Soliton perturbations and the random Kepler problem

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We consider the influence of randomly varying parameters on the propagation of solitons for the one dimensional nonlinear Schrödinger equation. This models, for example, optical soliton propagation in a fiber whose properties vary with distance along the fiber. By using an averaged Lagrangian approach we obtain a system of stochastic modulation equations for the evolution of the soliton parameters, which takes the form of a randomly perturbed Kepler problem. We use the action-angle formulation of the Kepler problem to calculate the statistics of the escape time. The mean escape time for the Kepler problem corresponds, in the optical context, to the expected distance until the soliton disintegrates.

1 Introduction

The propagation of solitons in a disordered medium has been extensively studied[1–9,15] but there are many important problems that require further analysis. One particular application of interest is optical pulse or soliton propagation in a disordered medium. The analysis of this problem is important for understanding the behavior of optical communications systems when the optical properties of the fiber vary with distance. The soliton dynamics in the case where the fiber properties change adiabatically has been considered by Abdullaev[1], Elgin [10] and Kodama [11]. Kath and Ueda[12,13] have considered the case where the birefringence of the fiber varies randomly. Finally

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Kutz and Wai[15] have considered the effect of noise on periodic pulses in a dispersion managed fiber. The survey article of Gredeskul and Kivshar[14] contains a general review.

In this paper we consider the randomly perturbed one-dimensional focusing nonlinear Schrödinger (NLS) equation,

$$iU_t + \frac{\kappa}{2}U_{xx} + \chi|U|^2U = f(U,t,x) \tag{1}$$

with the initial condition $U(x,0)$ given approximately by a soliton. Some perturbations of particular interest are

- Fluctuating quadratic potential term

$$f(U,x,t) = K\gamma(t)x^2U(t,x).$$

- Fluctuating nonlinearity

$$f(U,x,t) = \chi\gamma(t)|U|^2U.$$

- Fluctuating dispersion

$$f(U,x,t) = \frac{\kappa}{2}\gamma(t)U_{xx}.$$

Here $\gamma(t)$ is a Gaussian, white-noise process, with noise level σ^2.

In applications to optical fibers the variable which we have called t is actually distance along the fiber and the variable we denote by x is actually a time, so that NLS models time dispersion and nonlinearity. Thus the perturbations are due to random inhomogeneities along the length of the fiber. Throughout this paper we will use the variables as we have defined in Eqn (1) above, so it should be understood that when we calculate the mean time for an event to occur we are actually calculating the mean distance along the fiber until the event occurs.

In this paper we analyze the effect of random inhomogeneities on soliton propagation. In particular, we calculate explicit formulas for the mean time until the soliton disintegrates because of the weak random inhomogeneities. We also calculate the standard deviation of the escape time for the case of the quadratic potential, and give formulas for the moments in the general case.

We begin by using the well-known averaged Lagrangian [2] method [16,17,7]. We consider approximate solutions that look like a soliton whose parameters,

[2] Instead of the averaged Lagrangian method we could use soliton perturbation theory [19,18] or the conservation laws of NLS[18]

Perturbation	Mean Exit Time ($J_0 = 0$)	Mean Exit Time ($J_0 \gg 1$)
$\epsilon_1 \gamma(t) x^2 U$	$\frac{2(47-60\ln(2))}{15\pi^2}\epsilon_1^{-2}$	$\frac{256}{3\pi^8 \epsilon_1^2 J_0^6}$
$\epsilon_2 \gamma(t) U_{xx}$	$\propto \epsilon_2^{-2}$	$\frac{8}{15\pi^2 \epsilon_2^2 J_0^3}$
$\epsilon_2 \gamma(t) \lvert U \rvert^2 U$	$\frac{\pi^3}{4}\left(5\ln(5) - \sqrt{5}\ln(\frac{\sqrt{5}+1}{\sqrt{5}-1}) - 8\ln(2)\right)\epsilon_2^{-2}$	$\frac{8\pi}{21\epsilon_2^2 J_0}$

Table 1
The dimensionless mean escape time for an unchirped soliton. The dimensionless small parameters ϵ_1 and ϵ_2 are given by (3). The initial action J_0 is defined by (25).

the amplitude, width, phase and chirp, vary with t. In the standard way we substitute this trial solution into the Lagrangian and compute the variation of the resulting averaged Lagrangian with respect to the soliton parameters to find a closed system of ODE's. These stochastic modulation equations can be reduced to a randomly perturbed Kepler problem. This is important because the Kepler problem, a Hamiltonian system with two degrees of freedom, is *integrable* and there exists a well known transformation to action-angle variables. In these variables the Fokker-Planck equation for the evolution of the probability density of the soliton parameters has a simple form, allowing detailed analysis similar to the one carried out by Papanicolaou, Knapp and White[6] using elliptic functions. A linearized analysis, for oscillations near the bottom of the Kepler potential, has also been considered by Abdullaev [7] for the one dimensional NLS, and by Fibich and Papanicolaou[8] in connection with beam collapse in two-dimensional critical NLS.

In the action angle variables the Fokker-Planck equation for the probability distribution of soliton parameters becomes a one dimensional diffusion equation with a diffusivity which grows algebraically for large values of the action variable. Solution of the one dimensional diffusion equation provides explicit formulas for quantities such as the mean time to reach infinity. This mean exit time corresponds, in the optical setting, to the mean distance a soliton will travel along a fiber before breaking up. Thus we are able to explicitly compute the expected lifetime of a soliton under various random perturbations as a function of the the initial conditions and the noise level. The results are summarized in Table 2.

In this Table all quantities are dimensionless. The dimensionless unit of time is

$$\frac{\kappa}{\chi^2 N^4}, \tag{2}$$

where N^2 is the L^2 norm of the initial pulse $U(x,0)$, κ is the dispersion constant and χ is the coefficient of the nonlinearity. This choice of scaling is chosen so that the period of the unperturbed problem is one. In this scaling the dimensionless parameters ϵ_1 and ϵ_2, which are assumed to be small, are given by

$$\epsilon_1 = \frac{\sigma K \kappa^{5/2}}{\chi^3 N^6}, \quad \epsilon_2 = \frac{\sigma \chi N^2}{\kappa^{1/2}}. \tag{3}$$

We note that the mean exit time behaves like $\epsilon_{1,2}^{-2}$ which means that it is longer for smaller noise level σ, as one would expect. The dependence of the mean exit time on the initial L^2 norm N^2 is, however, different for the different perturbations of NLS. For the quadratic potential perturbation the larger the N^2 the longer the mean exit time, as can be seen from the form of ϵ_1 in (3). For perturbations of the dispersion and nonlinearity the dimensionless strength of the perturbation is ϵ_2, which is proportional to N^2. Thus, the mean exit time is shorter for larger N^2.

In the case of the quadratic perturbation it is clear that a stronger soliton, one with a larger N^2, will take longer to disintegrate. In the case of perturbations in the dispersion or the nonlinearity the situation is more complicated because a stronger soliton will make the perturbations stronger as well.

The main drawback of the averaged Lagrangian approach that we follow here is that it fails to include the effect of dispersive radiation, and in particular the effect of radiation damping. It is possible to include the effects of radiation in an averaged Lagrangian approach, as in the work of Kath and Smyth [20], but we do not do so here. We do, however, want to emphasize that near the bottom of the potential well, for strong solitons, the effect of dispersive radiation is expected to be negligible. It is only near the separatrix, for low, wide solitons, that the effects of dispersive radiation are expected to be important.

Our analysis makes no use of the exact integrability of the cubic NLS equation so it is in principle applicable to any dispersive, nonlinear wave equation. Generally, however, the stochastic modulation equations are difficult to analyze except when there exists a relatively simple transformation to action-angle variables, as in the perturbed Kepler problem. Of course, the exit time analysis of the random Kepler problem is of independent interest, independent of soliton perturbation theory.

2 Dimensionless variables and scaling

We begin with the perturbed NLS equation

$$iU_t + \frac{\kappa}{2} U_{xx} + \chi |U|^2 U = \gamma(t) f(U, x, t), \tag{4}$$

where $\gamma(t)$ is the mean-zero Gaussian white noise process satisfying

$$<\gamma(t)\gamma(t')>= 2\sigma^2\delta(t-t'). \tag{5}$$

and $f(U,x,t)$ is the perturbation, which we assume is such that the perturbed equations have a Lagrangian formulation, though this is not strictly necessary. The associated Lagrangian is given by

$$L = \frac{i}{2}(U_tU^* - U_t^*U) - \frac{\kappa}{2}|U_x|^2 + \frac{\chi}{2}|U|^4 - \gamma(t)F(U,x,t). \tag{6}$$

Some perturbations of particular interest include

- Random quadratic potential

$$\begin{aligned} f(U,x,t) &= Kx^2U(x,t), \\ F(U,x,t) &= Kx^2|U|^2. \end{aligned}$$

- Random nonlinearity

$$\begin{aligned} f(U,x,t) &= \chi|U|^2U, \\ F(U,x,t) &= \frac{\chi}{2}|U|^4. \end{aligned}$$

- Random dispersion

$$\begin{aligned} f(U,x,t) &= U_{xx}, \\ F(U,x,t) &= -|U_x|^2. \end{aligned}$$

It is convenient to nondimensionalize the problem, since this makes clear the role of the small parameter in the problem. We introduce the L^2 norm of the initial pulse, given by

$$2N^2 = \int |U|^2dx.$$

The parameters in NLS have the dimensions

$$[\kappa] = \frac{L^2}{T}, \quad [\chi] = \frac{1}{A^2T}, \quad [N^2] = A^2L, \quad [K] = \frac{1}{L^2T},$$

where A denotes the units of U, and L, T are the units of length and time respectively. The last of these only appears in the quadratic potential perturbation. From the first three dimensional quantities we can construct the nonlinear length, time and distance scales. If we make the following rescaling

$$t' = \left(\frac{N^4\chi^2}{\kappa}\right) t, \quad x' = \left(\frac{N^2\chi}{\kappa}\right) x, \quad U' = \left(\sqrt{\frac{\kappa}{\chi N^4}}\right) U$$

then the problem is nondimensionalized on the natural scales of the unperturbed problem. Note that in these variables the initial data has an L^2 norm of two:

$$\int |U'|^2 dx' = 2.$$

After this rescaling the dimensionless perturbed NLS is

$$iU'_{t'} + \frac{1}{2}U'_{x'x'} + |U'|^2 U' = \epsilon_i \gamma(t') f(U', x', t')$$

where $\epsilon_i = \epsilon_1$ for the quadratic potential perturbation and $\epsilon_i = \epsilon_2$ for the perturbation of the dispersion and the nonlinearity. Here ϵ_1 and ϵ_2 are the dimensionless parameters (3). The noise $\gamma(t')$ is now normalized to have unit level:

$$<\gamma(t'_1)\gamma(t'_2)>= 2\delta(t'_1 - t'_2).$$

We will drop the primes and work with dimensionless variables consistently in the sequel. The physical quantities can, of course, be recovered by multiplying by the appropriate scale factors in (2).

3 The modulation equations

In the averaged Lagrangian approach we assume that the solution is given by a soliton-like solution with parameters which vary in time. In particular we make the ansatz

$$U(t,x) = A(t)\mathrm{sech}\left(\frac{x}{a(t)}\right) \exp\left[i(\psi(t) + \phi(t)x^2)\right]. \tag{7}$$

Since the L^2 norm of U is conserved by the perturbation, the amplitude A and the width a are related as follows

$$2 = \int_{-\infty}^{\infty} |U|^2 dx = 2A^2 a. \tag{8}$$

To find the evolution equations for the parameters we substitute the ansatz into the Lagrangian L and integrate out the spatial dependence. This gives the averaged Lagrangian

$$
\begin{aligned}
\mathcal{L}[a,\phi,\psi] &= \int\limits_{-\infty}^{\infty} L[U]\,dx \qquad (9)\\
&= -2\psi_t - \frac{\pi^2}{6}\phi_t a^2 - \frac{1}{3a^2} - \frac{2\pi^2}{3}\phi^2 a^2 + \frac{4}{3a}.
\end{aligned}
$$

We then obtain the Euler-Lagrange equations for the functional $\mathcal{L}$

$$
\delta \int \mathcal{L}[a,\phi,\phi_t,\psi_t]dt = 0, \qquad (10)
$$

which give the stochastic modulation equations

$$
\begin{aligned}
a_t - 2a\phi &= \epsilon_i \gamma(t) f_1(a,\phi), \qquad (11)\\
a\phi_t - \frac{2}{\pi^2 a^3} + 2a\phi^2 + \frac{2}{\pi^2 a^2} &= \epsilon_i \gamma(t) f_2(a,\phi).
\end{aligned}
$$

Here f_1 and f_2 arise from the perturbations and have the following forms:

- Fluctuating quadratic potential

$$
f_1(a,\phi) = 0\ , \qquad f_2(a,\phi) = -\frac{a}{2}. \qquad (12)
$$

- Fluctuating nonlinear term

$$
f_1(a,\phi) = 0\ , \qquad f_2(a,\phi) = -\frac{2}{\pi^2 a^2}. \qquad (13)
$$

- Fluctuating group velocity dispersion

$$
f_1(a,\phi) = 2a\phi\ , \qquad f_2(a,\phi) = \frac{2}{\pi^2 a^3} - 2a\phi^2. \qquad (14)
$$

Note that in the first two cases [3] f_1 is identically zero and f_2 is a function only of the width a and not of the chirp ϕ. In the first two problems the phase ϕ can be eliminated from the modulation equations and the resulting equation for a is the randomly perturbed Kepler problem

$$
a_{tt} = -U'(a) + \epsilon\gamma(t) f_2(a, \frac{a_t}{2a}), \qquad (15)
$$

[3] In fact this holds for any perturbation which is a function of U alone, and not of the derivatives of U

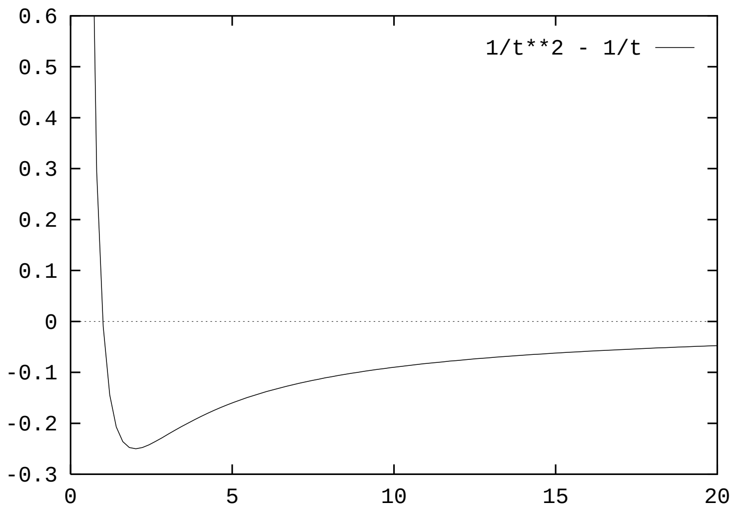

Fig. 1. The Kepler Potential

where $U(a)$ is the Kepler potential

$$U(a) = \frac{2}{\pi^2 a^2} - \frac{4}{\pi^2 a}. \tag{16}$$

The analysis of the case where the perturbation is in the group velocity dispersion is slightly more complicated and will be considered in a later section.

The Hamiltonian for the unperturbed Kepler problem is

$$H_0 = \frac{(a_t)^2}{2} + U(a). \tag{17}$$

The Hamiltonian for the perturbed Kepler problem (15) is

$$H \equiv H_0 + \epsilon\gamma(t)V \equiv \frac{(a_t)^2}{2} + U(a) + \epsilon\gamma(t)a^2, \tag{18}$$

for the fluctuating quadratic potential (12), and

$$H \equiv H_0 + \epsilon\gamma(t)V \equiv \frac{(a_t)^2}{2} + U(a) + \epsilon\gamma(t)\frac{2}{\pi^2 a}, \tag{19}$$

for the fluctuating nonlinear term (13).

4 Action-angle variables for the Kepler problem

In order to continue the analysis of the random Kepler problem, we must first transform the unperturbed Kepler problem to action-angle variables. We summarize here the relevant facts that we need, with details given in the texts by Landau and Lifschitz[21] or Goldstein[22].

The minimum of the potential $U(a)$ occurs at $a_c = 1$ and is equal to $U_0 = -2/\pi^2$. The frequency ω_0 of small oscillations about the minimum is given by $\omega_0 = 2/\pi$. This is the frequency of the width oscillations of an unperturbed soliton as it propagates down a homogeneous fiber.

For large oscillations the qualitative behavior of the orbits of the Kepler problem is determined by the energy

$$E_0 = 2a_0^2\phi_0^2 + \frac{2}{\pi^2 a_0^2} - \frac{4}{\pi^2 a_0},$$

where ϕ_0 is the initial chirp and a_0 is the initial width. When $E_0 < 0$ (i.e $\pi^2 a_0^2 \phi_0^2 < \frac{2}{a_0} - \frac{1}{a_0^2}$, corresponding to a sufficiently weak initial chirp) the orbits are closed, corresponding to oscillatory motion. In the the case $E_0 > 0$, corresponding to sufficiently large initial chirp, the orbits are unbounded, and the asymptotic motion is qualitatively like the motion of a free particle. In this regime the soliton does not persist, but instead spreads out and is lost ($a \to \infty$). The interesting questions in soliton propagation arise when the unpertubed motion is oscillatory, and so we consider the regime $E_0 < 0$.

We now make a change to action-angle variables. For the Kepler problem, where the phase space is two dimensional, there is one action variable J and the conjugate angle variable Θ. The action variable is given by

$$J = \frac{1}{2\pi} \oint a_t da.$$

This integral is easily evaluated by using the energy to express the momentum a_t in terms of the position. The resulting integral is elementary, and gives

$$J = \frac{2\sqrt{2}}{\pi^2\sqrt{-E}} - \frac{2}{\pi}. \tag{20}$$

Solving for the total energy E in terms of the action J gives the unperturbed Hamiltonian

$$H_0 = E = -\frac{8}{\pi^2(\pi J + 2)^2}, \tag{21}$$

which is a function of J only. The change of variables to action-angle variables is canonical, so that Hamilton's equations retain their form

$$\frac{dJ}{dt} = -\frac{\partial H_0(J)}{\partial \Theta} = 0,$$
$$\frac{d\Theta}{dt} = \frac{\partial H_0(J)}{\partial J} = \omega(J),$$

where

$$\omega(J) = \frac{16}{\pi(\pi J + 2)^3}. \tag{22}$$

The following implicit representation for the orbits is useful in the perturbation calculations. The position a and the time t can both be expressed parametrically in terms of a variable ξ (called the Kepler parameter) by

$$\begin{aligned} a &= b(1 - e_0 \cos \xi), \\ \Theta &= \omega(J)t = \xi - e_0 \sin \xi, \end{aligned} \tag{23}$$

where the eccentricity e_0 and the semi-major axis b are given in terms of the constant E as follows

$$\begin{aligned} e_0 &= (1 - \frac{\pi^2 \mid E \mid}{2})^{1/2} = \left(1 - \frac{4}{(\pi J + 2)^2}\right)^{1/2} \\ b &= \frac{2}{\pi^2 \mid E \mid} = \frac{(\pi J + 2)^2}{4} \end{aligned} \tag{24}$$

This parametric representation determines a implicitly as a function of t. Note that as Θ varies over $(0, 2\pi)$, ξ also varies over $(0, 2\pi)$. Figure (2) illustrates the geometric meaning of the Kepler parameter. The solid curve represents the Kepler ellipse while the dashed curve represents the circumscribing circle. The Kepler parameter, interpreted geometrically, is the angle of a point on the circumscribing circle, as measured from the center of the circle, to a point on the circle with the same x coordinate as the given point on the ellipse.

From equation (23) it follows that the width of the soliton oscillates between a minimum value of $a_{min} = b(1 - e_0)$ and a maximum value of $a_{max} = b(1 + e_0)$. The value of the action J varies from 0, for oscillations near the bottom of potential well, to ∞ for oscillations near the separatrix $E = 0$. The frequency $\omega(J)$ varies from $2/\pi$ for oscillations near the bottom of the potential well to 0 for oscillations near the separatrix.

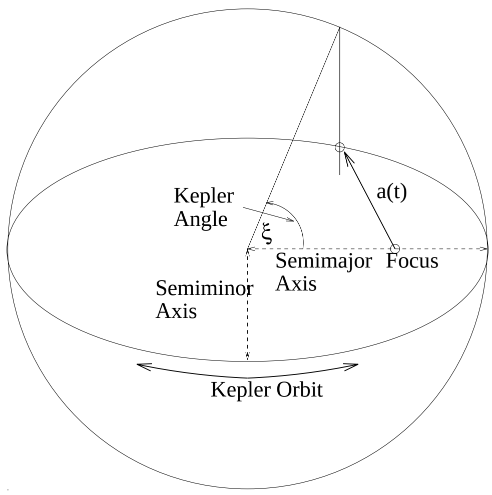

Fig. 2. The geometric interpretation of the Kepler parameter.

It is also useful to relate the initial action J_0 to the initial chirp ϕ_0 and initial width a_0:

$$\pi J_0 + 2 = \frac{2}{\sqrt{\frac{4}{a_0} - \frac{2}{a_0^2} - 2\pi^2 \phi_0^2 a_0^2}}. \tag{25}$$

5 The random Kepler problem - white noise case.

In the previous section we considered the oscillations in width of an unperturbed soliton, where the coefficients of the NLS are constant. In this section we address the case where the coefficients of the NLS vary randomly. In this case the soliton parameters evolve according to a set of random ODE's, and we can derive a Fokker-Planck equation for the evolution of the probability distribution for the soliton parameters.

The fluctuating quadratic potential perturbation (12) and the fluctuating nonlinear term perturbation (13) can be analyzed with white noise fluctuations directly, using the Stratonovich interpretation. In the case of fluctuating group velocity dispersion (14) the treatment is slightly more subtle, and is presented in a later section.

In action-angle variables, the perturbed Hamiltonian has the form

$$H = H_0(J) + \epsilon\gamma(t) V(J, \Theta), \tag{26}$$

where $V(J, \Theta)$ is obtained from (18) and (19). In general V, while simple in the original variables, is quite complicated in the action-angle variables. However for weak disorder we do not need the explicit form of V, but only averages over Θ. These averages will be simple to calculate using the implicit representation given in (23).

In the presence of weak disorder the action-angle variables for the unperturbed Kepler problem are a convenient framework for the analysis of the perturbed problem because the change in the action is proportional to the small parameter ϵ. Since the change to action-angle variables is canonical the Hamiltonian structure is preserved, and the perturbed equations become

$$\begin{aligned}\frac{dJ}{dt} &= -\epsilon\gamma(t)\frac{\partial V}{\partial\Theta}, \\ \frac{d\Theta}{dt} &= \omega(J) + \epsilon\gamma(t)\frac{\partial V}{\partial J}.\end{aligned} \tag{27}$$

Here $V(\Theta, J)$ is the perturbation Hamiltonian defined in equations (18,19). We assume that $\gamma(t)$ is Gaussian white noise, so the Stratonovich form of the Fokker-Planck equation for the evolution of the probability distribution of $P(J, \Theta, t)$, is given by

$$\begin{aligned}\frac{\partial P}{\partial t} &= \omega(J)\frac{\partial P}{\partial\Theta} + \epsilon^2\frac{\partial}{\partial J}(V_\Theta\frac{\partial}{\partial J}(V_\Theta P)) - \epsilon^2\frac{\partial}{\partial\Theta}(V_J\frac{\partial}{\partial J}(V_\Theta P)) \\ &- \epsilon^2\frac{\partial}{\partial J}(V_\Theta\frac{\partial}{\partial\Theta}(V_J P)) + \epsilon^2\frac{\partial}{\partial\Theta}(V_J\frac{\partial}{\partial\Theta}(V_J P))\end{aligned} \tag{28}$$

with initial condition $P(0, J, \Theta) = \delta(J - J_0)\delta(\Theta - \Theta_0)$.

In the absence of randomness, when $\epsilon = 0$, this is a first order PDE and can be solved simply by the method of characteristics. The characteristics are, of course, exactly the orbits of the Kepler problem, and the distribution is carried along these classical orbits with no spreading. When ϵ is small, so that the diffusion is weak, the Fokker-Planck equation can be simplified by averaging over the periodic orbits of the Kepler problem. Since the advective term $\omega\partial_\Theta$ acts on an $O(1)$ time scale, while the diffusion acts on the slower $O(\epsilon^2)$ time scale we introduce a relative angle variable Θ' defined by

$$\begin{aligned}\Theta' &= \Theta - \omega(J)t \\ \tau &= \epsilon^2 t.\end{aligned} \tag{29}$$

This transformation removes the advective term, giving a diffusion equation with rapidly varying coefficients. After averaging over the rapidly varying coefficients we are left with the simplified Fokker-Planck equation

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial J}(A(J)\frac{\partial P}{\partial J}) - \frac{\partial}{\partial J}B(J)\frac{\partial P}{\partial \Theta'} - \frac{\partial}{\partial \Theta'}B(J)\frac{\partial P}{\partial J} + \frac{\partial}{\partial \Theta'}C(J)\frac{\partial P}{\partial \Theta'}. \tag{30}$$

The coefficients $A(J), B(J), C(J)$ are given by averages of products of derivatives of V over the angular variable:

$$\begin{aligned} A(J) &= \frac{1}{2\pi}\int_0^{2\pi}(V_{\Theta'}(J,\Theta')^2 d\Theta', \\ B(J) &= \frac{1}{2\pi}\int_0^{2\pi}(V_{\Theta'}(J,\Theta')V_J(J,\Theta'))d\Theta', \\ C(J) &= \frac{1}{2\pi}\int_0^{2\pi}(V_J(J,\Theta'))^2 d\Theta'. \end{aligned} \tag{31}$$

These diffusion coefficients depend only on the action J, since the angular dependence has been averaged out.

The first of these terms is the most interesting, because if we assume that the initial probability distribution is independent of Θ' then the radial part of the distribution evolves according to

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial J}(A(J)\frac{\partial P}{\partial J}) \tag{32}$$

with $P(0, J) = \delta(J - J_0)$.

We note that this averaging calculation breaks down very near the separatrix, since $\omega(J)$ vanishes there, and the perturbations can no longer be considered small.

5.1 Exit Times

Since we are interested in calculating the distribution of exit times we will briefly review the relevant theory of exit times for a Fokker-Planck equation. For details we refer to the books of Feller[23] or Gardner[24].

Statistics of exit time can be understood easily in terms of the moment generating function (or Laplace transform) $\hat{P}(J, s)$, which is defined as

$$\hat{P}(J,s) = \int_0^\infty e^{-s\tau} P(J,\tau) d\tau$$

and which satisfies the transformed Fokker-Planck equation

$$s\hat{P}(J,s) + \delta(J - J_0) = \frac{\partial}{\partial J} A(J) \frac{\partial}{\partial J} \hat{P}(J,s). \tag{33}$$

The moment generating function provides a direct way to derive the differential equations satisfied by the moments of the exit time. The mean exit time is given by

$$\mu^{(1)}(J) = \int_0^\infty \tau P(J,\tau) d\tau = -\frac{\partial \hat{P}}{\partial s}, \tag{34}$$

with the last expression evaluated at $s = 0$. It is easy to show by differentiating (33) that $\mu^{(1)}_{(\alpha,\beta)}(J)$, the mean time to exit the interval (α, β) starting from the point J, satisfies the ordinary differential equation

$$\frac{\partial}{\partial J} A(J) \frac{\partial}{\partial J} \mu^{(1)}_{(a,b)}(J) = -1,$$

subject to vanishing boundary conditions $\mu^{(1)}_{(\alpha,\beta)}(\alpha) = \mu^{(1)}_{(\alpha,\beta)}(\beta) = 0$. Similarly the n^{th} moment satisfies

$$\frac{\partial}{\partial J} A(J) \frac{\partial}{\partial J} \mu^{(n)}(J) = -n\mu^{(n-1)}.$$

From these relations it is straightforward to calculate moments of the exit time. The solution for the equation for the mean time to exit the interval (α, β) is given by

$$\mu^{(1)}_{(\alpha,\beta)}(J) = \frac{\int_\alpha^J \frac{d\eta}{A(\eta)} \int_\alpha^\beta \frac{\eta d\eta}{A(\eta)} - \int_\alpha^J \frac{\eta d\eta}{A(\eta)} \int_\alpha^\beta \frac{d\eta}{A(\eta)}}{\int_\alpha^\beta \frac{d\eta}{A(\eta)}}, \tag{35}$$

and more generally the nth moment of the exit time from (α, β) is given by

$$\mu^{(n)}_{(\alpha,\beta)}(J) = n \frac{\int_\alpha^J \frac{d\eta}{A(\eta)} \int_\alpha^\beta \frac{\int \mu^{(n-1)}(\eta) d\eta}{A(\eta)} - \int_\alpha^J \frac{\int \mu^{(n-1)}(\eta) d\eta}{A(\eta)} \int_\alpha^\beta \frac{d\eta}{A(\eta)}}{\int_\alpha^\beta \frac{d\eta}{A(\eta)}}. \tag{36}$$

The averaged diffusivity $A(J)$ vanishes at $J = 0$, so the integrals diverge there. Upon taking the limits $\alpha \to 0$ and $\beta \to \infty$ we find that the following expressions for the mean time to reach infinity and the nth moment of the time to reach infinity starting at $J = J_0$:

$$\mu^{(1)}(J_0) = \int_{J_0}^{\infty} \frac{J dJ}{A(J)}, \tag{37}$$

$$\mu^{(n)}(J_0) = n \int_{J_0}^{\infty} \frac{\int \mu^{(n-1)}(J) dJ}{A(J)} \tag{38}$$

This is the exit time as measured in the *rescaled variable* τ. In the original variable, t, the expression is

$$\mu^{(1)}(J_0) = \frac{1}{\epsilon^2} \int_{J_0}^{\infty} dJ \frac{J}{A(J)}, \tag{39}$$

which is still dimensionless. In the next section we apply this formula to the specific perturbations which we have considered.

5.2 Fluctuating quadratic potential

For the case of a fluctuating quadratic potential the Hamiltonian is given by

$$H = \frac{a_t^2}{2} + U(a) + \frac{\epsilon\gamma(t)}{2} a^2,$$

so that the perturbation Hamiltonian V is given by

$$V(a) = \frac{a^2}{2}.$$

Here $\epsilon = \epsilon_1$ as defined by (3) in terms of the physical parameters. The quantity we are interested in is the diffusion coefficient $A(J)$ defined by

$$A(J) = \frac{1}{2\pi} \int_0^{2\pi} (V_\Theta)^2 d\Theta.$$

Rather than make the relatively complicated transformation to action-angle variable to evaluate this integral we make use of the following procedure. We have an implicit representation of solutions to the Kepler problem given in equation (21):

$$
\begin{aligned}
a &= b(1 - e_0 \cos \xi), \\
\Theta &= \omega(J)t = \xi - e_0 \sin \xi.
\end{aligned}
$$

As the angle variable Θ varies over $(0, 2\pi)$ the Kepler anomaly ξ also varies over $(0, 2\pi)$, and thus we can make a change of variable in the integral from Θ to ξ by making the replacements

$$
\begin{aligned}
V(a) &= \frac{a^2}{2} = \frac{b^2(1 - e_0 \cos(\xi))^2}{2}, \\
\frac{d}{d\Theta} &= (\frac{d\Theta}{d\xi})^{-1} \frac{d}{d\xi}, \\
d\Theta &= \frac{d\Theta}{d\xi} d\xi.
\end{aligned}
$$

After this change of variables the expression for $A(J)$ becomes

$$
\begin{aligned}
A(J) &= \frac{b^4 e_0^2}{2\pi\epsilon^2} \int \sin^2(\xi)(1 - e_0 \cos \xi) d\xi \\
&= \frac{b^4 e_0^2}{2\epsilon^2} \\
&= \frac{\pi J}{2^9 \epsilon^2} (\pi J + 4)(\pi J + 2)^6.
\end{aligned}
\tag{40}
$$

Note the behavior near the origin is given by $A'(0) = \frac{\pi}{2^9}$, so the Fokker-Planck equation is well-behaved, and that for large J the dispersion grows like J^8, so that the expected time to reach infinity is finite. Substituting into (37) when $J_0 = 0$, we find that the mean exit time for a soliton with no initial chirp is given

$$
\begin{aligned}
\mu^{(1)}(0) &= \frac{2^9}{\pi^8 \epsilon^2} \int_0^\infty \frac{dJ}{(J + \frac{\pi}{4})(J + \frac{2}{\pi})^6} \\
&= 2\frac{47 - 60\ln(2)}{15\pi^2 \epsilon^2} \approx \frac{.073}{\epsilon^2}.
\end{aligned}
\tag{41}
$$

In an optical setting the mean exit time corresponds to the mean distance until the soliton breaks up. Recall that here $\epsilon = \epsilon_1$, given by (3).

More generally we can consider the expected time to soliton breakup in the case where the initial soliton has a non-zero chirp ϕ_0. In this case the initial action J_0 is not equal to zero. Using equation (37) we can express the initial action in terms of the initial chirp. In this case the expected time to soliton dissolution is given by

$$\mu^{(1)}(J_0) = \frac{2^9}{\pi^8 \epsilon^2} \int\limits_{J_0}^{\infty} \frac{dJ}{(J+\frac{\pi}{4})(J+\frac{2}{\pi})^6}. \tag{42}$$

This integral is elementary, since it is the integral of a rational function, and can be done relatively easily using the identity

$$\frac{1}{(1+x)^n(2+x)} = \sum_{k=0}^{n-1} \frac{(-1)^k}{(1+x)^{n-k}} + \frac{(-1)^n}{x+2}.$$

giving the general result

$$\mu^{(1)}(J_0) = \frac{8}{\pi^2\epsilon^2}\left(\sum_{k=1}^{5} \frac{(-1)^{k+1}}{k(\frac{\pi}{2}J+1)^k} - \ln\left(\frac{(\frac{\pi}{2}J+2)}{(\frac{\pi}{2}J+1)}\right)\right)$$

In the case where the initial chirp J_0 is large, so that the soliton begins near the separatrix, the mean exit time is approximately given by

$$\mu^{(1)}(J_0) \approx \frac{256}{3\pi^8\epsilon^2 J_0^6} \qquad J_0 \gg 1.$$

It is also interesting to use the explicit formulas for the second moment to calculate the standard deviation of the escape time. If we substitute the above expressions for $\mu^{(1)}(J)$ and $A(J)$ into Equation (37) we find that the integral which gives the second moment is no longer elementary and involves the polylogarithm function. However the integral can easily be evaluated numerically. For the unchirped case, $J_0 = 0$, we find that

$$\mu^{(2)}(0) \approx \frac{.0073}{\epsilon^4}.$$

The standard deviation is given by

$$\sqrt{\mu^{(2)} - (\mu^{(1)})^2} \approx \frac{0.044}{\epsilon^2}$$

so the standard deviation is about 60% of the mean. For the rest of the cases we consider we do not calculate the variances/standard deviations, though of course they could be computed in exactly the same manner. It would be very interesting to calculate the asymptotic behavior of the moments, since intuition suggests that quantities such as exit times should have very large fluctuations.

5.3 Fluctuating nonlinearity.

For the case of a fluctuating nonlinear term the perturbation Hamiltonian is given by

$$V = \frac{4}{\pi^2 a}, \tag{43}$$

and $\epsilon = \epsilon_2$ with ϵ_2 given by (3) in terms of the physical parameters. We proceed in the same way as in the previous section, and make a change of variables from integration over the angle to integration over the Kepler parameter. The expression for $A(J)$ is

$$A(J) = \frac{8e_0^2}{\pi^5 b^2} \int_0^{2\pi} \frac{\sin^2(\xi)}{(1 - e_0 \cos\xi)^5} d\xi \tag{44}$$

$$= \frac{8e_0^2}{\pi^4 b^2} \frac{4 + e_0^2}{8(1 - e_0^2)^{7/2}} \tag{45}$$

$$= \frac{1}{16\pi^4}((\pi J + 2)^2 - 4)(5(\pi J + 2)^2 - 4)(\pi J + 2). \tag{46}$$

For the mean exit time $\mu^{(1)}$ for a particle starting from $J = 0$ we obtain

$$\mu^{(1)}(0) = \frac{1}{\epsilon^2} \int_0^{\infty} \frac{J dJ}{A(J)} \tag{47}$$

$$= \frac{\pi^3}{4\epsilon^2} \left(5\ln(5) - \sqrt{5}\ln(\frac{\sqrt{5}+1}{\sqrt{5}-1}) - \ln(256) \right).$$

In this example, as in the previous example, the diffusion $A(J)$ is a rational function of the action variable, so that the mean escape time can be computed for *any initial chirp*. Again the general result is somewhat lengthy, so we do not give it here. For large values of the initial chirp, however, the mean exit time is given by

$$\mu^{(1)}(J_0) = \frac{1}{\epsilon^2} \int_{J_0}^{\infty} \frac{J dJ}{A(J)} \tag{48}$$

$$\approx \frac{8\pi^3}{\epsilon^2} \int_{J_0}^{\infty} \frac{J dJ}{5(\pi J)^5}$$

$$\approx \frac{8}{15\pi^2\epsilon^2} J_0^{-3}$$

6 Other problems

The techniques we have used in the previous section are quite general, and are applicable to a wide variety of different perturbations. In particular the assumption that the perturbations be Hamiltonian and white noise can be relaxed. In this section we consider a number of other perturbations which can be be handled with techniques similar to the ones presented in the previous section.

6.1 Fluctuating dispersion.

The case of fluctuating dispersion is somewhat more subtle, and is treated in this section. For the first part of this calculation we do not assume that γ is white noise, but is instead some process with finite time correlations. After applying the averaged Lagrangian approach we find the following equations of motion for the soliton parameters:

$$a_t - 2(1+\epsilon\gamma(t))a\phi = 0, \tag{49}$$

$$a\phi_t - \frac{2(1+\epsilon\gamma(t))}{\pi^2 a^3} + 2(1+\epsilon\gamma(t))a\phi^2 + \frac{2}{\pi^2 a^2} = 0. \tag{50}$$

Here $\epsilon = \epsilon_2$ given by (3). Upon eliminating the variable ϕ we are led to a second order equation for a, the soliton width:

$$\frac{d}{dt}\frac{a_t}{2(1+\epsilon\gamma(t))} - \frac{2(1+\epsilon\gamma(t))}{\pi^2 a^3} + \frac{2}{\pi^2 a^2} = 0.$$

This equation comes from a Hamiltonian of the form

$$H = \frac{a_t^2}{2(1+\epsilon\gamma(t))} + \frac{2(1+\epsilon\gamma(t))}{\pi^2 a^2} - \frac{2}{\pi^2 a}.$$

We can write this as

$$H = \frac{a_t^2}{2}(1 - \epsilon\gamma(t) + \frac{\epsilon^2}{2} + O(\epsilon^2)) + \frac{2(1+\epsilon\gamma(t))}{\pi^2 a^2} - \frac{2}{\pi^2 a},$$

where the O(ϵ^2) terms have mean zero and thus do not contribute to the order we consider. We can eliminate the the $\epsilon^2/2$ term by making the rescaling of time $\tau = (1+\epsilon^2/2)^{1/2}t = \alpha t$. We get

$$
\begin{aligned}
H =& \frac{a_\tau^2}{2}(1-\epsilon\gamma((\alpha^{-1}\tau)+O(\epsilon^2))+\frac{2(1+\epsilon\gamma((\alpha^{-1}\tau))}{\pi^2 a^2}-\frac{2}{\pi^2 a} \\
=& \frac{a_\tau^2}{2}(1-\epsilon\gamma(\tau))+\frac{2(1+\epsilon\gamma(\tau))}{\pi^2 a^2}-\frac{2}{\pi^2 a}+\mathrm{O}(\epsilon^2),
\end{aligned} \tag{51}
$$

where, again, the $\mathrm{O}(\epsilon^2)$ terms have mean zero and thus do not contribute to leading order. At his point we can pass to the white noise limit and the arguments of the previous section apply. In particular the perturbation part of Hamiltonian in the random Kepler problem has the form

$$
\begin{aligned}
V =& -\frac{(a_x)^2}{2}+\frac{2}{\pi^2 a^2} \\
=& V_1+V_2.
\end{aligned} \tag{52}
$$

The diffusion coefficient $A(J)$ is given by

$$
A(J)=\frac{1}{2\pi}\int (V_\Theta)^2 d\Theta \tag{53}
$$
$$
=\frac{1}{2\pi}\int (V_{1\Theta})^2+2V_{1\Theta}V_{2\Theta}+(V_{2\Theta})^2 d\Theta.
$$

When we make the change of variable to the Kepler parameter ξ we obtain

$$
\begin{aligned}
V_1 =& -\frac{\omega^2 b^2 e_0^2 \sin\xi^2}{(1-e_0\cos\xi)^2}, \\
V_2 =& \frac{2}{\pi^2 b^2(1-e_0\cos\xi)^2}.
\end{aligned} \tag{54}
$$

The integral defining $A(J)$ is, as in the previous cases, elementary. We do not write the full result here, but we do note that for large J the dominant contribution to the integral comes from V_2. If we ignore the contributions of V_1 and the cross terms to the diffusion coefficient $A(J)$ we obtain the result

$$
A(J)=\frac{e_0^2(8+12e_0^2+e_0^4)}{\pi^4 b^4(1-e_0^2)^{11/2}} \tag{55}
$$
$$
=\frac{J(\pi J+4)(128+448\pi J+448\pi^2 J^2+168\pi^3 J^3+21\pi^4 J^4)}{8\pi^3(\pi J+2)^3} \tag{56}
$$

It is routine, though tedious, to calculate the contributions of V_1 and the cross terms, though we do not do this here. However we note that in the case where the soliton begins near the separatrix, so that J_0 is large we find that

$$
A(J)\approx \frac{21}{8\pi}J^3. \tag{57}
$$

Integrating from J_0 in (37), we obtain

$$\mu^{(1)} \approx \frac{8\pi}{21\epsilon^2 J_0}. \tag{58}$$

6.2 Exit times for the damped Kepler problem

The presence of damping terms in a stochastic differential equation can dramatically change the behavior of the solutions. The presence of damping can dramatically increase exit times, or even lead to infinite exit times, if the damping creates a steady-state.

It might be thought that radiative effects lead to damping in the Kepler problem, however this is not the case. The main effect of radiation is an adiabatic change in the L^2 norm, the mass of the soliton. However there do exist physical perturbations which can produce an effective damping term in the Kepler problem. One such perturbation arises in the work of Malomed [25]. Malomed considers the effects of weak amplification and filtering with the situation where the amplification is a constant plus a periodic array of delta function amplifiers. In the special case where the strength of the array of delta function amplifiers is zero, corresponding to constant amplification, the unperturbed model equation reduces to

$$iU_t + \frac{1}{2}U_{xx} + |U|^2 U = i\epsilon^2 \Gamma_0 \left(\frac{1}{3}U + U_{xx}\right).$$

Here we are working in dimensionless variables scaled on the natural scales of the unperturbed problem, so that U has unit L^2 norm. The factor of $1/3$ in front of the amplification guarantees that a soliton of unit L_2 norm will propagate without change. For our analysis it is important that the damping scales like ϵ^2, while the random fluctuations scale like ϵ, so that the model problem becomes

$$iU_t + \frac{1}{2}U_{xx} + |U|^2 U = i\epsilon^2 \Gamma_0 \left(\frac{1}{3}U + U_{xx}\right) + \epsilon f(U, x, t).$$

The standard variational calculation, as presented by Malomed[25], gives the following ODE for the soliton width

$$a_{tt} = a^{-3} - a^{-2} - \epsilon^2 \Gamma a_t + \epsilon \gamma(t) f(a),$$

which corresponds to the random Kepler problem with a linear damping term. Here Γ is related to Γ_0 by $\Gamma = \frac{8(6+\pi^2)}{3\pi^2}\Gamma_0 \approx 4.3\Gamma_0$. In the action-angle formulation of the problem this damping term leads to the presence of an extra term in the evolution equation for the action J, which again takes the form of linear damping.

$$\Theta_t = \omega(J) - \epsilon\gamma(t)\frac{\partial V}{\partial J},$$
$$J_t = \epsilon\gamma(t)\frac{\partial V}{\partial \Theta} - \epsilon^2 \Gamma J.$$

Note that these equations are no longer Hamiltonian, as the damping term breaks the Hamiltonian structure. The presence of this term leads to the presence of an additional term in the Fokker-Planck equation:

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial J} A(J)\frac{\partial P}{\partial J} + \Gamma\frac{\partial}{\partial J} JP.$$

The differential equation governing the mean exit time for this Fokker-Planck equation is easily seen to be

$$\epsilon^2 \left(\frac{\partial}{\partial J}\mu^{(1)} + \Gamma\frac{\partial}{\partial J}J\mu^{(1)} \right) = -1. \tag{59}$$

Integrating this equation once leads to

$$\epsilon^2 \left(A(J)\frac{\partial}{\partial J}\mu^{(1)} + \Gamma J\mu \right) = -J + C. \tag{60}$$

This equation has an integrating factor $\exp(\Gamma \int \frac{J}{A(J)})/A(J)$. We thus find the following expression for $\mu^{(1)}(J_0)$, the mean exit time to reach infinity starting at the point J_0:

$$\mu^{(1)}(J_0) = \frac{\exp[\Gamma \int_{J_0}^{\infty} \frac{JdJ}{A(J)}] - 1}{\Gamma\epsilon^2}.$$

Several commments are in order. First we note that since the escape time in the problem without amplification and filtering is given by $\int JdJ/A(J)$, there exists a surprisingly simple functional relationship between the escape time in the damped/driven case and the escape time in the undamped case. Since the exponential is convex it is easy to see that the escape time in the presence of weak amplification and filtering is strictly longer than the escape time in the absence of these terms. When J_0 is large or the damping coefficient Γ is small this expression is approximately the same as the one derived in the previous section. Thus when the soliton is near the separatrix the effect of damping is relatively unimportant. This is to be expected on physical grounds - for a low, wide soliton the effect of filtering is expected to be small. However when J_0 is not large and Γ is not small the expected escape time is *exponentially long*. Physically it is clear why this should be the case - the additional term in the Fokker-Planck equation corresponds to a mean drift towards the origin and away from the separatrix. It is well-known that the expected time to random walk a given distance against a mean flow is exponentially long. In an optical context this is a mathematical justification of the stabilizing effect of filters.

6.3 Colored Noise Perturbations

It is also possible to treat perturbatively the case where the randomness has a small but finite correlation time by expanding in a series in powers of the correlation time. We briefly sketch the derivation. Using standard arguments the effective diffusion coefficient $A(J)$ is given by

$$A(J) = \frac{1}{\tau_c} \int_0^\infty ds B(\frac{s}{\tau_c}) <V_\Theta(\Theta) V_\Theta(\Theta + \omega(J)s)>, \tag{61}$$

where $\tau_c^{-1} B(\frac{s}{\tau_c}) = <\gamma(s)\gamma(0)>$ is the correlation function of the colored noise process, and the brackets denote averaging over the angular variable. In the white-noise limit, when the correlation time τ_c is short the correlation function $\frac{B(\frac{s}{\tau_c})}{\tau_c}$ reduces to a delta function, and the above integral reduces to the previous expression. If we make the change of variable $\eta = \frac{s}{\tau_c}$ in the above expression for the diffusion coefficient $A(J)$ we get

$$A(J) = \int B(\eta) <V_\Theta(\Theta) V_\Theta(\Theta + \tau_c \omega(J)\eta)> .$$

Expanding $V_\Theta(\Theta + \tau_c \omega(J)\eta)$ in a Taylor series and integrating we get

$$\begin{aligned} A(J) \approx & <V_\Theta(\Theta) V_\Theta(\Theta)> - \frac{m_2 \tau_c^2 \omega(J)^2}{2} <V_{\Theta\Theta} V_{\Theta\Theta}> + \dots \\ & + (-1)^n \frac{\tau_c^{2n} \omega(J)^{2n} m_{2n}}{2n!} <\partial_\Theta^{n+1} V \partial_\Theta^{n+1} V> + \dots, \end{aligned}$$

Here m_n denotes the n^{th} moment of the correlation function $m_n = \int s^n B(s) ds$. Note that the odd terms in the perturbation series vanish, since they are proportional to angular averages of terms which are exact derivatives in the Θ variable.

Note also that the first order correction to the effective diffusivity is negative, leading to *increased* exit times. If this perturbation problem is to be regular we require that the first order correction to the white-noise result grow no faster than the leading order term for large J, otherwise the effective diffusivity would be negative for large J. Physically there is good reason to expect this to be true. When the action J is large the period of the orbit is large, and there are *many* correlation times in one period. Thus we expect that the effects of the finite time correlations should become less important near the separatrix. Of course this should still be checked for each example.

We calculate the form of the correction for the simplest case is the quadratic potential, which has a perturbation Hamiltonian $V(a) = a^2/2$. The calculation is essentially the same as the one for the leading order term. The second derivative $V_{\Theta\Theta}$, when expressed in terms of the Kepler parameter, is given by

$$V_{\Theta\Theta} = \frac{b^2 e_0 cos(\eta)}{1 - e_0 \cos(\eta)}$$

and thus the angular average of $V_{\Theta\Theta}^2$ is given by

$$<V_{\Theta\Theta}V_{\Theta\Theta}>= \frac{b^4 e_0^2}{2\pi} \int \frac{\cos^2(\eta)}{1 - e_0 \cos(\eta)} d\eta.$$

After doing the η integration we derive the following expression for the diffusion coefficient:

$$\begin{aligned} &A(J) \approx A_0(J) - \tau_c^2 A_1(J) \\ &A_0(J) = \frac{\pi J}{2^9 \epsilon^2}(\pi J + 4)(\pi J + 2)^6 \ , \quad A_1(J) = \frac{m_2(\pi J)(\pi J + 2)^2}{4\epsilon^2 \pi^2}. \end{aligned}$$

Note that the correction, which grows as J^3, does so much more slowly than the leading order term, which grows as J^8. Thus the mean exit time for the near-white-noise problem is given by

$$\begin{aligned} \mu_1(J_0) &\approx \int_{J_0}^{\infty} \frac{JdJ}{A_0(J) - \tau_c^2 A_1(J)} \\ &\approx \int_{J_0}^{\infty} \frac{JdJ}{A_0(J)} + \tau_c^2 \int_{J_0}^{\infty} \frac{JA_1(J)dJ}{A_0^2(J)}. \end{aligned}$$

As in previous examples these integrals involve rational functions, and can be done explicitly.

6.4 The external white noise.

It is interesting to consider the Kepler problem with a white noise driver. This is the equivalent of the standard random walk for a particle which moves in a Kepler potential rather than moving as a free particle. This particular case does not arise in a natural way from any soliton problem, but since the random Kepler problem is interesting in and of itself we include it.

The perturbation Hamiltonian takes the form

$$V = \gamma(t)a, \tag{62}$$

where $\gamma(t)$ is a white noise. The calculation gives the following expression for $A(J)$

$$A(J) = \frac{\pi J(\pi J + 2)^3}{16}. \tag{63}$$

For the mean exit time we find

$$\mu^{(1)} = \frac{8}{\pi^7 \epsilon^2}(J_0 + \frac{2}{\pi})^{-2}. \tag{64}$$

We note that for the white noise driver the mean exit time is proportional to the square of the initial power.

7 Conclusions

We have investigated the propagation of chirped solitons in Kerr media with randomly varying parameters. We have used a variational approach to reduce the analysis to that of a randomly perturbed Kepler problem. The fact that the underlying ODE is the Kepler problem means that the transformation to action-angle variables is known implicitly, and this transformation can be used, in the weak noise limit, to reduce the problem to a one-dimensional diffusion. For this one dimensional diffusion we can compute explicitly quantities such as the mean exit time, which corresponds physically to the expected time until the soliton disintegrates. This analysis is very general, and can be applied to many different types of perturbations, since the underlying ODE, the Kepler problem, is the same in all cases.

This same general theory can be applied to compute higher order moments of the escape time. One interesting computation, which we have not done, would be to calculate the behavior of the escape time for large moment number, a kind of large deviations theory. It would also be interesting to test the predictions of this theory with some numerical experiments.

Finally the most significant drawback of this approach is that it completely neglects radiation effects, which are expected to be important in many cases. It would be extremely interesting to do a similar exit time calculation including radiation effects, perhaps using an IST based perturbation theory, or the averaged Lagrangian approach ro radiation developed by Kath and Smythe[20]. However such a calculation seems extremely difficult.

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References

[1] F.Kh. Abdullaev, S.A. Darmanyan and P.K. Khabibullaev, *Optical Solitons*, (Springer-Verlag, Heidelberg, 1993).

[2] J.C. Bronski, J. Nonlinear Sci., **8**, 161 (1998).

[3] J.C. Bronski, J. Stat. Phys., **92**, 5/6 (1998).

[4] J. Garnier, SIAM J. Appl. Math, *In Press.*

[5] J. Garnier, Ph.D. Thesis, Ecole Polytechnique.

[6] R. Knapp, G. Papanicolaou, and B. White, In *Nonlinearity and Disorder* A.R.Bishop, D.Campbell, St.Pnevmatikos, (Springer-Verlag, Heidelberg, 1989).

[7] F.Kh. Abdullaev, In *"Fluctuations:Nonlinearity and Disorder"*, A.R.Bishop, L.Vasquez (Eds.) p. (Singapore, WS, 1995);F.Kh.Abdullaev,A.A.Abdumalikov and B.B.Baizakov, Opt.Commun. **138**,49 (1997).

[8] G. Fibich and G. Papanicolaou, SIAM, 1998, *In Press.*

[9] P.K. Wai, C.R Menyuk and H.H Chen, Opt. Lett. **16**, 1231 (1991).

[10] J.N. Elgin, Opt.Lett. **18**, 10 (1993).

[11] Yu. Kodama and A. Hasegawa, *Solitons in Optical Communications*, (Clarendon Press, Oxford, 1995).

[12] W. Kath and T. Ueda, Physica D,**55**, 166 (1992).

[13] W. Kath and T. Ueda, JOSA B,**11**, 818 (1994).

[14] S.A. Gredeskul and Yu.S. Kivshar, Phys.Rep., **216**, 1 (1992).

[15] J.N. Kutz, P.K. Wai, Opt. Lett. **23**, 13 (1998).

[16] G. Whitham, *Linear and Nonlinear Waves*, (Wiley-Interscience).

[17] D. Anderson, M. Lisak, and T. Reichel, JOSA **B5**, 207 (1988).

[18] V.I. Karpman, Physica Scripta, **20**, 462 (1978).

[19] J.P. Keener and D.W. McLaughlin, **16**, 777 (1977).

[20] W.L. Kath and N.F. Smyth, Phys. Rev. E **51**, 1484 (1995).

[21] L.D.Landau and E.M.Lifshitz, *Mechanics*, (Pergamon, New York, 1975).

[22] H. Goldstein, *Classical Mechanics*, (Wiley, New York, 1965).

[23] W. Feller, *Probability Theory and Its Applications*, (Wiley, New York, 1950)

[24] C.W. Gardner, *Handbook of Stochastic Methods*, (Springer-Verlag, Berlin, 1983).

[25] B.A. Malomed, J. Opt. Soc. Am. B, **13**, 677 (1996).