{,}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'$, is an input reflection coefficient whose value at transverse position $\mathbf{x}$ involves a superposition of scatter contributions from all other transverse points. The second term, $\zeta_0^{1/2}B(\boldsymbol{\kappa},\omega,0)$, is the spectral representation of the incident pressure; for the case of a point source, it is given by (2.20).

Now consider the transmitted pressure. From (2.17), (3.7) and the radiation condition in $z<-L$, we obtain the representation

$$p_{\text{trans}}(t,\mathbf{x},-L)\;=\;-(2\pi\varepsilon)^{-3}\iiint e^{i\frac{\omega}{\varepsilon}(\boldsymbol{\kappa}\cdot\mathbf{x}-t)}\zeta_2^{1/2}B(\boldsymbol{\kappa},\omega,-L^-)e^{-i\frac{\omega}{\varepsilon}\tau(-L)}\omega^2 d\omega d\boldsymbol{\kappa}$$

$$
\begin{aligned}
&= -(2\pi\varepsilon)^{-3} \iiint e^{i\frac{\omega}{\varepsilon}(\boldsymbol{\kappa}\cdot\mathbf{x}-t)} \zeta_2^{1/2} \cdot \\
&\quad \cdot \left[\iint \widetilde{T}(\boldsymbol{\kappa}, \mathbf{0}, \boldsymbol{\lambda}', \omega, 0) B(\boldsymbol{\kappa} + \varepsilon\boldsymbol{\lambda}', \omega, 0) d\boldsymbol{\lambda}' \right] \omega^2 d\omega d\boldsymbol{\kappa} \ . \qquad (4.3)
\end{aligned}
$$

To recast (4.3) into a more physically meaningful form, we again make the change of variables $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa} + \varepsilon\boldsymbol{\lambda}', \overline{\boldsymbol{\lambda}} = \boldsymbol{\lambda}'$ and use the fact that $\widetilde{T}(\boldsymbol{\kappa} - \varepsilon\boldsymbol{\lambda}, \mathbf{0}, \boldsymbol{\lambda}, \omega, 0) = e^{-i\omega\tau_{\boldsymbol{\kappa}}(-L)\cdot\boldsymbol{\lambda}}\, \widetilde{T}(\boldsymbol{\kappa}, -\boldsymbol{\lambda}, \mathbf{0}, \omega, 0)$. Since $\overline{T}$ and $\widetilde{T}$ also satisfy transform relation (3.3), we obtain

$$
\begin{aligned}
p_{\text{trans}}(t, \mathbf{x}, -L) &= -(2\pi\varepsilon)^{-3} \iiint e^{i\frac{\omega}{\varepsilon}(\boldsymbol{\kappa}\cdot\mathbf{x}-t-\tau(-L))} \left[\iint \overline{T}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, 0) d\mathbf{x}' \right] \zeta_2^{1/2} \cdot \\
&\quad \cdot B(\boldsymbol{\kappa}, \omega, 0) \omega^2 d\omega d\boldsymbol{\kappa} \ . \qquad (4.4)
\end{aligned}
$$

Thus, the transmitted pressure is likewise a spectral synthesis involving the product of two terms. $\overline{\overline{T}}(\boldsymbol{\kappa}, \mathbf{x}, \omega) \equiv \iint \overline{T}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, 0) d\mathbf{x}'$ is a slab transmission coefficient whose value at $\mathbf{x}$ is obtained by superposing contributions from all transverse starting points $\mathbf{x}'$. The term $-\zeta_2^{1/2} B(\boldsymbol{\kappa}, \omega, 0)$ equals $\sqrt{\frac{\zeta_2}{\zeta_0}}$ times the spectrally-resolved incident pressure (2.20). Note also that $-\tau(-L) = (1 - \kappa^2 c_1^2)^{1/2} c_1^{-1} L$ is the transit time required for a ray to travel a linear path, characterized by slowness κ, through the slab (*c.f.* 2.16).

5 Conservation Relations

We examine the conservation relations present in the model that we have developed. By direct computation, using equations (2.23), one can show that

$$
\frac{d}{dz} \iint \left[|\widetilde{B}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z)|^2 - |\widetilde{A}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z)|^2 \right] d\boldsymbol{\lambda} = 0 \ . \qquad (5.1)
$$

Use is made of the fact that ϕ and K_{11} are real-valued; thus, for example, $\hat{\phi}^*(\omega\boldsymbol{\lambda}, z) = \hat{\phi}(-\omega\boldsymbol{\lambda}, z)$. Noting (2.22), it immediately follows that

$$
\frac{d}{dz} \iint \left[|B(\boldsymbol{\kappa} + \varepsilon\boldsymbol{\lambda}, \omega, z)|^2 - |A(\boldsymbol{\kappa} + \varepsilon\boldsymbol{\lambda}, \omega, z)|^2 \right] d\boldsymbol{\lambda} = 0 \ . \qquad (5.2)
$$

Similarly, using Parseval's identity (or direct computation and equations (2.25)), we obtain

$$
\frac{d}{dz} \iint \left[|\overline{B}(\boldsymbol{\kappa}, \mathbf{x}, \omega, z)|^2 - |\overline{A}(\boldsymbol{\kappa}, \mathbf{x}, \omega, z)|^2 \right] d\mathbf{x} = 0 \ . \qquad (5.3)
$$

Thus, the resulting conservation law exists in a transversely integrated sense. The undulations and mean bulk modulus perturbations couple different slowness values; only the total integrated acoustic flux is conserved. From (2.17) and (5.2) it follows that

$$\begin{aligned}\iint Re\{\hat{p}(\boldsymbol{\kappa}+\varepsilon\boldsymbol{\lambda},\omega,0)\hat{u}^*(\boldsymbol{\kappa}+\varepsilon\boldsymbol{\lambda},\omega,0)\}d\boldsymbol{\lambda} \;&=\; \iint Re\{\hat{p}(\boldsymbol{\kappa}+\varepsilon\boldsymbol{\lambda},\omega,-L)\cdot \\ &\quad \cdot\hat{u}^*(\boldsymbol{\kappa}+\varepsilon\boldsymbol{\lambda},\omega,-L)\}d\boldsymbol{\lambda}\ . \qquad (5.4)\end{aligned}$$

Conservation relations (5.1) and (5.3) lead to analogous results for the reflection and transmission operators introduced in Section 3. Let $\widetilde{\Gamma}$, $\overline{\Gamma}$, $\widetilde{\mathbf{T}}$ and $\overline{\mathbf{T}}$ denote the integral operators defined by (3.1), (3.2), (3.7) and (3.8), respectively. Thus, for example (3.1) now becomes $\widetilde{A}=\widetilde{\Gamma}\widetilde{B}$. Let $\langle\cdot,\cdot\rangle$ and $||\cdot||$ denote the (complex) inner product and norm on $L_2(d\boldsymbol{\lambda})$. Then, it follows from (3.1), (3.7) and (5.1) that

$$\begin{aligned}\frac{d}{dz}\langle\widetilde{B}(z),(\mathbf{I}-\widetilde{\Gamma}^\dagger(z)\widetilde{\Gamma}(z))\widetilde{B}(z)\rangle \;&=\; \frac{d}{dz}\langle\widetilde{B}(-L^-),(\widetilde{\mathbf{T}}^{-1}(z))^\dagger(\mathbf{I}-\widetilde{\Gamma}^\dagger(z)\widetilde{\Gamma}(z))\cdot \\ &\quad \cdot\widetilde{\mathbf{T}}^{-1}(z)\widetilde{B}(-L^-)\rangle=0\ . \qquad (5.5)\end{aligned}$$

However, from (3.4), (3.5) and (3.9)

$$(\widetilde{\mathbf{T}}^{-1}(-L^+))^\dagger(\mathbf{I}-\widetilde{\Gamma}^\dagger(-L^+)\widetilde{\Gamma}(-L^+))\widetilde{\mathbf{T}}^{-1}(-L^+)=\mathbf{I}\ . \qquad (5.6)$$

Therefore, we obtain

$$\langle\widetilde{B}(-L^-),[(\widetilde{\mathbf{T}}^{-1}(z)^\dagger(\mathbf{I}-\widetilde{\Gamma}^\dagger(z)\widetilde{\Gamma}(z))\widetilde{\mathbf{T}}^{-1}(z)-\mathbf{I}]\widetilde{B}(-L^-)\rangle=0 \qquad (5.7)$$

for all possible transmitted fields $\widetilde{B}(-L^-)$. We thus obtain the conservation law

$$\widetilde{\Gamma}^\dagger(z)\widetilde{\Gamma}(z)+\widetilde{\mathbf{T}}^\dagger(z)\widetilde{\mathbf{T}}(z)=\mathbf{I}, \qquad (5.8)$$

and an analogous argument leads to the following spatial domain counterpart of (5.8)

$$\overline{\Gamma}^\dagger(z)\overline{\Gamma}(z)+\overline{\mathbf{T}}^\dagger(z)\overline{\mathbf{T}}(z)=\mathbf{I}. \qquad (5.9)$$

6 The Stochastic Limit; Ito Equations

Our goal is to describe the asymptotic behavior of quantities of interest, such as the reflected and transmitted pressure, in the limit as $\varepsilon\to 0$. Therefore, we need to characterize the limit reflection and transmission processes. A convenient way to describe these limit processes is by the use of Ito equations.

We shall first formulate operator versions of equations (2.23) and (2.25). We then pass to the limit for these linear equations, obtaining linear Ito equations for the operators which propagate the fields $\widetilde{A}, \widetilde{B}, \overline{A}$ and $\overline{B}$ upward from initial values at the slab bottom. Once these Ito equations have been obtained, the defining relations and Ito calculus enable us to readily derive corresponding Ito equations for the reflection and transmission operators $\widetilde{,}\, ,\, \overline{,}\, , \widetilde{\mathbf{T}}$ and $\overline{\mathbf{T}}$.

We define the operators $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}$ by the relations

$$\widetilde{\mathbf{A}}(z)\ (\widetilde{B}(-L^-)) = \widetilde{A}(z) \quad , \quad \widetilde{\mathbf{B}}(z)\ (\widetilde{B}(-L^-)) = \widetilde{B}(z) \quad , \quad -L^- \le z \le 0\ . \tag{6.1}$$

These operators propagate the transmitted field at $-L^-$ upward through the slab. If $\zeta_2 \neq \zeta_1$ (*i.e.* , $_I(-L) \neq 0$), the operators are discontinuous at $-L$. However, on the interval $-L^+ < z \le 0$, equations (2.23) imply that (prior to taking the ε-limit), the operators satisfy

$$\frac{d}{dz}\widetilde{\mathbf{A}} = i\frac{\omega}{\varepsilon}n\left[\widetilde{\mathbf{A}} - e^{-i2\frac{\omega}{\varepsilon}\tau}\widetilde{\mathbf{B}}\right] + i\omega\widetilde{\mathbf{Q}}_A\widetilde{\mathbf{A}}$$

$$\frac{d}{dz}\widetilde{\mathbf{B}} = i\frac{\omega}{\varepsilon}n\left[e^{i2\frac{\omega}{\varepsilon}\tau}\widetilde{\mathbf{A}} - \widetilde{\mathbf{B}}\right] + i\omega\widetilde{\mathbf{Q}}_B\widetilde{B} \quad , \quad -L < z < 0 \tag{6.2}$$

where, for brevity, we define

$$\begin{aligned}
\widetilde{\mathbf{Q}}_A(\widetilde{A})(z) \quad &\equiv \quad -\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}\widetilde{A}(\boldsymbol{\kappa},\boldsymbol{\lambda},\omega,z) - \frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2\iint\left[\hat{\phi}_z(\omega\boldsymbol{\lambda}',z) + i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}'\cdot\right.\\
&\quad \left.\cdot\hat{\phi}(\omega\boldsymbol{\lambda}',z)\ \right]\widetilde{A}(\boldsymbol{\kappa},\boldsymbol{\lambda}-\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}' + \frac{\zeta_1}{2}\left(\frac{\omega}{2\pi}\right)^2\iint(\widehat{K_{11}^{-1}})(\omega\boldsymbol{\lambda}',z)\widetilde{A}(\boldsymbol{\kappa},\boldsymbol{\lambda}-\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}'\\
\widetilde{\mathbf{Q}}_B(\widetilde{B})(z) \quad &\equiv \quad \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}\widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda},w,z) + \frac{\rho_1}{\zeta_1}\left(\frac{\omega}{2\pi}\right)^2\iint[\ \hat{\phi}_z(\omega\boldsymbol{\lambda}',z) - i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\boldsymbol{\lambda}'\cdot\\
&\quad \cdot\hat{\phi}(\omega\boldsymbol{\lambda}',z)\ ]\,\widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda}-\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}' - \frac{\zeta_1}{2}\left(\frac{\omega}{2\pi}\right)^2\iint\left(\widehat{K_{11}^{-1}}\right)(\omega\boldsymbol{\lambda}',z)\\
&\quad \widetilde{B}(\boldsymbol{\kappa},\boldsymbol{\lambda}-\boldsymbol{\lambda}',\omega,z)d\boldsymbol{\lambda}'\ .
\end{aligned} \tag{6.3}$$

If we apply the $\varepsilon \to 0$ limit to equations (6.2), we obtain operator-valued Markov processes which we can characterize in terms of linear Ito equations. Applying this limit is tantamount to making the following substitutions

$$
\begin{aligned}
\varepsilon^{-1} n\left(z/\varepsilon^2\right) dz &\to \sqrt{\alpha_{nn}}\, d\beta_1 \\
\varepsilon^{-1} n\left(z/\varepsilon^2\right) e^{\pm i2\frac{\omega}{\varepsilon}\tau(z)}\, dz &\to \sqrt{\alpha_{nn}} \left(\frac{d\beta_2 \pm i d\beta_3}{\sqrt{2}}\right),
\end{aligned}
\tag{6.4}
$$

where the β_j, $j = 1, 2, 3$, are real-valued, independent Brownian motions ($E\{d\beta_i d\beta_j\} = dz\delta_{ij}$) and (*cf.* (2.19), (2.28) and (3.13) in [15])

$$
\alpha_{nn} \equiv \frac{\int_0^\infty E\{\nu(s)\nu(0)\}ds}{4c_1^2(1-c_1^2\kappa^2)} \equiv \frac{\alpha}{c_1^2(1-c_1^2\kappa^2)} \ . \tag{6.5}
$$

The limit, roughly speaking, combines the dynamics of the Central Limit Theorem and the Method of Averaging. Therefore, the noise terms in (6.4) that are multiplied by the rapidly-varying phase factor become asymptotically independent of the noise term that has no such factor. Taking the $\varepsilon \to 0$ limit in (6.2) leads to the following operator-valued linear Ito equations

$$
\begin{aligned}
d\widetilde{\mathbf{A}} &= i\omega\sqrt{2\alpha_{nn}}\left[\widetilde{\mathbf{A}} d\beta_1 - \widetilde{\mathbf{B}}\left(\frac{d\beta_2 - i d\beta_3}{\sqrt{2}}\right)\right] + i\omega \widetilde{\mathbf{Q}}_A \widetilde{\mathbf{A}} dz \\
d\widetilde{\mathbf{B}} &= i\omega\sqrt{2\alpha_{nn}}\left[\widetilde{\mathbf{A}}\left(\frac{d\beta_2 + i d\beta_3}{\sqrt{2}}\right) - \widetilde{\mathbf{B}} d\beta_1\right] + i\omega \widetilde{\mathbf{Q}}_B \widetilde{\mathbf{B}} dz \ .
\end{aligned}
\tag{6.6}
$$

Having representations (6.6), we can use the operator relations

$$
\widetilde{\Gamma} = \widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^{-1} \quad , \quad \widetilde{\mathbf{T}} = \widetilde{\mathbf{B}}^{-1} \tag{6.7}
$$

and Ito calculus to derive the following initial value problems for the reflection and transmission operators

$$
\begin{aligned}
d\widetilde{\Gamma} &= i\omega\sqrt{2\alpha_{nn}}\left[2\widetilde{\Gamma} d\beta_1 - \mathbf{I}\left(\frac{d\beta_2 - i d\beta_3}{\sqrt{2}}\right) - \widetilde{\Gamma}^2\left(\frac{d\beta_2 + i d\beta_3}{\sqrt{2}}\right)\right] \\
&\quad + \left[-6\omega^2\alpha_{nn}\widetilde{\Gamma} + i\omega\left(\widetilde{\mathbf{Q}}_A\widetilde{\Gamma} - \widetilde{\Gamma}\widetilde{\mathbf{Q}}_B\right)\right] dz \\
\widetilde{\Gamma}|_{z=-L^+} &= \Gamma_I(-L) e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\mathbf{I}
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
d\widetilde{\mathbf{T}} &= i\omega\sqrt{2\alpha_{nn}}\left[\widetilde{\mathbf{T}} d\beta_1 - \widetilde{\mathbf{T}}\widetilde{\Gamma}\left(\frac{d\beta_2 + i d\beta_3}{\sqrt{2}}\right)\right] + \left[-2\omega^2\alpha_{nn}\widetilde{\mathbf{T}} - i\omega\widetilde{\mathbf{T}}\widetilde{\mathbf{Q}}_B\right] dz \\
\widetilde{\mathbf{T}}|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\mathbf{I} \ .
\end{aligned}
\tag{6.9}
$$

Note that the two Ito equations contain the drift terms $-6\omega^2\alpha_{nn}\widetilde{\Gamma}\, dz$ and $-2\omega^2\alpha_{nn}\widetilde{\mathbf{T}} dz$, respectively. Since α_{nn}, defined by (6.5), is positive, these terms, arising from the random layering, introduce exponential decay into the evolution of the reflection and transmission operators. From a physical point of view, such behavior is to be expected; it arises from the loss of coherence and localization induced by the random multiple scattering. We shall develop these ideas further in Sections 8 and 10.

Counterpart Ito equations for corresponding spatial domain operators $\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\boldsymbol{\Gamma}}$ and $\overline{\mathbf{T}}$ can be obtained by applying Fourier transforms (2.24) and (3.3) to (6.6), (6.8) and (6.9), or by repeating the development of this section, starting with equations (2.25). The defining relations are

$$\overline{\mathbf{A}}(z)(\overline{B}(-L^-)) = \overline{A}(z)\ ,\ \overline{\mathbf{B}}(z)(\overline{B}(-L^-)) = \overline{B}(z),\ \overline{\boldsymbol{\Gamma}} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}^{-1}\ ,\ \overline{\mathbf{T}} = \overline{\mathbf{B}}^{-1}, \tag{6.10}$$

and the corresponding Ito equations become

$$\begin{aligned} d\overline{\mathbf{A}} &= i\omega\sqrt{2\alpha_{nn}}\left[\overline{\mathbf{A}}d\beta_1 - \overline{\mathbf{B}}\left(\frac{d\beta_2 - id\beta_3}{\sqrt{2}}\right)\right] + \left[-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}} + i\omega\overline{q}_A(\mathbf{x},z)\mathbf{I}\right]\overline{\mathbf{A}}dz \\ d\overline{\mathbf{B}} &= i\omega\sqrt{2\alpha_{nn}}\left[\overline{\mathbf{A}}\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right) - \overline{\mathbf{B}}d\beta_1\right] + \left[\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}} + i\omega\overline{q}_B(\mathbf{x},z)\mathbf{I}\right]\overline{\mathbf{B}}dz, \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} \overline{q}_A(\mathbf{x},z) &\equiv -\frac{\rho_1}{\zeta_1}\left[\phi_z(\mathbf{x},z) + \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{x},z)\right] + \frac{\zeta_1}{2}K_{11}^{-1}(\mathbf{x},z) \\ \overline{q}_B(\mathbf{x},z) &\equiv \frac{\rho_1}{\zeta_1}\left[\phi_z(\mathbf{x},z) - \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\phi(\mathbf{x},z)\right] - \frac{\zeta_1}{2}K_{11}^{-1}(\mathbf{x},z), \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} d\overline{\boldsymbol{\Gamma}} &= i\omega\sqrt{2\alpha_{nn}}\left[2\overline{\boldsymbol{\Gamma}}d\beta_1 - \mathbf{I}\left(\frac{d\beta_2 - id\beta_3}{\sqrt{2}}\right) - \overline{\boldsymbol{\Gamma}}^2\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] + \left[-6\omega^2\alpha_{nn}\overline{\boldsymbol{\Gamma}}\right. \\ &\quad \left. -\frac{\zeta_1}{\rho_1}[(\boldsymbol{\kappa}\cdot\nabla)\overline{\boldsymbol{\Gamma}} + \overline{\boldsymbol{\Gamma}}(\boldsymbol{\kappa}\cdot\nabla)] + i\omega[\overline{q}_A\mathbf{I}\overline{\boldsymbol{\Gamma}} - \overline{\boldsymbol{\Gamma}}\overline{q}_B\mathbf{I}]\right]dz \\ \overline{\boldsymbol{\Gamma}}|_{z=-L^+} &= \Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\mathbf{I} \end{aligned} \tag{6.13}$$

$$\begin{aligned} d\overline{\mathbf{T}} &= i\omega\sqrt{2\alpha_{nn}}\left[\overline{\mathbf{T}}d\beta_1 - \overline{\mathbf{T}}\,\overline{\boldsymbol{\Gamma}}\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] + \left[-2\omega^2\alpha_{nn}\overline{\mathbf{T}} - \frac{\zeta_1}{\rho_1}\overline{\mathbf{T}}(\boldsymbol{\kappa}\cdot\nabla)\right. \\ &\quad \left. -i\omega\overline{\mathbf{T}}\overline{q}_B\mathbf{I}\right]dz \\ \overline{\mathbf{T}}|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\mathbf{I}\ . \end{aligned} \tag{6.14}$$

From these operator equations, one can readily obtain Ito equations for the fields themselves (*i.e.* $\widetilde{A}(z), ..., \overline{B}(z)$) and the reflection and transmission integral kernels. One can also verify that the conservation relations determined in Section 5 remain valid in the limit Ito setting.

7 Reduction to the Plane-Layered Case

Now that the locally layered model for the slab has been developed, we shall examine how this model reduces to the plane-layered description when the undulations and perturbations are removed.

If we set $\phi = 0$ and $K_{11}^{-1} = 0$, then the operators $\widetilde{\mathbf{Q}}_A$ and $\widetilde{\mathbf{Q}}_B$, defined by (6.3), reduce to $-\frac{\zeta_1}{\rho_1}(\boldsymbol{\kappa} \cdot \boldsymbol{\lambda})\mathbf{I}$ and $\frac{\zeta_1}{\rho_1}(\boldsymbol{\kappa} \cdot \boldsymbol{\lambda})\mathbf{I}$, respectively. In this case, the Ito equation and initial condition for $\tilde{\Gamma}$, the kernel of the reflection operator $\tilde{\boldsymbol{\Gamma}}$, reduce to (*c.f.* (6.8))

$$\begin{aligned} d\tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}', \omega, z) &= i\omega\sqrt{2\alpha_{nn}}\left[2\tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}', \omega, z)d\beta_1 - \delta(\boldsymbol{\lambda} - \boldsymbol{\lambda}')\left(\frac{d\beta_2 - id\beta_3}{\sqrt{2}}\right)\right. \\ &\quad \left. - \iint \tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}'', \omega, z)\tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}'', \boldsymbol{\lambda}', \omega, z)d\boldsymbol{\lambda}''\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] \\ &\quad + \left[-6\omega^2\alpha_{nn} - i\omega\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa} \cdot (\boldsymbol{\lambda} + \boldsymbol{\lambda}')\right]\tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}', \omega, z)dz \\ \tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}', \omega, -L^+) &= \Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\delta(\boldsymbol{\lambda} - \boldsymbol{\lambda}') \ . \end{aligned} \tag{7.1}$$

The solution of (7.1) has the form

$$\tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}', \omega, z) = \tilde{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z)\delta(\boldsymbol{\lambda} - \boldsymbol{\lambda}'), \tag{7.2}$$

and, noting (3.3), the kernel of the corresponding spatial domain operator assumes the form

$$\bar{\Gamma}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, z) = \bar{\Gamma}(\boldsymbol{\kappa}, \mathbf{x} - \mathbf{x}', \omega, z) \ . \tag{7.3}$$

The reflection coefficient appearing in the expression for the reflected pressure (4.2), *i.e.* $\bar{\bar{\Gamma}} = \iint \bar{\Gamma}(\boldsymbol{\kappa}, \mathbf{x} - \mathbf{x}', \omega, 0)d\mathbf{x}'$, thus becomes independent of $\mathbf{x}$. If we define

$$\bar{\bar{\Gamma}}(\boldsymbol{\kappa}, \omega, z) \equiv \iint \bar{\Gamma}(\boldsymbol{\kappa}, \mathbf{x}, \omega, z)d\mathbf{x}, \tag{7.4}$$

one can show that $\bar{\bar{\Gamma}}$ satisfies the initial value problem

$$\begin{aligned} & d\bar{\bar{\Gamma}} = i\omega\sqrt{2\alpha_{nn}}\left[2\bar{\bar{\Gamma}}d\beta_1 - \left(\frac{d\beta_2 - id\beta_3}{\sqrt{2}}\right) - \bar{\bar{\Gamma}}^2\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] - 6\omega^2\alpha_{nn}\bar{\bar{\Gamma}}dz \\ & \bar{\bar{\Gamma}}|_{z=-L^+} = \Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)} \ . \end{aligned} \tag{7.5}$$

This problem is, in fact, the limit problem for the plane-layered random slab ((2.32) and (3.1) of [15]).

The same argument can be used in the case of transmission. With $\phi = 0$ and $K_{11}^{-1} = 0$, we note from (3.9), (7.2) that the transmission operator kernel has the form

$$\widetilde{T}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\lambda}', \omega, z) = \widetilde{T}(\boldsymbol{\kappa}, \boldsymbol{\lambda}, \omega, z)\delta(\boldsymbol{\lambda} - \boldsymbol{\lambda}') \ . \tag{7.6}$$

The kernel of the spatial domain transmission kernel thus assumes the form

$$\overline{T}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, z) = \overline{T}(\boldsymbol{\kappa}, \mathbf{x} - \mathbf{x}', \omega, z) \tag{7.7}$$

and the transmission coefficient $\overline{\overline{T}} = \iint T(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, 0)d\mathbf{x}'$ appearing in the expression for the transmitted pressure (4.4) is no longer a function of $\mathbf{x}$. As in (7.4), let

$$\overline{\overline{T}}(\boldsymbol{\kappa}, \omega, z) \equiv \iint \overline{T}(\boldsymbol{\kappa}, \mathbf{x}, \omega, z)d\mathbf{x} \ . \tag{7.8}$$

Then, it can be shown that $\overline{\overline{T}}$ satisfies the following limit problem for the plane-layered slab

$$\begin{aligned} d\overline{\overline{T}} &= i\omega\sqrt{2\alpha_{nn}}\left[\overline{\overline{T}}d\beta_1 - \overline{\overline{T}}\,\overline{\overline{,}}\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] - 2\omega^2\alpha_{nn}\overline{\overline{T}}dz \\ \overline{\overline{T}}|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1 + \zeta_2}I \ . \end{aligned} \tag{7.9}$$

8 Coherent Fields

We begin our evaluation of the locally layered model by determining the mean or coherent reflected and transmitted pressure. From (4.2) and (4.4), it is clear that we must compute $E\{\overline{,}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, 0)\}$ and $E\{\overline{T}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, 0)\}$.

From (6.13), we note that the mean reflection operator satisfies the initial value problem

$$\begin{aligned} d\langle\overline{,}\rangle &= -\Big[6\omega^2\alpha_{nn}\langle\overline{,}\rangle + \frac{\zeta_1}{\rho_1}[(\boldsymbol{\kappa}\cdot\nabla)\langle\overline{,}\rangle + \langle\overline{,}\rangle(\boldsymbol{\kappa}\cdot\nabla)] - i\omega\left[\overline{q}_A\mathbf{I}\langle\overline{,}\rangle\right. \\ &\qquad \left. -\langle\overline{,}\rangle\overline{q}_B\mathbf{I}\ \right]\Big]\,dz \\ \langle\overline{,}\rangle|_{z=-L^+} &= ,_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\mathbf{I} \ . \end{aligned} \tag{8.1}$$

The corresponding kernel of this mean reflection operator, i.e. $\langle\overline{,}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, z)\rangle$, is thus a solution of

$$\begin{aligned} \partial_z\langle\overline{,}\rangle + \frac{\zeta_1}{\rho_1}\left[\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\langle\overline{,}\rangle - \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'}\langle\overline{,}\rangle\right] &= -\left[6\omega\alpha_{nn} - i\omega\left(\overline{q}_A(\mathbf{x}, z) - \overline{q}_B(\mathbf{x}', z)\right)\right]\langle\overline{,}\rangle \\ \langle\overline{,}\rangle|_{z=-L^+} &= ,_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\delta(\mathbf{x} - \mathbf{x}') \ . \end{aligned} \tag{8.2}$$

The solution of this initial value problem at $z = 0$ is

$$\langle \bar{,} (\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, 0)\rangle \;=\; ,_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)} \exp\Big\{ -6\omega^2\alpha_{nn}L + i\omega\frac{\zeta_1}{2}\int_{-L}^{0}\Big[K_{11}^{-1}(\mathbf{x}+ \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\sigma, \sigma) + K_{11}^{-1}\left(\mathbf{x}' - \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\sigma, \sigma\right)\Big]d\sigma\Big\}\delta\left(\mathbf{x} - \mathbf{x}' - 2\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}L\right) \tag{8.3}$$

and the coherent reflected pressure, assuming point source excitation, becomes (*c.f.* (2.20), (4.2))

$$\langle p_{\text{refl}}(t, \mathbf{x}, 0)\rangle \;=\; \frac{\varepsilon^{-3/2}}{2(2\pi)^3}\iiint e^{i\frac{\omega}{\varepsilon}[\boldsymbol{\kappa}\cdot\mathbf{x} - t + \tau(z_s) - 2\tau(-L)]}\exp\Big\{-6\omega^2\alpha_{nn}L + i\omega\frac{\zeta_1}{2}\int_{-L}^{0}\Big[K_{11}^{-1}(\mathbf{x}+\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\sigma, \sigma) + K_{11}^{-1}(\mathbf{x} - \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}(2L+\sigma), \sigma)\Big]d\sigma\Big\}\cdot \cdot ,_I(-L)\hat{f}(\omega)\omega^2 d\omega d\boldsymbol{\kappa}\ . \tag{8.4}$$

Because we have assumed $\zeta_1 = \zeta_0$, the slab effective medium is matched to the upper half-space. Coherent energy emitted by the source experiences no interface reflection at $z = 0$. It enters the slab, is partially reflected by the mismatch at $z = -L$ (if $0 < ,_I(-L) < 1$), and the reflected portion travels upward to $z = 0$. The factor $\tau(z_s) - 2\tau(-L) = (1 - \kappa^2 c_0^2)^{1/2}\ c_0^{-1} z_s + (1 - \kappa^2 c_1^2)^{1/2} c_1^{-1}(2L)$ accounts for the time delay associated with this two-way transit. The factor $e^{-6\omega^2\alpha_{nn}L}$ represents attenuation arising from loss of coherence. Multiple scattering by the random microstructure strips energy from the coherent field, converting it into incoherent energy. Because of our assumption that $\phi(x, y, 0) = \phi(x, y, -L) = 0$, the function ϕ defining the undulations does not contribute to the phase shift in (8.4). This phase shift amounts to an integration of the reciprocal bulk modulus perturbation along the piecewise-linear ray path associated with the reflection from the interface at $z = -L$.

We can make these observations more explicit by using a Stationary Phase approximation to evaluate (8.4). For simplicity, assume that $z_s = 0$, i.e. that the point source lies on the interface at the origin. In that case we obtain

$$\langle p_{\text{refl}}(t, \mathbf{x}, 0)\rangle \;\approx\; \frac{\sqrt{\varepsilon},_I(-L)\sin^2\theta}{8\pi L c_1}\frac{\partial}{\partial t}\Big[\frac{1}{2\pi}\int d\omega \hat{f}(\omega)e^{-i\frac{\omega}{\varepsilon}[t - 2L(c_1\sin\theta)^{-1}]}e^{-\omega^2\beta_1(L)}\cdot \cdot\exp\Big\{i\omega\rho_1 c_1(2\sin\theta)^{-1}\int_{-L}^{0}\big[K_{11}^{-1}(\mathbf{x}[1+\sigma/(2L)], \sigma) + K_{11}^{-1}(\mathbf{x}[-\sigma/(2L)], \sigma)\big]\,d\sigma\Big\}\Big], \tag{8.5}$$

where

$$\kappa c_1 \equiv \cos\theta \equiv |\mathbf{x}|/(|\mathbf{x}|^2+4L^2)^{1/2} \quad , \quad \beta_1(L) \equiv 6\alpha L/(c_1\sin\theta)^2 \ . \tag{8.6}$$

The interface reflection coefficient , $_I(-L)$ in (8.5) likewise depends upon θ. Let angles $\theta=\theta_1$ and θ_2 be related by Snell's Law, *i.e.*

$$\frac{c_1}{c_2}=\frac{\cos\theta_1}{\cos\theta_2} \ . \tag{8.7}$$

Then , $_I(-L)$ in (8.5) is given by

$$, _I(-L)=\frac{\rho_1c_1\sin\theta_2-\rho_2c_2\sin\theta_1}{\rho_1c_1\sin\theta_2+\rho_2c_2\sin\theta_1} \tag{8.8}$$

(*c.f.* (3.22)–(3.26) and Fig. 3.1 in [15]). The only difference between (8.5) and the comparable expression for the plane-layered coherent return is the phase shift caused by the bulk modulus perturbation K_{11}. In particular, the Gaussian spreading factor $\gamma(L)$ is not affected by the three dimensional deterministic perturbations.

From (6.14), we infer that the mean transmission operator satisfies the initial value problem

$$\begin{aligned} d\langle\overline{\mathbf{T}}\rangle &= \left[-2\omega^2\alpha_{nn}\langle\overline{\mathbf{T}}\rangle-\frac{\zeta_1}{\rho_1}\langle\overline{\mathbf{T}}\rangle(\boldsymbol{\kappa}\cdot\nabla)-i\omega\langle\overline{\mathbf{T}}\rangle\overline{q}_B\mathbf{I}\right]dz \\ \langle\overline{\mathbf{T}}\rangle|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\mathbf{I} \ . \end{aligned} \tag{8.9}$$

The corresponding kernel of this mean transmission operator, i.e. $\langle\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,z)\rangle$, therefore is a solution of

$$\begin{aligned} \frac{\partial}{\partial z}\langle\overline{T}\rangle-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'}\langle\overline{T}\rangle &= -\left[2\omega^2\alpha_{nn}+i\omega\overline{q}_B(\mathbf{x}',z)\right]\langle\overline{T}\rangle \\ \langle\overline{T}\rangle|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\delta(\mathbf{x}-\mathbf{x}') \ . \end{aligned} \tag{8.10}$$

The solution of this initial value problem at $z=0$ is

$$\begin{aligned} \langle\overline{T}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,0)\rangle &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\exp\left\{-2\omega^2\alpha_{nn}L+i\omega\frac{\zeta_1}{2}\cdot\right. \\ &\left.\cdot\int_{-L}^{0}K_{11}^{-1}(\mathbf{x}-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}(L+\sigma),\sigma)d\sigma\right\}\delta(\mathbf{x}-\mathbf{x}'-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}L) \ . \end{aligned} \tag{8.11}$$

The factor $e^{-2\omega^2\alpha_{nn}L}$ accounts for the attenuation of the coherent transmitted wave due to scattering by the random microstructure. The function ϕ, defining the undulations, again does not contribute to the phase integral, due to the fact that ϕ vanishes at both interfaces. The phase integral itself amounts to an integration of perturbation K_{11}^{-1} along a straight line path beginning at $\mathbf{x}' = \mathbf{x} - \boldsymbol{\kappa}\frac{\zeta_1}{\rho_1}L$ on the upper face ($z = 0$) and ending at $\mathbf{x}$ when $z = -L$.

For point source excitation (2.20) (with $z_s = 0$), we obtain from (4.4) that

$$\begin{aligned}\langle p_{\text{trans}}(t,\mathbf{x},-L)\rangle &= \frac{-\varepsilon^{-3/2}}{2(2\pi)^3}\iiint e^{i\frac{\omega}{\varepsilon}(\boldsymbol{\kappa}\cdot\mathbf{x}-t-\tau(-L))}\frac{2\zeta_2}{\zeta_1+\zeta_2}e^{-2\omega^2\alpha_{nn}L}\exp\Big\{i\omega\frac{\zeta_1}{2}\cdot \\ &\quad\cdot\int_{-L}^{0}K_{11}^{-1}(\mathbf{x}-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}(L+\sigma),\sigma)d\sigma\Big\}\hat{f}(\omega)\omega^2 d\omega d\boldsymbol{\kappa},\end{aligned} \tag{8.12}$$

and a Stationary Phase approximate evaluation of (8.11) leads to

$$\begin{aligned}\langle p_{\text{trans}}(t,\mathbf{x},-L)\rangle &\approx \frac{-\varepsilon^{-1/2}(sin\theta_1)^2}{4\pi L c_1}\left(\frac{2\rho_2 c_2\sin\theta_1}{\rho_1 c_1\sin\theta_2+\rho_2 c_2\sin\theta_1}\right)\frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int d\omega\hat{f}(\omega)\cdot\right. \\ &\cdot\exp\left\{-i\frac{\omega}{\varepsilon}[t-L(c_1\sin\theta_1)^{-1}]\right\}\exp\left\{-\frac{1}{3}\omega^2\beta_1(L)\right\}\cdot \\ &\cdot\left.\exp\left\{i\omega\rho_1 c_1(2\sin\theta_1)^{-1}\int_{-L}^{0}K_{11}^{-1}(\mathbf{x}[-\sigma/L],\sigma)d\sigma\right\}\right],\end{aligned} \tag{8.13}$$

where

$$\kappa c_1 \equiv \cos\theta_1 \equiv |\mathbf{x}|^2/(|\mathbf{x}|^2+L^2)^{1/2}\ , \tag{8.14}$$

$\beta_1(L)$ is defined by (8.6), and θ_2 is defined by (8.7). Note that in (8.12), the phase integral is taken along the linear ray path connecting the source at the origin and the exiting point at $(\mathbf{x},-L)$.

9 Pressure Correlations

We now consider two-point, two-time pressure correlations. To simplify matters, we shall, as in [16], assume plane wave excitation. In this case, the incident pressure at $z = 0$ has the form

$$p_{\text{inc}}(t,\mathbf{x},0) = \frac{-1}{\sqrt{\varepsilon}}\, f\left(\frac{t-\boldsymbol{\kappa}_0\cdot\mathbf{x}}{\varepsilon}\right), \tag{9.1}$$

where $\boldsymbol{\kappa}_0$ is the transverse slowness vector for the incident acoustic wave. The counterpart to (2.20) becomes

$$B(\boldsymbol{\kappa},\omega,0) = \left(\frac{2\pi}{\omega}\right)^2 \varepsilon^{5/2}\ \zeta_0^{-1/2}\hat{f}(\omega)\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}_0), \tag{9.2}$$

and, when substituted into (4.2), we obtain the following expression for the reflected pressure

$$p_{\text{refl}}(t,\mathbf{x},0) = \frac{1}{2\pi\sqrt{\varepsilon}}\int e^{i\omega(\boldsymbol{\kappa}_0\cdot\mathbf{x}-t)/\varepsilon}\left[\iint\overline{,}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\right]\hat{f}(\omega)d\omega\ . \tag{9.3}$$

In the random plane-layered problem, the reflected pressure would also be a plane wave. In our case, the undulations and bulk modulus perturbations impart a complex transverse spatial structure to the reflected pressure through the presence of $\overline{,}$ in (9.3). A similar statement can be made about the transmitted pressure. Substitution of (9.2) into (4.4) leads to

$$
\begin{aligned}
p_{\text{trans}}(t,\mathbf{x},-L) \quad = \quad & \frac{-1}{2\pi\sqrt{\varepsilon}} \int e^{i\omega[\boldsymbol{\kappa}_0\cdot\mathbf{x}-t+(1-\kappa_0^2c_1^2)^{1/2}c_1^{-1}L]/\varepsilon} \;\cdot \\
& \cdot \left[\iint \overline{T}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\right]\sqrt{\frac{\zeta_2}{\zeta_0}}\hat{f}(\omega)d\omega \;. \qquad (9.4)
\end{aligned}
$$

The two-point, two-time correlation function for the reflected pressure assumes the form

$$
\begin{aligned}
\langle p_{\text{refl}}(t+\frac{\varepsilon}{2}\overline{t},\ \mathbf{x}+\frac{\varepsilon}{2}\overline{\mathbf{x}},0)p_{\text{refl}}\,(t-\frac{\varepsilon}{2}\overline{t},\mathbf{x}-\frac{\varepsilon}{2}\overline{\mathbf{x}},0)\rangle = & \frac{1}{(2\pi)^2}\iint e^{i\omega(\boldsymbol{\kappa}_0\cdot\overline{\mathbf{x}}-\overline{t})}e^{-ih(\boldsymbol{\kappa}_0\cdot\mathbf{x}-t)}\;\cdot \\
& \cdot\left[\int\cdot\cdot\int\langle\overline{,}(\boldsymbol{\kappa}_0,\mathbf{x}+\frac{\varepsilon}{2}\overline{\mathbf{x}},\mathbf{x}',\omega-\frac{\varepsilon}{2}h,0)\;\cdot\right. \\
& \left.\cdot\overline{,}^*(\boldsymbol{\kappa}_0,\mathbf{x}-\frac{\varepsilon}{2}\overline{\mathbf{x}},\mathbf{x}'',\omega+\frac{\varepsilon}{2}h,0)\rangle d\mathbf{x}'d\mathbf{x}''\right]|\hat{f}(\omega)|^2d\omega dh\;.
\end{aligned}
$$

(9.5)

From (9.5) it's clear that we must determine the asymptotic (*i.e.* ε-limit) behavior of $\langle\overline{,_0}(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega-\frac{\varepsilon}{2}h,0)\overline{,}^*(\boldsymbol{\kappa}_0,\overline{\mathbf{x}},\overline{\mathbf{x}}',\omega+\frac{\varepsilon}{2}h,0)\rangle$. In [16] and [15], this was accomplished by solving an infinite system of transport equations (W-equations). In the present problem, for general oblique plane wave incidence, this approach is complicated by the presence of the $\boldsymbol{\kappa}_0\cdot\nabla$ drift operators present in the $\overline{,}$ Riccati equation. (*c.f.* (3.6), (6.13)). To obtain a closed system of moment equations, we must consider the asymptotic behavior of $\{\langle\overline{,}^{NM}\rangle\}_{N,M=0}^{\infty}$, where $\overline{,}^{NM}\equiv\prod_{n=0}^{N}\prod_{m=0}^{M}\overline{,}(\boldsymbol{\kappa}_0,\mathbf{x}_n,\mathbf{x}'_n,\omega-\frac{\varepsilon}{2}h,0)\overline{,}^*(\boldsymbol{\kappa}_0,\overline{\mathbf{x}}_m,\overline{\mathbf{x}}'_m,\omega+\frac{\varepsilon}{2}h,0)$. Thus, as N and M increase, the transverse spatial domain of $\overline{,}^{NM}$, *i.e.* $R^{4(N+M)}$, likewise increases.

Considerable simplification can be achieved, however, in the important special case of normal incidence, when $\boldsymbol{\kappa}_0=\mathbf{0}$. In this case, we infer from (6.13) that the limiting Ito equation for the integral operator kernel $\overline{,}(\mathbf{0},\mathbf{x},\mathbf{x}',\omega\pm\frac{\varepsilon}{2}h,z)\equiv\overline{,}^{(\pm)}$ is

$$
\begin{aligned}
d\overline{,}^{(\pm)} \quad = \quad & i\omega\sqrt{2\alpha_{nn}}\Bigg[2\overline{,}^{(\pm)}d\beta_1-e^{\mp ih\tau}\delta(\mathbf{x}-\mathbf{x}')\left(\frac{d\beta_2-id\beta_3}{\sqrt{2}}\right) \\
& -e^{\pm ih\tau}\iint\overline{,}^{(\pm)}(\mathbf{0},\mathbf{x},\mathbf{x}'',\omega\pm\frac{\varepsilon}{2}h,z)\overline{,}^{(\pm)}(\mathbf{0},\mathbf{x}'',\mathbf{x}',\omega\pm\frac{\varepsilon}{2}h,z)d\mathbf{x}''\left(\frac{d\beta_2+id\beta_3}{\sqrt{2}}\right)\Bigg] \\
& +\left[-6\omega^2\alpha_{nn}\overline{,}^{(\pm)}+i\omega(\overline{q}_A(\mathbf{x},z)-\overline{q}_B(\mathbf{x}',z))\overline{,}^{(\pm)}\right]dz, \qquad (9.6)
\end{aligned}
$$

with

$$\tau = c_1^{-1} z \qquad , \qquad \alpha_{nn} = \alpha c_1^{-2}$$

$$\overline{q}_A(\mathbf{x}, z) = -c_1^{-1} \phi_z(\mathbf{x}, z) + \frac{\rho_1 c_1}{2} K_{11}^{-1}(\mathbf{x}, z) = -\overline{q}_B(\mathbf{x}, z) \ . \tag{9.7}$$

Also

$$\overline{\Gamma}^{(\pm)}|_{z=-L^+} = \left[\frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}\right] e^{-i(2\frac{\omega}{\varepsilon} \pm h) c_1^{-1} L} \delta(\mathbf{x} - \mathbf{x}') \ . \tag{9.8}$$

The solution of (9.6)–(9.8) has the form

$$\overline{\Gamma}(\mathbf{0}, \mathbf{x}, \mathbf{x}', \omega \pm \frac{\varepsilon}{2} h, z) \equiv \overline{\Gamma}(\mathbf{x}, \omega, \pm h, z) \delta(\mathbf{x} - \mathbf{x}') \equiv \overline{\Gamma}^{(\pm)} \delta(\mathbf{x} - \mathbf{x}') \ . \tag{9.9}$$

Substitution of (9.9) leads to

$$\begin{aligned} d\overline{\Gamma}^{(\pm)} &= i\omega\sqrt{2\alpha_{nn}} \left[2\overline{\Gamma}^{(\pm)} d\beta_1 - e^{\mp ih\tau} \left(\frac{d\beta_2 - id\beta_3}{\sqrt{2}}\right) - e^{\pm ih\tau} \overline{\Gamma}^{(\pm)2} \left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] \\ &\quad + \left[-6\omega^2 \alpha_{nn} + i2\omega \overline{q}_A(\mathbf{x}, z)\right] \overline{\Gamma}^{(\pm)} dz \\ \overline{\Gamma}^{(\pm)}|_{z=-L^+} &= \left[\frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}\right] e^{-i(2\frac{\omega}{\varepsilon} \pm h) c_1^{-1} L} \end{aligned} \tag{9.10}$$

and correlation function (9.5) reduces to

$$\begin{aligned} \langle p_{\text{refl}}(t + \frac{\varepsilon}{2}\overline{t}, \mathbf{x} + \frac{\varepsilon}{2}\overline{\mathbf{x}}, 0) \ p_{\text{refl}}(t - \frac{\varepsilon}{2}\overline{t}, \mathbf{x} - \frac{\varepsilon}{2}\overline{\mathbf{x}}, 0)\rangle &= \frac{1}{(2\pi)^2} \iint e^{-i\omega(\omega \overline{t} - ht)} \ . \\ \cdot \langle \overline{\Gamma}(\mathbf{x} + \frac{\varepsilon}{2}\overline{\mathbf{x}}, \omega, -h, 0) \overline{\Gamma}^*(\mathbf{x} - \frac{\varepsilon}{2}\overline{\mathbf{x}}, \omega, h, 0)\rangle |\hat{f}(\omega)|^2 d\omega dh \ . \end{aligned} \tag{9.11}$$

In problem (9.10), transverse spatial dependence enters parametrically. We can obtain a system of moment equations very similar to those in [16] by defining

$$W^N(\mathbf{x}, \mathbf{x}', \omega, h, z) \equiv \langle \Big(\overline{\Gamma}(\mathbf{x}, \omega, -h, z) \overline{\Gamma}^*(\mathbf{x}', \omega, h, z)\Big)^N \rangle \quad N = 0, 1, 2, \ldots \ . \tag{9.12}$$

Using (9.10) and the Ito calculus, we obtain

$$\begin{aligned} \frac{d}{dz} W^N &= 2\omega^2 \alpha_{nn} N^2 \left[e^{i2h\tau} W^{N-1} - 2W^N + e^{-i2h\tau} W^{N+1}\right] + i2\omega N \Big[\overline{q}_A(\mathbf{x}, z) - \\ &\quad \overline{q}_A(\mathbf{x}', z)\Big] W^N, \quad -L < z \leq 0 \\ W^N|_{z=-L^+} &= \left[\frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}\right]^{2N} e^{i2Nhc_1^{-1} L} \ . \end{aligned} \tag{9.13}$$

The phase term involving $\overline{q}_A$ in (9.13) plays a role only when $\mathbf{x} \neq \mathbf{x}'$. If we collapse the spatial offset in (9.11), *i.e.* set $\overline{\mathbf{x}} = \mathbf{0}$ and consider $\langle p_{\text{refl}}(t + \frac{\varepsilon}{2}\overline{t}, \mathbf{x}, 0) p_{\text{refl}}(t - \frac{\varepsilon}{2}\overline{t}, \mathbf{x}, 0)\rangle$, the phase term vanishes and we obtain a system of W-equations identical to that arising in the plane-layered case (*c.f.* [16]). This observation is important since it identifies a robustness present in the plane-layered theory.

Similar observations can be made in the case of transmitted normal plane wave pressure correlations. From (9.4), with $\boldsymbol{\kappa}_0 = \mathbf{0}$

$$
\begin{aligned}
\langle p_{\text{trans}}(t + \frac{\varepsilon}{2}\overline{t}, \mathbf{x} + \frac{\varepsilon}{2}\overline{\mathbf{x}}, -L) p_{\text{trans}}(t - \frac{\varepsilon}{2}\overline{t}, \mathbf{x} - \frac{\varepsilon}{2}\overline{\mathbf{x}}, -L)\rangle &= \frac{1}{(2\pi)^2} \iint e^{-i(\omega\overline{t} - h[t - c^{-1}L])} \cdot \\
&\cdot \Big[\int \cdot\cdot \int \langle \overline{T}(\mathbf{0}, \mathbf{x} + \frac{\varepsilon}{2}\overline{\mathbf{x}}, \mathbf{x}', \omega - \frac{\varepsilon}{2}h, 0) \overline{T}^*(\mathbf{0}, \mathbf{x} - \frac{\varepsilon}{2}\overline{\mathbf{x}}, \mathbf{x}'', \\
&\omega + \frac{\varepsilon}{2}h, 0)\rangle d\mathbf{x}' d\mathbf{x}'' \Big] \frac{\zeta_2}{\zeta_0} \left| \hat{f}(\omega) \right|^2 d\omega dh \ . \qquad (9.14)
\end{aligned}
$$

As with the reflection coefficient, in the case of normal plane wave incidence, we infer from (6.14) and (9.9) that the kernel of the transmission integral operator has the form

$$
\overline{T}(\mathbf{0}, \mathbf{x}, \mathbf{x}', \omega \pm \frac{\varepsilon}{2}h, z) \equiv \overline{T}(\mathbf{x}, \omega, \pm h, z)\delta(\mathbf{x} - \mathbf{x}') \equiv \overline{T}^{(\pm)}\delta(\mathbf{x} - \mathbf{x}') \qquad (9.15)
$$

and $\overline{T}^{(\pm)}$ satisfies

$$
\begin{aligned}
d\overline{T}^{(\pm)} &= i\omega\sqrt{2\alpha_{nn}} \left[\overline{T}^{(\pm)} d\beta_1 - e^{\pm ih\tau} \overline{T}^{(\pm)} \overline{,}^{(\pm)} \left(\frac{d\beta_2 + i d\beta_3}{\sqrt{2}} \right) \right] - \Big[2\omega^2\alpha_{nn} + \\
&\qquad i\omega\overline{q}_B \Big] \overline{T}^{(\pm)} dz \\
\overline{T}^{(\pm)}|_{z=-L^+} &= \frac{2\sqrt{\rho_1 c_1 \rho_2 c_2}}{\rho_1 c_1 + \rho_2 c_2} \ . \qquad (9.16)
\end{aligned}
$$

where τ, α_{nn} and $\overline{q}_B$ are given by (9.7). The transmitted pressure correlation becomes

$$
\begin{aligned}
\langle p_{\text{trans}}(t + \frac{\varepsilon}{2}\overline{t}, \mathbf{x} + \frac{\varepsilon}{2}\overline{\mathbf{x}}, -L) p_{\text{trans}}(t - \frac{\varepsilon}{2}\overline{t}, \mathbf{x} - \frac{\varepsilon}{2}\overline{\mathbf{x}}, -L)\rangle &= \frac{1}{(2\pi)^2} \iint e^{-i(\omega\overline{t} - h[t - c^{-1}L])} \cdot \\
\langle \overline{T}(\mathbf{x} + \frac{\varepsilon}{2}\overline{\mathbf{x}}, \omega, -h, 0) \overline{T}^*(\mathbf{x} - \frac{\varepsilon}{2}\overline{\mathbf{x}}, \omega, h, 0)\rangle \frac{\zeta_2}{\zeta_0} \left| \hat{f}(\omega) \right|^2 d\omega dh \ . \qquad (9.17)
\end{aligned}
$$

The counterpart of (9.12) becomes the system of moments:

$$
\begin{aligned}
Z^N(\mathbf{x}, \mathbf{x}', \omega, h, z) &\equiv \langle \left(\overline{,}(\mathbf{x}, \omega, -h, z) \overline{,}^*(\mathbf{x}', \omega, h, z) \right)^N \overline{T}(\mathbf{x}, \omega, -h, z) \cdot \\
&\cdot \overline{T}^*(\mathbf{x}', \omega, h, z)\rangle \quad , \quad N = 0, 1, 2, \ldots
\end{aligned}
$$

which are solutions of the following initial value problem

$$
\begin{aligned}
\frac{d}{dz}Z^N &= 2\omega^2\alpha_{nn}\left[N^2 e^{i2h\tau}Z^{N-1} - (2N^2+2N+1)Z^N + e^{-i2h\tau}(N^2+2N+1)Z^{N+1}\right] \\
&\quad + i\omega(2N+1)\left[\overline{q}_A(\mathbf{x}) - \overline{q}_A(\mathbf{x}')\right]Z^N \\
Z^N|_{z=-L^+} &= \frac{4\rho_1 c_1 \rho_2 c_2 (\rho_1 c_1 - \rho_2 c_2)^{2N}}{(\rho_1 c_1 + \rho_2 c_2)^{2N+2}}\, e^{i2Nhc_1^{-1}L}\,. \qquad (9.18)
\end{aligned}
$$

As with the case of the reflected pressure correlations, problem (9.18) reduces to the plane-layered problem if we collapse the spatial offset.

10 Localization

We now show that for time harmonic, normally incident plane wave excitation, the limiting behavior of the locally-layered perturbed random slab localizes the energy in the same manner as that of the plane-layered slab. Let $\boldsymbol{\kappa}_0 \to \mathbf{0}, (2\pi\sqrt{\varepsilon})^{-1}\hat{f}(\omega) \to \delta(\omega-\omega_0)$ and use (9.15) in (9.4). Then

$$
p_{\mathrm{trans}}(t,\mathbf{x},-L) = -\sqrt{\frac{\rho_2 c_2}{\rho_1 c_1}}\; e^{-i\omega_0(t-c_1^{-1}L)/\varepsilon}\,\overline{T}(\mathbf{x},\omega_0,0). \qquad (10.1)
$$

In the discussion of localization, it is convenient to relocate the origin at $z=-L$. Therefore, let $\zeta \equiv z+L$ and let $\overline{T}(\mathbf{x},\omega_0,z) \to \overline{T}(\mathbf{x},\omega_0,\zeta)$. It follows from (9.16) and (9.7) that

$$
d\left[|\ln \overline{T}(\mathbf{x},\omega_0,\zeta)|\right] = \omega\sqrt{\alpha_{nn}}\left[Im\{\overline{\Gamma}\}d\beta_2 + Re\{\overline{\Gamma}\}d\beta_3\right] - \omega^2\alpha_{nn}d\zeta. \qquad (10.2)
$$

Using the boundedness of $\overline{\Gamma}$, we conclude that

$$
\lim_{\zeta\to\infty}\zeta^{-1}|\ln\overline{T}(\mathbf{x},\omega_0,\zeta)| = -\omega^2\alpha_{nn} \qquad (10.3)
$$

with probability one, which is identical to the result obtained in the plane-layered case.

11 O'Doherty-Anstey Theory

In this section, we shall establish the robustness of O'Doherty-Anstey theory for both transmission and reflection. We restrict attention to plane wave pulses. The point source case is analyzed by [13] for plane layered random media and in [14] for a different class of locally layered random media. Related work on the O'Doherty-Anstey theory can be found in [10, 11, 12].

Recall that (*c.f.* (6.14))

$$
\begin{aligned}
d\overline{\mathbf{T}} &= i\omega\sqrt{2\alpha_{nn}}\left[\overline{\mathbf{T}}d\beta_1 - \overline{\mathbf{T}}\text{,}^{-}\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right] + \left[-2\omega^2\alpha_{nn}\overline{\mathbf{T}} - \frac{\zeta_1}{\rho_1}\overline{\mathbf{T}}(\boldsymbol{\kappa}\cdot\nabla)\right. \\
&\quad \left. -i\omega\overline{\mathbf{T}}\overline{q}_B\mathbf{I}\right]dz \\
\overline{\mathbf{T}}|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\mathbf{I}\ . \qquad (11.1)
\end{aligned}
$$

Define

$$
\widehat{\mathbf{T}} = e^{-i\omega\sqrt{2\alpha_{nn}}\beta_1(z)}\overline{\mathbf{T}}, \qquad (11.2)
$$

where β_1 is normalized so that $\beta_1(-L) = 0$. Then

$$
\begin{aligned}
d\widehat{\mathbf{T}} &= -i\omega\sqrt{2\alpha_{nn}}\widehat{\mathbf{T}}\text{,}^{-}\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right) + \left[-\omega^2\alpha_{nn}\widehat{\mathbf{T}} - \frac{\zeta_1}{\rho_1}\widehat{\mathbf{T}}(\boldsymbol{\kappa}\cdot\nabla) - i\omega\widehat{\mathbf{T}}\overline{q}_B\mathbf{I}\right]dz \\
\widehat{\mathbf{T}}|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\mathbf{I}\ . \qquad (11.3)
\end{aligned}
$$

Equation (9.4) is an expression for the transmitted pressure. We shall now consider $\sqrt{\varepsilon}p_{\text{trans}}$ and show that this quantity has an asymptotic limit. From (11.2), we write a corresponding relation between kernels as

$$
\overline{T} = e^{-i\omega\sqrt{2\alpha_{nn}}\beta_1(z)}\widehat{T}. \qquad (11.4)
$$

Inserting (11.4) into (9.4) yields

$$
\begin{aligned}
\sqrt{\varepsilon}p_{\text{trans}}(t,\mathbf{x},-L) &= \frac{-1}{2\pi}\int e^{i\omega[\boldsymbol{\kappa}_0\cdot\mathbf{x}-t+(1-\kappa_0^2c_1^2)^{1/2}c_1^{-1}L+\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0)]/\varepsilon}\ \cdot \\
&\quad \cdot\left[\iint \widehat{T}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\right]\sqrt{\frac{\zeta_2}{\zeta_0}}\hat{f}(\omega)d\omega\ . \qquad (11.5)
\end{aligned}
$$

Define

$$
\widehat{T}_0 = E\{\widehat{T}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,z)|\beta_1(z)\}. \qquad (11.6)
$$

Then

$$
\begin{aligned}
E\{\sqrt{\varepsilon}p_{\text{trans}}(t,\mathbf{x},-L)|\beta_1(0)\} &= \frac{-1}{2\pi}\int e^{i\omega[\boldsymbol{\kappa}_0\cdot\mathbf{x}-t+(1-\kappa_0^2c_1^2)^{1/2}c_1^{-1}L+\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0)]/\varepsilon}\ \cdot \\
&\quad \cdot\left[\iint \widehat{T}_0(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\right]\sqrt{\frac{\zeta_2}{\zeta_0}}\hat{f}(\omega)d\omega\ . \qquad (11.7)
\end{aligned}
$$

Note in particular that $E\{\left(\frac{d\beta_2(z)+id\beta_3(z)}{\sqrt{2}}\right)|\beta_1(z)\} = 0$, so that

$$\begin{aligned} d\widehat{\mathbf{T}}_0 &= \left[-\omega^2\alpha_{nn}\widehat{\mathbf{T}}_0 - \frac{\zeta_1}{\rho_1}\widehat{\mathbf{T}}_0(\boldsymbol{\kappa}\cdot\nabla) - i\omega\widehat{\mathbf{T}}_0\overline{q}_B\mathbf{I}\right]dz \\ \widehat{\mathbf{T}}_0|_{z=-L^+} &= \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\mathbf{I}\,. \end{aligned} \tag{11.8}$$

We can solve (11.8) by simply noting that replacing α_{nn} by $2\alpha_{nn}$ in this equation gives equation (8.9) for the mean transmission operator. Therefore, the kernel $\widehat{T}_0$ can be obtained by replacing α_{nn} by $\alpha_{nn}/2$ in (8.11) to obtain

$$\widehat{T}_0 = \frac{2\sqrt{\zeta_1\zeta_2}}{\zeta_1+\zeta_2}\exp\left\{-\omega^2\alpha_{nn}L + i\omega\frac{\zeta_1}{2}\int_{-L}^{0}K_{11}^{-1}(\mathbf{x}-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}(L+\sigma),\sigma)d\sigma\right\}\delta(\mathbf{x}-\mathbf{x}'-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}L)\,. \tag{11.9}$$

The δ-function reduces integration over $\mathbf{x}'$ to an evaluation. Therefore, substituting (11.9) into (11.7) yields

$$\begin{aligned} E\{\sqrt{\varepsilon}p_{\text{trans}}(t,\mathbf{x},-L)|\beta_1(0)\} &= \frac{-1}{2\pi}\int\exp\{i\omega[\boldsymbol{\kappa}_0\cdot\mathbf{x} - t + (1-\kappa_0^2c_1^2)^{1/2}c_1^{-1}L + \varepsilon\sqrt{2\alpha_{nn}}\beta_1(0) \\ &\quad +\varepsilon\frac{\zeta_1}{2}\int_{-L}^{0}K_{11}^{-1}(\mathbf{x}-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}(L+\sigma),\sigma)d\sigma]/\varepsilon - \omega^2\alpha_{nn}L\}\cdot \\ &\quad \cdot\frac{2\zeta_2}{\zeta_1+\zeta_2}\hat{f}(\omega)d\omega \end{aligned} \tag{11.10}$$

It is instructive to compare the mean transmitted pressure field (11.10) with the random time shift $\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0)$ and the mean transmitted pressure field without the shift (8.12). For the comparison we have to convert (8.12) to the plane wave case by ignoring the $\boldsymbol{\kappa}$ integration, setting $\boldsymbol{\kappa} = \boldsymbol{\kappa}_0$ and adjusting the constants in front of the ω integral appropriately (compare (2.20) and (9.2)). We find that when we average (11.10) with respect to the shift we recover the coherent pressure field (8.12). Thus, when the pulse is observed in a random time frame that is defined by $t-\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0)$, then the Gaussian spreading factor is one half that of the coherent field. Moreover, in this time frame the pulse stabilizes, that is its variance tends to zero.

We now show that the conditional variance of $\sqrt{\varepsilon}p_{\text{trans}}$, given $\beta_1(0)$, is zero. From (11.5) we have that

$$\left(\sqrt{\varepsilon}p_{\text{trans}}(t,\mathbf{x},-L)\right)^2 = \frac{1}{4\pi^2}\int\int\exp\{i(\omega_1+\omega_2)[\boldsymbol{\kappa}_0\cdot\mathbf{x} - t + (1-\kappa_0^2c_1^2)^{1/2}c_1^{-1}L$$

$$+\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0)]/\varepsilon\} \left[\int\int\int\int \widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}_1', \omega_1, 0)\widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}_2', \omega_2, 0)d\mathbf{x}_1' d\mathbf{x}_2'\right] \cdot$$
$$\cdot \frac{\zeta_2}{\zeta_0}\hat{f}(\omega_1)d\omega_1 \hat{f}(\omega_2)d\omega_2. \tag{11.11}$$

We will show that, for $\omega_1 \neq \omega_2$,

$$E\{\widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}_1, \mathbf{x}_1', \omega_1, z)\widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}_2, \mathbf{x}_2', \omega_2, z)|\beta_1(z)\} = E\{\widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}_1, \mathbf{x}_1', \omega_1, z)|\beta_1(z)\} \cdot \cdot E\{\widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}_2, \mathbf{x}_2', \omega_2, z)|\beta_1(z)\}. \tag{11.12}$$

From (11.11) and (11.12) it will follow that

$$E\{(\sqrt{\varepsilon}p_{\text{trans}}(t, \mathbf{x}, -L))^2|\beta_1(0)\} = (E\{\sqrt{\varepsilon}p_{\text{trans}}(t, \mathbf{x}, -L)|\beta_1(0)\})^2, \tag{11.13}$$

and the vanishing of the conditional variance of $\sqrt{\varepsilon}p_{\text{trans}}$ given $\beta_1(0)$ will have been shown.

It remains to show (11.12). From (11.3), we can write equations for the kernels as

$$d\widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}', \omega, z) = -i\omega\sqrt{2\alpha_{nn}} \int \widehat{T}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}'', \omega, z)\overline{\,}(\boldsymbol{\kappa}_0, \mathbf{x}'', \mathbf{x}', \omega, z) \left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right) -\omega^2\alpha_{nn}\widehat{T}dz + \frac{\zeta_1}{\rho_1}(\boldsymbol{\kappa}\cdot\nabla_{x'})\widehat{T}dz - i\omega\overline{q}_B(\mathbf{x}', z)\widehat{T}dz. \tag{11.14}$$

Let

$$\widehat{T}^{(j)} = \widehat{T}((\boldsymbol{\kappa}_0, \mathbf{x}_j, \mathbf{x}_j', \omega_j, z), \quad j = 1, 2. \tag{11.15}$$

We shall use the fact that $\frac{\beta_2(\omega_1,z)+i\beta_3(\omega_1,z)}{\sqrt{2}}$ is independent of $\frac{\beta_2(\omega_2,z)+i\beta_3(\omega_2,z)}{\sqrt{2}}$ for $\omega_1 \neq \omega_2$ to obtain that

$$d(\widehat{T}^{(1)}\widehat{T}^{(2)}) = \widehat{T}^{(1)}(d\widehat{T}^{(2)}) + (d\widehat{T}^{(1)})\widehat{T}^{(2)}. \tag{11.16}$$

There is no Ito term since $d\left[\frac{\beta_2(\omega_1,z)+i\beta_3(\omega_1,z)}{\sqrt{2}}\right] d\left[\frac{\beta_2(\omega_2,z)+i\beta_3(\omega_2,z)}{\sqrt{2}}\right] = 0$. It follows that

$$dE\{\widehat{T}^{(1)}\widehat{T}^{(2)}|\beta_1(z)\} = \{-(\omega_1^2+\omega_2^2)\alpha_{nn} + \frac{\zeta_1}{\rho_1}(\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_1'} + \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_2'}) -i[\omega_1\overline{q}_B(\mathbf{x}_1', z) + \omega_1\overline{q}_B(\mathbf{x}_2', z))]\}E\{\widehat{T}^{(1)}\widehat{T}^{(2)}|\beta_1(z)\}dz. \tag{11.17}$$

However, from (11.14)

$$dE\{\widehat{T}^{(j)}|\beta_1(z)\} = \left[-\omega_j^2\alpha_{nn} + \frac{\zeta_1}{\rho_1}(\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_j'}) - i\omega_j\overline{q}_B(\mathbf{x}_j', z)\right]E\{\widehat{T}^{(j)}|\beta_1(z)\}dz, \tag{11.18}$$

so that

$$d[E\{\widehat{T}^{(1)}|\beta_1(z)\}E\{\widehat{T}^{(2)}|\beta_1(z)\}] = \Big[-(\omega_1^2+\omega_2^2)\alpha_{nn} + \frac{\zeta_1}{\rho_1}(\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'_1} + \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'_2})$$
$$-i[\omega_1\overline{q}_B(\mathbf{x}'_1,z) + \omega_1\overline{q}_B(\mathbf{x}'_2,z))]\Big] E\{\widehat{T}^{(1)}|\beta_1(z)\}E\{\widehat{T}^{(2)}|\beta_1(z)\}dz. \qquad (11.19)$$

A comparison of (11.17) and (11.19), together with the fact that both equations have the same initial conditions, verifies (11.12) and establishes the O'Doherty-Anstey result for the case of transmission.

The argument establishing the O'Doherty-Anstey result for reflection proceeds in much the same way (see [11, 10] for the plane layered case). We show that the conditional variance of $\sqrt{\varepsilon}p_{\text{refl}}(t,\mathbf{x},0)$, given $\beta_1(0)$, is zero. Noting (6.13), let

$$\widehat{\mathbf{\Gamma}} \;=\; e^{-i2\omega\sqrt{2\alpha_{nn}}\beta_1(z)}\overline{\mathbf{\Gamma}}. \qquad (11.20)$$

Then, the initial value problem for $\widehat{\mathbf{\Gamma}}$ becomes

$$d\widehat{\mathbf{\Gamma}} \;=\; -i\omega\sqrt{2\alpha_{nn}}\left[e^{-i2\omega\sqrt{2\alpha_{nn}}\beta_1}\mathbf{I}\left(\frac{d\beta_2 - id\beta_3}{\sqrt{2}}\right) + e^{i2\omega\sqrt{2\alpha_{nn}}\beta_1}\widehat{\mathbf{\Gamma}}^2\left(\frac{d\beta_2 + id\beta_3}{\sqrt{2}}\right)\right]$$
$$+\left[-2\omega^2\alpha_{nn}\widehat{\mathbf{\Gamma}} - \frac{\zeta_1}{\rho_1}[(\boldsymbol{\kappa}\cdot\nabla)\,\widehat{\mathbf{\Gamma}} + \widehat{\mathbf{\Gamma}}(\boldsymbol{\kappa}\cdot\nabla)] + i\omega[\overline{q}_A\mathbf{I}\widehat{\mathbf{\Gamma}} - \widehat{\mathbf{\Gamma}}\overline{q}_B\mathbf{I}\,]\,dz\right.$$
$$\widehat{\mathbf{\Gamma}}|_{z=-L^+} \;=\; \Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\mathbf{I}. \qquad (11.21)$$

Noting (9.3), the equation for $\sqrt{\varepsilon}p_{\text{refl}}$ can be written as

$$\sqrt{\varepsilon}p_{\text{refl}}(t,\mathbf{x},0) \;=\; \frac{1}{2\pi}\int e^{i\omega(\boldsymbol{\kappa}_0\cdot\mathbf{x}-t)/\varepsilon}\left[\iint \overline{\Gamma}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\right]\hat{f}(\omega)d\omega\;. \qquad (11.22)$$

From (11.20), it follows that the relationship between kernels is

$$\overline{\Gamma} \;=\; e^{i2\omega\sqrt{2\alpha_{nn}}\beta_1}\widehat{\Gamma}. \qquad (11.23)$$

If we define

$$\widehat{\Gamma}_0 = E\{\widehat{\Gamma}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,z)|\beta_1(z)\}, \qquad (11.24)$$

then it follows from (11.22) - (11.24) that

$$E\{\sqrt{\varepsilon}p_{\text{refl}}(t,\mathbf{x},0)|\beta_1(0)\} \;=\; \frac{1}{2\pi}\int e^{i\omega(\boldsymbol{\kappa}_0\cdot\mathbf{x}-t+2\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0))/\varepsilon}\;\cdot$$
$$\cdot\left[\iint \widehat{\Gamma}_0(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}',\omega,0)d\mathbf{x}'\right]\hat{f}(\omega)d\omega\;. \qquad (11.25)$$

Using (11.24), (11.21) and the fact that $(\beta_2 + i\beta_3)/\sqrt{2}$ is independent of β_1, we obtain the following initial value problem for the operator $\widehat{\mathbf{\Gamma}}_0$ (having kernel $\widehat{\Gamma}_0$).

$$
\begin{aligned}
d\widehat{\mathbf{\Gamma}}_0 &= \Big[-2\omega^2\alpha_{nn}\widehat{\mathbf{\Gamma}}_0 - \frac{\zeta_1}{\rho_1}[(\boldsymbol{\kappa}\cdot\nabla)\,\widehat{\mathbf{\Gamma}}_0 + \widehat{\mathbf{\Gamma}}_0(\boldsymbol{\kappa}\cdot\nabla)] \\
&\quad + i\omega[\overline{q}_A\mathbf{I}\widehat{\mathbf{\Gamma}}_0 - \widehat{\mathbf{\Gamma}}_0\overline{q}_B\mathbf{I}]\Big]dz \\
\widehat{\mathbf{\Gamma}}_0|_{z=-L^+} &= \Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\mathbf{I}.
\end{aligned} \tag{11.26}
$$

A solution for the kernel $\widehat{\Gamma}_0$ can be obtained by observing that (11.26) becomes identical to problem (8.1) for $\langle\widehat{\mathbf{\Gamma}}\rangle$ if α_{nn} is replaced by $3\alpha_{nn}$. Therefore, replacing α_{nn} by $\alpha_{nn}/3$ in (8.3) yields the following expression for $\widehat{\Gamma}_0$

$$
\begin{aligned}
\langle\widehat{\Gamma}_0(\boldsymbol{\kappa},\mathbf{x},\mathbf{x}',\omega,0)\rangle &= \Gamma_I(-L)e^{-i2\frac{\omega}{\varepsilon}\tau(-L)}\exp\Big\{-2\omega^2\alpha_{nn}L + i\omega\frac{\zeta_1}{2}\int_{-L}^{0}\Big[K_{11}^{-1}\,(\mathbf{x}+ \\
&\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}_0\sigma,\sigma) + K_{11}^{-1}\left(\mathbf{x}' - \frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}_0\sigma,\sigma\right)\Big]d\sigma\Big\}\,\delta\left(\mathbf{x}-\mathbf{x}'-2\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}L\right)
\end{aligned} \tag{11.27}
$$

When (11.27) is substituted into (11.25), the δ-function reduces the $\mathbf{x}'$ integration to an evaluation and we obtain

$$
\begin{aligned}
E\{\sqrt{\varepsilon}p_{\text{refl}}(t,\mathbf{x},0)|\beta_1(0)\} &= \frac{1}{2\pi}\int\exp\Big\{i\frac{\omega}{\varepsilon}\Big[\boldsymbol{\kappa}_0\cdot\mathbf{x} - t - 2\tau(-L) + 2\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0) \\
&+\frac{\zeta_1}{2}\int_{-L}^{0}\Big[K_{11}^{-1}\left(\mathbf{x}+\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}_0\sigma,\sigma\right) + K_{11}^{-1}\left(\mathbf{x}'-\frac{\zeta_1}{\rho_1}\boldsymbol{\kappa}_0\sigma,\sigma\right)\Big]d\sigma\Big] \\
&-2\omega^2\alpha_{nn}L\Big\}\,\Gamma_I(-L)\hat{f}(\omega)d\omega.
\end{aligned} \tag{11.28}
$$

It is instructive here again to compare the mean reflected field (11.28) with the coherent reflected field (8.4), specialized to plane wave pulses as explained above for the transmitted pressure. When (11.28) is averaged over the random time shift $2\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0)$ we obtain exactly the coherent reflected field (8.4). The Gaussian spreading factor is now three times bigger. The reflected pulse stabilizes as well.

We next show that the conditional variance of $\sqrt{\varepsilon}p_{\text{refl}}$, given $\beta_1(0)$, is zero. From (11.22)

$$
\begin{aligned}
(\sqrt{\varepsilon}p_{\text{refl}}(t,\mathbf{x},0))^2 &= \frac{1}{4\pi^2}\int\int\exp\Big\{i(\omega_1+\omega_2)(\boldsymbol{\kappa}_0\cdot\mathbf{x} - t + 2\varepsilon\sqrt{2\alpha_{nn}}\beta_1(0))/\varepsilon\Big\}\cdot \\
&\Big[\int\int\int\int\widehat{\Gamma}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}_1',\omega_1,0)\widehat{\Gamma}(\boldsymbol{\kappa}_0,\mathbf{x},\mathbf{x}_2',\omega_2,0)d\mathbf{x}_1'd\mathbf{x}_2'\Big]\hat{f}(\omega_1)\hat{f}(\omega_2)d\omega_1 d\omega_2
\end{aligned} \tag{11.29}
$$

We will show that, for $\omega_1 \neq \omega_2$,

$$E\left\{\hat{\Gamma}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}_1', \omega_1, z)\hat{\Gamma}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}_2', \omega_2, z)|\beta_1(z)\right\} \;=\; E\left\{\hat{\Gamma}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}_1', \omega_1, z)|\beta_1(z)\right\} E\left\{\hat{\Gamma}(\boldsymbol{\kappa}_0, \mathbf{x}, \mathbf{x}_2', \omega_2, z)|\beta_1(z)\right\} \tag{11.30}$$

It will then follow from (11.30) and (11.29) that

$$E\{(\sqrt{\varepsilon}p_{\text{refl}}(t, \mathbf{x}, 0))^2|\beta_1(0)\} \;=\; \left(E\{\sqrt{\varepsilon}p_{\text{refl}}(t, \mathbf{x}, 0)|\beta_1(0)\}\right)^2, \tag{11.31}$$

and the vanishing of the conditional variance of $\sqrt{\varepsilon}p_{\text{refl}}$, given $\beta_1(0)$, will thus have been shown.

It remains to show (11.30). From (11.21), we can write the equation of the kernel $\hat{\Gamma}$ as

$$\begin{aligned} d\hat{\Gamma}(\boldsymbol{\kappa}, \mathbf{x}, \mathbf{x}', \omega, z) \;=\; & -i\omega\sqrt{2\alpha_{nn}}\left(e^{-i2\omega\sqrt{2\alpha_{nn}}\beta_1}\delta(\mathbf{x}-\mathbf{x}')\left(\frac{d\beta_2(z) - d\beta_3(z)}{\sqrt{2}}\right)\right. \\ & \left. + e^{i2\omega\sqrt{2\alpha_{nn}}\beta_1}\left[\int\int\hat{\Gamma}(\mathbf{x}, \mathbf{x}'', \omega, z)\hat{\Gamma}(\mathbf{x}'', \mathbf{x}', \omega, z)d\mathbf{x}''\right]\delta(\mathbf{x}-\mathbf{x}')\left(\frac{d\beta_2(z) + d\beta_3(z)}{\sqrt{2}}\right)\right) \\ & + \left[-2\alpha_{nn}\omega^2\hat{\Gamma} - \frac{\zeta_1}{\rho_1}[\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}}\hat{\Gamma} - \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}'}\hat{\Gamma}] + i\omega[\overline{q}_A(\mathbf{x}, z) - \overline{q}_B(\mathbf{x}', z)]\hat{\Gamma}\right]dz. \end{aligned} \tag{11.32}$$

Let

$$\hat{\Gamma}^{(j)} \;=\; \hat{\Gamma}(\boldsymbol{\kappa}_0, \mathbf{x}_j, \mathbf{x}_j', \omega_j, z), \qquad j = 1, 2. \tag{11.33}$$

We use the fact that $\frac{\beta_2(\omega_1,z)+i\beta_3(\omega_1,z)}{\sqrt{2}}$ is independent of $\frac{\beta_2(\omega_2,z)+i\beta_3(\omega_2,z)}{\sqrt{2}}$, for $\omega_1 \neq \omega_2$, so that $d\frac{\beta_2(\omega_1,z)+i\beta_3(\omega_1,z)}{\sqrt{2}}d\frac{\beta_2(\omega_2,z)+i\beta_3(\omega_2,z)}{\sqrt{2}} = 0$ to obtain

$$d\left(\hat{\Gamma}^{(1)}\hat{\Gamma}^{(2)}\right) \;=\; \hat{\Gamma}^{(1)}d\hat{\Gamma}^{(2)} + \left(d\hat{\Gamma}^{(1)}\right)\hat{\Gamma}^{(2)}. \tag{11.34}$$

It follows that

$$\begin{aligned} dE\left\{\hat{\Gamma}^1\hat{\Gamma}^2|\beta_1(z)\right\} \;=\; & \left[-2\alpha_{nn}(\omega_1^2 + \omega_2^2) - \frac{\zeta_1}{\rho_1}[\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_1} + \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_2} - \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_1'} + \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_2'}]\right. \\ & \left. + [\omega_1\overline{q}_A(\mathbf{x}_1, z) + \omega_2\overline{q}_A(\mathbf{x}_2, z) - \omega_1\overline{q}_B(\mathbf{x}_1', z) - \omega_2\overline{q}_B(\mathbf{x}_2', z)\right] \\ & E\left\{\hat{\Gamma}^1\hat{\Gamma}^2|\beta_1(z)\right\}dz. \end{aligned} \tag{11.35}$$

However, from (11.32)

$$\begin{aligned} dE\left\{\hat{\Gamma}^{(j)}|\beta_1(z)\right\} \;=\; & \left[-2\alpha_{nn}\omega_j^2 - \frac{\zeta_1}{\rho_1}[\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_j} - \boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_j'}]\right. \\ & \left. + i\omega_j\left(\overline{q}_A(\mathbf{x}_j, z) - \overline{q}_B(\mathbf{x}_j', z)\right)\right]E\left\{\hat{\Gamma}^j|\beta_1(z)\right\}dz, \end{aligned} \tag{11.36}$$

so that

$$
\begin{aligned}
d\Big[E\left\{\widehat{,}^1|\beta_1(z)\right\}E\left\{\widehat{,}^2|\beta_1(z)\right\}\Big] \quad = \quad & \Big[-2\alpha_{nn}(\omega_1^2+\omega_2^2)-\frac{\zeta_1}{\rho_1}[\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_1}+\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_2}-\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_1'}+\boldsymbol{\kappa}\cdot\nabla_{\mathbf{x}_2'}] \\
& +i[\omega_1\overline{q}_A(\mathbf{x}_1,z)+\omega_2\overline{q}_A(\mathbf{x}_2,z)-\omega_1\overline{q}_B(\mathbf{x}_1',z)-\omega_2\overline{q}_B(\mathbf{x}_2',z)]\Big]\cdot \\
& \cdot E\left\{\widehat{,}^1|\beta_1(z)\right\}E\left\{\widehat{,}^2|\beta_1(z)\right\}dz. \qquad (11.37)
\end{aligned}
$$

Comparing (11.35) and (11.37), and noting that both equations have the same initial conditions, establishes (11.30).

If proper allowance is made for the differences between point source and plane wave excitation (*c.f.* (2.20) and (9.2)), the conditional means considered in this section can be seen to agree with their counterparts in Section 8. Using the fact that $E\{e^{i\omega\sqrt{2\alpha_{nn}}\beta_1(0)}\} = e^{-\omega^2\alpha_{nn}L}$, expressions (11.10) and (11.28) can be put into agreement with (8.12) and (8.4), respectively.

12 Conclusions

We have confirmed the robustness of the random plane layered theory of acoustics by obtaining qualitatively similar results from a model that has three dimensional perturbations. This perturbed model allows for small spatial dependence of the background bulk modulus, and for small spatial deviations in the normal to the random layers. These deviations from plane layers produce $O(1)$ effects on many of the statistics of interest.

For the generalized O'Doherty-Anstey theory, and for the reflected and transmitted coherent fields, only the variations of the background have an effect, to leading order and not on the Gaussian spreading rate. However, all the spatial variations have a leading order effect in computing the space and time correlation functions of the reflected and transmitted pressure waves at non-normal incidence. For these correlations, a system of moment equations was derived. These moment equations are qualitatively similar to what is obtained in the plane layered case, but with new terms that are dependent on the three-dimensional effects.

However, when the incident wave is normal to the layering, a remarkable robustness is obtained for the coherent field and for the time autocorrelation function of reflected or transmitted pressure at a single point in space. In these cases all three-dimensional effects vanish, and leading order results identical to the plane layered case are reproduced exactly. A related robustness result is obtained for the localization length, which, at normal incidence, is identical to that in the strictly plane layered theory. Thus despite the nontrivial deviation from a one-dimensional model, the localization phenomenon, at normal incidence, is preserved quantitatively as well as qualitatively.

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References

[1] P. Sheng, Editor, *Scattering and Localization of Classical Waves in Random Media*, World Scientific, Singapore (1990).

[2] I. M. Lifshits, S. A. Gredeskul and L. A. Pastur, *Introduction to the Theory of Disordered Systems*, J. Wiley, New York (1988).

[3] A. Crisanti, G. Paladin and A. Vulpiani, *Products of Random Matrices in Statistical Physics*, Springer, New York (1993).

[4] I. J. Goldsheid, S. A. Molchanov and L. A. Pastur, *A Random One-Dimensional Schroedinger Operator Has Pure Point Spectrum*, Functional Anal. Appl. 11, 1-10 (1977).

[5] V. I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media*, Nauka, Moscow (1980) (in Russian).

[6] R. F. O'Doherty and N. A. Anstey, *Reflections on Amplitudes*, Geophysical Prospecting 19, 430-458 (1971).

[7] R. Burridge, B. White and G. Papanicolaou, *One-Dimensional Wave Propagation in a Highly Discontinuous Medium*, Wave Motion 10, 19 - 44 (1988).

[8] J.F. Clouet and J.P. Fouque, *Spreading of a pulse travelling in a random medium*, Annals of Applied Probability, 4, 1083-1097, 1994.

[9] P. Lewicki, R. Burridge and G. Papanicolaou, *Pulse Stabilization in a Strongly Heterogeneous Layered Medium*, Wave Motion 20, 177-196 (1994).

[10] P. Lewicki, *Long time evolution of wavefronts in random media*, SIAM J. Appl. Math. 54, 907-934, 1994.

[11] P. Lewicki and G. Papanicolaou, *Reflection of wavefronts by randomly layered media*, Wave Motion 20, 245-266, 1994.

[12] P. Lewicki, R. Burridge and M.V. de Hoop, *Beyond effective medium theory: Pulse stabilization for multimode wave propagation in high contrast media*, SIAM J. Appl. Math. 56, 256-276, 1996.

[13] J. Chillan and J.P. Fouque, *Pressure fields generated by acoustic pulses propagating in randomly layered media*, SIAM J. Appl. Math. 58, 1532-1546, 1998.

[14] K. Solna, *Stable spreading of acoustic pulses due to laminated microstructure*, Ph.D. thesis, Stanford University, 1997.

[15] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. White, *Frequency Content of Randomly Scattered Signals*, SIAM Review 33, 519-625 (1991).

[16] R. Burridge, G. Papanicolaou, P. Sheng and B. White, *Probing a Random Medium with a Pulse*, SIAM J. Appl. Math. 49, 582-607 (1989).

[17] W. Kohler, G. Papanicolaou and B. White, *Localization and Mode Conversion for Elastic Waves in Randomly Layered Media*, Wave Motion 23, 1-22 and 181-201, 1996.

[18] M. Asch, W. Kohler, G. Papanicolaou M. Postel and B. White, *Statistical Inversion from Reflections of Spherical Waves By A Randomly Layered Medium*, Waves in Random Media 6, 293-334, 1996.