Variational Principles for Propagation Speeds in Inhomogeneous Media

S. Heinze* G. Papanicolaou † A. Stevens‡

∗ *Max-Planck-Institute for Mathematics in the Sciences (MIS), D-04103 Leipzig, Germany.*
†*Stanford University, Department of Mathematics, Stanford, CA 94305, USA.*
‡*Max-Planck-Institute for Mathematics in the Sciences (MIS), D-04103 Leipzig, Germany.*

Abstract

An important problem in reactive flows is how to estimate the speed of front propagation, especially when inhomogeneities are present. Here we prove a variational characterization of the front speed for reaction-diffusion-advection equations in periodically varying heterogeneous media. This formulation makes it possible to calculate sharp estimates for the speed explicitly. The method can be applied to any problem obeying a maximum principle. Three examples will be analyzed in detail: a shear flow problem, a problem with rapidly oscillating coefficients and a discretized diffusion problem. In all cases the effects of the inhomogeneous medium on the speed are discussed in comparison to the homogeneous problem. For the shear flow problem, enhancement of the speed results.

*heinze@mis.mpg.de

†papanico@math.stanford.edu

‡stevens@mis.mpg.de Supported by the *Deutsche Forschungsgemeinschaft* during a stay at Stanford University, Stanford, CA 94305, USA.

1 Introduction

Front propagation occurs in many applied problems, such as chemical kinetics, combustion, transport in porous media, and in biology. The basic phenomena can often be described by reaction-diffusion-advection equations. In homogeneous media front propagation has been studied for a long time. However, the study of fronts in inhomogeneous media has begun more recently. For details and an account on the existing literature compare Chapter 2.

Since heterogeneities occur in every natural environment, understanding their influence on the location of fronts, on their profile, and on their speed is of great importance. Also for the description of moving interfaces in the large space-time scaling limit, precise information on the wave speed is needed. Since the normal velocity of the interface is equal to the speed of the wave in normal direction, even the formulation of interface motion requires a priori knowledge about the wave speed, compare [2].

In this paper we study the effects of a periodically varying environment on the speed of fronts. First, in Chapter 2 the basic model is introduced. Then, in Section 3.1 we prove a min/max characterization of the traveling wave velocities of reactive flows in media with spatially varying diffusion and drift coefficients. Our variational approach generalizes the corresponding 1-D result, given by Vol'pert et al. in [22], to higher space dimensions and to inhomogeneous problems. The latter is a genuine extension even in one dimension. We also explain the connection between the existence of variational principles for wave speeds and the stability of waves. Our proofs are based on the maximum principle and therefore further extensions are possible:

- periodically perforated domains,
- domains with variable cross section,
- monotone systems in higher dimensions,
- discretized diffusion.

In Section 3.2 we use the weak version of the maximum principle to derive a variational principle for the wave speed in integral form.

Applications of our results are discussed in Chapter 4. We calculate explicit upper and lower bounds for the wave speed for three different types of problems and estimate the influence of the respective heterogeneities on the speed of the front. Section 4.1 deals with shear flows in cylinders. They appear e.g. in the study of premixed flame propagation where an underlying flow field is given. As a heuristic rule it is known that turbulence increases the effectiveness of combustion. We prove rigorously that the introduction of a small amplitude drift coefficient always enhances the front speed and provide explicit speed estimates which are accurate in the small amplitude limit and the rapid oscillation limit.

Diffusion in heterogeneous media or multiscale problems are very common in applications and can frequently be described by their effective behavior. They appear for instance in models which describe the effective permeability of flows through porous

media or the effective conductivity of a composite material. This means that there is a homogeneous medium, the effective medium, whose diffusive properties are close to those of the real medium when measured on large space-time scales. During an averaging or homogenization process the often complicated small scale structure of the material is replaced by an asymptotically equivalent homogeneous structure. In Section 4.2 we analyze reaction-diffusion models with general rapidly oscillating diffusion and drift coefficients. With our theory we can rigorously justify the formal asymptotic expansion of the speed in the fast oscillation limit. We also compute the deviation of the speed in comparison to the homogenized problem.
In Section 4.3 we consider the discretized version of the Nagumo equation. This is a simplification of the discretized classical Fitzhugh-Nagumo system which is used as a model in nerve physiology to describe the dynamics of coupled cells. If the propagation of action potentials in coupled nerve cells is slowed down and finally even blocked, the whole system may break down with possibly fatal consequences. So there is a major interest in a detailed understanding of the mechanisms by which action potentials propagate. Here the coupling strength of the cells plays a crucial role. We apply a modified version of our variational principle and justify an asymptotic expansion of the speed as the step size of the spatial discretization of the Nagumo equation tends to zero. For an exact cubic reaction term we prove that a weaker coupling of the cells slows down the wave in comparison to the continuous model. The expansion of the speed is calculated explicitly up to second order.

2 The basic problem

We consider the following reaction diffusion equation in an infinite cylinder in R^n.

$$u_t = \nabla_x\Big(A(x)\nabla_x u\Big) + f(u) \quad \text{for } t>0, \quad x \in D = I\!R \times \Omega \subset I\!R^n \tag{1}$$

with initial data

$$u(0,x) = u_0(x) \ ,$$

where the cross-section $\Omega \subset I\!R^{n-1}$ is a bounded domain with $C^{1,\alpha}$ boundary. We assume Neumann boundary conditions on $\partial\Omega$. In a modified form our results also hold for Dirichlet or periodic boundary conditions. We will always use the notation $x = (x_1, \tilde{x})$ for any vector x in $I\!R \times \Omega$. Let $A = A(x)$ be a positive definite matrix, 1-periodic in x_1, with $C^{1,\alpha}(D)$ coefficients. We do not assume symmetry of $A(x)$. Decomposing $A(x) = A^s(x) + A^a(x)$ into its symmetric and antisymmetric part and defining the vector $B(x)$ by

$$B_j(x) = \sum_{i=1}^{n} \partial_i A^a_{ij}(x)$$

we can also rewrite (1) with an explicit convective term

$$u_t = \nabla_x\Big(A^s(x)\nabla_x u\Big) + B(x)\cdot\nabla_x u + f(u) \ . \tag{2}$$

The vector field $B(x)$ has zero divergence and the mean value of $B(x)$ over $(0,1) \times \Omega$ vanishes because of the periodicity of $A(x)$. In applications this models a given underlying incompressible flow field. We assume that the nonlinearity $f(u)$ is C^1 and has zeros at 0 and 1. Three important classes of nonlinearities are given by

(i) KPP case:

$$f(u) > 0 \text{ in } (0,1), \quad \dot{f}(0) > 0 \ , \tag{3}$$

e.g. $f(u) = u(1-u)$.

(ii) Bistable case:

$$f(u) < 0 \quad \text{in } (0,\mu), \quad f(u) > 0 \text{ in } (\mu, 1) \ , \tag{4}$$

e.g. $f(u) = u(1-u)(u-\mu)$.

(iii) Combustion case

$$f(u) = 0 \text{ in } (0,\mu), \quad f(u) > 0 \text{ in } (\mu, 1) \ . \tag{5}$$

In this paper we restrict ourselves to the bistable and combustion cases, since we will assume the existence of a unique stable wave. For KPP nonlinearities a continuum of waves exist and their stability is a subtle matter. Generalizations for the KPP case will be studied by the authors in a forthcoming paper.
The existence of a global solution of (1) for uniformly continuous initial data u_0 with values in the interval $[0,1]$ follows by standard arguments. This set of initial data will be denoted by I. Let $e_1 = (1, 0, \ldots, 0) \in I\!R^n$. Depending on the properties of $A(x)$ we distinguish two different definitions of a wave u^* with velocity $c \neq 0$ connecting the two zeros 0 and 1 of f.

Case (a): If A is independent of x_1, i.e. $A = A(\tilde{x})$, then a wave solution (u^*, c) of equation (1) in direction e_1 can be written in the form

$$u^*(t,x) = U(x_1 + ct, \tilde{x}) \tag{6}$$

with

$$U(-\infty, \tilde{x}) = 0, \quad U(\infty, \tilde{x}) = 1 \ .$$

In the moving frame $\xi = (\xi_1, \tilde{\xi}) = x + cte_1$, the function $U(\xi)$ satisfies the elliptic equation

$$\nabla_\xi \Big(A(\tilde{\xi}) \nabla_\xi U \Big) - cU_{\xi_1} + f(U) = 0 \ . \tag{7}$$

Case (b): If $A(x)$ is $1-$*periodic* in x_1 then equation (1) is invariant only for discrete shifts. A wave with velocity $c \neq 0$ in direction e_1 should satisfy

$$u^*(t + \frac{1}{c}, x - e_1) = u^*(t,x) \quad \text{for all } t \in I\!R, \quad x \in D \ , \tag{8}$$

$$u^*(-\infty, x) = 0, \quad u^*(\infty, x) = 1 \ .$$

Equivalently we have

$$u^*(t, x) = U(x_1 + ct, x) \ , \tag{9}$$

and U satisfies in the moving coordinate $\xi = x_1 + ct$

$$(\nabla_x + e_1\partial_\xi)\Big(A(x)(\nabla_x + e_1\partial_\xi)U\Big) - c\partial_\xi U + f(U) = 0 \ , \tag{10}$$

$$U(\xi, x + e_1) = U(\xi, x), \tag{11}$$

$$U(-\infty, x) = 0, \quad U(\infty, x) = 1 \ .$$

This equation is still parabolic, but due to condition (11) should be regarded as degenerate elliptic in $I\!R^{n+1}$. Waves with velocity zero are just time independent solutions of (1) connecting 0 and 1.

The variational principle which will be introduced in the next chapter can be applied to the velocity of reactive fronts in cases where a unique stable wave exists. There is a vast amount of literature available on the existence and stability of waves for the different nonlinearities f, compare the survey paper [26] for an excellent reference. Here we will review only those results which will be used in Chapter 4.
In case (a) the shear flow problem has been studied most. Existence of a unique monotone wave for problem (1) with a bistable nonlinearity and a convex cross section Ω has been shown in [6], and for the combustion nonlinearity in [5]. In [21] the global stability of these waves is proven, which is the multidimensional extension of the classical one-dimensional result of Fife and McLeod [11]. So this class of problems will satisfy our assumptions A1 and A2 given in Chapter 3.
For case (b) Xin has shown existence of a wave for combustion nonlinearities in [24], and for bistable nonlinearities where the matrix $A(x)$ is close to a constant in [23]. In the later case he could prove local stability in one space dimension. It has also been shown that waves do not exist in the bistable case if the oscillations of the matrix $A(x)$ are too large, compare [25]. In [16] existence of traveling waves for systems in a periodically perforated domain close to the homogenization limit is proven. This method also covers the case of rapidly oscillating coefficients in equation (1). To our knowledge a global stability result for case (b) has not yet been proven.
The existence proofs contain only very little information about the quantitative dependence of the speed on parameters which are characteristic for the respective problem. Therefore explicit formulas of variational type for the speed are of great importance. Variational principles for the asymptotic speed of propagation of compactly supported initial data for KPP nonlinearities have first been derived by Gärtner and Freidlin [12]. This coincides with the minimal speed of the traveling waves, if they exist. These and related results for KPP type problems are the only higher dimensional variational principles in inhomogeneous media we are aware of. In one space dimension variational principles for the minimal wave speed for this case have first been derived by Hadeler and Rothe [13].

There are a few variational principles that minimize integral-functionals. Only a limited class of problems appear as the Euler-Lagrange equations of suitable functionals, compare [4] in the 1-dimensional case and [15] in the multidimensional case. Especially for variable coefficients the equations governing traveling waves are not of variational type. Nevertheless, we will provide an integral min/max principle for the wave speed in Section 3.2 which is based on the weak maximum principle.

3 Min-Max Principles for the Wave Speed

3.1 Strong Formulation

Roughly speaking, the variational formula will define the actual speed of the wave as the speed of the fastest sub-solution and as the speed of the slowest super-solution in the domain of attraction of the wave. For the variational formulation we need the following two assumptions.

A1: Existence
Assume the existence of a unique wave solution (u^*, c) of (1) in the sense defined above. We remark that we do not impose a sign restriction on the velocity. Even standing waves are allowed.

A2: Stability
The wave is stable w.r.t. some subset $I_S \subset I$ of initial data. More precisely, in case (a) we assume that for the solution of (1) with $u_0 \in I_S$

$$\lim_{t\to\infty} \sup_{x\in D} |u(t,x) - U(x_1 + ct + \theta, \tilde{x})| = 0 \tag{12}$$

holds for some shift θ.
In case (b) we require

$$\lim_{t\to\infty} \sup_{x\in D} |u(t,x) - U(x_1 + ct + \theta, x)| = 0 \tag{13}$$

for some shift $\theta \in I\!R$.

The subset I_S depends on the problem and will be specified in the applications in Chapter 4. Assumptions A1, A2 do not hold for KPP nonlinearities, since a continuum of waves exist. But for bistable nonlinearities they are known to be true, at least in some nontrivial cases.

With this stability assumption the formula for the wave speed given below also provides the asymptotic speed for every initial value in I_S.

We remark that in all cases studied in this paper the stability assumption automatically implies the monotonicity of the wave, see [19]. So we additionally have for $c \neq 0$

$$c\partial_t u^*(t,x) > 0 \ .$$

Now we first concentrate on case (a). We essentially follow the line of arguments used by Vol'pert et al. [22] for traveling waves of monotone systems of ODE's. Case (b) is not treated by them and requires some modifications. Below, this will be discussed separately.

Before we can state the variational characterization of the wave velocity c we define the sets of admissible comparison functions:

$$K = \{v \in C^2(D) \mid \partial_{x_1} v(x) > 0,\ 0 < v(x) < 1,\ v \in I_S\} \ . \tag{14}$$

Of course v also should satisfy the corresponding boundary conditions on ∂D. Define the following function for $v \in K$:

$$\psi(v(x)) = \frac{\nabla_x (A(\tilde{x}) \nabla_x v(x)) + f(v(x))}{\partial_{x_1} v(x)} \ . \tag{15}$$

We then have the following characterization of the wave speed

Theorem 1 *Let A be independent of x_1 and assume that A1 and A2 hold. Then the wave velocity c is given by*

$$\sup_{v \in K} \inf_{x \in D} \psi(v(x)) = c = \inf_{v \in K} \sup_{x \in D} \psi(v(x)) \ . \tag{16}$$

Proof:
The wave (u^*, c) obviously satisfies $\psi(u^*) = c$. Therefore it suffices to show, that for all $v \in K$

$$\inf_{x \in D} \psi(v(x)) \leq c \leq \sup_{x \in D} \psi(v(x))$$

holds. Suppose the second inequality is false. The other case is treated similarly. Then there exist $v \in K$, $c_0 < c$, s.t. for all $x \in D$

$$\psi(v)(x) < c_0$$

or

$$c_0 \partial_{x_1} v > \nabla_x \Big(A(\tilde{x}) \nabla_x v \Big) + f(v)$$

holds.
The function $\tilde{v}(t,x) = v(x + c_0 t e_1)$ satisfies

$$\partial_t \tilde{v} > \nabla_x \Big(A(\tilde{x}) \nabla_x \tilde{v} \Big) + f(\tilde{v}) \ ,$$

$$\partial_t \tilde{v} = c_0 \partial_{x_1} \tilde{v}, \quad \partial_{x_1} \tilde{v} > 0 \ .$$

Now let $w(t,x)$ be the solution of

$$\partial_t w = \nabla_x \Big(A(\tilde{x}) \nabla_x w \Big) + f(w) \ ,$$

$$w(0,x) = \tilde{v}(0,x) = v(x) \ .$$

Then the parabolic maximum principle implies

$$w(t,x) \le \tilde{v}(t,x) \ ,$$

and especially

$$w(t, x - c_0 t e_1) \le \tilde{v}(t, x - c_0 t e_1) = v(x) < 1 \ , \tag{17}$$

for all $t > 0$. However, we have assumed stability of the wave, i.e.

$$\lim_{t \to \infty} \sup_{x \in D} |w(t,x) - U(x_1 + ct + \theta, \tilde{x})| = 0 \ .$$

So in particular we have

$$\Big| w(t, x - c_0 t e_1) - U(x_1 + (c - c_0) t e_1, \tilde{x}) \Big| \to 0 \text{ as } t \to \infty \ . \tag{18}$$

Since $c > c_0$ and $U(\infty, \tilde{x}) = 1$ we arrive at

$$w(t, x - c_0 t e_1) \to 1 \text{ as } t \to \infty \ ,$$

which contradicts (17) and completes the proof.

Now we state the corresponding theorem for case (b). We recover the analog of Theorem 1 for nonzero speeds. But we obtain upper and lower bounds in any case.
Here the comparison functions have to be time-dependent and actually be defined also for negative times. This seems reasonable because the traveling waves have this property. Let

$$K = \{ v \in C^1(I\!R, C^2(D)) \mid 0 < v(t,x) < 1, \ v(0,\cdot) \in I_S,$$

$$\partial_t v(t,x) > 0, \ v(t+1, x - e_1) = v(t,x) \} \ . \tag{19}$$

Again v also satisfies appropriate boundary conditions on ∂D. Define the function

$$\psi(v(t,x)) = \frac{\nabla_x (A(x) \nabla_x v(t,x)) + f(v(t,x))}{\partial_t v(t,x)} \ , \tag{20}$$

for $v \in K$. Note that the occurrence of the time derivative prevents us from using standing waves as comparison functions in (20). We have

Theorem 2 *Let $A(x)$ be $1-$periodic in x_1 and assume conditions A1 and A2 to hold. Then we have the following estimates for the velocity.*

$$\sup_{v\in K}\inf_{(t,x)\in I\!R\times D}\psi(v)(t,x)\ \le\ c\ \le\ \inf_{v\in K}\sup_{(t,x)\in I\!R\times D}\psi(v)(t,x)\ . \tag{21}$$

For nonzero speeds c, equality holds in (21).

Proof:
Here we modify the arguments of the previous case. We only prove the second inequality. Suppose that it is false. Then there exists $v\in K$ and $0\neq c_0<c$, s.t. for all t, $x\in I\!R^n$

$$\psi(v(t,x))<c_0$$

holds. Introduce $\tilde v(t,x)=v(c_0t,x)$. Then we get

$$\partial_t\tilde v>\nabla_x\Big(A(x)\nabla_x\tilde v\Big)+f(\tilde v)\ , \tag{22}$$

$$\tilde v(0,x)=v(0,x)\ ,$$

$$\tilde v(t+1/c_0,x-e_1)=\tilde v(t,x)=\tilde v(t-1/c_0,x+e_1) \tag{23}$$

or

$$\tilde v(t+1/|c_0|,x-sign(c_0)e_1)=\tilde v(t,x)\ .$$

Let w be the solution of the initial value problem

$$\partial_t w=\nabla_x\Big(A(x)\nabla_x w\Big)+f(w)\ , \tag{24}$$

$$w(0,x)=\tilde v(0,x)\ .$$

The maximum principle implies for all $t\ge 0$

$$w(t,x)\le\tilde v(t,x)\ . \tag{25}$$

Iterating (23) we conclude from (25)

$$w(m/|c_0|,x-sign(c_0)me_1)\le\tilde v(m/|c_0|,x-sign(c_0)me_1)=\tilde v(0,x)=v(0,x)<1 \tag{26}$$

for all $m\in\mathbf{N}$. On the other hand, since the wave U is stable, i.e.

$$\lim_{\tau\to\infty}\sup_{x\in D}\Big|w(t,x)-U(x_1+ct+\theta,x)\Big|=0\ ,$$

and because of the periodicity of U we obtain

$$\lim_{m\to\infty}\Big|w(m/|c_0|,x-sign(c_0)me_1)-U(x_1+m\frac{c-c_0}{|c_0|}+\theta,x)\Big|=0\ .$$

From $c > c_0$, we conclude

$$U(x_1 + m\frac{c - c_0}{|c_0|} + \theta, x) \to 1 \ ,$$

since $U(t, \cdot)$ tends to 1 uniformly. Hence

$$w(m/|c_0|, x - sign(c_0)me_1) \to 1 \text{ as } m \to \infty \ .$$

This contradicts (26).

For the equalities in (21) with $c \neq 0$ let $\tilde{u}(t,x) = u^*(t/c, x)$ be the rescaled waves. Obviously $\psi(\tilde{u}(t,x)) = c$ holds identically. For $c = 0$ this is not possible, since $\partial_t u^*$ vanishes.
This completes the proof.

Further results:
Since before we have only used the maximum principle, we obtain the following extensions of our results. The proofs are similar to the ones given for Theorems 1 and 2.

E1: Theorem 2 can be extended in a straightforward way to periodically perforated domains. Here one uses a slight modification of the set K in (19) whereas the function ψ in (20) remains the same.

E2: Another extension to domains with variable cross section is possible, i.e.

$$\Omega = \Omega(x_1) \quad \text{ is 1-periodic .}$$

E3: We can also consider monotone systems in higher dimensions in the same way as Vol'pert et al. did in one dimension. These are systems of equations as in (1) where the Jacobian of the nonlinearity satisfies

$$\frac{\partial F_i(u)}{\partial u_j} \geq 0, \quad for\ i \neq j,\ i,j = 1, ..., m \ . \tag{27}$$

Analogous theorems hold for this case, if one defines a function ψ for each component and takes the inf/sup over all those functions.

E4: The discretized diffusion equation with a bistable nonlinearity will be discussed in Section 4.3 . Also in this case we can 'translate' Theorem 1.

3.2 Weak form of the variational principle

Now we want to employ the weak form of the maximum principle. Hence we need less regularity of the coefficients:

$$A \in L^\infty(D) \; .$$

Also for $\partial\Omega$ we can drop the smoothness assumption. The definitions of a wave (7) and (10) are then meant to hold almost everywhere. In both cases (a) and (b), the definitions of the sets K are the same as before, but one has to interpret all the equalities and inequalities in a weak sense for functions v in $H^1_{loc}(D))$ and in $L^2_{loc}(I\!R; H^1_{loc}(D))$ respectively. For the space of testfunctions we choose for case (a)

$$L = \{\phi \in C_0^\infty(D) | \phi \geq 0, \phi \neq 0\} \; ,$$

and for case (b)

$$L = \{\phi \in C_0^\infty(I\!R \times D) | \phi \geq 0, \phi \neq 0\} \; ,$$

but less regularity is sufficient. The weak form of the functions ψ are the functionals

$$\psi(v,\phi) = \frac{\int\limits_D -\nabla_x \phi A(x) \nabla_x v + \phi f(v) \; dx}{-\int\limits_D \partial_{x_1}\phi(x) v(x) \; dx} \; , \tag{28}$$

and

$$\psi(v,\phi) = \frac{\int\limits_{I\!R \times D} -\nabla_x \phi A(x) \nabla_x v + \phi f(v) \; dx \; dt}{-\int\limits_D \partial_t \phi v \; dx \; dt + \int\limits_D \phi(0,x) v(0,x) \; dx} \; . \tag{29}$$

The proof of the following theorem is almost identical to those of Theorems 1 and 2. One checks easily, that the rescaled wave fulfills the equality and applies the weak maximum principle instead of its strong form. The contradiction is then obtained in the same way as before.

Theorem 3 *Assume case (a), or case (b) with $c \neq 0$. Let the assumptions A1, A2 hold. Then the wave velocity is given by the variational formula*

$$\sup_{v \in K} \inf_{\phi \in L} \psi(v,\phi) = c = \inf_{v \in K} \sup_{\phi \in L} \psi(v,\phi) \; . \tag{30}$$

If $c = 0$ in case (b), only the inequalities hold.

This provides a variational principle in integral form. In the applications the same bounds for c result as compared with the strong form, but under less smoothness assumptions.

4 Applications

For all examples given below, we construct suitable comparison functions in order to apply Theorems 1 and 2. These comparison functions are based on modified one-dimensional wave profiles.

4.1 Shear flows in cylinders

In this section we analyze a model for shear flows in cylinders with combustion or bistable nonlinearity. The main example we have in mind is the propagation of premixed flames. Here the drift vector only depends on the direction which is transversal to the wave and describes a prescribed incompressible fluid velocity with zero mean value. Due to the inhomogeneous drift one expects an enhancement of the speed. This results from the enlargement of the reaction zone when compared to the planar front in case no advection is present. We will prove that a small amplitude drift coefficient with zero mean increases the magnitude of the front speed. The first order deviation of the speed in terms of the drift coefficient only depends on the drift and not on the particular form of the reaction term. Explicit speed estimates are given, which are optimal for the small amplitude limit and the rapid oscillation limit.
Our model is the following special form of (2). The matrix $A^s(x)$ is the identity matrix and the vector $B(x)$ is given by $B(x) = (\delta b(\tilde{x}), 0..., 0)$ where $b(\tilde{x})$ has mean value zero.

$$u_t = \Delta u + \delta b(\tilde{x}) \partial_{x_1} u + f(u) \ . \tag{31}$$

This corresponds to case (a). We included a parameter δ, since we want to study the asymptotics of the speed for small and large δ.

As mentioned earlier, assumptions A1 and A2 are fulfilled for the bistable and combustion nonlinearity and a convex cross section Ω.

In the bistable case (i) let μ be the middle zero of f.
In the combustion case (iii) let $\mu = \sup\{u < 1; f(u) = 0\}$.

In [21] the class I_S of initial data converging to the wave is described by the set of uniformly continuous functions $v : I\!R \times \Omega \to (0, 1)$ which satisfy

in the bistable case (i):

$$\limsup_{x_1 \to -\infty} v(x_1, \tilde{x}) < \mu, \qquad \liminf_{x_1 \to \infty} v(x_1, \tilde{x}) > \mu, \tag{32}$$

and in the combustion case (iii)

$$\liminf_{x_1 \to \infty} v(x_1, \tilde{x}) > \mu \ , \tag{33}$$

and $v(x_1, \tilde{x})$ decays exponentially for $x_1 \to -\infty$.

For this class of problems Theorem 1 can be applied. The unique wave profile U for $b = 0$ satisfies the ODE

$$U'' - c_0 U' + f(U) = 0 \ , \qquad (34)$$
$$U(-\infty) = 0, \quad U(\infty) = 1.$$

The linearization of (31) at 0 and 1 shows that U approaches the rest states exponentially. Furthermore the derivatives of U tend to zero with same exponential rate. Hence every derivative of U is a bounded multiple of $U' > 0$. This will become crucial below. We first take $v(x) = U(x_1)$ as a comparison function. Applying Theorem 1 with

$$\psi(v(x)) = \frac{U'' + \delta b U' + f(U)}{U'} = c_0 + \delta b \ ,$$

we get the trivial bounds

$$c_0 - \delta |b^-|_\infty \leq c \leq c_0 + \delta |b^+|_\infty \ ,$$

where $b^\pm$ are the positive and negative part of b. This shows already, that the speed grows at most linearly in δ. Next we derive an improved estimate which has the correct asymptotics for small and large δ. For $a > 0$ we choose the comparison function

$$v(x) = U(a(x_1 + \delta\chi(\tilde{x})) \ ,$$

where $\chi(\tilde{x})$ solves

$$\Delta_{\tilde{x}} \chi(\tilde{x}) = -b(\tilde{x}) \ , \qquad (35)$$

with Neumann boundary conditions and zero mean value. This choice is motivated by the first order formal expansion of the wave as δ tends to zero. We calculate

$$\begin{aligned} \psi(v(x)) &= \frac{a^2 U''(1 + \delta^2 |\nabla_{\tilde{x}}\chi|^2) + a\delta U'(\Delta_{\tilde{x}}\chi + b) + f(U)}{aU'} \\ &= \frac{U''}{aU'}\Big(a^2(1 + \delta^2|\nabla_{\tilde{x}}\chi|^2) - 1\Big) + c_0/a \ . \end{aligned}$$

We choose

$$a = (1 + \delta^2 \overline{|\nabla_{\tilde{x}}\chi|^2})^{-1/2} \ ,$$

where the bar denotes the average over Ω and get

$$\psi(v(x)) = \frac{U''}{U'} \delta^2 a (|\nabla_{\tilde{x}}\chi|^2 - \overline{|\nabla_{\tilde{x}}\chi|^2}) + c_0/a \ .$$

Using the boundedness of U''/U' and applying Theorem 1 we obtain

$$|c_\delta - c_0/a| \leq M\delta^2 a \Big| \ |\nabla_{\tilde{x}}\chi|^2 - \overline{|\nabla_{\tilde{x}}\chi|^2} \ \Big|_{L^\infty(\Omega)} \ .$$

So we can formulate

Theorem 4 *The wave velocity c_δ for the shear flow problem is estimated by*

$$c_0/a - Ma\delta^2 \left| \, |\nabla_{\tilde{x}}\chi|^2 - \overline{|\nabla_{\tilde{x}}\chi|^2} \right|_{L^\infty(\Omega)} \leq c_\delta \leq c_0/a + Ma\delta^2 \left| \, |\nabla_{\tilde{x}}\chi|^2 - \overline{|\nabla_{\tilde{x}}\chi|^2} \right|_{L^\infty(\Omega)} \quad (36)$$

with

$$a = (1 + \delta^2 \overline{|\nabla_{\tilde{x}}\chi|^2})^{-1/2}$$

and χ solves (35). The constant M only depends on the nonlinearity.

This estimate has the asymptotic form

$$\begin{aligned} |c_\delta - c_0| &\leq M\delta^2 \quad \text{for } \delta \to 0 \ , \\ \text{and } |c_\delta| &\leq M\delta \quad \text{for } \delta \to \infty \ . \end{aligned}$$

Similar bounds for the asymptotic speed of propagation for KPP nonlinearities have been obtained in [8].

We give an explicit example for $\Omega = (0, 2\pi)$ and $b(\tilde{x}) = cos(n\tilde{x})$. For χ we get

$$\chi(\tilde{x}) = \frac{1}{n^2} cos(n\tilde{x}) \text{ and } \overline{\chi'^2} = \frac{1}{2n^2} \ .$$

Hence we obtain

$$c_0/a - Ma\frac{\delta^2}{2n^2} \leq c_{\delta,n} \leq c_0/a + Ma\frac{\delta^2}{2n^2} \quad (37)$$

with

$$a = (1 + \frac{\delta^2}{2n^2})^{-1/2}.$$

This estimate is optimal for

$$\delta/n \to 0 \ ,$$

which implies $c_{\delta,n} \to c_0$.
So the drift has no influence on the speed, although its sup-norm might tend to infinity. The limit $\delta/n \to \kappa \neq 0$ corresponds to homogenization and (37) still gives reasonable bounds. The exact asymptotics in this case will be studied in Section 4.2.

Now we want to justify the expansion for $\delta \to 0$ up to second order. In [20] this expansion was formally derived if f is an exact cubic nonlinearity. There it was found, that c_δ is larger than c_0 for small δ. Below we rigorously derive this expansion for an arbitrary bistable or combustion nonlinearity by choosing v essentially to be the second order formal expansion. The expansion is most easily performed in coordinates, which already incorporate the first order term. Define

$$\xi_1 \;=\; x_1 + \delta\chi(\tilde{x}) \ , \quad (38)$$

where χ solves (35). In the coordinates $\xi_1, \tilde{x}$ equation (31) transforms to

$$u_t = (1 + \delta^2 |\nabla_{\tilde{x}}\chi|^2)\partial_{\xi_1\xi_1} u + 2\delta\nabla_{\tilde{x}}\chi\nabla_{\tilde{x}}\partial_{\xi_1} u + \Delta_{\tilde{x}} u + f(u) \ . \quad (39)$$

As a comparison function in Theorem 1 we choose functions of the form

$$v(t,x) = U(t+\xi_1) + \delta^2 \tilde{u}(t+\xi_1, \tilde{x}) \tag{40}$$

with U defined as above. The function $\tilde{u}$ and its derivatives will be constructed as bounded multiples of $U' > 0$ and will be specified later. So every higher order term which appears below is a bounded multiple of U'. For small δ we also have

$$\partial_t v = U' + \delta^2 \partial_{\xi_1} \tilde{u} > 0 \ ,$$

and

$$\frac{1}{\partial_t v} = \frac{1}{U'}\Big(1 - \delta^2 \frac{\partial_{\xi_1}\tilde{u}}{U'} + O(\delta^4)\Big) \ .$$

In particular v is an admissible comparison function in Theorem 1. We calculate

$$\begin{aligned} \Delta_x v + \delta b \partial_{x_1} v + f(v) &= U'' + f(U) + \delta^2 \Big(|\nabla_{\tilde{x}}\chi|^2 U'' + \partial_{\xi_1\xi_1}\tilde{u} + \Delta_{\tilde{x}}\tilde{u} + \dot{f}(U)\tilde{u}\Big) \\ &\quad + U' O(\delta^3) \\ &= c_0 U' + \delta^2 \Big(|\nabla_{\tilde{x}}\chi|^2 U'' + \partial_{\xi_1\xi_1}\tilde{u} + \Delta_{\tilde{x}}\tilde{u} + \dot{f}(U)\tilde{u}\Big) + U'O(\delta^3) \ , \end{aligned}$$

where for the last equality (34) is used. For $\psi(v)$ we now obtain the expansion up to order δ^2

$$\psi(v) = c_0 + \frac{1}{U'}\delta^2 \Big(\Delta_{\tilde{x}}\tilde{u} + \partial_{\xi_1\xi_1}\tilde{u} - c_0 \partial_{\xi_1}\tilde{u} + \dot{f}(U)\tilde{u} + |\nabla_{\tilde{x}}\chi|^2 U''\Big) + O(\delta^3) \ . \tag{41}$$

We want the term in brackets to be a multiple of U', i.e.

$$\partial_{\xi_1\xi_1}\tilde{u} + \Delta_{\tilde{x}}\tilde{u} - c_0\partial_{\xi_1}\tilde{u} + \dot{f}(U)\tilde{u} = -|\nabla_{\tilde{x}}\chi|^2 U'' + \gamma U' \ .$$

This problem can be solved by an expansion into eigenfunction $\phi_j(\tilde{x})$ of $-\Delta_{\tilde{x}}$ with eigenvalues λ_j. Let

$$\begin{aligned} \tilde{u}(\xi,\tilde{x}) &= \sum_j u_j(\xi)\phi_j(\tilde{x}) \ , \\ |\nabla_{\tilde{x}}\chi|^2 &= \sum_j a_j \phi_j(\tilde{x}) \ . \end{aligned}$$

Due to the Neumann boundary conditions, and since $\phi_1 = 1$, $\lambda_1 = 0$, and

$$a_1 = \frac{1}{|\Omega|}\int_\Omega |\nabla_{\tilde{x}}\chi(\tilde{x})|^2 \, d\tilde{x} \ ,$$

we get

$$L_j u_j := u_j'' - \lambda_j u_j - c_0 u_j' + \dot{f}(U) u_j = -a_j U'' + \delta_{1,j}\gamma U' \ , \tag{42}$$

where $\delta_{1,j}$ denotes the Kronecker symbol. The operator L_1 is the linearization of (34) at U and has the simple eigenvalue zero with eigenfunction U'. A calculation shows, that the kernel of the adjoint of L_1 is spanned by $U^* = U' e^{-c_0 \xi}$. So the solvability condition for (42) with $j = 1$ gives

$$a_1 \int_{I\!R} U'' U^* \, d\xi = \gamma \int_{I\!R} U' U^* \, d\xi \ .$$

After partial integration we see that

$$\gamma = c_0 a_1 / 2$$

holds. The equations for u_j, $j > 1$ are uniquely solvable since $\lambda_j > 0$ is in the resolvent set of L_1. Furthermore the decay rate at $\pm\infty$ of $\tilde{u}$ and its derivatives is the same as that of the right hand side in (42), i.e. that of U'. Coming back to $\psi(v)$ we see that

$$\psi(v(t,x)) = c_0 + \delta^2 \gamma + O(\delta^3) \ .$$

So the variational principle implies

Theorem 5 *Consider the shear flow problem (31) with bistable or combustion nonlinearity. Let $\chi(\tilde{x})$ be the solution of (35). Then the velocity has the following second order expansion:*

$$c_\delta = c_0 + \frac{c_0 \delta^2}{2|\Omega|} \int_\Omega |\nabla_{\tilde{x}} \chi(\tilde{x})|^2 \, d\tilde{x} + O(\delta^3) \ . \tag{43}$$

In particular, if c_0 is positive, then c_δ is always larger than c_0 for small δ.

This proves that at least for small δ the drift always accelerates the wave. Up to order two the deviation of the speed only depends on the drift and not on the particular nonlinearity. Because of definition (35), the integral in (43) is equal to the square of the H^{-1} norm of b.

4.2 Homogenization Limit

Usually inhomogeneities in reactive flows appear on very different length scales. Here we consider rapidly oscillating diffusion and drift coefficients as they appear e.g. in models for composite materials. There exists a well established theory on how to pass to the effective homogenized equations as the period of the oscillations tends to zero. But a general theory on how to compare qualitative properties, as front propagation, in the original model with the homogenized equation is missing. This is of course important to know for applications. Using the variational principle we will justify the formal asymptotic expansions of the wave speed rigorously. Then we discuss three examples which cover layered media and two dimensional shear flows. In the examples for layered media acceleration or slowing down of the wave speed could not be decided upon the first order expansion of the wave speed. Whereas for the two dimensional

shear flow the first order coefficient is nonzero in general. Hence, in contrast to the perturbation from a zero drift, the wave speed can be smaller or larger as compared to the homogenized medium. This depends on the particular nonlinearity as well as on the drift term.
We consider

$$u_t = \nabla_x\Big(A(x/\epsilon)\nabla_x u\Big) + f(u), \quad x \in I\!R^n, \quad t > 0 \ , \tag{44}$$

where $A(z)$ is 1-periodic in all variables. So here the cross section Ω is a unit torus. Suppose that this equation admits a unique monotone and stable traveling wave u^ϵ, c^ϵ in direction e_1. Waves do exist in the bistable case if ϵ is sufficiently small [16]. But their stability has not yet been proven. Of course the wave now should satisfy

$$u^\epsilon(t + \epsilon/c^\epsilon, x - \epsilon e_1) = u(t, x) \ .$$

We want to derive an asymptotic expansion of c^ϵ. For this we will need the unique periodic solutions χ_i with zero mean value of the cell problems of order 1 up to order 3, which are

$$\begin{aligned}
\nabla_z(A(z)(\nabla_z\chi_1(z) + e_1)) &= 0 \ , && (45)\\
\nabla_z(A\nabla_z\chi_2) &= A_0 - \nabla_z(Ae_1\chi_1) - e_1A(\nabla_z\chi_1 + e_1) \ , && (46)\\
\nabla_z(A\nabla_z\chi_3) &= A_0\chi_1 + A_1 - \nabla_z(Ae_1\chi_2) - e_1(A\nabla_z\chi_2 + e_1\chi_1) && (47)
\end{aligned}$$

with

$$\begin{aligned}
A_0 &= \int_{(0,1)^n} e_1\ A(z)(\nabla_z\chi_1(z) + e_1)\ dz \ , && (48)\\
A_1 &= \int_{(0,1)^n} e_1A(z)(\nabla_z\chi_2(z) + e_1\chi_1(z))\ dz \ , && (49)
\end{aligned}$$

where A_0 is the homogenized diffusion coefficient. The right hand sides in (46), (47) have zero average. Hence the functions χ_i are periodic with zero mean value. We assume that the χ_i are bounded C^2 functions. The homogenized traveling wave equation in direction e_1 is the ODE

$$A_0u_0'' - c_0u_0' + f(u_0) = 0 \ . \tag{50}$$

We assume, that this has a solution with $c_0 \neq 0$. Using u_0 and χ_i we will construct a comparison function for the variational principle of Theorem 2. We remark, that our analysis is valid, if ϵ is small compared to c_0. For larger values of ϵ standing waves can occur, which block propagation. This has been shown in [17]. Let

$$\xi = x_1 + t, \quad z = x/\epsilon \ ,$$

where z is in the unit torus. Define the comparison function

$$v(t, x) = u_0(\xi) + \epsilon u_1(\xi, z) + \epsilon^2u_2(\xi, z) + \epsilon^3u_3(\xi, z) \ , \tag{51}$$

where u_0 solves (50) and

$$\begin{aligned} u_1(\xi,z) &= \chi_1(z)u_{0\xi}(\xi)+\phi(\xi)\ , \\ u_2(\xi,z) &= \chi_2(z)u_{0\xi\xi}(\xi)+\chi_1(z)\phi(\xi)_\xi\ , \\ u_3(\xi,z) &= \chi_3(z)u_0^{(3)}(\xi)+\chi_2(z)\phi(\xi)_{\xi\xi}\ . \end{aligned}$$

We have

$$v(t+\epsilon,x-\epsilon)=v(t,x)\ .$$

Near $\pm\infty$ all derivatives of u_0 have the same exponential decay rate as u_0'. Also ϕ, which will be specified below, has this decay rates. Hence $\partial_t v>0$ holds for small ϵ and v is admissible. We calculate

$$\begin{aligned} &\nabla_x(A\nabla_x v)+f(v)=(e_1\partial_\xi+\frac{1}{\epsilon}\nabla_z)(A(e_1\partial_\xi v+\frac{1}{\epsilon}\nabla_z v))+f(v) \qquad (52)\\ =\ &\frac{1}{\epsilon}\Big(\nabla_z(A\nabla_z u_1)+\nabla_z(Ae_1)u_{0\xi}\Big)\\ &+\nabla_z(A\nabla_z u_2)+\nabla_z(Ae_1u_{1\xi})+e_1A(\nabla_z u_{1\xi})+e_1Ae_1u_{0\xi\xi}+f(u_0)\\ &+\epsilon\Big(\nabla_z(A\nabla_z u_3)+\nabla_z(Ae_1u_{2\xi})+e_1A\nabla_z u_{2\xi}+e_1Ae_1u_{1\xi\xi}+\dot f(u_0)u_1\Big)+O(\epsilon^2)\\ =\ &c_0u_{0\xi}+\epsilon\Big((A_1+\chi_1A_0)u_0^{(3)}+A_0\phi_{\xi\xi}+\dot f(u_0)(\chi_1u_{0\xi}+\phi)\Big)+O(\epsilon^2)\\ =\ &c_0u_{0\xi}+\epsilon\Big(A_1u_0^{(3)}+c_0u_{1\xi}-c_0\phi_\xi+A_0\phi_{\xi\xi}+\dot f(u_0)\phi\Big)+O(\epsilon^2)\ , \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\partial_t v} &= \frac{1}{u_{0\xi}+\epsilon u_{1\xi}+\epsilon^2u_{2\xi}+\epsilon^3u_{3\xi}}\\ &= \frac{1}{u_{0\xi}}-\epsilon\frac{u_{1\xi}}{u_{0\xi}^2}+O(\epsilon^2)\ , \end{aligned}$$

where we have used the cell problems (46), (47) the definition of u_i and (50). Now we obtain

$$\psi(v)(t,x)=c_0+\frac{\epsilon}{u_{0\xi}}\Big(A_1u_0^{(3)}+A_0\phi_{\xi\xi}-c_0\phi_\xi+\dot f(u_0)\phi\Big)+O(\epsilon^2)\ .$$

The term in brackets should be a multiple of $u_{0\xi}$, i.e.

$$A_0\phi_{\xi\xi}-c_0\phi_\xi+\dot f(u_0)\phi=c_1u_{0\xi}-A_1u_0^{(3)}\ .$$

Then the solvability condition defines c_1:

$$c_1\int_{I\!R}u_{0\xi}^2e^{-\frac{c_0}{A_0}\xi}\,d\xi\ =\ A_1\int_{I\!R}u_{0\xi}u_0^{(3)}e^{-\frac{c_0}{A_0}\xi}\,d\xi\ . \qquad (53)$$

The variational principle results in

Theorem 6 *There exists $M > 0$, depending only on f and A, s.t*

$$|c^\epsilon - c_0 - \epsilon c_1| \leq M\epsilon^2 \ . \tag{54}$$

This justifies the homogenization limit for stable waves. Higher order expansions are possible, but become very messy.

Examples
Now we calculate the first order coefficient c_1 for some specific examples.

1. Layered media
1.a. Consider the spatially inhomogeneous one-dimensional problem

$$\partial_t u = (A(\frac{x}{\epsilon})u_x)_x + f(u) \ .$$

In this case A_0 is the harmonic mean of A and

$$A(z)(\chi_1'(z) + 1) = A_0$$

holds. For χ_2 we get from (46)

$$\chi_2'(z) = -\chi_1(z) \ ,$$

and from (49) $A_1 = 0$ and $c_1 = 0$ follows.

1.b. Transversal layering
Now let A be a diagonal matrix independent of x_1, i.e. $A = a(\frac{\tilde{x}}{\epsilon})I$. Then by uniqueness $\chi_1 = 0$, and A_0 is the mean value of a. Since all χ_i do not depend on z_1 we again obtain from (49) that $A_1 = 0$ and $c_1 = 0$.

In these two cases Theorem 6 gives no information whether the wave is accelerated or slowed down by the inhomogeneous medium. Higher order expansions are needed. In principle these can be calculated but the analysis becomes extremely messy.

2. Two-dimensional shear flows
For this problem the existence of a unique globally stable wave is known. Here the first order coefficient does play a role. Let

$$A(z) = \begin{pmatrix} 1 & -\beta(z_2) \\ \beta(z_2) & 1 \end{pmatrix} \ ,$$

where β has zero average. This corresponds to a drift vector $(\frac{1}{\epsilon}\beta'(z_2/\epsilon), 0)$. All χ_i only depend on z_2. From (45) we get for χ_1

$$\chi_1'' + \beta' = 0 \quad , \quad \chi_1' = -\beta \ ,$$

and with these expressions we get from (49)

$$A_0 = 1 + \int_0^1 \chi_1'(z_2)^2 \, dz_2 \; .$$

Now (46) reads

$$\chi_2'' = A_0 - 1 - \beta' \chi_1 \; .$$

Finally by partial integration and using (49) we get

$$\begin{aligned} A_1 &= \int_0^1 (\beta(z_2)\chi_2'(z_2) + \chi_1(z_2)) \, dz_2 - \int_0^1 \chi_1(z_2)'' \chi_2'(z_2) \, dz_2 \\ &= \int_0^1 \chi_1(z_2)' \chi_2''(z_2) \, dz_2 = \int_0^1 \chi_1(z_2) \chi_1'(z_2)^2 \, dz_2 \; , \end{aligned}$$

which is in general nonzero and has no definite sign. So unlike the perturbation from a zero drift, waves near the homogenization limit can be slower or faster compared to the homogenized wave. Actually, by formula (53) the sign of c_1 depends on the drift and on the nonlinearity because of the influence of A_1 and of u_0.

4.3 Discretized Diffusion

Propagation of action potentials in systems of nerve cells is a fundamental property. It is of great interest, e.g. in neurophysiology or cardiophysiology, to understand the phenomena in detail. Different models for nerve cells exist, one of them is the Fitzhugh-Nagumo system, where an equation for the electric potential and one for a recovery variable is given. Coupled cells can be modeled by the related discrete system. When ignoring the recovery variable one ends up with the discrete Nagumo equation

$$\frac{d}{dt} u_n(t) = \frac{a}{h^2} \Big(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t) \Big) + f(u_n(t)) \; , \tag{55}$$

for $n \in \mathbf{Z}$ and $t \geq 0$. Here u_n describe the transmembrane potential of the nth cell, $\frac{a}{h^2}$ is the coupling coefficient or the inverse of the intercellular resistance and f is the bistable nonlinearity. In this section we give an asymptotic expansion for the wave speed in the large coupling strength limit. So we deal with a nerve system which can still transmit information.

It is well known that for weak coupling of the cells wave pinning or quenching can occur and information is no longer transmitted. This is due to the existence of standing waves, which prevent propagation. The situation was analyzed by e.g. [18] for equation (55) and [1] for the full Fitzhugh-Nagumo system. In [3] two reaction diffusion systems with jump discontinuities in the diffusion coefficient are discussed which model propagation of action potentials along nonuniform excitable fibers. Here wave pinning takes place with a thickening of the fiber. Wave pinning for a discrete bistable equation with a piecewise linear reaction term was analyzed in [10]. Pinning in more general discrete systems and for two dimensions has been studied e.g. in [7], [9].

Here we are interested in an asymptotic expansion which explains how the wave speed is slowed down in dependence on the coupling strength of the cells. In order to have a nonzero speed we assume that

$$\int_0^1 f(u)\,du \neq 0\ .$$

A traveling wave solution of (55) satisfies

$$u_{n-1}(t + c/h) = u_n(t)\ ,$$

or equivalently

$$u_n(t) = U(nh + ct)\ ,$$

where $U(x)$ fulfills

$$cU'(x) = \frac{a}{h^2}\Big(U(x+h) - 2U(x) + U(x-h)\Big) + f(U(x))\ ,$$

$$U(-\infty) = 0, \quad U(\infty) = 1\ .$$

For small $h > 0$ unique existence and global stability of a wave with $c > 0$ has been shown in [27], [28]. Like its continuous version equation (55) satisfies the maximum principle. So Theorem 1 can be adapted to cover also this case. Let $v(t)$ denote a sequence of C^1 functions $v_n(t)$. Then the set K of comparison functions is defined by all $v \in K$ which fulfill

$$0 < v_{n-1}(t+h) = v_n(t) < 1, \quad v_n'(t) > 0$$

for all n and

$$\limsup_{n\to-\infty} v_n(0) < \mu < \liminf_{n\to\infty} v_n(0)\ .$$

The corresponding function ψ is

$$\psi(v)(t,n)) = \frac{\frac{a}{h^2}\Big(v_{n+1}(t) - 2v_n(t) + v_{n-1}(t)\Big) + f(v_n(t))}{v_n'(t)}\ .$$

Then the analog of Theorem 1 can be formulated as follows:

Theorem 7 *For small $h > 0$ the unique wave velocity c^h is given by*

$$\sup_{v\in K}\inf_{n,t} \psi(v)(n,t) = c^h = \inf_{v\in K}\sup_{n,t} \psi(v)(n,t)\ .$$

Again we will use this characterization for the justification of the limit $h \to 0$. Let u_0 be the solution of

$$au_0'' - c_0u_0' + f(u_0) = 0\ , \qquad (56)$$
$$u_0(-\infty) = 0, u_0(\infty) = 1\ .$$

As a comparison function we take

$$v_n(t) = u_0(nh+t) + h^2 w(nh+t) \ ,$$

where w will be determined later. For simplicity let $a=1$. Again w and its derivatives are constructed as bounded multiples of u_0'. We have

$$v_n'(t) = u_0(nh+ht) + h^2 w'(nh+t) > 0$$

for small h. By Taylor expansion with $x = nh+t$ we get

$$\begin{aligned}
&\frac{1}{h^2}\Big(u_0(x+h) - 2u_0(x) + u_0(x-h)\Big) + w(x+h) - 2w(x) + w(x-h) \\
&\quad + f(u_0(x) + h^2 w(x)) \\
&= u_0''(x) + \frac{h^2}{12}u_0^{(4)}(x) + h^2 w''(x) + f(u_0) + h^2 \dot f(u_0(x))w(x) + O(h^4)u_0'(x) \\
&= c_0 u_0'(x) + \frac{h^2}{12}u_0^{(4)}(x) + h^2 w''(x) + h^2 \dot f(u_0)w(x) + O(h^4)u_0'(x) \ .
\end{aligned}$$

Now let w be the solution of

$$w'' - c_0 w' + \dot f(u_0) w = c_1 u_0' - \frac{1}{12}u_0^{(4)} \tag{57}$$

with $w(\pm\infty) = 0$.
The solvability condition for (57) determines c_1

$$c_1 \int\limits_{I\!R} u_0'^2(x) e^{-c_0 x}\, dx = \frac{1}{12}\int\limits_{I\!R} u_0^{(4)}(x) u_0'(x) e^{-c_0 x}\, dx \ .$$

Hence we get

$$\begin{aligned}
\psi(v)(t,n) &= \frac{c_0(u_0'(x) + h^2 w(x)) + c_1 h^2 u_0'(x) + O(h^4)u_0'(x)}{u_0'(x) + h^2 w'(x)} \\
&= c_0 + c_1 h^2 + O(h^4) \ .
\end{aligned}$$

And finally

$$|c^h - c_0 - h^2 c_1| \le M h^4 \ .$$

After some manipulations we arrive at the following expression for c_1

$$c_1 = \frac{c_0 \int\limits_{I\!R} (c_0^2 u_0'^2(x) - 3u_0''^2(x)) e^{-c_0 x}\, dx}{24 \int\limits_{I\!R} u_0'^2(x) e^{-c_0 x}\, dx} \ .$$

For an exact cubic nonlinearity these integrals can be evaluated explicitly and c_1/c_0 turns out to be always negative So spatial discretization slows down the wave in this case. We conjecture that this holds in general for $h > 0$ and any bistable nonlinearity.

Addendum:
During the process of publication the authors learned to know, that in a paper by F. Hamel [14], corresponding to our case (a), the same formula for the wave speed is derived. Case (b) is not treated therein and no applications are given.

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