TRANSPORT EQUATIONS FOR WAVES IN A HALF SPACE

Leonid Ryzhik, George C. Papanicolaou and Joseph B.Keller
Department of Mathematics
Stanford University
Stanford CA 94305
Internet: ryzhik@math.stanford.edu,
papanico@math.stanford.edu, keller@math.stanford.edu

Abstract

We derive boundary conditions for the phase space energy density of acoustic waves in a half space, in the high frequency limit. These boundary conditions generalize the usual reflection-transmission relations for plane waves and are well suited for the study of wave propagation in bounded random media in the radiative transport approximation [15]. The high frequency analysis is based on direct calculations with Fourier integrals in the case of constant coefficients and Wigner measures in general, and it is presented in detail.

1 Introduction

Energy propagation for high frequency waves can be described by the phase-space energy density function (measure) that satisfies the Liouville equation (transport equation). This is well known to researchers in wave propagation in random media [1, 9] and was first analyzed mathematically for a deterministic medium by L. Tartar [18] and P. Gérard [4, 5] as well as by P.L. Lions and T. Paul [10]. High frequency asymptotic analysis (geometrical optics) [2, 8] deals with the phase and amplitude of the waves, which leads to special forms of the phase-space energy density. In this paper we analyze the phase-space energy density in the presence of boundaries and interfaces, generalizing the well known results of geometrical optics. This is of interest in wave propagation in random media because it provides appropriate boundary conditions for the radiative transport equation [15, 16].

We recall briefly the transmission and reflection of time harmonic, acoustic plane waves, at a plane interface. The homogeneous acoustic equations are

$$\begin{aligned} \rho\frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0 \\ \kappa\frac{\partial p}{\partial t} + \mathrm{div}\mathbf{u} = 0 \end{aligned} \tag{1.1}$$

for the velocity $\mathbf{u}$ and pressure p. They admit plane wave solutions of the form

$$\mathbf{u} = \mathbf{u}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \quad p = p_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}. \tag{1.2}$$

The frequency ω and the wave vector $\mathbf{k}$ are related by the dispersion law

$$\omega = v|\mathbf{k}|, \tag{1.3}$$

and with $v = 1/\sqrt{\kappa\rho}$ the sound speed. The amplitude vector $(\mathbf{u}_0, p_0)$ has the form

$$\mathbf{u}_0 = A\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, \quad p_0 = A\frac{1}{\sqrt{2\kappa}}, \tag{1.4}$$

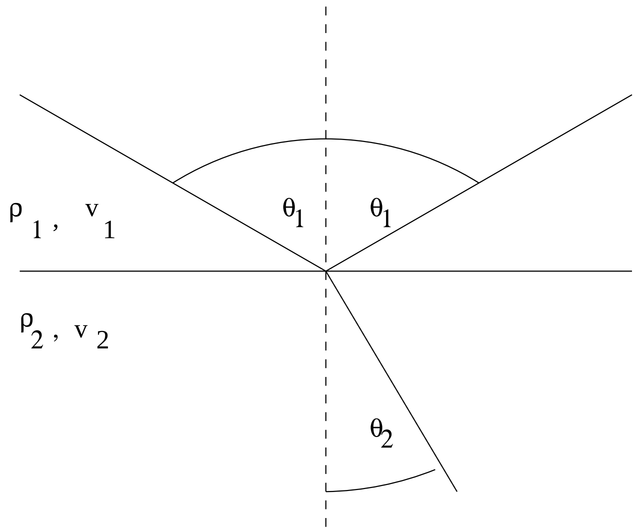

Figure 1.1: Reflection and transmission of acoustic waves.

where $\hat{\mathbf{k}} = \dfrac{\mathbf{k}}{|\mathbf{k}|}$ and A is a scalar amplitude.

We consider the reflection and transmission between media 1 and 2 of a plane wave incident at angle θ_1 on an interface, as is shown in Figure 1.1. The interface conditions for the waves are continuity of the normal velocity and pressure. This implies that the angle of incidence equals the angle of reflection, and Snell's law holds:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}. \tag{1.5}$$

Also, if A, B and C are the amplitudes of the incident, reflected and transmitted waves, respectively, then the reflection coefficient is

$$R(\theta_1) = \frac{B}{A} = \frac{\zeta_2\cos\theta_1 - \zeta_1\cos\theta_2}{\zeta_2\cos\theta_1 + \zeta_1\cos\theta_2}, \tag{1.6}$$

where $\zeta_i = \rho_i v_i$ is the impedance of medium i, $i = 1, 2$, and the transmission coefficient is

$$T(\theta_1) = \frac{C}{A} = 2\sqrt{\frac{\rho_2}{\rho_1}}\frac{\zeta_1\cos\theta_1}{\zeta_1\cos\theta_2 + \zeta_2\cos\theta_1}. \tag{1.7}$$

High frequency asymptotics, geometrical optics [2, 8], generalizes this plane wave reflection-transmission analysis to very general classes of waves that have a rapidly varying phase. The general theory of phase-space energy densities [4, 5, 10, 18] does not address boundary and interface phenomena, except in the special circumstances treated in [6]. In this paper we analyze in detail the behavior of phase-space energy densities in the presence of boundaries and interfaces.

We begin in Section 2 with a brief review of the Wigner distribution and its properties. In the high frequency limit it tends to the phase-space energy density, the object of our study. More details about the Wigner distribution and its limit are given in Appendix A. In Section 3 we analyze reflection and transmission of time harmonic, high frequency acoustic waves in a *homogeneous* space with a *plane* interface. Here, incident, reflected, and transmitted waves are given explicitly by Fourier integrals and the high frequency analysis can be done with direct calculations. The results are stated in Section 3.2 and are what one expects from geometrical optics. Sections 3.3 and 3.4 contain the asymptotic analysis. In Sections 3.5-3.7 we apply the results to the derivation of boundary conditions for the phase-space energy density (the limiting Wigner measure). Section 4 contains the high frequency analysis of reflection-transmission by a plane interface when the medium is *inhomogeneous*, so that we do not have explicit Fourier integral representations for the waves. The general theory of the Wigner distribution (and semiclassical operators) is used here, along with the necessary modifications to handle boundaries and interfaces. The results are stated in Section 4.2 and the derivations are given in Section 4.3.

Our interest in the Wigner distribution and its high frequency limit is motivated by our experience with waves in random media [15], where it plays an essential role. We are interested in spatial energy density, considered by Francfort and Murat [3], which is one of the main objectives of geometrical optics [2, 8], but also in the angularly resolved wave energy density. We do not deal with general H-measures [4, 18], because in the application to random media different spatial scales often appear explicitly. For example, the wavelength may be comparable to the size of the scatterers and small compared to the deterministic features of the medium and to the overall propagation distance.

After this paper was submitted L. Miller obtained by a different technique the results on the reflection-transmission problem for the time dependent Schrödinger and wave equations [11, 12, 13], including part of the grazing rays region.

2 The Wigner distribution and its high frequency limit

2.1 The Wigner distribution

We recall some basic facts about Wigner distributions which are useful in the analysis of high frequency energy propagation. They can be found in [6, 7, 10] and references therein. Given a function $f(\mathbf{x})$ in $\mathcal{S}(R^d)$, the Schwartz space of test functions, its Wigner distribution $W(\mathbf{x},\mathbf{k})$ is defined by

$$W(\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{y}} f(\mathbf{x}-\frac{\mathbf{y}}{2}) f^*(\mathbf{x}+\frac{\mathbf{y}}{2}). \tag{2.1}$$

The function $f(\mathbf{x})$ can be scalar or vector-valued; in the latter case W is a matrix. The Wigner distribution is real valued, or a self-adjoint matrix in the vector case, and its integral over all wave vectors $\mathbf{k}$ is

$$\int d\mathbf{k} W(\mathbf{x},\mathbf{k}) = f(\mathbf{x}) f^*(\mathbf{x}). \tag{2.2}$$

In the next section we will extend the definition (2.1) to $f \in \mathcal{S}'$, the Schwartz distributions. We may think of W as energy density in phase space. For example, the Wigner distribution of a plane wave with wave vector $\boldsymbol{\xi}$

$$f(\mathbf{x}) = A e^{i\boldsymbol{\xi}\cdot\mathbf{x}} \tag{2.3}$$

is supported on $\mathbf{k} = \boldsymbol{\xi}$:

$$W(\mathbf{x}, \mathbf{k}) = |A|^2 \delta(\mathbf{k} - \boldsymbol{\xi}). \tag{2.4}$$

In general, however, it is not positive, i.e. not a measure, and so this interpretation is not quite right.

2.2 High frequency limit

We want to consider the Wigner distributions of high frequency waves, i.e. of functions $f_\varepsilon(\mathbf{x})$ which are oscillating on a scale ε as $\varepsilon \to 0$. In order that W have a nontrivial limit we rescale it:

$$W_\varepsilon(\mathbf{x}, \mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{y}} f_\varepsilon(\mathbf{x} - \frac{\varepsilon \mathbf{y}}{2}) f_\varepsilon^*(\mathbf{x} + \frac{\varepsilon \mathbf{y}}{2}). \tag{2.5}$$

In terms of the Fourier transform $\hat{f}$ of f, the scaled form of W is

$$W_\varepsilon(\mathbf{x}, \mathbf{k}) = \int \frac{d\mathbf{p}}{(2\pi\sqrt{\varepsilon})^{2d}} e^{i\mathbf{p}\cdot\mathbf{x}} \hat{f}_\varepsilon(\frac{\mathbf{k}}{\varepsilon} + \frac{\mathbf{p}}{2}) \hat{f}_\varepsilon^*(\frac{\mathbf{k}}{\varepsilon} - \frac{\mathbf{p}}{2}). \tag{2.6}$$

Here $\hat{f}$ is

$$\hat{f}(\mathbf{k}) = \int d\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}). \tag{2.7}$$

The duality between (2.5) and (2.6) can be expressed succinctly by

$$W_\varepsilon[f_\varepsilon(\cdot)](\mathbf{x}, \mathbf{k}) = W_\varepsilon[\frac{1}{(2\pi\varepsilon)^{d/2}} \hat{f}_\varepsilon\left(\frac{\cdot}{\varepsilon}\right)](\mathbf{k}, -\mathbf{x}). \tag{2.8}$$

We are interested in the weak limit of W_ε as $\varepsilon \to 0$. We first define W_ε weakly for $f_\varepsilon \in \mathcal{S}'$, the space of tempered distributions. Let $a(\mathbf{x}, \mathbf{k})$ be a test function in $\mathcal{S}(R^d \times R^d)$. Then

$$< a, W_\varepsilon >= (a^w(\mathbf{x}, \varepsilon D) f_\varepsilon, f_\varepsilon), \tag{2.9}$$

where $<,>$ is the usual inner product on $R^d \times R^d$, $(,)$ is inner product on R^d, and the Weyl operator $a^w(\mathbf{x}, \varepsilon D)$ is defined by

$$a^w(\mathbf{x}, \varepsilon D) f(\mathbf{x}) = \iint \frac{d\mathbf{y} d\mathbf{k}}{(2\pi)^d} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} a(\frac{\mathbf{x}+\mathbf{y}}{2}, \varepsilon \mathbf{k}) f(\mathbf{y}) \tag{2.10}$$

$$= \int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d} \hat{a}(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{y}-\mathbf{x}}{\varepsilon}) f(\mathbf{y}).$$

Here $\hat{a}$ is the Fourier transform of $a(\mathbf{x},\mathbf{k})$ in the variable $\mathbf{k}$ only. This operator is bounded on L^2, uniformly in ε (see Appendix A)

$$||a^w(\mathbf{x},\varepsilon D)||_{L^2\to L^2} \leq C(a). \tag{2.11}$$

We note that if the functions f_ε are uniformly bounded in L^2, then there is a μ in $\mathcal{S}'(R^d\times R^d)$ and a subsequence of their Wigner distributions W_ε that converges weakly to it. This follows from the general theory in Chapter VII of [17].

An alternative way to get to limits of Wigner distributions is to introduce

$$\widetilde{W}_\varepsilon(\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{y}} f_\varepsilon(\mathbf{x}-\varepsilon\mathbf{y}) f^*_\varepsilon(\mathbf{x}) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon}}{(2\pi\varepsilon)^d}\hat{f}_\varepsilon(\frac{\mathbf{k}}{\varepsilon}) f^*_\varepsilon(\mathbf{x}). \tag{2.12}$$

and weakly, for $a(\mathbf{x},\mathbf{k})\in\mathcal{S}(R^d\times R^d)$,

$$< a, \widetilde{W}_\varepsilon >= (a(\mathbf{x},\varepsilon D) f_\varepsilon, f_\varepsilon). \tag{2.13}$$

Now the semiclassical operators $a(\mathbf{x},\varepsilon D)$ are defined by

$$\begin{aligned} a(\mathbf{x},\varepsilon D)f(\mathbf{x}) &= \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{x},\varepsilon\mathbf{k})\hat{f}(\mathbf{k}) \\ &= \int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d}\hat{a}(\mathbf{x},\frac{\mathbf{y}-\mathbf{x}}{\varepsilon}) f(\mathbf{y}). \end{aligned} \tag{2.14}$$

and they are frequently more convenient for calculations than (2.10). The main fact shown in Appendix A is that (2.14) and (2.10) are asymptotically equivalent as $\varepsilon\to 0$

$$||a^w(\mathbf{x},\varepsilon D) - a(\mathbf{x},\varepsilon D)||_{L^2\to L^2} \to 0. \tag{2.15}$$

Thus the family $\widetilde{W}_\varepsilon$ has the same weak limits as W_ε, as $\varepsilon\to 0$, and if μ is a limit Wigner distribution of f_ε then

$$\lim_{\varepsilon\to 0}(a(\mathbf{x},\varepsilon D) f_\varepsilon, f_\varepsilon) =< a,\mu >= \mathrm{Tr}\int a(\mathbf{x},\mathbf{k})\mu(d\mathbf{x}d\mathbf{k}) \tag{2.16}$$

with the limit taken along the subsequence ε_j corresponding to the subsequence W_{ε_j} converging weakly to μ.

The high frequency analog of (2.4) is the limit Wigner distribution of the inhomogeneous high frequency plane wave

$$f_\varepsilon = A(\mathbf{x})e^{iS(\mathbf{x})/\varepsilon}, \tag{2.17}$$

where $S(\mathbf{x})$ is the phase (real valued and smooth) and $A(\mathbf{x})$ is the amplitude (complex valued and continuous). The limit Wigner measure is easily seen to be

$$\mu(\mathbf{x}, \mathbf{k}) = |A(\mathbf{x})|^2 \, \delta(\mathbf{k} - \nabla S(\mathbf{x})). \tag{2.18}$$

2.3 Some properties of limit Wigner distributions

A basic tool in Fourier integral operator calculus, proved in Appendix A along with some other properties of semiclassical operators, is

Property 1 *The product of two operators $a(\mathbf{x}, \varepsilon D)$, $b(\mathbf{x}, \varepsilon D)$ is*

$$b(\mathbf{x}, \varepsilon D)a(\mathbf{x}, \varepsilon D) = (ba)(\mathbf{x}, \varepsilon D) + \frac{\varepsilon}{i}(\nabla_{\mathbf{k}} b \cdot \nabla_{\mathbf{x}} a)(\mathbf{x}, \varepsilon D) + \varepsilon^2 Q_\varepsilon, \tag{2.19}$$

where the operators Q_ε are uniformly bounded on L^2.

The Weyl operators satisfy the same product rule (2.19). The adjoint Weyl operators $a^w(\mathbf{x}, \varepsilon D)$ are given by

$$a^w(\mathbf{x}, \varepsilon D)^* = a^{*w}(\mathbf{x}, \varepsilon D). \tag{2.20}$$

A basic property of Wigner distributions is that their high frequency limits are nonnegative distributions, that is, measures.

Property 2 (Positivity) *Let the family f_ε be uniformly bounded on L^2. Then any limit Wigner distribution is a measure, called a Wigner measure.*

Proof: It is sufficient to verify that $< a, \mu >\geq 0$ for all test functions $a(\mathbf{x}, \mathbf{k})$ of the form $a(\mathbf{x}, \mathbf{k}) = |b(\mathbf{x}, \mathbf{k})|^2$ because those are dense in the set of positive test functions. Then (2.20) and the product rule (2.19) implies that for such test functions

$$
\begin{aligned}
< a, \mu > &= \lim_{\varepsilon \to 0} (a^w(\mathbf{x}, \varepsilon D) f_\varepsilon, f_\varepsilon) \\
&= \lim_{\varepsilon \to 0} (b^w(\mathbf{x}, \varepsilon D) f_\varepsilon, b^w(\mathbf{x}, \varepsilon D) f_\varepsilon) \geq 0,
\end{aligned} \tag{2.21}
$$

where the limit is along a convergent subsequence.

Another important property is the following.

Property 3 (Localization) *Let $f_\varepsilon(\mathbf{x})$ be a family of uniformly bounded functions in L^2 and let $\mu_f(\mathbf{x}, \mathbf{k})$ be any limit Wigner measure. Let $\phi(\mathbf{x})$ be a smooth function. Then the Wigner measure of the family $g_\varepsilon(\mathbf{x}) = \phi(\mathbf{x}) f_\varepsilon(\mathbf{x})$ is $|\phi(\mathbf{x})|^2 \mu_f(\mathbf{x}, \mathbf{k})$. Moreover, let f_ε, g_ε be two uniformly bounded families of L^2 functions which coincide in an open neighbourhood of a point $\mathbf{x}_0$. Then any limit Wigner measures μ_f and μ_g coincide in this neighborhood.*

Proof. Let $a(\mathbf{x}, \mathbf{k})$ be a test function compactly supported in $\mathbf{x}$, such that its Fourier transform in $\mathbf{k}$ is compactly supported. Then

$$
\begin{aligned}
&(a(\mathbf{x}, \varepsilon D) g_\varepsilon, g_\varepsilon) - (|\phi(\mathbf{x})|^2 a(\mathbf{x}, \varepsilon D) f_\varepsilon, f_\varepsilon) \\
&= \int d\mathbf{x} d\mathbf{z} f_\varepsilon^*(\mathbf{x}) \ \phi^*(\mathbf{x}) \hat{a}(\mathbf{x}, \mathbf{z}) (\phi(\mathbf{x} + \varepsilon \mathbf{z}) - \phi(\mathbf{x})) f_\varepsilon(\mathbf{x} + \varepsilon \mathbf{z}) \to 0
\end{aligned}
$$

as $\varepsilon \to 0$. This proves the first statement which implies in turn the second statement.

The localization property is quite useful because it allows the consideration of Wigner measures for families of functions f_ε that are uniformly bounded in L^2_{loc}.

Another useful and intuitively clear property is that the Wigner measure of waves going in different directions is the sum of the individual Wigner measures.

Property 4 (Orthogonality) *Let f_ε, g_ε be two families of functions with Wigner measures μ_f and μ_g, which are mutually singular. Then the Wigner measure of the sum $f_\varepsilon + g_\varepsilon$ is $\mu_f + \mu_g$.*

Proof. Let $a(\mathbf{x},\mathbf{k})$ be a positive test function of the form $a(\mathbf{x},\mathbf{k}) = |b(\mathbf{x},\mathbf{k})|^2$, then by the product rule (2.19) and the Schwartz inequality we have

$$\begin{aligned}|(a(\mathbf{x},\varepsilon D)\ f_\varepsilon, g_\varepsilon)| &= |(b(\mathbf{x},\varepsilon D)f_\varepsilon, b(\mathbf{x},\varepsilon D)g_\varepsilon)| + O(\varepsilon)\\ &\le |(b(\mathbf{x},\varepsilon D)f_\varepsilon, b(\mathbf{x},\varepsilon D)f_\varepsilon)|^{1/2}|(b(\mathbf{x},\varepsilon D)g_\varepsilon, b(\mathbf{x},\varepsilon D)g_\varepsilon)|^{1/2} + O(\varepsilon)\\ &\to < a, \mu_f >^{1/2} < a, \mu_g >^{1/2}\ . \end{aligned} \tag{2.22}$$

Then, since μ_f and μ_g are mutually singular, we can split $a(\mathbf{x},\mathbf{k})$ into a sum of three terms:

$$a = a_1 + a_2 + a_3, \tag{2.23}$$

so that a_1 is orthogonal to μ_f, a_2 is orthogonal to μ_g, and the integral of a_3 with respect to both measures can be made arbitrarily small. Then the right side of (2.22) is arbitrarily small and thus its left side goes to zero in the limit $\varepsilon \to 0$.

2.4 Convergence of energy

The Wigner distribution is well suited for studying high frequency limits and, in particular, families of functions that depend on a small parameter in an oscillatory manner, the ε-oscillatory families of [6, 7]. The oscillatory property is conveniently characterized by the following definition.

Definition 1 *A family of functions f_ε that is uniformly bounded in L^2_{loc} is said to be ε-oscillatory if for every smooth and compactly supported function $\phi(\mathbf{x})$*

$$\limsup_{\varepsilon\to 0} \int_{|\boldsymbol{\xi}|\ge R/\varepsilon} |\widehat{\phi f_\varepsilon}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \to 0 \quad as \quad R \to +\infty. \tag{2.24}$$

A simple and intuitive sufficient condition for (2.24) is that there exist a positive integer j and a constant C independent of ε such that

$$\varepsilon^j \left\| \frac{\partial^j f_\varepsilon}{\partial x^j} \right\|_{L^2_{loc}} \leq C. \tag{2.25}$$

This condition holds, for instance, for high frequency plane waves

$$f_\varepsilon(\mathbf{x}) = A e^{i\boldsymbol{\xi}\cdot\mathbf{x}/\varepsilon}. \tag{2.26}$$

Another natural example of ε-oscillatory functions is

$$g_\varepsilon(\mathbf{x}) = g(\frac{\mathbf{x}}{\varepsilon}), \tag{2.27}$$

where $g(\mathbf{x})$ is a periodic function with bounded gradient.

The main reason for introducing ε-oscillatory functions is the following theorem concerning weak convergence of energy, i.e. of the integral of the square of the wave function.

Proposition 1 *Let f_ε be a bounded family in L^2_{loc} with limit Wigner measure $\mu(\mathbf{x}, \mathbf{k})$. Then for any smooth function of compact support $\theta(\mathbf{x})$*

$$\int\!\!\int |\theta(\mathbf{x})|^2 \mu(d\mathbf{x}, d\mathbf{k}) \leq \limsup_{\varepsilon\to 0} \int_{R^d} |\theta(\mathbf{x}) f_\varepsilon(\mathbf{x})|^2 d\mathbf{x} \tag{2.28}$$

with equality holding if and only if f_ε is ε-oscillatory. In this case lim sup *can be replaced by* lim *on the right side of (2.28).*

The proof can be found in [6, 7].

With this proposition and the positivity property we can interpret $\mu(\mathbf{x}, \mathbf{k})$ as a phase space energy density, that is, energy density resolved over directions and wavenumbers. The relation between the limit Wigner measure and the high frequency limit of the energy of the acoustic waves [15] is given in the next Section.

3 Transport equation and boundary conditions for acoustic waves in a uniform half space

3.1 The acoustic equations

The equations for the acoustic velocity $\mathbf{u}$ and pressure p in the half space $x^n \geq 0$ are

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0 \tag{3.1}$$
$$\kappa \frac{\partial p}{\partial t} + \mathrm{div}\mathbf{u} = 0,$$

where $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{x} = (\mathbf{x}', x^n)$, $\mathbf{x}' \in R^{n-1}$, $x^n \geq 0$, with $n = 2$ or 3. The density ρ and compressibility κ are constants. We consider time harmonic high frequency solutions of (3.1) of the form

$$\mathbf{w}_\varepsilon(t, \mathbf{x}) = \mathbf{w}_\varepsilon(\mathbf{x}) e^{-i\omega t/\varepsilon}, \tag{3.2}$$

where $\mathbf{w}_\varepsilon = (\mathbf{u}_\varepsilon, p_\varepsilon)^t$, and ω/ε is the frequency. Then (3.1) becomes

$$-i\omega\rho\mathbf{u}_\varepsilon + \varepsilon\nabla p_\varepsilon = 0 \tag{3.3}$$
$$-i\omega\kappa p_\varepsilon + \varepsilon\mathrm{div}\mathbf{u}_\varepsilon = 0$$

and thus any uniformly bounded in L^2_{loc} family of solutions is ε-oscillatory by (2.25). Equations (3.3) can be written as a time-reduced symmetric hyperbolic system

$$\varepsilon D^j \frac{\partial \mathbf{w}_\varepsilon}{\partial x^j} - i\omega A\mathbf{w}_\varepsilon = 0, \tag{3.4}$$

where the matrix A is

$$A = \mathrm{diag}(\rho, \rho, \rho, \kappa), \tag{3.5}$$

the symmetric matrices D^j correspond to the divergence and gradient operators in (3.3), and repeated indices are summed. Let

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \mathbf{v} e^{i\mathbf{k}\cdot\mathbf{x}/\varepsilon} \tag{3.6}$$

be a high frequency plane wave solution of (3.3). Then the vector $\mathbf{v}$ must be an eigenvector of the dispersion matrix

$$L(\mathbf{k}) = A^{-1} k_j D^j. \tag{3.7}$$

The eigenvalues of L are

$$\omega_+ = v|\mathbf{k}|, \qquad \omega_- = -v|\mathbf{k}|, \qquad \omega_3 = 0, \tag{3.8}$$

where

$$v = \frac{1}{\sqrt{\kappa\rho}} \tag{3.9}$$

is the speed of sound. The first two eigenvalues are simple and the eigenvalue zero has multiplicity $n-1$. The corresponding eigenvectors are

$$\mathbf{b}_+ = (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, -\frac{1}{\sqrt{2\kappa}}), \quad \mathbf{b}_- = (\frac{\hat{\mathbf{k}}}{\sqrt{2\rho}}, \frac{1}{\sqrt{2\kappa}}), \tag{3.10}$$

$$\mathbf{b}_j^3 = (\frac{\mathbf{z}^j}{\sqrt{\rho}}, 0), \quad j = 1, ..., n-1$$

where $\hat{\mathbf{k}} = \dfrac{\mathbf{k}}{|\mathbf{k}|}$ is the unit vector in the direction of $\mathbf{k}$ and vectors $\mathbf{z}^j$ and $\hat{\mathbf{k}}$ form an orthonormal frame. The $\mathbf{b}_\alpha$ are normalized so that

$$(A\mathbf{b}_\alpha, \mathbf{b}_\beta) = \delta_{\alpha\beta}. \tag{3.11}$$

The energy density of an acoustic wave is

$$\mathcal{E}_\varepsilon = \frac{\rho(\mathbf{u}_\varepsilon)^2}{2} + \frac{\kappa(p_\varepsilon)^2}{2} = \frac{1}{2}\int (A\mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon) d\mathbf{k} \tag{3.12}$$

and since $\mathbf{w}_\varepsilon$ is ε-oscillatory, its weak high frequency limit is

$$\mathcal{E}(\mathbf{x}) = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\mathbf{x}) = \frac{1}{2}\mathrm{Tr}\int A\boldsymbol{\mu}(\mathbf{x}, d\mathbf{k}), \tag{3.13}$$

where $\boldsymbol{\mu}$ is the Wigner measure of $\mathbf{w}^\varepsilon$. The energy flux is

$$\mathcal{F}_{j\varepsilon} = \frac{1}{2} p_\varepsilon u_{j\varepsilon} = \frac{1}{2}\int (D^j \mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon) d\mathbf{k} \tag{3.14}$$

and its high frequency limit is

$$\mathcal{F}_j(\mathbf{x}) = \lim_{\varepsilon\to 0} \mathcal{F}_{j\varepsilon} = \frac{1}{2}\mathrm{Tr}\int D^j \boldsymbol{\mu}(\mathbf{x}, d\mathbf{k}). \tag{3.15}$$

All integrals are over R^n.

3.2 Statement of the results

Let $\mathbf{w}_\varepsilon$ be a family of solutions of (3.4) uniformly bounded in L^2_{loc}. An arbitrary solution of (3.4) can be decomposed into a sum of waves incident on and reflected from the plane $x^n = 0$,

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \mathbf{w}_\varepsilon^I(\mathbf{x}) + \mathbf{w}_\varepsilon^R(\mathbf{x}). \tag{3.16}$$

Here

$$\mathbf{w}_\varepsilon^I(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}'\cdot\mathbf{x}'/\varepsilon + ik_n^- x^n/\varepsilon} \alpha_\varepsilon(\mathbf{k}')\mathbf{b}_+(\mathbf{k}^-) \tag{3.17}$$

$$\mathbf{w}_\varepsilon^R(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}'\cdot\mathbf{x}'/\varepsilon + ik_n^+ x^n/\varepsilon} \beta_\varepsilon(\mathbf{k}')\mathbf{b}_+(\mathbf{k}^+). \tag{3.18}$$

The vector $\mathbf{k}' \in R^{n-1}$ is called a horizontal wave vector and

$$\mathbf{k}^\pm(\mathbf{k}') = (\mathbf{k}', k_n^\pm), \quad k_n^\pm = \pm\sqrt{\frac{\omega^2}{v^2} - \mathbf{k}'^2}, \tag{3.19}$$

with the square root chosen so that the imaginary part of $k_n^\pm$ is nonnegative since $x^n \geq 0$. We deal with general waves in inhomogeneous media in Section 4 and so, for simplicity, we will analyze here only propagating waves. This means that the amplitudes $\alpha_\varepsilon(\mathbf{k}')$ and $\beta_\varepsilon(\mathbf{k}')$ of the incident and reflected waves are tempered distributions (in $\mathcal{S}'$) with support uniformly in $\{|\mathbf{k}'| < \omega/v\}$. We also assume that $\mathbf{w}_\varepsilon^I$ and $\mathbf{w}_\varepsilon^R$ are bounded in L^2_{loc} and have boundary values on $x^n = 0$ that are bounded in L^2_{loc}. In the next section we will prove

Proposition 2 *The Wigner measure* $\boldsymbol{\mu}$ *of the family* $\mathbf{w}_\varepsilon$ *(3.16), with* $\mathbf{w}_\varepsilon^I(\mathbf{x})$ *and* $\mathbf{w}_\varepsilon^R(\mathbf{x})$ *given by (3.17) and (3.18), respectively, has the form*

$$\boldsymbol{\mu}(\mathbf{x},\mathbf{k}) = \mu(\mathbf{x},\mathbf{k})\mathbf{b}_+(\mathbf{k})\mathbf{b}_+^*(\mathbf{k}). \tag{3.20}$$

The scalar measure $\mu(\mathbf{x},\mathbf{k})$ *is supported on the sphere*

$$U = \{\mathbf{k} : v|\mathbf{k}| = \omega\}, \tag{3.21}$$

and satisfies weakly the transport equation

$$v\hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \mu = 0 \tag{3.22}$$

in the half space $x^n > 0$. The boundary value of μ is

$$\mu(\mathbf{x}', 0, \mathbf{k}) = \nu_\alpha(\mathbf{k}', -\mathbf{x}')\delta(k_n - k_n^-) + \nu_\beta(\mathbf{k}', -\mathbf{x}')\delta(k_n - k_n^+), \tag{3.23}$$

where ν_α and ν_β are the $n-1$-dimensional Wigner measures of $\alpha_\varepsilon(\mathbf{k}')$ and $\beta_\varepsilon(\mathbf{k}')$, respectively.

The second statement (3.21) is a general form of the eiconal equation in geometrical optics [2, 8]. The transport equation of geometrical optics $\nabla \cdot (|A|^2 \nabla S) = 0$ follows from (3.22) for Wigner measures of the form (2.18). The scalar measure $\mu(\mathbf{x}, \mathbf{k})$ is the energy density in phase space since (3.11) implies that the limit energy density (3.13) is

$$\mathcal{E}(\mathbf{x}) = \frac{1}{2} \int \mu(\mathbf{x}, \mathbf{k}) d\mathbf{k} \tag{3.24}$$

and the energy density flux (3.15) is

$$\mathcal{F}(\mathbf{x}) = \frac{1}{2} \int v\hat{\mathbf{k}} \mu(\mathbf{x}, \mathbf{k}) d\mathbf{k} \tag{3.25}$$

since

$$(D^i \mathbf{b}_+, \mathbf{b}_+) = v\hat{k}_i. \tag{3.26}$$

3.3 The transport equation in the interior

The limit Wigner distribution of the wave (3.16) can be computed directly in a straightforward but tedious manner. By using Properties 2, 3 and 4 of Section 2.3, this computation can be simplified and systematized so that it can be generalized to inhomogeneous media, in Section 4.

Let $\mathbf{x}$ be a point inside the upper half space. Then the Wigner measure of $\mathbf{w}_\varepsilon$ at $\mathbf{x}$ is the sum of four terms

$$\boldsymbol{\mu}(\mathbf{x},\mathbf{k}) = \boldsymbol{\mu}_I(\mathbf{x},\mathbf{k}) + \boldsymbol{\mu}_R(\mathbf{x},\mathbf{k}) + \boldsymbol{\mu}_{IR}(\mathbf{x},\mathbf{k}) + \boldsymbol{\mu}_{RI}(\mathbf{x},\mathbf{k}). \tag{3.27}$$

The function $\mathbf{w}_\varepsilon^I(\mathbf{x})$ can be extended to all of R^n by (3.17) because we have excluded evanescent waves. This extension will not change the values of the Wigner measure $\boldsymbol{\mu}_I(\mathbf{x},\mathbf{k})$ inside the upper half space by Property 3. The same is true for $\mathbf{w}_\varepsilon^R(\mathbf{x})$ and $\boldsymbol{\mu}_R(\mathbf{x},\mathbf{k})$. We denote by $\mathbf{w}_\varepsilon^I$ and $\mathbf{w}_\varepsilon^R$ these extended functions. The Fourier transform of $\mathbf{w}_\varepsilon^I$ is

$$\hat{\mathbf{w}}_\varepsilon^I(\frac{\mathbf{k}}{\varepsilon}) = (2\pi\varepsilon)^{(n+1)/2}\alpha_\varepsilon(\mathbf{k}')\mathbf{b}_+(\mathbf{k}^-)\delta(k_n - k_n^-). \tag{3.28}$$

Thus the Wigner measure μ_I is supported on the lower hemisphere of U in (3.21) away from the equator, while the Wigner measure μ_R is supported on the upper hemisphere. Then the cross Wigner measures μ_{IR} and μ_{RI} vanish by Property 4 and

$$\boldsymbol{\mu}(\mathbf{x},\mathbf{k}) = \boldsymbol{\mu}_I(\mathbf{x},\mathbf{k}) + \boldsymbol{\mu}_R(\mathbf{x},\mathbf{k}). \tag{3.29}$$

Property 3 and (2.8) imply that the Wigner measure $\boldsymbol{\mu}_I(\mathbf{x},\mathbf{k})$ has the form

$$\boldsymbol{\mu}_I(\mathbf{x},\mathbf{k}) = \mu_I(\mathbf{x},\mathbf{k})\mathbf{b}_+(\mathbf{k})\mathbf{b}_+^*(\mathbf{k}), \tag{3.30}$$

where $\mu_I(\mathbf{x},\mathbf{k})$ is the scalar Wigner measure of the family v_ε^I given by

$$w_\varepsilon^I(\mathbf{x}) = \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}'\cdot\mathbf{x}'/\varepsilon + ik_n^- x^n/\varepsilon}\alpha_\varepsilon(\mathbf{k}'). \tag{3.31}$$

This function satisfies the reduced wave equation

$$\varepsilon^2\Delta w_\varepsilon^I + \frac{\omega^2}{v^2}w_\varepsilon^I = 0 \tag{3.32}$$

and hence $\mu_I(\mathbf{x},\mathbf{k})$ satisfies weakly the transport equation [7, 10, 15]

$$\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\mu_I = 0. \tag{3.33}$$

This is verified using (2.9), (3.32) and a test function $a(\mathbf{x}, \mathbf{k})$,

$$< a, \mathbf{k}\cdot \nabla_{\mathbf{x}} W_\varepsilon^I >= \int d\mathbf{x} w_\varepsilon^*(\mathbf{x}) \iint \frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^n} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} \mathbf{k}\cdot\nabla_{\mathbf{x}} a(\frac{\mathbf{x}+\mathbf{y}}{2}, \varepsilon\mathbf{k}) w_\varepsilon(\mathbf{y})$$

$$= -\frac{1}{2i} \int d\mathbf{x} w_\varepsilon^*(\mathbf{x}) \iint \frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^n} \Delta_{\mathbf{y}} \left[e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} a(\frac{\mathbf{x}+\mathbf{y}}{2}, \varepsilon\mathbf{k}) \right] w_\varepsilon(\mathbf{y})$$

$$+\frac{1}{2i} \int d\mathbf{x} w_\varepsilon^*(\mathbf{x}) \Delta_{\mathbf{x}} \left[\iint \frac{d\mathbf{y}d\mathbf{k}}{(2\pi)^n} e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}} a(\frac{\mathbf{x}+\mathbf{y}}{2}, \varepsilon\mathbf{k}) w_\varepsilon(\mathbf{y}) \right] = 0.$$

Similarly the Wigner measure μ_R has the form

$$\boldsymbol{\mu}_R(\mathbf{x}, \mathbf{k}) = \mu_R(\mathbf{x}, \mathbf{k}) \mathbf{b}_+(\mathbf{k}) \mathbf{b}_+^*(\mathbf{k}). \tag{3.34}$$

The scalar measure $\mu_R(\mathbf{x}, \mathbf{k})$ is supported on the upper hemisphere of U (3.21) and satisfies the transport equation (3.33). Then (3.30) implies that $\boldsymbol{\mu}(\mathbf{x}, \mathbf{k})$ has the form (3.20)

$$\boldsymbol{\mu}(\mathbf{x}, \mathbf{k}) = \mu(\mathbf{x}, \mathbf{k}) \mathbf{b}_+(\mathbf{k}) \mathbf{b}_+^*(\mathbf{k}). \tag{3.35}$$

The scalar measure μ is supported on the sphere U in (3.21) and satisfies the transport equation (3.22).

3.4 Boundary values of the Wigner distribution in the high frequency limit

In order to compute $\mu(\mathbf{x}', 0, \mathbf{k})$ let

$$r_\varepsilon^I(\mathbf{x}') = w_\varepsilon^I(\mathbf{x}', 0) \tag{3.36}$$

with w_ε^I defined by (3.31). Its Fourier transform is

$$\frac{1}{(2\pi\varepsilon)^{(n-1)/2}} \hat{r}_\varepsilon^I(\frac{\mathbf{k}'}{\varepsilon}) = \alpha_\varepsilon(\mathbf{k}'). \tag{3.37}$$

Then (2.12) and (2.8) imply that

$$\widetilde{W}^\varepsilon[w_\varepsilon^I](\mathbf{x}', 0, \mathbf{k}) = \frac{e^{i\mathbf{k}'\cdot\mathbf{x}'/\varepsilon}}{(2\pi\varepsilon)^n} \hat{w}_\varepsilon^I(\frac{\mathbf{k}}{\varepsilon}) w_\varepsilon^{I*}(\mathbf{x}', 0) = \widetilde{W}_\varepsilon[r_\varepsilon^I](\mathbf{x}', \mathbf{k}') \delta(k_n - k_n^-) \tag{3.38}$$
$$= \widetilde{W}_\varepsilon[\alpha_\varepsilon^I](\mathbf{k}', -\mathbf{x}') \delta(k_n - k_n^-),$$

and thus

$$\mu_I(\mathbf{x}', 0, \mathbf{k}) = \nu_\alpha(\mathbf{k}', -\mathbf{x}')\delta(k_n - k_n^-), \tag{3.39}$$

where $\nu_\alpha(\mathbf{k}', \mathbf{x}')$ is the Wigner measure of $\alpha_\varepsilon(\mathbf{k}')$. A similar computation shows that

$$\mu_R(\mathbf{x}', 0, \mathbf{k}) = \nu_\beta(\mathbf{k}', -\mathbf{x}')\delta(k_n - k_n^+). \tag{3.40}$$

The boundary value of $\mu(\mathbf{x}, \mathbf{k})$ is therefore

$$\mu(\mathbf{x}', 0, \mathbf{k}) = \nu_\alpha(\mathbf{k}', -\mathbf{x}')\delta(k_n - k_n^-) + \nu_\beta(\mathbf{k}', -\mathbf{x}')\delta(k_n - k_n^+), \tag{3.41}$$

as claimed in (3.23).

3.5 Boundary conditions for a reflecting boundary

We consider first the acoustic equations (3.3) with reflecting boundary conditions, Neumann

$$u_n \,|_{x^n=0} \, = 0 \tag{3.42}$$

or Dirichlet

$$p \,|_{x^n=0} \, = 0. \tag{3.43}$$

These boundary conditions imply that the amplitudes $\alpha^\varepsilon(\mathbf{k}')$ and $\beta^\varepsilon(\mathbf{k}')$ are related by

$$\alpha^\varepsilon(\mathbf{k}') = \beta^\varepsilon(\mathbf{k}') \tag{3.44}$$

or

$$\alpha^\varepsilon(\mathbf{k}') = -\beta^\varepsilon(\mathbf{k}'), \tag{3.45}$$

respectively. Both (3.44) and (3.45) imply that

$$\nu_\alpha(\mathbf{k}', \mathbf{x}') = \nu_\beta(\mathbf{k}', \mathbf{x}') \tag{3.46}$$

and hence the Wigner measure $\mu(\mathbf{x}, \mathbf{k})$ satisfies the boundary condition

$$\mu(\mathbf{x}', 0, \mathbf{k}', k_n) = \mu(\mathbf{x}', 0, \mathbf{k}', -k_n). \tag{3.47}$$

This is what one expects from physical considerations since (3.47) implies that all the energy is reflected under the boundary conditions (3.42) or (3.43), and the normal energy flux at the boundary vanishes:

$$\mathcal{F}_n(\mathbf{x}', 0) = \frac{u_n p}{2}(\mathbf{x}', 0) = \frac{1}{2}\int v\hat{k}_n \mu(\mathbf{x}', 0, \mathbf{k}) d\mathbf{k} = 0. \tag{3.48}$$

3.6 Interface conditions

Consider reflection and transmission at the interface $x^n = 0$ between two homogeneous half spaces. In the upper half space $x^n > 0$ the acoustic wave $\mathbf{w}(\mathbf{x})$ has the form

$$\begin{aligned}\mathbf{w}^{(1)}(\mathbf{x}) &= \mathbf{w}_I^{(1)}(\mathbf{x}) + \mathbf{w}_R^{(1)}(\mathbf{x}) \\ &= \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}' \cdot \mathbf{x}'/\varepsilon + ik_n^{(1)-} x^n/\varepsilon} \alpha_\varepsilon^{(1)}(\mathbf{k}') \mathbf{b}_+^{(1)}(\mathbf{k}^-) \\ &+ \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}' \cdot \mathbf{x}'/\varepsilon + ik_n^{(1)+} x^n/\varepsilon} \beta_\varepsilon^{(1)}(\mathbf{k}') \mathbf{b}_+^{(1)}(\mathbf{k}^+)\end{aligned} \tag{3.49}$$

and in the lower half space $x^n < 0$

$$\begin{aligned}\mathbf{w}^{(2)}(\mathbf{x}) &= \mathbf{w}_I^{(2)}(\mathbf{x}) + \mathbf{w}_R^{(2)}(\mathbf{x}) \\ &= \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}' \cdot \mathbf{x}'/\varepsilon + ik_n^{(2)-} x^n/\varepsilon} \alpha_\varepsilon^{(2)}(\mathbf{k}') \mathbf{b}_+^{(2)}(\mathbf{k}^-) \\ &+ \int \frac{d\mathbf{k}'}{(2\pi\varepsilon)^{(n-1)/2}} e^{i\mathbf{k}' \cdot \mathbf{x}'/\varepsilon + ik_n^{(2)+} x^n/\varepsilon} \beta_\varepsilon^{(2)}(\mathbf{k}') \mathbf{b}_+^{(2)}(\mathbf{k}^+).\end{aligned} \tag{3.50}$$

Here the quantities with the superscripts one and two correspond to the media above and below the interface, respectively, and we omit the subscript ε. The interface conditions are continuity of the normal velocity and pressure:

$$\begin{aligned}u_n^{(1)}(\mathbf{x}', 0) &= u_n^{(2)}(\mathbf{x}', 0) \\ p^{(1)}(\mathbf{x}', 0) &= p^{(2)}(\mathbf{x}', 0).\end{aligned} \tag{3.51}$$

This implies that the wave amplitudes in two media are related by

$$\begin{aligned}\beta_\varepsilon^{(1)}(\mathbf{k}') &= R^{(1)}(\mathbf{k}')\alpha_\varepsilon^{(1)}(\mathbf{k}') + T^{(2)}(\mathbf{k}')\beta_\varepsilon^{(2)}(\mathbf{k}') \\ \alpha_\varepsilon^{(2)}(\mathbf{k}') &= T^{(1)}(\mathbf{k}')\alpha_\varepsilon^{(1)}(\mathbf{k}') + R^{(2)}(\mathbf{k}')\beta_\varepsilon^{(2)}(\mathbf{k}'),\end{aligned} \tag{3.52}$$

where $R^1(\mathbf{k}')$ and $T^1(\mathbf{k}')$ are the plane waves reflection and transmission coefficients from the first medium into the second, and $R^2(\mathbf{k}')$ and $T^2(\mathbf{k}')$ are those from the second medium into the first. They are given by (1.6) and (1.7) with $|\mathbf{k}'| = \omega \sin\theta / v$. It follows that the boundary Wigner measures $\nu_\alpha^{(1,2)}$ and $\nu_\beta^{(1,2)}$ are related by

$$\begin{aligned}\nu_\beta^{(1)}(\mathbf{k}',\mathbf{x}') &= |R^{(1)}(\mathbf{k}')|^2\nu_\alpha^{(1)} + |T^{(2)}(\mathbf{k}')|^2\nu_\beta^{(2)} + 2\mathrm{Re}\left[R^{(1)}(\mathbf{k}')\bar{T}^{(2)}(\mathbf{k}')\nu_{\alpha\beta}^{(12)}\right] \\ \nu_\alpha^{(2)}(\mathbf{k}',\mathbf{x}') &= |T^{(1)}(\mathbf{k}')|^2\nu_\alpha^{(1)} + |R^{(2)}(\mathbf{k}')|^2\nu_\beta^{(2)} + 2\mathrm{Re}\left[T^{(1)}(\mathbf{k}')\bar{R}^{(2)}(\mathbf{k}')\nu_{\alpha\beta}^{(12)}\right].\end{aligned} \tag{3.53}$$

We see from these expressions that the values of $\nu_\alpha^{(1)}$ and $\nu_\beta^{(2)}$, which determine the Wigner measures of the waves incident on the interface from each side, do not determine the values of $\nu_\beta^{(1)}$ and $\nu_\alpha^{(2)}$. The additional information needed is the cross Wigner measure $\nu_{\alpha\beta}^{(12)}$ of the waves incident from each side. This determines the phase coherence of the incident waves. We now give some examples of situations in which this information is known.

3.7 Coherent and incoherent incident waves

Example 1. Waves incident from one side.

We assume that

$$\beta_\varepsilon^{(2)} = 0. \tag{3.54}$$

If the medium above is slower than the one below the interface, that is if

$$v_1 < v_2. \tag{3.55}$$

then we can have total internal reflection. Consider first a horizontal wave vector $\mathbf{k}'$ for which there is no total internal reflection,

$$|\mathbf{k}'| \leq \frac{\omega}{v_2}. \tag{3.56}$$

Then (3.53) and (3.41) imply that for $k_n < 0$

$$\mu^{(1)}(\mathbf{x}', 0, \mathbf{k}', -k_n) = |R^{(1)}(\mathbf{k}')|^2 \mu^{(1)}(\mathbf{x}', 0, \mathbf{k}', k_n) \tag{3.57}$$

$$\mu^{(2)}(\mathbf{x}', 0, \mathbf{k}', k_n^{(2)}) = |T(\mathbf{k}')|^2 \mu^{(1)}(\mathbf{x}', 0, \mathbf{k}', k_n). \tag{3.58}$$

The normal component of the transmitted wave vector $k_n^{(2)}$ is given by

$$k_n^{(2)} = -\sqrt{\frac{v_1^2}{v_2^2}\mathbf{k}'^2 - \mathbf{k}'^2 + \frac{v_1^2}{v_2^2}k_n^2}. \tag{3.59}$$

The results (3.57) and (3.58) are the usual energy reflection and transmission boundary conditions.

There is total internal reflection if the horizontal wave vector $\mathbf{k}'$ is such that

$$\frac{\omega}{v_2} < |\mathbf{k}'| < \frac{\omega}{v_1}. \tag{3.60}$$

Then $k_n^{(2)}$ is imaginary and

$$\mu^{(2)}(\mathbf{x}', 0, \mathbf{k}', k_n) = 0 \tag{3.61}$$

for all k_n. Moreover, (1.6) implies that

$$|R^{(1)}(\mathbf{k}')| = 1 \tag{3.62}$$

and thus

$$\mu^{(1)}(\mathbf{x}', 0, \mathbf{k}', -k_n) = \mu^{(1)}(\mathbf{x}', 0, \mathbf{k}', k_n), \tag{3.63}$$

as expected.

Example 2. Uncorrelated random incident waves.

Assume that the incident amplitudes $\alpha_\varepsilon^{(1)}$ and $\beta_\varepsilon^{(2)}$ are independent random processes with mean zero and let $\mathbf{k}'$ be a horizontal wave vector for which there is no total internal reflection. Then the expectation of the cross Wigner measure $\nu_{\alpha\beta}^{(12)}$ vanishes and (3.53) becomes

$$
\begin{aligned}
<\nu_\beta^{(1)}> &= |R^{(1)}(\mathbf{k}')|^2 <\nu_\alpha^{(1)}> + |T^{(2)}(\mathbf{k}')|^2 <\nu_\beta^{(2)}> \\
<\nu_\alpha^{(2)}> &= |T^{(1)}(\mathbf{k}')|^2 <\nu_\alpha^{(1)}> + |R^{(2)}(\mathbf{k}')|^2 <\nu_\beta^{(2)}>,
\end{aligned} \tag{3.64}
$$

where $<,>$ denotes ensemble average. In this case the averaged Wigner measures satisfy the usual transport-theoretic boundary conditions

$$
\begin{aligned}
<\mu^{(1)}(\mathbf{x}',0,\mathbf{k}',-k_n)> =& |R^{(1)}(\mathbf{k}')|^2 <\mu^{(1)}(\mathbf{x}',0,\mathbf{k}',k_n)> \\
&+ |T^{(2)}(\mathbf{k}')|^2 <\mu^{(2)}(\mathbf{x}',0,\mathbf{k}',-k_n^{(2)})> \\
<\mu^{(2)}(\mathbf{x}',0,\mathbf{k}',k_n^{(2)})> =& |T^{(1)}(\mathbf{k}')|^2 <\mu^{(1)}(\mathbf{x}',0,\mathbf{k}',k_n)> \\
&+ |R^{(2)}(\mathbf{k}')|^2 <\mu^{(2)}(\mathbf{x}',0,\mathbf{k}',-k_n^{(2)})> .
\end{aligned}
$$

Here the wave vector k_n of the incident wave is negative and $k_n^{(2)}$ is the vertical wave vector of the transmitted wave given by (3.59).

Example 3. Correlated spatially homogeneous random beams.

We now consider wave amplitudes $\alpha_\varepsilon^{(1)}(\mathbf{k}')$ and $\beta_\varepsilon^{(2)}(\mathbf{k}')$ of the incident waves that have the form

$$
\alpha_\varepsilon^{(1)}(\mathbf{k}') = \varepsilon^{(n-1)/2}\alpha(\mathbf{k}') \tag{3.65}
$$

and

$$
\beta_\varepsilon^{(2)}(\mathbf{k}') = \varepsilon^{(n-1)/2}\beta(\mathbf{k}'), \tag{3.66}
$$

with $\alpha(\mathbf{k}')$ and $\beta(\mathbf{k}')$ the Fourier transforms of real valued spatially homogeneous random processes. Then $\alpha^*(\mathbf{k}') = \alpha(-\mathbf{k}')$, $\beta^*(\mathbf{k}') = \beta(-\mathbf{k}')$, and the power spectra of $\alpha(\mathbf{k}')$ and $\beta(\mathbf{k}')$ are given by

$$
<\alpha(\mathbf{k}')\alpha(\mathbf{p}')> = (2\pi)^{n-1}\hat{Q}_{\alpha\alpha}(\mathbf{k}')\delta(\mathbf{k}'+\mathbf{p}')
$$

$$< \beta(\mathbf{k}')\beta(\mathbf{p}') > = (2\pi)^{n-1}\hat{Q}_{\beta\beta}(\mathbf{k}')\delta(\mathbf{k}'+\mathbf{p}') \tag{3.67}$$

$$< \alpha(\mathbf{k}')\beta(\mathbf{p}') > = (2\pi)^{n-1}\hat{Q}_{\alpha\beta}(\mathbf{k}')\delta(\mathbf{k}'+\mathbf{p}').$$

In this case the average of the Wigner measures ν_α is

$$\begin{aligned} < \nu_\alpha^{(1)}(\mathbf{k}', -\mathbf{x}') >=&< \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}} e^{-i\mathbf{x}'\cdot\mathbf{p}'} \alpha_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\alpha_\varepsilon^*(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2}) > \\ &= Q_{\alpha\alpha}(\mathbf{k}'). \end{aligned} \tag{3.68}$$

Similarly

$$\begin{aligned} < \nu_\beta^{(2)}(\mathbf{k}', -\mathbf{x}') >=&< \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}} e^{-i\mathbf{x}'\cdot\mathbf{p}'} \beta_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\beta_\varepsilon^*(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2}) > \\ &= Q_{\beta\beta}(\mathbf{k}') \end{aligned} \tag{3.69}$$

and

$$\begin{aligned} < \nu_{\alpha\beta}^{(12)}(\mathbf{k}', -\mathbf{x}') >=&< \int \frac{d\mathbf{p}'}{(2\pi)^{n-1}} e^{-i\mathbf{x}'\cdot\mathbf{p}'} \alpha_\varepsilon(\mathbf{k}' - \frac{\varepsilon\mathbf{p}'}{2})\beta_\varepsilon^*(\mathbf{k}' + \frac{\varepsilon\mathbf{p}'}{2}) > \\ &= Q_{\alpha\beta}(\mathbf{k}'). \end{aligned} \tag{3.70}$$

The boundary values of the average Wigner measures above and below the interface are now given by

$$\begin{aligned} &< \mu^{(1)}(\mathbf{x}', 0, \mathbf{k}', k_n) >= \hat{Q}_{\alpha\alpha}(\mathbf{k}')\delta(k_n - k_n^{(1)-}) \\ &+ \left\{ |R^{(1)}|^2\hat{Q}_{\alpha\alpha}(\mathbf{k}') + |T^{(2)}|^2\hat{Q}_{\beta\beta}(\mathbf{k}') + 2\mathrm{Re}[R^{(1)}\bar{T}^{(2)}\hat{Q}_{\alpha\beta}] \right\} \delta(k_n - k_n^{(1)+}) \end{aligned} \tag{3.71}$$

and

$$\begin{aligned} &< \mu^{(2)}(\mathbf{x}', 0, \mathbf{k}', k_n) >= \left\{ |R^{(2)}|^2\hat{Q}_{\beta\beta}(\mathbf{k}') + |T^{(2)}|^2\hat{Q}_{\alpha\alpha}(\mathbf{k}') \right. \\ &\left. + 2\mathrm{Re}[R^{(2)}\bar{T}^{(1)}\hat{Q}_{\alpha\beta}] \right\} \delta(k_n - k_n^{(2)-}) \\ &+ \hat{Q}_{\beta\beta}(\mathbf{k}')\delta(k_n - k_n^{(2)+}). \end{aligned} \tag{3.72}$$

4 Transport equation and boundary conditions waves in an inhomogeneous half space

4.1 The acoustic equations in an inhomogeneous medium

We will now consider waves in an inhomogeneous medium, where the density $\rho(\mathbf{x})$ and compressibility $\kappa(\mathbf{x})$ are uniformly positive, bounded and smooth functions. The acoustic equations (3.3) are

$$-i\omega\rho(\mathbf{x})\mathbf{u}_\varepsilon + \varepsilon\nabla p_\varepsilon = 0 \qquad (4.1)$$
$$-i\omega\kappa(\mathbf{x})p_\varepsilon + \varepsilon \mathrm{div}\mathbf{u}_\varepsilon = 0.$$

This is a time-reduced symmetric hyperbolic system

$$\varepsilon D^j \frac{\partial \mathbf{w}_\varepsilon}{\partial x^j} - i\omega A(\mathbf{x})\mathbf{w}_\varepsilon = 0, \qquad (4.2)$$

where the matrix $A(\mathbf{x})$ is given by (3.5) and D^j are as before. The eigenvalues and eigenvectors of the principal symbol

$$L(\mathbf{x},\mathbf{k}) = A^{-1}(\mathbf{x})k_j D^j \qquad (4.3)$$

are given by (3.8) and (3.10) as in the constant coefficients case, with $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$ depending on $\mathbf{x}$.

The results that we obtained in Section 3 generalize without essential differences to solutions of the inhomogeneous equations (4.1). However, now we do not have Fourier integral representations for the solutions and we must deduce the results from the equations directly. This requires use of the theory of Wigner measures outlined in Section 2 and it is carried out in this Section.

4.2 Statement of results

Let $\mathbf{w}_\varepsilon$ be a family of solutions of (4.2), uniformly bounded in L^2_{loc} , with boundary values

$$\mathbf{r}_\varepsilon(\mathbf{x}') = \mathbf{w}_\varepsilon(\mathbf{x}', 0) \tag{4.4}$$

that are also uniformly bounded in L^2_{loc}. Given a tangential wave vector $\mathbf{k}' \in R^{n-1}$, we define the wave vectors $\mathbf{k}^\pm(\mathbf{k}') = (\mathbf{k}', k_n^\pm)$, where

$$k_n^\pm(\mathbf{x}', 0) = \pm\sqrt{\frac{\omega^2}{v(\mathbf{x}', 0)^2} - \mathbf{k}'^2}. \tag{4.5}$$

We assume that the singular support of the Wigner measure of $r_n(\mathbf{x}') = u_n(\mathbf{x}', 0)$ is away from the sphere $|\mathbf{k}'| = \frac{\omega}{v}$.

Lemma 1 *Any $n-1$-dimensional Wigner measure $\boldsymbol{\nu}(\mathbf{x}', \mathbf{k}')$ for the boundary value $\mathbf{r}_\varepsilon(\mathbf{x}')$ has the form*

$$\begin{aligned}\boldsymbol{\nu} = &\nu_\beta\, \mathbf{b}_+(\mathbf{k}^+)\mathbf{b}_+^*(\mathbf{k}^+) + \nu_{\beta\alpha}\mathbf{b}_+(\mathbf{k}^+)\mathbf{b}_+^*(\mathbf{k}^-) \\ &+ \nu_{\beta\alpha}^*\mathbf{b}_+(\mathbf{k}^-)\mathbf{b}_+^*(\mathbf{k}^+) + \nu_\alpha\mathbf{b}_+(\mathbf{k}^-)\mathbf{b}_+^*(\mathbf{k}^-),\end{aligned} \tag{4.6}$$

where all ν's on the right side of (4.6) are scalar measures, and the eigenvector $\mathbf{b}_+(\mathbf{x}, \mathbf{k})$ is given by (3.10).

The analog of Proposition 2 is as follows.

Theorem 1 *Any Wigner measure $\boldsymbol{\mu}$ of the family $\mathbf{w}_\varepsilon$ has the form*

$$\boldsymbol{\mu}(\mathbf{x}, \mathbf{k}) = \mu(\mathbf{x}, \mathbf{k})\mathbf{b}_+(\mathbf{x}, \mathbf{k})\mathbf{b}_+^*(\mathbf{x}, \mathbf{k}). \tag{4.7}$$

The scalar measure μ is supported on the set

$$U = \{(\mathbf{x}, \mathbf{k}) : v(\mathbf{x})|\mathbf{k}| = \omega\} \tag{4.8}$$

and satisfies weakly the transport equation

$$\nabla_\mathbf{k}\omega_+ \cdot \nabla_\mathbf{x}\mu - \nabla_\mathbf{x}\omega_+ \cdot \nabla_\mathbf{k}\mu = v\hat{k}_n(\nu_\alpha\delta(k_n - k_n^-) + \nu_\beta\delta(k_n - k_n^+))\delta(x^n) \tag{4.9}$$

Here the measures ν_α and ν_β are defined by (4.6), the weak limit $\boldsymbol{\nu}$ corresponding to the same subsequence $\varepsilon_j \to 0$.

The second statement (4.8) generalizes the eiconal equation of geometrical optics. It implies that the phase function $S(\mathbf{x})$ for the geometrical optics solutions (2.17), with the Wigner measure given by (2.18), satisfies the eiconal equation [2, 8]

$$|\nabla S(\mathbf{x})|^2 = \frac{\omega^2}{v^2(\mathbf{x})}. \tag{4.10}$$

The transport equation (4.9) in the interior of the upper half space does not depend on the presence of boundaries and is the same as for acoustics waves in all of R^n [7, 15], as expected. It reduces to $\nabla \cdot (\nabla S(\mathbf{x})|A(\mathbf{x})|^2) = 0$, the usual transport equation of geometrical optics, for Wigner measures of the form (2.18).

Note that if μ satisfies the transport equation

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = 0 \tag{4.11}$$

for $x^n > 0$, $\mu = 0$ for $x^n < 0$, and $\mu(\mathbf{x}, \mathbf{k})$ is continuous up to the boundary, so that $\mu(\mathbf{x}', 0, \mathbf{k})$ is defined, then the weak form of (4.11) is

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = \frac{\partial \omega_+}{\partial k_n}\mu(\mathbf{x}', 0, \mathbf{k})\delta(x^n). \tag{4.12}$$

Since $\dfrac{\partial \omega_+}{\partial k_n} = v\hat{k}_n$, (4.9) is equivalent in that case to the boundary value problem

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = 0, \quad x^n > 0 \tag{4.13}$$

$$\mu(\mathbf{x}', 0, \mathbf{k}) = \nu_\beta(\mathbf{x}', \mathbf{k}')\delta(k_n - k_n^+) + \nu_\alpha(\mathbf{x}', \mathbf{k}')\delta(k_n - k_n^-). \tag{4.14}$$

The boundary value (4.14) of $\mu(\mathbf{x}, \mathbf{k})$ has the same form as (3.23), the difference being that the vertical wave number $k_n^\pm$ varies with $\mathbf{x}$. Furthermore, ν_α and

ν_β, defined by (4.6), are not given explicitly as limits of Fourier integrals as in (3.17) and (3.18).

The form (4.14) of the Wigner measure on the boundary can be used to derive boundary and interface conditions for reflection and transmission as in Sections 3.5, 3.6 and 3.7. The results in the inhomogeneous case are identical to the ones in the homogeneous case but they are derived in a different way. We now illustrate this with reflecting boundaries.

Consider the acoustic equations (4.1) with Dirichlet boundary conditions (3.43). The last component of the vector $\mathbf{r}_\varepsilon(\mathbf{x}')$ vanishes identically and hence the last row and column of the matrix $\boldsymbol{\nu}$ vanish. Applying this condition to a matrix of the form (4.6) and using the explicit form (3.10) of the eigenvectors $\mathbf{b}_+(\mathbf{x},\mathbf{k})$ we obtain

$$\nu_\alpha = \nu_\beta = \nu_{\beta\alpha}. \tag{4.15}$$

The form (4.14) of the boundary Wigner measure implies now that

$$\mu(\mathbf{x}', 0, \mathbf{k}', k_n) = \mu(\mathbf{x}', 0, \mathbf{k}', -k_n) \tag{4.16}$$

as in (3.47).

4.3 Derivation of the transport equation and boundary conditions

4.3.1 Outline of the derivation

We cannot derive (4.7-4.14) using explicit solutions, as for a homogeneous medium (3.16-3.18), because there aren't any in general. We use instead a modification of the technique of Gerard and Leichtnam [6], which can be applied to any symmetric time-harmonic hyperbolic system such as Maxwell's equations and the elastic equations. First we prove Theorem 1, postponing the proof of Lemma 1 till the end. In the next section we restrict solutions to a compact set and show that any Wigner measure $\boldsymbol{\mu}$ has the form (4.7) with μ supported on the set U defined by (4.8). We derive the weak form (4.43) of the transport equation in Section 4.3.3. In Section 4.3.4 we evaluate the limit on the right side of (4.43) in terms of the matrix-valued boundary Wigner measure $\boldsymbol{\nu}(\mathbf{x}', \mathbf{k}')$ and in Section 4.3.5 we use expression (4.6) for this measure to obtain (4.14). The proof of Lemma 1 is in Section 4.3.6.

4.3.2 The eiconal equation

We set $\mathbf{w}_\varepsilon$ to zero in the lower half space $x^n < 0$ and, following [6], rewrite (4.2) in weak form in R^n:

$$\varepsilon D^j \frac{\partial \mathbf{w}_\varepsilon}{\partial x^j} - i\omega A(\mathbf{x})\mathbf{w}_\varepsilon = \varepsilon D^n \mathbf{r}_\varepsilon(\mathbf{x}') \otimes \delta(x^n). \tag{4.17}$$

Here $\mathbf{r}_\varepsilon(\mathbf{x}')$ are the boundary values given by (4.4). The Wigner measure of families of functions uniformly bounded in L^2_{loc} is defined in $\mathcal{D}'(R^n \times R^n)$ using the Localization Property 3 as follows. To define a Wigner measure $\boldsymbol{\mu}$ on a compact set K we multiply $\mathbf{w}_\varepsilon$ by a test function $\theta(\mathbf{x})$ of compact support that is equal to one on K. The Wigner measure of the family $\theta\mathbf{w}_\varepsilon$ on K is independent of the choice of the cutoff function θ. Let

$$\mathbf{w}^\theta_\varepsilon(\mathbf{x}) = \theta(\mathbf{x})\mathbf{w}_\varepsilon(\mathbf{x}) \tag{4.18}$$

and let $\mathbf{r}^\theta_\varepsilon(\mathbf{x}')$ be its boundary value. Then $\mathbf{w}^\theta_\varepsilon$ and $\mathbf{r}^\theta_\varepsilon$ are uniformly bounded in L^2. The system with cutoff is

$$\varepsilon D^j \frac{\partial \mathbf{w}^\theta_\varepsilon}{\partial x^j} - i\omega A(\mathbf{x})\mathbf{w}^\theta_\varepsilon - \varepsilon D^j \frac{\partial \theta}{\partial x^j}\mathbf{w}_\varepsilon = \varepsilon D^n \mathbf{r}^\theta_\varepsilon(\mathbf{x}') \otimes \delta(x^n). \tag{4.19}$$

Let $a(\mathbf{x}, \mathbf{k})$ be a matrix-valued test function having support with respect to $\mathbf{x}$ inside K. We apply the operator $a_\varepsilon = a(\mathbf{x}, \varepsilon D)$, given by (2.14), to both sides of (4.19) and take inner product with $\mathbf{w}^\theta_\varepsilon$ to obtain

$$(a_\varepsilon(\varepsilon D^j \frac{\partial \mathbf{w}^\theta_\varepsilon}{\partial x^j} - i\omega A\mathbf{w}^\theta_\varepsilon - \varepsilon D^j \frac{\partial \theta}{\partial x^j}\mathbf{w}_\varepsilon), \mathbf{w}^\theta_\varepsilon) = \varepsilon(a_\varepsilon(D^n \mathbf{r}_\varepsilon \otimes \delta), \mathbf{w}^\theta_\varepsilon). \tag{4.20}$$

The family $\mathbf{r}_\varepsilon^\theta \otimes \delta$ is uniformly bounded in $H^{-1/2-\alpha}$ for any $\alpha > 0$, as can be seen by taking Fourier transforms, and thus estimate (5.4) implies that the right side of (4.20) vanishes as $\varepsilon \to 0$. To evaluate the limit on the left side we note that the third and fourth terms go to zero and in the first two terms we use the product rule (2.19). Thus

$$\mathrm{Tr} \int a(\mathbf{x}, \mathbf{k})(ik_j D^j - i\omega A)\boldsymbol{\mu}(d\mathbf{x}d\mathbf{k}) = 0 \tag{4.21}$$

for all test functions $a(\mathbf{x}, \mathbf{k})$. This implies that

$$(k_j D^j - \omega A)\boldsymbol{\mu} = 0 \tag{4.22}$$

and, since $\boldsymbol{\mu}$ is self-adjoint,

$$\boldsymbol{\mu}(k_j D^j - \omega A) = 0. \tag{4.23}$$

Therefore the Wigner measure $\boldsymbol{\mu}$ is supported on the set $U = \{\mathbf{k} : v(\mathbf{x})|\mathbf{k}| = \omega\}$, where the dispersion matrix $L(\mathbf{x}, \mathbf{k})$ in (4.3) has eigenvalue ω. It has the form

$$\boldsymbol{\mu}(\mathbf{x}, \mathbf{k}) = \mu(\mathbf{x}, \mathbf{k})\mathbf{b}_+(\mathbf{x}, \mathbf{k})\mathbf{b}_+^*(\mathbf{x}, \mathbf{k}) \tag{4.24}$$

with the scalar measure μ supported on the set U, as claimed in (4.7).

4.3.3 Weak form of the transport equation

We now derive the transport equation (4.9) for $\mu(\mathbf{x}, \mathbf{k})$. Equations (4.19) imply the identity

$$\begin{aligned} 0 =& \, i\omega(a_\varepsilon \mathbf{w}_\varepsilon^\theta, \mathbf{w}_\varepsilon^\theta) - i\omega(a_\varepsilon \mathbf{w}_\varepsilon^\theta, \mathbf{w}_\varepsilon^\theta) = (a_\varepsilon i\omega \mathbf{w}_\varepsilon, \mathbf{w}_\varepsilon^\theta) + (a_\varepsilon \mathbf{w}_\varepsilon^\theta, i\omega \mathbf{w}_\varepsilon^\theta) \\ =& \, \varepsilon(a_\varepsilon \left[A^{-1}D^j \frac{\partial \mathbf{w}_\varepsilon^\theta}{\partial x^j}\right], \mathbf{w}_\varepsilon^\theta) - \varepsilon(a_\varepsilon \left[A^{-1}D^n \mathbf{r}_\varepsilon^\theta \otimes \delta\right], \mathbf{w}_\varepsilon^\theta) \\ & -\varepsilon(\frac{\partial}{\partial x^j}\left[D^j A^{-1} a_\varepsilon \mathbf{w}_\varepsilon^\theta\right], \mathbf{w}_\varepsilon^\theta) - \varepsilon(a_\varepsilon \mathbf{w}_\varepsilon^\theta, A^{-1}D^n \mathbf{r}_\varepsilon^\theta \otimes \delta) \\ & -\varepsilon(a_\varepsilon \left[A^{-1}D^j \frac{\partial \theta}{\partial x^j}\mathbf{w}_\varepsilon\right], \mathbf{w}_\varepsilon^\theta) - \varepsilon(a_\varepsilon \mathbf{w}_\varepsilon^\theta, A^{-1}D^j \frac{\partial \theta}{\partial x^j}\mathbf{w}_\varepsilon). \end{aligned} \tag{4.25}$$

The last term in the last line vanishes because the function θ equals one identically on the support of $a(\mathbf{x},\mathbf{k})$. We rewrite (4.25) as

$$\begin{aligned}
&\varepsilon(a_\varepsilon\left[A^{-1}D^j\frac{\partial \mathbf{w}_\varepsilon^\theta}{\partial x^j}\right],\mathbf{w}_\varepsilon^\theta)-\varepsilon(\frac{\partial}{\partial x^j}\left[D^jA^{-1}a_\varepsilon\mathbf{w}_\varepsilon^\theta\right],\mathbf{w}_\varepsilon^\theta)\\
&-\varepsilon(a_\varepsilon\left[A^{-1}D^j\frac{\partial\theta}{\partial x^j}\mathbf{w}_\varepsilon\right],\mathbf{w}_\varepsilon^\theta)\\
&=\varepsilon(a_\varepsilon\left[A^{-1}D^n\mathbf{r}_\varepsilon^\theta\otimes\delta\right],\mathbf{w}_\varepsilon^\theta)+\varepsilon(a_\varepsilon\mathbf{w}_\varepsilon^\theta,A^{-1}D^n\mathbf{r}_\varepsilon^\theta\otimes\delta). \qquad (4.26)
\end{aligned}$$

The product rule (2.19) implies that

$$\varepsilon a_\varepsilon A^{-1}D^j\frac{\partial}{\partial x^j}-\varepsilon\frac{\partial}{\partial x^j}D^jA^{-1}a_\varepsilon=\phi_0(\mathbf{x},\varepsilon D)+\varepsilon\phi_1(\mathbf{x},\varepsilon D)+\varepsilon^2R_\varepsilon, \quad (4.27)$$

where ϕ_0 is

$$\phi_0(\mathbf{x},\mathbf{k})=ia(\mathbf{x},\mathbf{k})A^{-1}(\mathbf{x})k_jD^j-ik_jD^jA^{-1}(\mathbf{x})a(\mathbf{x},\mathbf{k}), \qquad (4.28)$$

ϕ_1 is

$$\phi_1(\mathbf{x},\mathbf{k})=\frac{\partial a}{\partial k_m}\frac{\partial A^{-1}}{\partial x^m}D^jk_j-D^j\frac{\partial A^{-1}}{\partial x^j}a-D^jA^{-1}\frac{\partial a}{\partial x^j} \qquad (4.29)$$

and the operators R_ε are uniformly bounded on L^2. To evaluate the limit of (4.26) as $\varepsilon\to 0$ we use (4.27), (5.4) and (5.5), as in the previous section. We obtain

$$\mathrm{Tr}\int\phi_0(\mathbf{x},\mathbf{k})\boldsymbol{\mu}(d\mathbf{x}d\mathbf{k})=0 \qquad (4.30)$$

for all $a(\mathbf{x},\mathbf{k})$. This holds automatically because of the eiconal equation (4.22) or (4.23).

Next, we divide (4.26) by ε, and take the limit $\varepsilon\to 0$. This gives

$$\mathrm{Tr}\int\phi_1(\mathbf{x},\mathbf{k})\boldsymbol{\mu}(d\mathbf{x}d\mathbf{k})+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\phi_0(\mathbf{x},\varepsilon D)\mathbf{w}_\varepsilon^\theta,\mathbf{w}_\varepsilon^\theta)=\lim_{\varepsilon\to 0}M_\varepsilon(a), \quad (4.31)$$

where

$$M_\varepsilon(a)=(a_\varepsilon[A^{-1}(\mathbf{x})D^n\mathbf{r}_\varepsilon^\theta\otimes\delta],\mathbf{w}_\varepsilon^\theta)+(a_\varepsilon\mathbf{w}_\varepsilon^\theta,A^{-1}(\mathbf{x})D^n\mathbf{r}_\varepsilon^\theta\otimes\delta). \quad (4.32)$$

The term involving $\partial\theta/\partial x^j$ in (4.26) vanishes in this limit since $a(\mathbf{x},\mathbf{k})$ and $\partial\theta/\partial x^j$ have disjoint support. To derive the transport equation for the scalar measure μ it is sufficient to consider test functions $a(\mathbf{x},\mathbf{k})$ of the form

$$a(\mathbf{x},\mathbf{k}) = a_+(\mathbf{x},\mathbf{k})\mathbf{c}_+(\mathbf{x},\mathbf{k})\mathbf{c}_+^*(\mathbf{x},\mathbf{k}), \tag{4.33}$$

with $a_+(\mathbf{x},\mathbf{k})$ scalar valued, and $\mathbf{c}_+$ the left eigenvector of the dispersion matrix (4.3) corresponding to the eigenvalue ω_+. It is given by

$$\mathbf{c}_+(\mathbf{x},\mathbf{k}) = A(\mathbf{x})\mathbf{b}_+(\mathbf{x},\mathbf{k}). \tag{4.34}$$

Then ϕ_0 vanishes and ϕ_1 becomes

$$\begin{aligned}\phi_1 = {} & \frac{\partial a_+}{\partial k_s}\mathbf{c}_+\mathbf{c}_+^*\frac{\partial A^{-1}}{\partial x^s}D^j k_j - \frac{\partial a_+}{\partial x^j}D^j A^{-1}\mathbf{c}_+\mathbf{c}_+^* + a_+\left\{\frac{\partial \mathbf{c}_+}{\partial k_s}\mathbf{c}_+^*\frac{\partial A^{-1}}{\partial x^s}D^j k_j\right. \\ & \left. + \mathbf{c}_+\frac{\partial \mathbf{c}_+^*}{\partial k_s}\frac{\partial A^{-1}}{\partial x^s}D^j k_j - D^j\frac{\partial A^{-1}}{\partial x^j}\mathbf{c}_+\mathbf{c}_+^* - D^j A^{-1}\frac{\partial \mathbf{c}_+}{\partial x^j}\mathbf{c}_+^* - D^j A^{-1}\mathbf{c}_+\frac{\partial \mathbf{c}_+^*}{\partial x^j}\right\} \\ & = \phi_{11} + \phi_{12} + \phi_{13}.\end{aligned} \tag{4.35}$$

We calculate $< \phi_1, \boldsymbol{\mu} >$ by evaluating $\boldsymbol{\mu}$ against each term in (4.35) separately. This is similar to the calculation performed in R^n in [7] and [15]. Using (4.24), the normalization (3.11) and (3.26), we first compute

$$\begin{aligned}< \phi_{11}, \boldsymbol{\mu} > &= < \frac{\partial a_+}{\partial k_s}(\mathbf{b}_+, \mathbf{c}_+)(k_j D^j\frac{\partial A^{-1}}{\partial x^s}\mathbf{c}_+, \mathbf{b}_+), \mu > \\ &= < \frac{\partial a_+}{\partial k_s}(\frac{\partial \omega_+}{\partial x^s}\mathbf{c}_+ + \omega_+\frac{\partial \mathbf{c}_+}{\partial x^s} - k_j D^j A^{-1}\frac{\partial \mathbf{c}_+}{\partial x^s}, \mathbf{b}_+), \mu > \\ &= < \frac{\partial \omega_+}{\partial x^s}\frac{\partial a_+}{\partial k_s}, \mu >\end{aligned} \tag{4.36}$$

and

$$< \phi_{12}, \boldsymbol{\mu} > = - < \frac{\partial a^{\alpha}_{im}}{\partial x^j}(\mathbf{b}_+, \mathbf{c}_+)(D^j A^{-1}\mathbf{c}_+, \mathbf{b}_+), \mu > = - < \frac{\partial \omega_+}{\partial k_j}\frac{\partial a_+}{\partial x^j}, \mu(> \tag{4.37}$$

We now show that

$$< \phi_{13}, \boldsymbol{\mu} > = 0. \tag{4.38}$$

We write

$$< \phi_{13}, \boldsymbol{\mu} >=< a_+ T, \mu >, \tag{4.39}$$

where

$$\begin{aligned} T= & (\mathbf{b}_+, \frac{\partial \mathbf{c}_+}{\partial k_s})(\mathbf{b}_+, k_j D^j \frac{\partial A^{-1}}{\partial x^s}\mathbf{c}_+) + (\mathbf{b}_+, \mathbf{c}_+)(\mathbf{b}_+, k_j D^j \frac{\partial A^{-1}}{\partial x^s}\frac{\partial \mathbf{c}_+}{\partial k_s}) \\ & -(\mathbf{b}_+, D^j \frac{\partial A^{-1}}{\partial x^j}\mathbf{c}_+)(\mathbf{b}_+, \mathbf{c}_+) - (\mathbf{b}_+, D^j A^{-1}\frac{\partial \mathbf{c}_+}{\partial x^j})(\mathbf{b}_+, \mathbf{c}_+) \\ & -(\mathbf{b}_+, D^j A^{-1}\mathbf{c}_+)(\mathbf{b}_+, \frac{\partial \mathbf{c}_+}{\partial x^j}). \end{aligned} \tag{4.40}$$

The first term in T vanishes because of the normalization (3.11). We transform the last term using (3.26) and (4.34) to get

$$T = (\mathbf{b}_+, k_j D^j \frac{\partial A^{-1}}{\partial x^s}\frac{\partial \mathbf{c}_+}{\partial k_s} - D^j \frac{\partial A^{-1}}{\partial x^j}\mathbf{c}_+ - D^j A^{-1}\frac{\partial \mathbf{c}_+}{\partial x_j} - \frac{\partial \omega_+}{\partial k_j}\frac{\partial \mathbf{c}_+}{\partial x^j}). \tag{4.41}$$

Now we use the fact that $\mathbf{b}_+$ is an eigenvector of (4.3) and (4.34) to transform the first term. We also use (4.34) to combine the second and third terms, and use (3.11) to transform the last one in (4.41). This gives

$$T = -\omega_+(\mathbf{b}_+, \frac{\partial A}{\partial x^s}\frac{\partial \mathbf{b}_+}{\partial k_s}) - (D^j \mathbf{b}_+, \frac{\partial \mathbf{b}_+}{\partial x^j}) + \frac{\partial \omega_+}{\partial k_j}(\mathbf{b}_+, A\frac{\partial \mathbf{b}_+}{\partial x^j}). \tag{4.42}$$

If we use again the fact that $\mathbf{b}_+$ is an eigenvector of $L(\mathbf{x}, \mathbf{k})$ we can rewrite the last two terms and then collect them together to obtain

$$\begin{aligned} T= & -\omega_+(\mathbf{b}_+, \frac{\partial A}{\partial x^j}\frac{\partial \mathbf{b}_+}{\partial k_j}) + (\frac{\partial \mathbf{b}_+}{\partial x^s}, k_j D^j \frac{\partial \mathbf{b}_+}{\partial k_s} - A\omega_+ \frac{\partial \mathbf{b}_+}{\partial k_s}) \\ = & \frac{\partial \omega_+}{\partial x^j}(A\mathbf{b}_+, \frac{\partial \mathbf{b}_+}{\partial k_j}) = 0, \end{aligned}$$

which is (4.38) . Then inserting (4.36), (4.37) and (4.38) into (4.31) we obtain

$$< a_+, \nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu >= \lim_{\varepsilon \to 0} M_\varepsilon(a). \tag{4.43}$$

If the support of the test function $a(\mathbf{x}, \mathbf{k})$ lies inside the upper half space $x^n > 0$, then $\lim_{\varepsilon \to 0} M_\varepsilon(a) = 0$. In fact, then the second term in (4.32) vanishes identically, and the first vanishes in the limit $\varepsilon \to 0$, because a is supported away from the support of the δ-function. Thus we obtain the transport equation (4.9) in the interior of the upper half space.

4.3.4 The boundary term in the transport equation

We now show that the limit on the right side of (4.43) is zero for a_+ supported in the interior, which gives the transport equation (4.9). It also gives the weak form of the boundary conditions (4.14). We recall that $M_\varepsilon(a)$ is defined by (4.32)

$$M_\varepsilon(a) = (a_\varepsilon[A^{-1}(\mathbf{x})D^n\mathbf{r}_\varepsilon^\theta \otimes \delta], \mathbf{w}_\varepsilon^\theta) + (a_\varepsilon \mathbf{w}_\varepsilon^\theta, A^{-1}(\mathbf{x})D^n\mathbf{r}_\varepsilon^\theta \otimes \delta) \quad (4.44)$$

and that a has the form (4.33).

Each term in (4.44) is of order $\varepsilon^{-1/2-\alpha}$ for any $\alpha > 0$, as can be seen from the H^s estimates (5.4) and (5.5) since $\mathbf{r}_\varepsilon^\theta \otimes \delta$ is uniformly bounded in H^s for $s = -\frac{1}{2} - \alpha$ for any $\alpha > 0$. Thus cancellation of divergent terms should occur if $M_\varepsilon(a)$ is to have a finite limit.

Let us first consider the limit of $M_\varepsilon(a)$ for a special class of matrices $a(\mathbf{x}, \mathbf{k})$ of the form

$$a(\mathbf{x}, \mathbf{k}) = \tilde{a}(\mathbf{x}, \mathbf{k})[L(\mathbf{x}, \mathbf{k}) - \omega I]. \quad (4.45)$$

Here $L(\mathbf{x}, \mathbf{k})$ is the dispersion matrix (4.3), and the matrix $\tilde{a}(\mathbf{x}, \mathbf{k})$ satisfies

$$\tilde{a}(\mathbf{x}, \mathbf{k})[L(\mathbf{x}, \mathbf{k}) - \omega I] = [L^*(\mathbf{x}, \mathbf{k}) - \omega I]\tilde{a}(\mathbf{x}, \mathbf{k}). \quad (4.46)$$

Then

$$M_\varepsilon(a) \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad (4.47)$$

which is verified using the product rule (2.19) and the acoustic equations (4.19) as follows

$$\begin{aligned} M_\varepsilon(a) = & (([L^* - \omega I]\tilde{a})(\mathbf{x}, \varepsilon D)(A^{-1}D^n\mathbf{r}_\varepsilon^\theta \otimes \delta, \mathbf{w}_\varepsilon^\theta) \\ & +((\tilde{a}[L - \omega I])(\mathbf{x}, \varepsilon D)\mathbf{w}_\varepsilon^\theta, A^{-1}D^n\mathbf{r}_\varepsilon^\theta \otimes \delta) \\ & \sim (\tilde{a}(\mathbf{x}, \varepsilon D)(A^{-1}D^n\mathbf{r}_\varepsilon^\theta \otimes \delta), \frac{\varepsilon}{i}A^{-1}D^n\mathbf{r}_\varepsilon^\theta \otimes \delta + \frac{\varepsilon}{i}A^{-1}D^j\frac{\partial\theta}{\partial x^j}\mathbf{w}_\varepsilon) \quad (4.48) \\ & +(\tilde{a}(\mathbf{x}, \varepsilon D)(\frac{\varepsilon}{i}A^{-1}D^n\mathbf{r}_\varepsilon^\theta \otimes \delta + \frac{\varepsilon}{i}A^{-1}D^j\frac{\partial\theta}{\partial x^j}\mathbf{w}_\varepsilon^\theta, A^{-1}D^n\mathbf{r}_\varepsilon^\theta \otimes \delta) \to 0. \end{aligned}$$

Here $\sim$ means that two expressions have the same limit as $\varepsilon \to 0$.

The scalar operators $a_+(\mathbf{x}, \varepsilon D)$ in the expression (4.33) for a act in (4.44) approximately, by the product rule (2.19), on solutions $\mathbf{w}_\varepsilon^\theta$ of the acoustic equations. If we could write a_+ in the form $\tilde{a}_+(v|\mathbf{k}| - \omega)$ then $M_\varepsilon(a) \to 0$ because $\tilde{a}_+(v|\mathbf{k}|-\omega)c_+c_+^* = \tilde{a}_+c_+c_+^*[L-\omega I]$. This means that $a = a_+c_+c_+^*$ has the form (4.45) with $\tilde{a} = \tilde{a}_+c_+c_+^*$ and (4.46) holds. However, this is impossible because the dispersion matrix $L - \omega I$ is not invertible. It is singular on the set $U = \{(\mathbf{x}, \mathbf{k}) : v(\mathbf{x})|\mathbf{k}| = \omega\}$ which means that not every matrix a is in the range of $L - \omega I$. It is, however, possible to write every test function a_+ in the form

$$a_+(\mathbf{x}, \mathbf{k}) = a_0(\mathbf{x}, \mathbf{k}') + a_1(\mathbf{x}, \mathbf{k}')k_n + a_2(\mathbf{x}, \mathbf{k})(v|\mathbf{k}| - \omega) \tag{4.49}$$

with $\mathbf{k} = (\mathbf{k}', k_n)$, the tangential and normal components of the wave vector, and a_0, a_1 and a_2 test functions that are uniquely determined by a_+. Then any a of the form (4.33) can be written as

$$\begin{aligned} a(\mathbf{x}, \mathbf{k}) = &(a_0(\mathbf{x}, \mathbf{k}') + a_1(\mathbf{x}, \mathbf{k}')k_n)A(\mathbf{x}) \\ &+ \left(a_2(\mathbf{x}, \mathbf{k})\mathbf{c}_+\mathbf{c}_+^* + \frac{a_0(\mathbf{x}, \mathbf{k}') + a_1(\mathbf{x}, \mathbf{k}')k_n}{v(\mathbf{x})|\mathbf{k}| + \omega}\mathbf{c}_-\mathbf{c}_-^* \right. \\ &\left. + \frac{a_0(\mathbf{x}, \mathbf{k}') + a_1(\mathbf{x}, \mathbf{k}')k_n}{\omega} \sum_{j=1}^{n-1} \mathbf{c}_0^j \mathbf{c}_0^{j*} \right) [L - \omega I](\mathbf{x}, \mathbf{k}). \end{aligned} \tag{4.50}$$

This follows from the spectral representation of $L - \omega I$

$$L - \omega I = (\omega_+ - \omega)\mathbf{b}_+\mathbf{c}_+^* + (\omega_- - \omega)\mathbf{b}_-\mathbf{c}_-^* - \omega \sum_{j=1}^{n-1} \mathbf{b}_0^j \mathbf{c}_0^{j*} \tag{4.51}$$

and the resolution of the identity in the form

$$A = \mathbf{c}_+\mathbf{c}_+^* + \mathbf{c}_-\mathbf{c}_-^* + \sum_{j=1}^{n-1} \mathbf{c}_0^j \mathbf{c}_0^{j*} \tag{4.52}$$

The last term in the expression (4.50) for a has the form $a = \tilde{a}[L - \omega I]$ of (4.45) and the relation (4.46) holds. Therefore, in calculating the limit of

$M_\varepsilon(a)$ it is enough to consider the first two terms, the ones multiplied by A, in (4.50).

Let a' denote the first two terms in (4.50). Since it does not decay in k_n we regularize it by multiplying it by a suitable cutoff function

$$a'' = a'\phi(\varepsilon^3 k_n). \tag{4.53}$$

The function $\phi(k_n)$ is compactly supported and equal to one in a neighbourhood of zero and the scaling in ε is faster than the oscillations so the limits are not affected (see[6] for the details). The first term in $M_\varepsilon(a'')$ is, after using the product rule to cancel A with its inverse,

$$\begin{aligned}
&\int d\mathbf{x}\mathbf{w}_\varepsilon^\theta(\mathbf{x})\int \frac{d\boldsymbol{\xi}}{(2\pi)^n} e^{i\mathbf{k}\cdot\mathbf{x}} a_0(\mathbf{x},\varepsilon\mathbf{k}')\phi(\varepsilon^3 k_n) D^n \hat{\mathbf{r}}_\varepsilon^\theta(\mathbf{k}')\\
&+\int d\mathbf{x}'\mathbf{r}_\varepsilon^{\theta *}\int \frac{d\mathbf{k}}{(2\pi)^n} e^{i\mathbf{k}'\cdot\mathbf{x}'} a_0(\mathbf{x}',0,\mathbf{k}')\phi(\varepsilon^3 k_n) D^n \hat{\mathbf{w}}_\varepsilon^\theta(\mathbf{k})\\
&\to \mathrm{Tr}\int d\boldsymbol{\nu} D^n a_0(\mathbf{x}',0,\mathbf{k}').
\end{aligned} \tag{4.54}$$

Here $\boldsymbol{\nu}$ is the Wigner measure of $\mathbf{r}_\varepsilon^\theta(\mathbf{x}')$. Similarly, the second term in $M_\varepsilon(a'')$ is approximately

$$\begin{aligned}
&\int d\mathbf{x}\mathbf{w}_\varepsilon^{\theta *}(\mathbf{x})\int \frac{d\boldsymbol{\xi}}{(2\pi)^n} e^{i\mathbf{k}\cdot\mathbf{x}} a_1(\mathbf{x},\varepsilon\mathbf{k}')\varepsilon k_n\phi(\varepsilon^3 k_n) D^n \hat{\mathbf{r}}_\varepsilon^\theta(\mathbf{k}')\\
&+\int d\mathbf{x}'\mathbf{r}_\varepsilon^{\theta *}\int \frac{d\mathbf{k}}{(2\pi)^n} e^{i\mathbf{k}'\cdot\mathbf{x}'} a_1(\mathbf{x}',0,\mathbf{k}')\varepsilon k_n\phi(\varepsilon^3 k_n) D^n \hat{\mathbf{w}}_\varepsilon^\theta(\mathbf{k})\\
&\sim -\frac{\varepsilon}{i}\int d\mathbf{x}\frac{\partial \mathbf{w}_\varepsilon^{\theta *}}{\partial x^n}(\mathbf{x})\int \frac{d\boldsymbol{\xi}}{(2\pi)^n} e^{i\mathbf{k}\cdot\mathbf{x}} a_1(\mathbf{x},\varepsilon\mathbf{k}')\phi(\varepsilon^3 k_n) D^n \hat{\mathbf{r}}_\varepsilon^\theta(\mathbf{k}')\\
&+\frac{\varepsilon}{i}\int d\mathbf{x}'\mathbf{r}_\varepsilon^{\theta *}\int \frac{d\mathbf{k}}{(2\pi)^n} e^{i\mathbf{k}'\cdot\mathbf{x}'} a_1(\mathbf{x}',0,\mathbf{k}')\phi(\varepsilon^3 k_n) D^n \widehat{\frac{\partial \mathbf{w}_\varepsilon^\theta}{\partial x^n}}(\mathbf{k})\\
&\to -\mathrm{Tr}\int d\boldsymbol{\nu}[\sum_{j=1}^{n-1}\xi_j D^j - \omega A(\mathbf{x}',0)]a_1(\mathbf{x}',0,\mathbf{k}').
\end{aligned} \tag{4.55}$$

Thus we have

$$\begin{aligned}
M_\varepsilon(a) \to \mathrm{Tr}\int d\boldsymbol{\nu}[&D^n a_0(\mathbf{x}',0,\mathbf{k}')\\
&-(\sum_{j=1}^{n-1}\xi_j D^j - \omega A(\mathbf{x}',0))a_1(\mathbf{x}',0,\mathbf{k}')].
\end{aligned} \tag{4.56}$$

This is zero if the test function $a(\mathbf{x},\mathbf{k})$ is supported away from $x^n = 0$ and thus the transport equation (4.9) in the interior follows from (4.43).

4.3.5 Boundary conditions for the transport equation

We first recall the relation (3.26), $(D^n \mathbf{b}_+(\mathbf{k}), \mathbf{b}_+(\mathbf{k})) = v\hat{k}_n$ and

$$\begin{aligned} &(D^n \mathbf{b}_+(\mathbf{k}^+), \mathbf{b}_+(\mathbf{k}^-)) = 0, \\ &(\sum_{j=1}^{n-1} \xi_j D^j - \omega A)\mathbf{b}_+(\mathbf{k}^\pm) = -k_n^\pm D^n \mathbf{b}_+(\mathbf{k}^\pm). \end{aligned} \tag{4.57}$$

We next use these relations and the form (4.6) of the boundary Wigner measure $\boldsymbol{\nu}$ in the limit (4.56) to get, after using (4.50),

$$\begin{aligned} \lim_{\varepsilon\to 0} M_\varepsilon(a) =& \int \nu_\alpha(d\mathbf{x}'d\mathbf{k}')[v\hat{k}_n^- a_0 + vk_n^- \hat{k}_n^- a_1] + \\ & \int \nu_\beta(d\mathbf{x}'d\mathbf{k}')[v\hat{k}_n^+ a_0 + vk_n^+ \hat{k}_n^+ a_1] \\ =& \int \nu_\alpha(d\mathbf{x}'d\mathbf{k}') a_+(\mathbf{x}', 0, \mathbf{k}', k_n^-) v\hat{k}_n^- \\ & + \int \nu_\beta(d\mathbf{x}'d\mathbf{k}') a_+(\mathbf{x}', 0, \mathbf{k}', k_n^+) v\hat{k}_n^+. \end{aligned} \tag{4.58}$$

The cross term $\nu_{\alpha\beta}$ has dropped out because of (4.57). Combining (4.43) and (4.58), we obtain the weak form of the transport equation for the Wigner measure $\mu(\mathbf{x},\mathbf{k})$ in the upper half space

$$\nabla_{\mathbf{k}}\omega_+ \cdot \nabla_{\mathbf{x}}\mu - \nabla_{\mathbf{x}}\omega_+ \cdot \nabla_{\mathbf{k}}\mu = v\hat{k}_n(\nu_\alpha \delta(k_n - k_n^-) + \nu_\beta \delta(k_n - k_n^+))\delta(x^n) \tag{4.59}$$

The idea of using the representation (4.50) for the test matrix a comes from a similar one used for the scalar wave equation by Gérard and Leichtman [6].

4.3.6 Form of the boundary Wigner measure

We prove now Lemma 1, which is independent of the results in Sections 4.3.2-4.3.5.

The function $\mathbf{r}_\varepsilon(\mathbf{x}')$ satisfies the system

$$\sum_{j=1}^{n-1} \varepsilon D^j \frac{\partial \mathbf{r}_\varepsilon}{\partial x^j} - i\omega A \mathbf{r}_\varepsilon = -\varepsilon D^n \frac{\partial \mathbf{w}_\varepsilon}{\partial x^n}\bigg|_{x^n=0}. \tag{4.60}$$

Thus, for any matrix P such that

$$PD^n = 0 \tag{4.61}$$

we have

$$\sum_{j=1}^{n-1} \varepsilon P D^j \frac{\partial \mathbf{r}_\varepsilon}{\partial x^j} - i\omega P A \mathbf{r}_\varepsilon = 0 \tag{4.62}$$

and hence

$$\sum_{j=1}^{n-1} P(k_j D^j - \omega A)\boldsymbol{\nu} = 0. \tag{4.63}$$

From the form of D^n

$$D^n = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{4.64}$$

and (4.62) we conclude that the first $n-1$ rows of the matrix $(\sum_{j=1}^{n-1} k_j D^j - \omega A)\boldsymbol{\nu}$ are zero. Such matrices form a $2(n+1)$-dimensional subspace of the space of $(n+1)\times(n+1)$ matrices.

We distinguish two cases. First, we assume that $|\mathbf{k}'| \neq \dfrac{\omega}{v}$. Then the matrix $(\sum_{j=1}^{n-1} k_j D^j - \omega A)$ is invertible and $\boldsymbol{\nu}$ belongs to the $2(n+1)$-dimensional subspace of matrices of the form $\mathbf{b}_+(\mathbf{k}^\pm)\mathbf{u}^*$ for some vector $\mathbf{u}$. Such matrices satisfy (4.63) because

$$(k_j' D^j - \omega A)\mathbf{b}_+(\mathbf{k}^\pm) = -k_n^\pm D^n \mathbf{b}_+(\mathbf{k}^\pm) \tag{4.65}$$

Therefore, since the Wigner matrix $\boldsymbol{\nu}$ is self-adjoint, it has the form (4.6)

$$\begin{aligned}\boldsymbol{\nu} &= \nu_\beta \mathbf{b}_+(\mathbf{k}^+)\mathbf{b}_+^*(\mathbf{k}^+) + \nu_{\beta\alpha}\mathbf{b}_+(\mathbf{k}^+)\mathbf{b}_+^*(\mathbf{k}^-) \\ &+ \nu_{\alpha\beta}\mathbf{b}_+(\mathbf{k}^-)\mathbf{b}_+^*(\mathbf{k}^+) + \nu_\alpha \mathbf{b}_+(\mathbf{k}^-)\mathbf{b}_+^*(\mathbf{k}^-),\end{aligned}$$

where all ν's are scalar valued measures and $\nu_{\beta\alpha} = (\nu_{\alpha\beta})^*$.

Consider now the case when $|\mathbf{k}'| = \frac{\omega}{v}$. Then $\mathbf{k}^+ = \mathbf{k}^-$ and the space of matrices of the form $\mathbf{b}_+(\mathbf{k}^\pm)\mathbf{u}^*$ is $(n+1)$-dimensional. There are two other self-adjoint matrices, in addition to $\mathbf{b}_+(\mathbf{k}^\pm)\mathbf{b}_+^*(\mathbf{k}^\pm)$, satisfying (4.63)

$$Q_1 = \mathbf{e}\mathbf{e}^*, \quad \text{where} \quad \mathbf{e} = (0, 0, \ldots, 1, 0) \tag{4.66}$$

and

$$Q_2 = \begin{bmatrix} 0 & 0 & \ldots & \xi_1 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \xi_1 & \xi_2 & \ldots & 0 & \omega\rho \\ 0 & 0 & \ldots & \omega\rho & 0 \end{bmatrix}. \tag{4.67}$$

However, both of these matrices, are non-zero only in the n-th column and row, and, since we assumed that the Wigner measure of $u_n(\mathbf{x}', 0)$ is not singular on the set $|\mathbf{k}'| = \frac{\omega}{v}$, they do not contribute to $\boldsymbol{\nu}$. This completes the proof of Lemma 1.

5 Basic properties of semiclassical operators

We derive some of the properties of the semiclassical operators $a(\mathbf{x}, \varepsilon D)$ stated in Section 2.3. Most of the proofs are either given or outlined in [6, 7]. We recall first Schur's lemma.

Lemma 2 *If $K = K(\mathbf{x}, \mathbf{y})$ is a continuous function on $R^d \times R^d$ and there exists C such that*

$$\int_{R^d} |K(\mathbf{x}, \mathbf{y})| d\mathbf{y} \leq C, \quad \int_{R^d} |K(\mathbf{x}, \mathbf{y})| d\mathbf{x} \leq C, \tag{5.1}$$

then the operator

$$Af(\mathbf{x}) = \int_{R^d} K(\mathbf{x}, \mathbf{y}) f(y) d\mathbf{y} \tag{5.2}$$

is bounded on L^2 and $||A||_{L^2} \leq C$.

The next Lemma proves the L^2-bound (2.11) and the H^s-estimates used in Section 4.3.

Lemma 3 *Let* $a(\mathbf{x},\mathbf{k}) \in \mathcal{S}(R^d_x \times R^d_k)$. *Then for* $s > 0$

$$||a(\mathbf{x},\varepsilon D)||_{L^2\to L^2} \le C(a) \tag{5.3}$$

$$\varepsilon^s\,||a(\mathbf{x},\varepsilon D)||_{H^{-s}\to L^2} \le C_s(a) \tag{5.4}$$

$$\varepsilon^s\,||a(\mathbf{x},\varepsilon D)||_{L^2\to H^s} \le C_s(a). \tag{5.5}$$

Proof. We have

$$a(\mathbf{x},\varepsilon D)f = \int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d}\hat{a}(\mathbf{x},\frac{\mathbf{y}-\mathbf{x}}{\varepsilon})f(\mathbf{y}), \tag{5.6}$$

where hat denotes the Fourier transform in $\mathbf{k}$. Since

$$\int \frac{d\mathbf{y}}{(2\pi\varepsilon)^d}\left|\hat{a}(\mathbf{x},\frac{\mathbf{y}-\mathbf{x}}{\varepsilon})\right| = \int \frac{d\mathbf{y}}{(2\pi)^d}\left|\hat{a}(\mathbf{x},\mathbf{y})\right| \tag{5.7}$$

and

$$\int \frac{d\mathbf{x}}{(2\pi\varepsilon)^d}\left|\hat{a}(\mathbf{x},\frac{\mathbf{y}-\mathbf{x}}{\varepsilon})\right| = \int \frac{d\mathbf{z}}{(2\pi)^d}\left|\hat{a}(\mathbf{y}+\varepsilon\mathbf{z},\mathbf{z})\right| \le \int \frac{d\mathbf{z}}{(2\pi)^d}\sup_{\mathbf{x}}\left|\hat{a}(\mathbf{x},\mathbf{z})\right|,$$

the estimate (5.3) follows by Lemma 2. To show that (5.4) holds, we define a function $g(\mathbf{x})$ by its Fourier transform

$$\hat{g}(\mathbf{k}) = \frac{\hat{f}(\mathbf{k})}{(1+|\mathbf{k}|^2)^{s/2}}, \tag{5.8}$$

so that $||f||_{H^{-s}} = ||g||_{L^2}$. Then

$$\begin{aligned}\varepsilon^s a(\mathbf{x},\varepsilon D)f &= \varepsilon^s\int \frac{d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{x}}a(\mathbf{x},\varepsilon\mathbf{k})(1+|\mathbf{k}|^2)^{s/2}\hat{g}(\mathbf{k}) \\ &= \int \frac{d\mathbf{k}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{x}}a(\mathbf{x},\varepsilon\mathbf{k})(\varepsilon^2+\varepsilon^2|\mathbf{k}|^2)^{s/2}\hat{g}(\mathbf{k}),\end{aligned} \tag{5.9}$$

and (5.4) follows by Schur's Lemma. The estimate (5.5) for integer s is obtained by differentiating (2.14) s times with respect to $\mathbf{x}$.

Lemma 4 *The Weyl operators $a^w(\mathbf{x}, \varepsilon D)$ and the semiclassical operators $a(\mathbf{x}, \varepsilon D)$ are asymptotically equivalent, that is*

$$||a(\mathbf{x}, \varepsilon D) - a^w(\mathbf{x}, \varepsilon D)||_{L^2 \to L^2} \to 0 \tag{5.10}$$

as $\varepsilon \to 0$.

Proof. Given a function $f \in L^2$ we have

$$\begin{aligned} a(\mathbf{x}, \varepsilon D)f &= \int \frac{d\mathbf{y}}{\varepsilon^d} \tilde{a}(\mathbf{x}, \frac{\mathbf{x} - \mathbf{y}}{\varepsilon}) f(\mathbf{y}) \\ a^w(\mathbf{x}, \varepsilon D)f &= \int \frac{d\mathbf{y}}{\varepsilon^d} \tilde{a}(\frac{\mathbf{x} + \mathbf{y}}{2}, \frac{\mathbf{x} - \mathbf{y}}{\varepsilon}) f(\mathbf{y}) \end{aligned} \tag{5.11}$$

and thus

$$(a^w(\mathbf{x}, \varepsilon D) - a(\mathbf{x}, \varepsilon D))f = \int \frac{d\mathbf{y}}{\varepsilon^d} \{\tilde{a}(\frac{\mathbf{x} + \mathbf{y}}{2}, \frac{\mathbf{x} - \mathbf{y}}{\varepsilon}) - \tilde{a}(\mathbf{x}, \frac{\mathbf{x} - \mathbf{y}}{\varepsilon})\} f(y).$$

Then (5.10) follows by Schur's Lemma and the dominated convergence theorem.

We now show that the product of two semiclassical operators corresponds to the operator which is nearly the product of their symbols.

Lemma 5 *The product of two operators $a(\mathbf{x}, \varepsilon D)$, $b(\mathbf{x}, \varepsilon D)$ is*

$$b(\mathbf{x}, \varepsilon D)a(\mathbf{x}, \varepsilon D) = (ba)(\mathbf{x}, \varepsilon D) + \frac{\varepsilon}{i}(\nabla_{\mathbf{k}} b \cdot \nabla_{\mathbf{x}} a)(\mathbf{x}, \varepsilon D) + \varepsilon^2 Q_\varepsilon, \tag{5.12}$$

where the operators Q_ε are uniformly bounded on L^2.

Proof. We have

$$b(\mathbf{x}, \varepsilon D)a(\mathbf{x}, \varepsilon D)f = \iint \frac{d\mathbf{p} d\mathbf{q} e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{q})}}{(2\pi)^{2d}} b(\mathbf{x}, \varepsilon\mathbf{p} + \varepsilon\mathbf{q})\hat{a}(\mathbf{p}, \varepsilon\mathbf{q})\hat{f}(\mathbf{q}), \tag{5.13}$$

where hat denotes the Fourier transform in $\mathbf{x}$ now. Then (5.12) follows by expanding (5.13) in the powers of ε. A more detailed proof is given in the Appendix to [7].

ACKNOWLEDGEMENTS

We are grateful to Luc Miller for many useful discussions and for bringing to our attention the papers [6, 7].

This work was partially sponsored by a grant from the NSF, DMS 9308471, and by the Air Force Office of Scientific Research, Air Force Materials Command, USAF, under grant number F49620-95-1-0315. The US Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the US Government.

References

[1] Yu.Barabanenkov, A.Vinogradov, Yu.Kravtsov and V.Tatarskii, Application of the theory of multiple scattering of waves to the derivation of the radiative transfer equation for a statistically inhomogeneous medium, Radiofizika, **15** 1972, 1852-1860. English translation pp. 1420-1425.

[2] N. Bleistein, Mathematical Methods for Wave Phenomena, Academic Press, 1984.

[3] G. Frankfort, F. Murat, Oscillations and energy densities in the wave equation, Comm. PDEs, **17**, 1992, 1785-1865.

[4] P. Gérard, Microlocal defect measures, Comm. PDEs, **16**, 1991, 1761-1794.

[5] P. Gérard, Mesures Semi-Classiques et Ondes de Bloch, Sém. Ecole Polytechnique, exposé **XVI**, 1990-91, 1-19.

[6] P.Gérard, E.Leichtnam, Ergodic properties of eigenfunctions for the Dirichlet problem, Duke Math. J., **71**, 1993, 559-607.

[7] P.Gérard, P.Markovich, N.Mauser and F.Poupaud, Homogenization limits and Wigner transforms, Comm.Pure Appl. Math., **50**, 1997, 323-380.

[8] J.B.Keller and R.Lewis, Asymptotic methods for partial differential equations: The reduced wave equation and Maxwell's equations, in Surveys in applied mathematics, eds. J.B.Keller, D.McLaughlin and G.Papanicolaou, Plenum Press, New York, 1995.

[9] C.W.Law and K.Watson, Radiation transport along curved ray paths, Jour. Math. Phys., **11**, 1970, 3125-3137.

[10] P.L. Lions and T. Paul, Sur les Mesures de Wigner, Revista Mat. Iberoamericana, **9**, 1993, 553-618.

[11] L.Miller, Réfraction d'ondes semi-classiques par des interfaces franches, C. R. Acad. Sci. Paris, Série I, 1997, to appear.

[12] L. Miller, Short Waves Through Thin Interfaces and 2-microclocal measures, Journées Equations aux Dérivées Partielles, Saint-Jean-de-Monts, 1997.

[13] L. Miller, Propagation d'ondes semi-classiques à travers une interface et mesures 2-microlocales, Doctorat de l'École Polytechnique, Palaiseau, 1996.

[14] P.Sheng Introduction to wave scattering, localization, and mesoscopic phenomena, Academic Press, San Diego, 1995.

[15] L.Ryzhik, G.Papanicolaou and J.Keller, Transport equations for elastic and other waves in random media, Wave Motion, **24**, 1996, 327-370.

[16] L.Ryzhik, G.Papanicolaou and J.Keller, Boundary conditions for transport equations for waves in random media, in preparation.

[17] L. Schwartz, Théorie des Distributions, Herman, Paris, 1966.

[18] L. Tartar, H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh, **115A**, 1990, 193-230.

[19] E.Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev., **40**, 1932, 749-759.